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# The structure of fractional spaces generated by multi-dimensional elliptic difference operator

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## Abstract

In the present paper, the positivity of a multi-dimensional difference operator in the half-space is considered. The structure of fractional spaces generated by this operator is investigated. The equivalence of the norms of these fractional spaces and Hölder spaces is proved. In applications, the stability of difference schemes for elliptic differential equations is presented.

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**Keywords:** positive operator; fractional spaces; Green's function; Hölder spaces

## 1 Introduction

The role played by the positivity of differential and difference operators in Banach spaces in the study of the stability of solutions of boundary value problems for partial differential equations and the stability of difference schemes for partial differential equations, and of summation Fourier series converging in $C$-norm is well known (see [1–3]).

Recall that an operator $A$ densely defined in a Banach space $E$ with domain $D(A)$ is called positive in $E$, if its spectrum $\sigma_A$ lies in the interior of the sector of angle $\varphi$, $0 < \varphi < \pi$, symmetric with respect to the real axis, and, moreover, on the edges of this sector $S_1(\varphi) = \{\rho e^{i\varphi} : 0 \le \rho \le \infty\}$ and $S_2(\varphi) = \{\rho e^{-i\varphi} : 0 \le \rho \le \infty\}$, and outside of the sector the resolvent $(A - \lambda I)^{-1}$ is subject to the bound (see [2])

$$\left\| (A - \lambda I)^{-1} \right\|_{E \to E} \le \frac{M}{1 + |\lambda|},$$

where $I$ is the identity operator. The infimum of all such angles $\varphi$ is called the spectral angle of the positive operator $A$ and is denoted by $\varphi(A) = \varphi(E, A)$.

Throughout the present work, we will indicate with $M$ positive constants which can be different from time to time and we are not interested in precise. We will write $M(\alpha, \beta, \dots)$ to express the fact that the constant depends only on $\alpha, \beta, \dots$.

The theory of differential and difference operators in Banach spaces and their related applications have been investigated by many scientists (see, for example, [4–30]).

Great progress has been made in the study of structure of interpolation spaces generated by positive operators from the view-point of the stability analysis of high-order accuracy

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difference schemes for various partial differential equations. However, the positivity of difference operators and structure of fractional spaces generated by these operators in Banach spaces are not well investigated in general. Therefore, the investigation of structure of fractional spaces generated by positive difference operators in Banach spaces and its applications to the stability of difference schemes for various partial differential equations is an important subject.

For a positive operator $A$ in the Banach space $E$, let us introduce the fractional spaces $E_\alpha = E_\alpha(E, A)$ $(0 < \alpha < 1)$ consisting of those $v \in E$ for which the norms

$$\|v\|_{E_\alpha} = \sup_{\lambda > 0} \lambda^\alpha \|A(A + \lambda I)^{-1} v\|_E$$

are finite. Clearly, the positive operator commutes $A$ and its resolvent $(A - \lambda I)^{-1}$. By the definition of the norm in the fractional space $E_\alpha = E_\alpha(E, A)$ $(0 < \alpha < 1)$, we get

$$\|(A - \lambda I)^{-1}\|_{E_\alpha \to E_\alpha} \le \|(A - \lambda I)^{-1}\|_{E \to E}. \tag{1}$$

Thus, from the positivity of operator $A$ in the Banach space $E$ follows the positivity of this operator in fractional spaces $E_\alpha = E_\alpha(E, A)$ $(0 < \alpha < 1)$.

In [8], Simirnitskii and Sobolevskii considered the difference operator $A_{1h}^x$ which is an elliptic difference operator of an arbitrary high order of accuracy approximating the multi-dimensional elliptic operator $A_1^x = B^x + \delta I$.

Let us define the grid space $\mathbb{R}_h^{n-1}$ $(0 < h \le h_0)$ as the set of all points of the Euclidean space $\mathbb{R}^{n-1}$ whose coordinates are given by

$$x_k = s_k h, \quad s_k = 0, \pm 1, \pm 2, \ldots, k = 1, \ldots, n-1.$$

The number $h$ is called the step of the grid space. A function defined on $\mathbb{R}_h^{n-1}$ will be called a grid function. To the differential operator $B$ with constant coefficients of the form

$$B^x = \sum_{|r|=2m} b_r \frac{\partial^{r_1 + \cdots + r_{n-1}}}{\partial_{x_1}^{r_1} \cdots \partial_{x_{n-1}}^{r_{n-1}}},$$

we assign the difference operator

$$B_h^x = h^{-m} \sum_{2m \le |s| \le S} d_s \Delta_{1-}^{s_1} \Delta_{1+}^{s_2} \cdots \Delta_{(n-1)-}^{s_{2n-3}} \Delta_{(n-1)+}^{s_{2n-2}}, \tag{2}$$

which acts on functions defined on the entire space $\mathbb{R}_h^{n-1}$. Here, $s \in \mathbb{R}^{2(n-1)}$ is a vector with nonnegative integer coordinates,

$$\Delta_{k\pm} f^h(x) = \pm(f^h(x \pm e_k h) - f^h(x)),$$

and $e_k$ is the unit vector of the axis $x_k$.

An infinitely differentiable function of the continuous argument $y \in \mathbb{R}^{n-1}$ that is continuous and bounded together with all its derivatives is said to be smooth. Let $\varphi(y)$ be a

smooth function on $\mathbb{R}^{n-1}$. Using the Taylor expansion of $\varphi(y)$, one can show that

$$\sup_{x \in \mathbb{R}_h^{n-1}} \left| h^{-1} \Delta_{k\pm} \varphi(x) - \frac{\partial}{\partial y_k} \varphi(x) \right| \leq M(\varphi)h.$$

Here, the grid function $\varphi(x)$ and $\frac{\partial}{\partial y_k} \varphi(x)$ are the traces of the functions $\varphi(y)$ and $\frac{\partial}{\partial y_k} \varphi(y)$, respectively. The last inequality means that the difference operator $h^{-1} \Delta_{k\pm}$ is a first-order approximation for the differential operator $\frac{\partial}{\partial y_k}$.

We say that the difference operator $B_h^x$ is a $\lambda$th-order ($\lambda > 0$) approximation of the differential operator $B^x$ if the inequality

$$\sup_{x \in R_h^{n-1}} \left| B_h^x \varphi(x) - B^x \varphi(x) \right| \leq M(\varphi)h^\lambda$$

holds for any smooth function $\varphi(y)$. We shall assume that the operator $B_h^x$ approximates the differential operator $B^x$ with any prescribed order.

A function of a continuous [resp., discrete] argument that decays at infinity faster than any negative power of $|y|$ [resp., $|x|$] is said to be rapidly decreasing. Let us define the Fourier transform of a grid function $f^h(x)$ by the formula

$$\tilde{f}(\xi) = (2\pi)^{-(n-1)} \sum_{x \in R_h^{n-1}} \exp\{-i(x,\xi)\} f^h(x) h^{n-1}, \quad \xi \in \mathbb{R}^{n-1}. \tag{3}$$

This formula defines a $2\pi h^{-1}$-periodic smooth function of the continuous argument $\xi$ whenever $f^h(x)$ is a rapidly decreasing grid function. In this last case, (3) is just a Fourier series expansion of the function $\tilde{f}(\xi)$ and the numbers $f^h(x)$ are the Fourier coefficients, given by the formula

$$f^h(x) = \int_{|\xi_1| \leq \pi h^{-1}} \cdots \int_{|\xi_{n-1}| \leq \pi h^{-1}} \exp\{i(x,\xi)\} \tilde{f}(\xi) \, d\xi_1 \cdots d\xi_{n-1}. \tag{4}$$

The inverse Fourier transform of a $2\pi h^{-1}$-periodic function $\varphi(\xi)$ is defined to be the grid function $\tilde{\varphi}^h(x)$ given by the formula

$$\tilde{\varphi}^h(x) = \int_{|\xi_1| \leq \pi h^{-1}} \cdots \int_{|\xi_{n-1}| \leq \pi h^{-1}} \exp\{i(x,\xi)\} \varphi(\xi) \, d\xi_1 \cdots d\xi_{n-1}. \tag{5}$$

Equations (4) and (5) establish a one-to-one correspondence between rapidly decreasing grid functions of a continuous argument. In particular, if $f^h(x)$ is a rapidly decreasing grid function, then

$$\widehat{[\tilde{f}]}^h (x) = f^h(x).$$

If $f^h(x)$ is a rapidly decreasing grid function, then the grid function $B_h^x f^h(x)$ exists and is given by (3) and we have the equality

$$\widetilde{B^x_h f}(\xi) = B(\xi h, h) \tilde{f}(\xi).$$

The function $B(\xi h, h)$ is obtained by replacing the operator $\Delta_{k\pm}$ in the right-hand side of equality (2) with the expression $\pm(\exp\{\pm i\xi_k h\} - 1)$, respectively, and is called the symbol of the difference operator. Since $\exp\{\pm i\xi_k h\}$ is a bounded analytic $2\pi h^{-1}$-periodic function, the symbol $B(\xi h, h)$ is a bounded analytic $2\pi h^{-1}$-periodic function. It follows that for large $|\xi|$ one has the estimate

$$\left|B(\xi h, h)\right| \le M(h)|\xi|^m, \qquad |\xi|^2 = |\xi_1|^2 + \cdots + |\xi_{n-1}|^2.$$

Let us give the difference operator $A_{1h}^x$ by the formula

$$A_{1h}^x u^h(x) = \sum_{2m \le |r| \le S} a_r^x D_h^r u^h(x) + \delta u^h(x). \tag{6}$$

The coefficients are chosen in such a way that the operator $A_{1h}^x$ approximates in a specified way the operator

$$\sum_{|r|=2m} a_r(x) \frac{\partial^{|r|}}{\partial x_1^{r_1} \cdots \partial x_{n-1}^{r_{n-1}}} + \delta.$$

We shall assume that for $|\xi_k h| \le \pi$ and fixed $x$ the symbol $A_1^x(\xi h, h)$ of the operator $A_{1h}^x - \delta$ satisfies the inequalities

$$(-1)^m A_1^x(\xi h, h) \ge M_1 |\xi|^{2m}, \qquad \left|\arg A_1^x(\xi h, h)\right| \le \phi < \phi_0 \le \frac{\pi}{2}.$$

In [12], Danelich considered the difference elliptic operator $A_h^x$ which is to an arbitrary high order of accuracy approximating the multi-dimensional elliptic operator $A^x$ defined by

$$A^x = \sum_{|r|=2m} a_r(x) \frac{\partial^{|r|}}{\partial x_1^{r_1} \cdots \partial x_{n-1}^{r_{n-1}}} + (-1)^m a(x) \frac{\partial^{2m}}{\partial x_n^{2m}} + \delta I \tag{7}$$

with the domain

$$D(A^x) = \left\{ u \in C^{2m}(\mathbb{R}_n^+) : u(x)|_{x_n=0} = \left.\frac{\partial u(x)}{\partial N}\right|_{x_n=0} = \cdots = \left.\frac{\partial^{m-1} u(x)}{\partial N^{m-1}}\right|_{x_n=0} = 0 \right\}.$$

Here, $\delta > 0$ is the sufficiently large number and $a(x)$ is a continuous function defined on $\mathbb{R}^+ = \{x : x \ge 0\}$ with

$$0 < s_1 \le a(x) \le s_2 < \infty. \tag{8}$$

Let us define the grid space $\mathbb{R}_{hn}^+$ $(0 < h \le h_0)$ as the set of all points of the space $\mathbb{R}_n^+$ whose coordinates are given by

$$x = (j_1 h, \ldots, j_n h) \in \mathbb{R}_n^+, \qquad j_k \in \mathbb{Z}, \quad k = \overline{1,n}, j_n = 0, 1, 2, \ldots.$$

The number $h$ is called the step of the grid space. A function $\varphi^h(x)$ defined on $\mathbb{R}_{hn}^+$ will be called a grid function. To the differential operator $A^x$ defined by (7), we assign the difference operator

$$A_h^x \varphi^h(x) = A_{1h}^x \varphi^h(x) + A_{2h}^x \varphi^h(x) + \delta \varphi^h(x), \quad x \in \mathbb{R}_{hn}^+,$$

$$A_{1h}^x \varphi^h(x) = h^{-2m} \sum_{2m \le |p| \le S} a_r^x \Delta_{1-}^{p_1} \Delta_{1+}^{p_2} \cdots \Delta_{(n-1)-}^{p_{2n-3}} \Delta_{(n-1)+}^{p_{2n-2}} \varphi^h(x), \quad x \in \mathbb{R}_{hn}^+, \qquad (9)$$

$$A_{2h}^x \varphi^h(x) = (-1)^m h^{-2m} \sum_{\substack{2m \le r+s \le S \\ s \le m}} a_{r,s}^x \Delta_{n+}^r \Delta_{n-}^s \varphi^h(x), \quad x \in \mathbb{R}_{hn}^+,$$

which acts on functions defined on the entire space $\mathbb{R}_{hn}^+$. Here, $p \in \mathbb{R}_{2n-2}^+$ is a vector with nonnegative integers coordinates. The coefficients $a_r^x$ are chosen in such a way that the operator $A_{1h}^x$ approximates in a specified way the operator

$$\sum_{|r|=2m} a_r(x) \frac{\partial^{|r|}}{\partial x_1^{r_1} \cdots \partial x_{n-1}^{r_{n-1}}}.$$

We shall assume that for $|\xi_k h| \le \pi$ and fixed $x$ the symbol $A_1^x(\xi h, h)$ of the operator $A_{1h}^x$ satisfies the inequalities

$$(-1)^m A_1^x(\xi' h, h) \ge M_1 |\xi'|^{2m},$$
$$|\arg A_1^x(\xi' h, h)| \le \phi < \phi_0 < \pi, \quad \xi' = (\xi_1, \ldots, \xi_{n-1}). \qquad (10)$$

For the definition of the operator $A_{2h}^x$, we will define $\varphi^h(x)$ which is extended to $\overline{\varphi}^h(x)$ defined on $\mathbb{R}_{hn}^+$ and additionally on the points

$$(x', x_k) = (x', kh) \notin \mathbb{R}_{hn}^+, \quad x' \in \mathbb{R}_{h(n-1)}, k = -1, -2, \ldots, -(m-1),$$

and

$$\varphi^h(x', 0) = 0, \qquad h^{-k} \sum_{-k \le s \le s_k} \alpha_{s,k} \overline{\varphi}^h(x', sh) = 0, \quad k = \overline{1, m-1}, \alpha_{-k,k} \neq 0. \qquad (11)$$

The coefficients $\alpha_{s,k}$ are chosen in such a way that the expression

$$h^{-k} \sum_{-k \le s \le s_k} \alpha_{s,k} \overline{\varphi}^h(x', sh)$$

approximates in a specified way the expression $\varphi^{(k)}(x', 0)$. The coefficients $a_{r,s}^x$ are chosen in such a way that the operator $h^{-k} \sum_{-k \le s \le s_k} \alpha_{s,k} \overline{\varphi}^h(x', sh)$ approximates in a specified way the operator $A_2^x$. We shall assume that the operator $A_{2h}^x$ approximates the differential operator $A_2^x$ with any prescribed order.

In the present paper, a Green's function is assigned. The organization of the present paper is as follows. In Section 3, the main theorem on the structure of fractional spaces $E_\alpha(\dot{C}_h(\mathbb{R}_{hn}^+), A_h^x)$ generated by $A_h^x$ is investigated. In Section 4, applications on theorems on well-posedness in a Hölder space of parabolic and elliptic problems are presented. Finally, the conclusion is given.

## 2 Green's function

**Theorem 1** [12] *Suppose that assumptions* (8). *Let the function $a(x)$ satisfy the estimate*

$$\left| a(x) - a(y) \right| \leq s_3 |x - y|^\alpha \quad \left( 0 < \alpha \leq 1; x, y \in \mathbb{R}_{h1}^+ \right). \tag{12}$$

*Then, for sufficiently large positive $\delta$ and $\operatorname{Re} \lambda \geq 0$, there exists a unique solution of the resolvent equation*

$$A_h^x u^h(x) + \lambda u^h(x) = f^h(x), \quad x \in \mathbb{R}_{hn}^+ \tag{13}$$

*and the following formula*:

$$f^h(y) h^n \left( A_h^x + \lambda I \right)^{-1} f^h(x) = \sum_{y \in \mathbb{R}_{hn}^+} G_h(x, y; \lambda) \tag{14}$$

*holds. Here, $G_h(x, y; \lambda)$ is the Green's function of the resolvent equation* (13).

**Lemma 1** *The following identities hold*:

$$\sum_{y \in \mathbb{R}_{hn}^+} G_h(0, y; \lambda) h^n = 0, \tag{15}$$

$$\sum_{y \in \mathbb{R}_{hn}^+} G_h(x, y; \lambda) h^n = \frac{1}{\delta + \lambda} \left( 1 - v^h(x) \right), \quad x_n > h, x \in \mathbb{R}_{hn}^+, \tag{16}$$

*where $v^h(x)$ is the solution of the following problem*:

$$\begin{cases} h^{-2m} \sum_{2m \leq |p| \leq S} a_r^x \Delta_{1-}^{p_1} \Delta_{1+}^{p_2} \cdots \Delta_{(n-1)-}^{p_{2n-3}} \Delta_{(n-1)+}^{p_{2n-2}} v^h(x) \\ \quad + (-1)^m h^{-2m} \sum_{\substack{2m \leq r+s \leq S \\ s \leq m}} a_{r,s}^x \Delta_+^r \Delta_-^s v^h(x) + (\delta + \lambda) v^h(x) = 0, \quad x \in \mathbb{R}_{hn}^+, \\ v^h(x', 0) = 1, \qquad h^{-k} \sum_{-k \leq s \leq s_k} \alpha_{s,k} \overline{v}^h(x', sh) = 0, \quad k = \overline{1, m-1}, \alpha_{-k,k} \neq 0, \\ x' \in \mathbb{R}_{h(n-1)}. \end{cases} \tag{17}$$

*Proof* We consider the problem

$$\begin{cases} h^{-2m} \sum_{2m \leq |p| \leq S} a_r^x \Delta_{1-}^{p_1} \Delta_{1+}^{p_2} \cdots \Delta_{(n-1)-}^{p_{2n-3}} \Delta_{(n-1)+}^{p_{2n-2}} v^h(x) \\ \quad + (-1)^m h^{-2m} \sum_{\substack{2m \leq r+s \leq S \\ s \leq m}} a_{r,s} \Delta_+^r \Delta_-^s u^h(x) + (\delta + \lambda) u^h(x) = f^h(x), \quad x \in \mathbb{R}_{h1}^+, \\ u^h(x', 0) = 1, \qquad h^{-k} \sum_{-k \leq s \leq s_k} \alpha_{s,k} \overline{u}^h(x', sh) = 0, \quad k = \overline{1, m-1}, \alpha_{-k,k} \neq 0, \\ x' \in \mathbb{R}_{h(n-1)}. \end{cases}$$

We have

$$u^h(x) = \sum_{y \in \mathbb{R}_{hn}^+} G_h(x, y; \lambda) f^h(y) h^n + v^h(x) \frac{1}{\delta + \lambda}, \quad x \in \mathbb{R}_{hn}^+.$$

Since for $f^h(x) = 1$, $u^h(x) = \frac{1}{\delta + \lambda}$ is the solution of (13), we get

$$\frac{1}{\delta + \lambda} = \sum_{y \in \mathbb{R}_{hn}^+} G_h(x, y; \lambda) h^n + v^h(x) \frac{1}{\delta + \lambda}, \quad x \in \mathbb{R}_{hn}^+.$$

Then

$$\sum_{y \in \mathbb{R}_{hn}^+} G_h(x, y; \lambda) h^n = \frac{1}{\delta + \lambda} \big(1 - v^h(x)\big), \quad x \in \mathbb{R}_{hn}^+.$$

Identity (16) is proved. Since $v^h(0) = 1$, we have

$$\frac{1}{\delta + \lambda} = \sum_{y \in \mathbb{R}_{hn}^+} G_h(0, y; \lambda) h^n + \frac{1}{\delta + \lambda}.$$

Thus follows (15). Lemma 1 is proved. □

For any $x_0 \in \mathbb{R}_{hn}^+$, we have

$$G_h(x, y; \lambda) = G_h^{x_0}(x, y; \lambda) + \sum_{y \in \mathbb{R}_{hn}^+} G_h(x, z; \lambda) \big[A_h^{x_0} - A_h^z\big] G_h^{x_0}(z, y; \lambda) h^n, \tag{18}$$

where $G_h^{x_0}(x, y; \lambda)$ is the Green's function of the operator $A_h^{x_0}$.

Moreover, applying (8), (10), (11), and (12) conditions, one proved the following pointwise estimates of the Green's function $G_h(x, y; \lambda)$ of the resolvent equation and its difference derivatives:

$$\big|G_h(x, y; \lambda)\big| \leq M_1 \exp\big\{-b|\lambda + \delta|^p |x - y|\big\} |\lambda + \delta|^{np-1} \tag{19}$$

for $2m > n$,

$$\big|G_h(x, y; \lambda)\big| \leq M_1 \exp\big\{-b|\lambda + \delta|^p |x - y|\big\} \big[1 + \ln\big\{\big(|x - y||\lambda + \delta|^p\big)^{-1} + 1\big\}\big] \tag{20}$$

for $2m = n$,

$$\big|G_h(x, y; \lambda)\big| \leq M_1 \exp\big\{-b|\lambda + \delta|^p |x - y|\big\} |x - y|^{2m-n} \tag{21}$$

for $2m < n$, and its derivatives respect to $x$

$$\big|D_h^r G_h(x, y; \lambda)\big| \leq M_1 \exp\big\{-b|\lambda + \delta|^p |x - y|\big\} |\lambda + \delta|^{(n+r)p-1} \tag{22}$$

for $2m - r > n$,

$$\big|D_h^r G_h(x, y; \lambda)\big| \leq M_1 \exp\big\{-b|\lambda + \delta|^p |x - y|\big\} \big[1 + \ln\big\{\big(|x - y||\lambda + \delta|^p\big)^{-1} + 1\big\}\big] \tag{23}$$

for $2m - r = n$,

$$\big|D_h^r G_h(x, y; \lambda)\big| \leq M_1 \exp\big\{-b|\lambda + \delta|^p |x - y|\big\} |x - y|^{2m-n-r} \tag{24}$$

for $2m - r < n$, where $p = \frac{1}{2m}$, $a > 0$, $M_1 > 0$ for $0 < |\lambda| \leq Lh^{-2m}$, $\lambda \in Q_\psi \cup \{0\}$, and

$$\big|G_h(x, y; \lambda)\big| \leq M_1 \exp\big\{-bh^{-1}|x - y|\big\} h^{2m-n} \big(|\delta + \lambda|h^{2m} + 1\big)^{-1}, \tag{25}$$

$$\left|D_h^r G_h(x, y; \lambda)\right| \le M_1 \exp\{-bh^{-1}|x - y|\} h^{2m-n-r} \left(|\delta + \lambda|h^{2m} + 1\right)^{-1} \tag{26}$$

for $|\lambda| > Lh^{-2m}$. Here $p = \frac{1}{2m}$, $b > 0$, $M_1 > 0$.

Note that under assumptions (8) and (10) there exists a unique solution $v^h(x)$ of problem (17) and the following estimate holds:

$$\left|v^h(x)\right| \le M_1 \exp\{-b|\lambda + \delta|^p|x|\}, \tag{27}$$

for $0 < |\lambda| \le Lh^{-2m}$, $\lambda \in Q_\psi \cup \{0\}$ and

$$\left|v^h(x)\right| \le M_1 \exp\{-bh^{-1}|x|\} \left(|\delta + \lambda|h^{2m} + 1\right)^{-1}, \tag{28}$$

for $|\lambda| > Lh^{-2m}$.

**Lemma 2** [2] $|\lambda| > Lh^{-2m}$, *where* $L > 0$ *large enough. Then*

$$\left|G_h(x, y; \lambda)\right| \le M_1 \exp\{-bh^{-1}|x - y|\} h^{2m} \left(|\delta + \lambda|h^{2m} + 1\right)^{-2} |x - y|^{-n}, \quad x \neq y. \tag{29}$$

*By* (25) *and* (29), *we obtain the estimate*

$$\begin{aligned}
\left|G_h(x, y; \lambda)\right| \\
&\le M_1 \exp\{-bh^{-1}|x - y|\} h^{-1+2m(1+\mu)} \\
&\quad \times \left(|\delta + \lambda|h^{2m} + 1\right)^{-(1+2m\mu)} |x - y|^{-2m\mu}, \quad x \neq y.
\end{aligned} \tag{30}$$

Moreover, applying the Green's function of $A_h^x$ the following results were proved.

**Theorem 2** [12] $A_h^x$ *is a positive operator in the space* $\dot{C}(\mathbb{R}_{hn}^+)$ *of all mesh functions* $\varphi^h(x)$ *defined on* $\mathbb{R}_{hn}^+$ *with the norm*

$$\left\|\varphi^h\right\|_{\dot{C}_h} = \sup_{x \in \mathbb{R}_{hn}^+} \left|\varphi^h(x)\right|$$

*with the spectral angle* $\varphi(A_h^x, \dot{C}_h) = \pi - \psi$.

Let $C_h^\beta = C^\beta(\mathbb{R}_{hn}^+)$ be the Hölder space of all mesh functions $\varphi^h(x)$ defined on $\mathbb{R}_{hn}^+$ satisfying a Hölder condition with the indicator $\beta \in (0, 1)$ with the norm

$$\left\|\varphi^h\right\|_{C_h^\beta} = \left\|\varphi^h\right\|_{C_h} + \sup_{\substack{x, y \in \mathbb{R}_{hn}^+ \\ x \neq y}} \frac{|\varphi^h(x) - \varphi^h(y)|}{|x - y|^\beta}.$$

One proved the strong positivity of $A_h^x$ in the Banach space $\dot{C}_h = \dot{C}(\mathbb{R}_{hn}^+)$ (difference analog of $\dot{C}(\mathbb{R}_n^+)$) for sufficiently large positive $\delta$. Passing to limit when $h \to 0$, we can get the strong positivity of differential operator $A^x$ in the Banach space $\dot{C}(\mathbb{R}_n^+)$.

## 3 The structure of fractional spaces $E_\alpha(\dot{C}_h(\mathbb{R}_{hn}^+), A_h^x)$

**Theorem 3** *Suppose $2m\alpha \in (0, 1)$. Then the norms of the spaces $E_\alpha(\dot{C}_h, A_h^x)$ and $\dot{C}^{2m\alpha}(\mathbb{R}_{hn}^+)$ are equivalent. Here,*

$$\dot{C}_h^{2m\alpha} = \left\{ \varphi^h : \varphi^h \in C_h^{2m\alpha}, \varphi^h(x', 0) = 0, h^{-k} \sum_{-k \le s \le s_k} \alpha_{s,k} \overline{\varphi}^h(x', sh) = 0, \right.$$

$$\left. k = \overline{1, m-1}, \alpha_{-k,k} \ne 0, x' \in \mathbb{R}_{h(n-1)} \right\}.$$

*Proof* Assume that $f^h \in \dot{C}^{2m\alpha}(\mathbb{R}_{hn}^+)$. Let $x \in \mathbb{R}_{hn}^+$ and $\lambda > 0$ be fixed. Using equation (14) and identity (16), we can write

$$A_h^x(A_h^x + \lambda)^{-1} f^h(x)$$

$$= \frac{\delta}{\lambda + \delta} f^h(x) + \lambda \sum_{y \in \mathbb{R}_n^+} G_h(x, y; \lambda) \big[ f^h(x) - f^h(y) \big] h^n$$

$$+ \frac{\lambda}{\lambda + \delta} v^h(x) \big( f^h(x) - f^h(x_1, \dots, x_{n-1}, 0) \big) + \frac{\lambda}{\lambda + \delta} v^h(x) f^h(x_1, \dots, x_{n-1}, 0).$$

Since $f^h(x_1, \dots, x_{n-1}, 0) = 0$, we have

$$A_h^x(A_h^x + \lambda)^{-1} f^h(x) = \frac{\delta}{\lambda + \delta} f^h(x) + \lambda \sum_{y \in \mathbb{R}_n^+} G_h(x, y; \lambda) \big[ f^h(x) - f^h(y) \big] h^n$$

$$+ \frac{\lambda}{\lambda + \delta} v^h(x) \big( f^h(x) - f^h(x_1, \dots, x_{n-1}, 0) \big). \tag{31}$$

Using equation (31) and the triangle inequality, and the definition of the $\dot{C}^{2m\alpha}(\mathbb{R}_{hn}^+)$-norm, we have

$$\big| \lambda^\alpha A_h^x(A_h^x + \lambda)^{-1} f^h(x) \big|$$

$$\le \left[ \frac{\lambda^\alpha \delta}{\lambda + \delta} + \lambda^{\alpha+1} \sum_{y \in \mathbb{R}_n^+} \big| G_h(x, y; \lambda) \big| |x - y|^{2m\alpha} h^n + \frac{\lambda^{\alpha+1}}{\lambda + \delta} \big| v(x) \big| |x|^{2m\alpha} \right] \big\| f^h \big\|_{\dot{C}^{2m\alpha}(\mathbb{R}_{hn}^+)}$$

$$= [I_1 + I_2 + I_3] \big\| f^h \big\|_{\dot{C}^{2m\alpha}(\mathbb{R}_{hn}^+)}.$$

We will estimate $I_i$, $i = 1, 2, 3$, separately. First, let us estimate $I_1$. Clearly, using the estimate

$$\frac{\lambda^\alpha \delta^{1-\alpha}}{\lambda + \delta} \le 1$$

we get

$$I_1 \le \delta^\alpha \tag{32}$$

for any $\lambda > 0$.

We will consider two cases: $\lambda > Lh^{-2m}$ and $\lambda \leq Lh^{-2m}$. First, let $\lambda > Lh^{-2m}$. Using estimate (30), we have

$$
\begin{aligned}
I_2 &\leq M_1 \lambda^{\alpha+1} \sum_{y \in \mathbb{R}_n^+} e^{-bh^{-1}|x-y|} h^{-n+2m(1+\alpha)} \left((\lambda+\delta)h^{2m}+1\right)^{-(1+2m\alpha)} h^n \\
&\leq M_2 \frac{(\lambda h^{2m})^\alpha \lambda}{((\lambda+\delta)h^{2m})^{2m\alpha}(\lambda+\delta)} \leq M_3
\end{aligned}
\tag{33}
$$

for any $\lambda > 0$ and $x \in \mathbb{R}_{hn}^+$. From estimate (27) and the following inequality:

$$
(at)^\theta e^{-at} \leq M, \quad a > 0, \theta \in [0,1],
\tag{34}
$$

it follows that

$$
\begin{aligned}
I_3 &\leq M_1 \frac{\lambda^{\alpha+1}}{\lambda+\delta} e^{-ah^{-1}|x|} \left((\lambda+\delta)h^{2m}+1\right)^{-1} |x|^{2m\alpha} \\
&\leq M_2 \frac{\lambda^{\alpha+1} h^{2m\alpha}}{(\lambda+\delta)((\lambda+\delta)h^{2m}+1)} \leq M_3
\end{aligned}
\tag{35}
$$

for any $\lambda > 0$ and $x \in \mathbb{R}_{hn}^+$. Combining estimates (32)-(35), we get

$$
\left| \lambda^\alpha A_h^x \left(A_h^x + \lambda\right)^{-1} f^h(x) \right| \leq M(\alpha)
\tag{36}
$$

for any $\lambda > 0$ and $x \in \mathbb{R}_{hn}^+$.

Second, let $\lambda \leq Lh^{-2m}$. We consider three cases: $2m > n$, $2m = n$, and $2m < n$.

In the first case, using estimate (19), applying inequality (34), and the definition of the $\dot{C}^{2m\alpha}(\mathbb{R}_{hn}^+)$-norm, we have

$$
\begin{aligned}
I_2 &\leq M_1 \lambda^{\alpha+1} \sum_{y \in \mathbb{R}_{hn}^+} e^{-b(\lambda+\delta)^{\frac{1}{2m}}|x-y|} (\lambda+\delta)^{\frac{n}{2m}-1} \\
&\leq M_2 \frac{\lambda^{\alpha+1}}{(\lambda+\delta)^{1-\frac{n}{2m}+\alpha+\frac{\beta}{2m}+\frac{n}{2m}}} \leq M_3
\end{aligned}
\tag{37}
$$

for any $\lambda > 0$ and $x \in \mathbb{R}_{hn}^+$.

In the second case, applying estimate (20), and inequality (34), we have

$$
\begin{aligned}
I_2 &\leq M_1 \lambda^{\alpha+1} \sum_{y \in \mathbb{R}_{hn}^+} e^{-b(\lambda+\delta)^{\frac{1}{2m}}|x-y|} \left[1 + \ln\left\{\left(|x-y|(\lambda+\delta)^{\frac{1}{2m}}\right)^{-1}+1\right\}\right] |x-y|^{2m\alpha+\beta} h^n \\
&\leq M_2 \frac{\lambda^{\alpha+1}}{(\lambda+\delta)^{\alpha+\frac{n}{2m}+\frac{\beta}{2m}}} \leq M_3
\end{aligned}
\tag{38}
$$

for any $\lambda > 0$ and $x \in \mathbb{R}_{hn}^+$.

In the third case, using estimate (21) we obtain

$$
\begin{aligned}
I_2 &\leq M_1 \lambda^{\alpha+1} \sum_{y \in \mathbb{R}_{hn}^+} e^{-b(\lambda+\delta)^{\frac{1}{2m}}|x-y|} (\lambda+\delta)^{\frac{n}{2m}-1} |x-y|^{2m-n} |x-y|^{2m\alpha+\beta} h^n \\
&\leq M_2 \frac{\lambda^{\alpha+1}}{(\lambda+\delta)^{\frac{2m-n+2m\alpha+\beta}{2m}-\frac{n}{2m}}} \leq M_3
\end{aligned}
\tag{39}
$$

for any $\lambda > 0$ and $x \in \mathbb{R}_{hn}^+$. Estimates (37)-(39) result in

$$I_2 \leq M \tag{40}$$

for any $\lambda > 0$ and $x \in \mathbb{R}_{hn}^+$. From estimate (27) and inequality (34) it follows that

$$I_3 \leq M_1 \frac{\lambda^{\alpha+1}}{\lambda + \delta} e^{-b(\lambda+\delta)^{\frac{1}{2m}}|x|} |x|^{2m\alpha+\beta} \leq M_2 \tag{41}$$

for any $\lambda > 0$ and $x \in \mathbb{R}_{hn}^+$. Combining estimates (32), (40), and (41), we have

$$\left| \lambda^\alpha A_h^x (A_h^x + \lambda)^{-1} f^h(x) \right| \leq M(\alpha) \|f^h\|_{\dot{C}^{2m\alpha}} \tag{42}$$

for any $\lambda > 0$ and $x \in \mathbb{R}_{hn}^+$.

Combining estimates (36) and (42), we get

$$\sup_{\lambda > 0} \sup_{x \in \mathbb{R}_{hn}^+} \left| \lambda^\alpha A_h^x (A_h^x + \lambda)^{-1} f^h(x) \right| \leq M(\alpha) \|f^h\|_{\dot{C}^{2m\alpha}}. \tag{43}$$

Now, we will prove the opposite inequality. Using the definition of $E_\alpha(\dot{C}_h, A_h^x)$, we obtain

$$\sup_{x \in \mathbb{R}_n^+} \left| f^h(x) \right| \leq \|f^h\|_{E_\alpha(\dot{C}_h, A_h^x)} \tag{44}$$

for any $x \in \mathbb{R}_{hn}^+$. By Theorem 2, $A_h^x$ is a positive operator in the Banach space $E_\alpha(\dot{C}_h, A_h^x)$. Hence, for $f^h \in E_\alpha(\dot{C}_h, A_h^x)$, we have

$$f^h = \int_0^\infty A_h^x (A_h^x + \lambda)^{-2} V \, d\lambda. \tag{45}$$

It follows from equation (14) and equation (45) that

$$
\begin{aligned}
f^h(x) &= \int_0^\infty (A_h^x + \lambda)^{-1} A_h^x (A_h^x + \lambda)^{-1} f^h(x) \, d\lambda \\
&= \int_0^\infty \sum_{y \in \mathbb{R}_{hn}^+} G_h(x, y; \lambda) A_h^x (A_h^x + \lambda)^{-1} f^h(y) h^n \, d\lambda.
\end{aligned}
\tag{46}
$$

Let $x, \tau \in \mathbb{R}_{hn}^+$ be fixed. We will consider two cases: $\lambda > Lh^{-2m}$ and $\lambda \leq Lh^{-2m}$. First, let $\lambda > Lh^{-2m}$. Using equation (46) and Lemma 2, we have

$$\frac{f^h(x+\tau) - f^h(x)}{|\tau|^{2m\alpha}} = \int_0^\infty \sum_{y \in \mathbb{R}_{hn}^+} \frac{[G_h(x+\tau, y; \lambda) - G_h(x, y; \lambda)]}{|\tau|^{2m\alpha} \lambda^\alpha} A_h^x (A_h^x + \lambda)^{-1} f^h(y) h^n \, d\lambda.$$

From the triangle inequality and the definition of the $E_\alpha(\dot{C}_h, A_h^x)$-norm it follows that

$$\left| \frac{f^h(x + \tau) - f^h(x)}{|\tau|^{2m\alpha}} \right| \leq \left[ \int_0^{\frac{1}{|\tau|^{2m}}} \sum_{y \in \mathbb{R}_{hn}^+} \frac{|G_h(x + \tau, y; \lambda) - G_h(x, y; \lambda)|}{|\tau|^{2m\alpha} \lambda^\alpha} h^n \, d\lambda \right.$$

$$\left. + \int_{\frac{1}{|\tau|^{2m}}}^\infty \sum_{y \in \mathbb{R}_{hn}^+} \frac{|G_h(x + \tau, y; \lambda) - G_h(x, y; \lambda)|}{|\tau|^{2m\alpha} \lambda^\alpha} h^n \, d\lambda \right] \|f^h\|_{E_\alpha(\dot{C}_h, A_h^x)}$$

$$= [L_1 + L_2] \|f^h\|_{E_\alpha(\dot{C}_h, A_h^x)}.$$

We will estimate $L_1$ and $L_2$. Let us estimate $L_1$. From the Lagrange theorem and estimate (26), we have, for some $x^*$ between $x$, $x + \tau$,

$$L_1 \leq M_1 \int_0^{\frac{1}{|\tau|^{2m}}} \sum_{y \in \mathbb{R}_{hn}^+} \frac{|\tau| D_h^1 G_h(x^*, y; \lambda)|}{|\tau|^{2m\alpha} \lambda^\alpha} h^n \, d\lambda$$

$$\leq M_2 \int_0^{\frac{1}{|\tau|^{2m}}} \sum_{y \in \mathbb{R}_{hn}^+} \frac{|\tau| e^{-bh^{-1}|x^* - y|} h^{2m-1-n} ((\lambda + \delta)h^{2m} + 1)^{-1}}{|\tau|^{2m\alpha} \lambda^\alpha} h^n \, d\lambda$$

$$\leq M_3(\alpha) \int_0^{\frac{1}{|\tau|^{2m}}} \frac{|\tau|}{|\tau|^{2m\alpha} \lambda^{\alpha + 1 - \frac{1}{2m}}} \, d\lambda \leq M_4(\alpha) \tag{47}$$

for any $\lambda > 0$ and $x \in \mathbb{R}_{hn}^+$. Let us estimate $L_2$. The Lagrange theorem, estimate (25), and the inequality (34) yield

$$L_2 \leq M_1 \left[ \int_{\frac{1}{|\tau|^{2m}}}^\infty \sum_{y \in \mathbb{R}_{hn}^+} \frac{e^{-bh^{-1}|x + \tau - y|} h^{2m-n} ((\lambda + \delta)h^{2m} + 1)^{-1}}{|\tau|^{2m\alpha} \lambda^\alpha} h^n \, d\lambda \right.$$

$$\left. + \int_{\frac{1}{|\tau|^{2m}}}^\infty \sum_{y \in \mathbb{R}_{hn}^+} \frac{e^{-bh^{-1}|x - y|} h^{2m-n} ((\lambda + \delta)h^{2m} + 1)^{-1}}{|\tau|^{2m\alpha} \lambda^\alpha} h^n \, d\lambda \right]$$

$$\leq M_2 \int_{\frac{1}{|\tau|^{2m}}}^\infty \frac{1}{|\tau|^{2m\alpha} \lambda^{\alpha + 1}} \, d\lambda \leq M_3(\alpha) \tag{48}$$

for any $\lambda > 0$ and $x \in \mathbb{R}_{hn}^+$. Then, combining estimates (47) and (48), we get

$$\left| \frac{f^h(x + \tau) - f^h(x)}{|\tau|^{2m\alpha}} \right| \leq M \|f^h\|_{E_\alpha}. \tag{49}$$

Second, let $\lambda \leq L h^{-2m}$. We will estimate $L_1$ and $L_2$. Let us estimate $L_1$. We consider three cases: $2m > n$, $2m = n$, and $2m < n$. When $2m > n$ we consider two separate cases: $2m - 1 = n$ and $2m - 1 > n$. First, let $2m - 1 > n$. From the Lagrange theorem, estimate (22), the inequality (34) we have, for some $x^*$ between $x$, $x + \tau$,

$$L_1 \leq M_1 \int_0^{\frac{1}{|\tau|^{2m}}} \sum_{y \in \mathbb{R}_{hn}^+} \frac{|\tau| e^{-a(\lambda + \delta)^{\frac{1}{2m}} |x^* - y|}}{|\tau|^{2m\alpha} \lambda^\alpha (\lambda + \delta)^{1 - \frac{n+1}{2m}}} h^n \, d\lambda$$

$$\leq M_2 \int_0^{\frac{1}{|\tau|^{2m}}} \frac{|\tau|}{|\tau|^{2m\alpha} \lambda^{\alpha + 1 - \frac{1}{2m}}} \, d\lambda \leq M_3(\alpha) \tag{50}$$

for any $\lambda > 0$ and $x \in \mathbb{R}_{hn}^+$. Second, let $2m - 1 = n$. From the Lagrange theorem, estimate (23), and the inequality (34), we have, for some $x^*$ between $x, x + \tau$,

$$
\begin{aligned}
L_1 &\leq M_1 \int_0^{\frac{1}{|\tau|^{2m}}} \sum_{y \in \mathbb{R}_{hn}^+} \frac{|\tau|}{|\tau|^{2m\alpha} \lambda^\alpha} e^{-b(\lambda+\delta)^{\frac{1}{2m}}|x^*-y|} \Big[1 + \ln\big\{\big(|x^*-y|(\lambda+\delta)^{\frac{1}{2m}}\big)^{-1} + 1\big\}\Big] h^n \, d\lambda \\
&\leq M_2 \int_0^{\frac{1}{|\tau|^{2m}}} \frac{|\tau|}{|\tau|^{2m\alpha} \lambda^{\alpha+1-\frac{n}{2m}}} \, d\lambda \leq M_3(\alpha)
\end{aligned}
\tag{51}
$$

for any $\lambda > 0$ and $x \in \mathbb{R}_{hn}^+$. In the cases $2m = n$ and $2m < n$, we have a situation where $2m - 1 < n$. Using the Lagrange theorem, estimate (24), we have, for some $x^*$ between $x$, $x + \tau$,

$$
\begin{aligned}
L_1 &\leq M_1 \int_0^{\frac{1}{|\tau|^{2m}}} \frac{|\tau|}{|\tau|^{2m\alpha} \lambda^\alpha} \sum_{y \in \mathbb{R}_{hn}^+} e^{-a(\lambda+\delta)^{\frac{1}{2m}}|x^*-y|} |x^*-y|^{2m-n-1} h^n \, d\lambda \\
&\leq M_2 \int_0^{\frac{1}{|\tau|^{2m}}} \frac{|\tau|}{|\tau|^{2m\alpha} \lambda^{\alpha+1-\frac{1}{2m}}} \, d\lambda \leq M_3(\alpha)
\end{aligned}
\tag{52}
$$

for any $\lambda > 0$ and $x \in \mathbb{R}_{hn}^+$. Then, combining estimates (50)-(52), we get

$$
L_1 \leq M(\alpha)
\tag{53}
$$

for any $\lambda > 0$ and $x \in \mathbb{R}_{hn}^+$. Let us estimate $L_2$. We consider three cases: $2m > n$, $2m = n$, and $2m < n$. First, let $2m > n$. Using the triangle inequality, estimate (19), we get

$$
\begin{aligned}
L_2 &\leq M_1 \Bigg[ \int_{\frac{1}{|\tau|^{2m}}}^\infty \sum_{y \in \mathbb{R}_{hn}^+} \frac{e^{-b(\lambda+\delta)^{\frac{1}{2m}}|x+\tau-y|}}{|\tau|^{2m\alpha} \lambda^\alpha (\lambda+\delta)^{1-\frac{n}{2m}}} h^n \, d\lambda \\
&\quad + \int_{\frac{1}{|\tau|^{2m}}}^\infty \sum_{y \in \mathbb{R}_{hn}^+} \frac{e^{-b(\lambda+\delta)^{\frac{1}{2m}}|x-y|}}{|\tau|^{2m\alpha} \lambda^\alpha (\lambda+\delta)^{1-\frac{n}{2m}}} h^n \, d\lambda \Bigg] \\
&\leq M_2 \int_{\frac{1}{|\tau|^{2m}}}^\infty \frac{1}{|\tau|^{2m\alpha} \lambda^{\alpha+1}} \, d\lambda \leq M_3(\alpha)
\end{aligned}
\tag{54}
$$

for any $\lambda > 0$ and $x \in \mathbb{R}_{hn}^+$. Second, let $2m = n$. From the triangle inequality, estimate (20) it follows that

$$
\begin{aligned}
L_2 &\leq M_1 \Bigg[ \int_{\frac{1}{|\tau|^{2m}}}^\infty \sum_{y \in \mathbb{R}_{hn}^+} \frac{1}{|\tau|^{2m\alpha} \lambda^\alpha} e^{-b(\lambda+\delta)^{\frac{1}{2m}}|x+\tau-y|} \\
&\quad \times \Big[1 + \ln\big\{\big(|x+\tau-y|(\lambda+\delta)^{\frac{1}{2m}}\big)^{-1} + 1\big\}\Big] h^n \, d\lambda \\
&\quad + \int_{\frac{1}{|\tau|^{2m}}}^\infty \sum_{y \in \mathbb{R}_{hn}^+} \frac{1}{|\tau|^{2m\alpha} \lambda^\alpha} e^{-b(\lambda+\delta)^{\frac{1}{2m}}|x-y|} \Big[1 + \ln\big\{\big(|x-y|(\lambda+\delta)^{\frac{1}{2m}}\big)^{-1} + 1\big\}\Big] h^n \, d\lambda \Bigg] \\
&\leq M_2 \int_{\frac{1}{|\tau|^{2m}}}^\infty \frac{1}{|\tau|^{2m\alpha} \lambda^{\alpha+1}} \, d\lambda \leq M_3(\alpha)
\end{aligned}
\tag{55}
$$

for any $\lambda > 0$ and $x \in \mathbb{R}_{hn}^+$. Third, let $2m < n$. Using the triangle inequality, estimate (21), we have

$$
L_2 \leq M_1 \Bigg[ \int_{\frac{1}{|\tau|^{2m}}}^{\infty} \sum_{y \in \mathbb{R}_{hn}^+} \frac{e^{-b(\lambda+\delta)^{\frac{1}{2m}}|x+\tau-y|}}{|\tau|^{2m\alpha}\lambda^\alpha} |x+\tau-y|^{2m-n} h^n \, d\lambda
$$

$$
+ \int_{\frac{1}{|\tau|^{2m}}}^{\infty} \sum_{y \in \mathbb{R}_{hn}^+} \frac{e^{-b(\lambda+\delta)^{\frac{1}{2m}}|x-y|}}{|\tau|^{2m\alpha}\lambda^\alpha} |x-y|^{2m-n} h^n \, d\lambda \Bigg]
$$

$$
\leq M_2 \int_{\frac{1}{|\tau|^{2m}}}^{\infty} \frac{1}{|\tau|^{2m\alpha}\lambda^{\alpha+1}} \, d\lambda \leq M_3(\alpha). \tag{56}
$$

Then, combining estimates (54)-(56), we have

$$
L_2 \leq M(\alpha). \tag{57}
$$

Estimates (53) and (57) yield

$$
\left| \frac{f^h(x+\tau) - f^h(x)}{|\tau|^{2m\alpha}} \right| \leq M(\alpha) \left\| f^h \right\|_{E_\alpha}. \tag{58}
$$

Combining estimates (49) and (58), we obtain

$$
\sup_{\substack{x \in \mathbb{R}_{hn}^+ \\ \tau \neq (0,\ldots,0)}} \left| \frac{f^h(x+\tau) - f^h(x)}{|\tau|^{2m\alpha}} \right| \leq M(\alpha) \left\| f^h \right\|_{E_\alpha}. \tag{59}
$$

From estimates (43) and (59) it follows that

$$
E_\alpha\big(\dot{C}_h, A_h^x\big) \subset \dot{C}^{2m\alpha}\big(\mathbb{R}_{hn}^+\big).
$$

This is the end of the proof of Theorem 3. □

From Theorem 2 and Theorem 3 follows the following result.

**Theorem 4** *$A_h^x$ is a positive operator in the space $\dot{C}^\beta(\mathbb{R}_{hn}^+)$, $\beta \in (0,1)$.*

## 4 Applications

Now, we present some applications of Theorems 2-3.

First, we consider the difference schemes for the approximate solution of problem

$$
\begin{cases}
-\frac{\partial^2 u(t,x)}{\partial t^2} + \sum_{|r|=2m} a_r(x) \frac{\partial^{|r|} u(t,x)}{\partial x_1^{r_1} \cdots \partial x_{n-1}^{r_{n-1}}} \\
\quad + (-1)^m a(x) \frac{\partial^{2m} u(t,x)}{\partial x^{2m}} + \delta u(t,x) = f(t,x), \quad 0 < t < T, x \in \mathbb{R}_n^+, \\
u(0,x) = \varphi(x), \qquad u(T,x) = \psi(x), \quad x \in \mathbb{R}_n^+, \\
u(t,x)|_{x_n=0} = \frac{\partial u(t,x)}{\partial N}|_{x_n=0} = \cdots = \frac{\partial^{m-1} u(t,x)}{\partial N^{m-1}}|_{x_n=0} = 0, \quad x \in \mathbb{R}_n^+, 0 \leq t \leq T.
\end{cases} \tag{60}
$$

Here, $a_r(x), a(x), \varphi(x), \psi(x)$, and $f(t,x)$ are sufficiently smooth functions and they satisfy all compatibility conditions which guarantee that problem (60) has a smooth solution $u(t,x)$.

Assume that uniform ellipticity holds. Note that problem (60) can be written in the form of the abstract boundary value problem

$$
\begin{cases}
-u''(t) + Au(t) = f(t), & 0 < t < T, \\
u(0) = \varphi, & u(T) = \psi
\end{cases}
\tag{61}
$$

in a Banach space $E = \dot{C}_h$ with positive operator $A = A_h^x$ defined by (9) and (11). The discretization of problem (60) is carried out in two steps. In the first step, let us give the difference operator $A_h^x$ by equation (9). With the help of $A_h^x$ we arrive at the boundary value problem

$$
\begin{cases}
-\frac{d^2 u^h(t,x)}{dt^2} + A_h^x u^h(t,x) = f^h(t,x), & 0 < t < T, x \in \mathbb{R}_{hn}^+, \\
u^h(0,x) = \varphi^h(x), & u^h(T,x) = \psi^h(x), \quad x \in \mathbb{R}_{hn}^+,
\end{cases}
\tag{62}
$$

for an infinite system of ordinary differential equations.

In the second step, we replace problem (62) by the difference scheme

$$
\begin{cases}
-\frac{1}{\tau^2}(u_{k+1}^h(x) - 2u_k^h(x) + u_{k-1}^h(x)) + A_h^x u_k^h(x) = f_k^h(x), \\
f_k^h(x) = f^h(t_k, x), \quad t_k = k\tau, 1 \le k \le N-1, N\tau = T, x \in \mathbb{R}_{hn}^+, \\
u_0^h(x) = \varphi^h(x), \quad u_N^h(x) = \psi^h(x), \quad x \in \mathbb{R}_{hn}^+,
\end{cases}
\tag{63}
$$

for the approximate solution of boundary value problem (60).

**Theorem 5** *Let $0 \le \mu < 1$. Then the solution of difference scheme* (63) *satisfies the following stability estimate*:

$$
\max_{0 \le k \le N} \| u_k^h \|_{\dot{C}_h^\mu} \le M(\mu) \Big[ \| \varphi^h \|_{\dot{C}_h^\mu} + \| \varphi^h \|_{\dot{C}_h^\mu} + \max_{1 \le k \le N-1} \| f_k^h \|_{\dot{C}_h^\mu} \Big].
$$

The proof of Theorem 5 is based on Theorem 2 and Theorem 4, on the positivity of the difference operator $A_h^x$ in the Banach space $\dot{C}_h^\mu$ for $0 \le \mu < 1$, and on the following abstract theorem on the stability of the difference scheme

$$
\begin{cases}
-\frac{1}{\tau^2}(u_{k+1} - 2u_k + u_{k-1}) + Au_k = f_k, \\
f_k = f(t_k), \quad t_k = k\tau, 1 \le k \le N-1, N\tau = T, u_0 = \varphi, u_N = \psi,
\end{cases}
\tag{64}
$$

for the approximate solution of the abstract boundary value problem (61).

**Theorem 6** [3] *Let $A$ be a positive operator in a Banach space $E$. Then, for the solution of difference scheme* (64), *the following stability inequality holds*:

$$
\max_{0 \le k \le N} \| u_k \|_{E_\alpha} \le M(\alpha) \Big[ \| \varphi \|_{E_\alpha} + \| \psi \|_{E_\alpha} + \max_{1 \le k \le N-1} \| f_k \|_{E_\alpha} \Big].
$$

**Theorem 7** *The solution of difference scheme* (63) *satisfies the following almost coercive stability estimate*:

$$
\max_{1 \le k \le N-1} \left\| \frac{1}{\tau^2} \left( u_{k+1}^h - 2u_k^h + u_{k-1}^h \right) \right\|_{\dot{C}_h}
$$

$$
+ h^{-2m} \sum_{2m \le |p| \le S} \left\| \Delta_{1-}^{p_1} \Delta_{1+}^{p_2} \cdots \Delta_{(n-1)-}^{p_{2n-3}} \Delta_{(n-1)+}^{p_{2n-2}} u^h \right\|_{\dot{C}_h} + h^{-2m} \sum_{\substack{2m \le r+s \le S \\ s \le m}} \left\| \Delta_+^r \Delta_-^s u^h \right\|_{\dot{C}_h}
$$

$$
\le M \ln \frac{1}{h} \left[ h^{-2m} \sum_{2m \le |p| \le S} \left\| \Delta_{1-}^{p_1} \Delta_{1+}^{p_2} \cdots \Delta_{(n-1)-}^{p_{2n-3}} \Delta_{(n-1)+}^{p_{2n-2}} \varphi^h \right\|_{\dot{C}_h} \right.
$$

$$
+ h^{-2m} \sum_{\substack{2m \le r+s \le S \\ s \le m}} \left\| \Delta_+^r \Delta_-^s \varphi^h \right\|_{\dot{C}_h} + h^{-2m} \sum_{2m \le |p| \le S} \left\| \Delta_{1-}^{p_1} \Delta_{1+}^{p_2} \cdots \Delta_{(n-1)-}^{p_{2n-3}} \Delta_{(n-1)+}^{p_{2n-2}} \psi^h \right\|_{\dot{C}_h}
$$

$$
+ h^{-2m} \sum_{\substack{2m \le r+s \le S \\ s \le m}} \left\| \Delta_+^r \Delta_-^s \psi^h \right\|_{\dot{C}_h} \left. \right] + M \ln \frac{1}{\tau + h} \max_{1 \le k \le N-1} \left\| f_k^h \right\|_{\dot{C}_h}.
$$

The proof of Theorem 7 is based on Theorem 2 on the positivity of an elliptic difference operator $A_h^x$ in the Banach space $\dot{C}_h$ and on the estimate

$$
\min \left\{ \ln \frac{1}{\tau}, 1 + \left| \ln \left\| A_h^x \right\|_{\dot{C}_h \to \dot{C}_h} \right| \right\} \le M \ln \frac{1}{\tau + h} \tag{65}
$$

and on the following theorems on the almost coercive stability of difference scheme (64) and on the almost coercive stability of the elliptic difference problem.

**Theorem 8** [3] *Let A be a positive operator in a Banach space E. Then, for the solution of difference scheme* (64), *the following almost coercive stability inequality holds*:

$$
\max_{1 \le k \le N-1} \left\| \frac{1}{\tau^2} \left( u_{k+1} - 2u_k + u_{k-1} \right) \right\|_E + \max_{0 \le k \le N} \left\| A u_k \right\|_E
$$

$$
\le M \left[ \left\| A\varphi \right\|_E + \left\| A\psi \right\|_E + \min \left\{ \ln \frac{1}{\tau}, 1 + \left| \ln \left\| A \right\|_{E \to E} \right| \right\} \max_{1 \le k \le N-1} \left\| f_k \right\|_E \right].
$$

**Theorem 9** *Under assumptions* (8) *and* (10), *the solution of the difference elliptic problem*

$$
A_h^x u^h(x) = f^h(x), \quad x \in \mathbb{R}_{hn}^+, \tag{66}
$$

*satisfies the following almost coercive inequality*:

$$
h^{-2m} \sum_{2m \le |p| \le S} \left\| \Delta_{1-}^{p_1} \Delta_{1+}^{p_2} \cdots \Delta_{(n-1)-}^{p_{2n-3}} \Delta_{(n-1)+}^{p_{2n-2}} u^h \right\|_{\dot{C}_h} + h^{-2m} \sum_{\substack{2m \le r+s \le S \\ s \le m}} \left\| \Delta_+^r \Delta_-^s u^h \right\|_{\dot{C}_h}
$$

$$
\le M \ln \frac{1}{h} \left\| f^h \right\|_{\dot{C}_h}.
$$

The proof of Theorem 9 uses the techniques introduced in [2] and it is based on estimates for the Green's function of the operator $A_h^x$ defined by (7).

**Theorem 10** *Let* $0 < 2m\alpha < 1$. *Then, for the solution of difference scheme* (63), *the following coercive stability estimate holds*:

$$
\max_{1 \leq k \leq N-1} \left\| \frac{1}{\tau^2} \left( u_{k+1}^h - 2u_k^h + u_{k-1}^h \right) \right\|_{\dot{C}_h^{2m\alpha}}
$$

$$
+ h^{-2m} \sum_{2m \leq |p| \leq S} \left\| \Delta_{1-}^{p_1} \Delta_{1+}^{p_2} \cdots \Delta_{(n-1)-}^{p_{2n-3}} \Delta_{(n-1)+}^{p_{2n-2}} u^h \right\|_{\dot{C}_h^{2m\alpha}}
$$

$$
+ h^{-2m} \sum_{\substack{2m \leq r+s \leq S \\ s \leq m}} \left\| \Delta_+^r \Delta_-^s u^h \right\|_{\dot{C}_h^{2m\alpha}}
$$

$$
\leq M(\alpha) \Bigg[ h^{-2m} \sum_{2m \leq |p| \leq S} \left\| \Delta_{1-}^{p_1} \Delta_{1+}^{p_2} \cdots \Delta_{(n-1)-}^{p_{2n-3}} \Delta_{(n-1)+}^{p_{2n-2}} \varphi^h \right\|_{\dot{C}_h^{2m\alpha}}
$$

$$
+ h^{-2m} \sum_{\substack{2m \leq r+s \leq S \\ s \leq m}} \left\| \Delta_+^r \Delta_-^s \varphi^h \right\|_{\dot{C}_h^{2m\alpha}}
$$

$$
+ h^{-2m} \sum_{2m \leq |p| \leq S} \left\| \Delta_{1-}^{p_1} \Delta_{1+}^{p_2} \cdots \Delta_{(n-1)-}^{p_{2n-3}} \Delta_{(n-1)+}^{p_{2n-2}} \psi^h \right\|_{\dot{C}_h^{2m\alpha}}
$$

$$
+ h^{-2m} \sum_{\substack{2m \leq r+s \leq S \\ s \leq m}} \left\| \Delta_+^r \Delta_-^s \psi^h \right\|_{\dot{C}_h^{2m\alpha}} + M(\alpha) \max_{1 \leq k \leq N-1} \left\| f_k^h \right\|_{\dot{C}_h^{2m\alpha}} \Bigg].
$$

The proof of Theorem 10 is based on Theorem 3 on the structure of the fractional spaces $E_\alpha(\dot{C}_h, A_h^x)$, on Theorem 2 on the positivity of an elliptic difference operator $A_h^x$ in the Banach space $\dot{C}_h$, and on the following theorems on the structure of the fractional space $E'_\alpha = E_\alpha(E, A^{1/2})$, on the coercive stability of difference scheme (64), and on the coercive stability of the elliptic difference problem.

**Theorem 11** [16] *The spaces* $E_\alpha(E, A)$ *and* $E'_{2\alpha}(A^{1/2}, E)$ *coincide for any* $0 < \alpha < \frac{1}{2}$, *and their norms are equivalent*.

**Theorem 12** [3] *Let* $A$ *be a strongly positive operator in a Banach space* $E$. *Then, for the solution of difference scheme* (64), *the following coercive stability inequality*:

$$
\max_{1 \leq k \leq N-1} \left\| \frac{1}{\tau^2} (u_{k+1} - 2u_k + u_{k-1}) \right\|_{E_\alpha} + \max_{0 \leq k \leq N} \| A u_k \|_{E_\alpha}
$$

$$
\leq M(\alpha) \Big[ \| A\varphi \|_{E_\alpha} + \| A\psi \|_{E_\alpha} + \max_{1 \leq k \leq N-1} \| f_k \|_{E_\alpha} \Big]
$$

*is valid*.

**Theorem 13** *Let* $0 < \mu < 1$. *Then, under assumptions* (8) *and* (10) *for the solution of difference elliptic problem* (66), *we have the following coercive inequality*:

$$
h^{-2m} \sum_{2m \leq |p| \leq S} \left\| \Delta_{1-}^{p_1} \Delta_{1+}^{p_2} \cdots \Delta_{(n-1)-}^{p_{2n-3}} \Delta_{(n-1)+}^{p_{2n-2}} u^h \right\|_{\dot{C}_h^\mu}
$$

$$
+ h^{-2m} \sum_{\substack{2m \leq r+s \leq S \\ s \leq m}} \left\| \Delta_+^r \Delta_-^s u^h \right\|_{\dot{C}_h^\mu} \leq M(\mu) \left\| f^h \right\|_{\dot{C}_h^\mu}.
$$

The proof of Theorem 13 uses the techniques introduced in [3] and it is based on estimates for the Green's function of operator $A_h^x$ defined by (7).

Second, we consider the difference schemes for the approximate solution of problem

$$
\begin{cases}
-\frac{\partial^2 u(t,x)}{\partial t^2} + \sum_{|r|=2m} a_r(x)\frac{\partial^{|r|} u(t,x)}{\partial x_1^{r_1} \cdots \partial x_{n-1}^{r_{n-1}}} \\
\quad + (-1)^m a(x)\frac{\partial^{2m} u(t,x)}{\partial x^{2m}} + \delta u(t,x) = f(t,x), \quad 0 < t < T, x \in \mathbb{R}_n^+, \\
u(0,x) = u(T,x), \qquad u_t(0,x) = u_t(T,x), \quad x \in \mathbb{R}_n^+, \\
u(t,x)|_{x_n=0} = \frac{\partial u(t,x)}{\partial N}|_{x_n=0} = \cdots = \frac{\partial^{m-1} u(t,x)}{\partial N^{m-1}}|_{x_n=0} = 0, \quad x \in \mathbb{R}_n^+, 0 \le t \le T.
\end{cases}
\tag{67}
$$

Here, $a_r(x)$, $a(x)$, and $f(t,x)$ are sufficiently smooth functions and they satisfy every compatibility conditions which guarantee that problem (67) has a smooth solution $u(t,x)$. Assume that the assumption of the uniform ellipticity holds. The discretization of problem (67) is carried out in two steps. In the first step, let us give the difference operator $A_h^x$ by equation (9). With the help of $A_h^x$ we arrive at the boundary value problem

$$
\begin{cases}
-\frac{d^2 u^h(t,x)}{dt^2} + A_h^x u^h(t,x) = f^h(t,x), \quad 0 < t < T, x \in \mathbb{R}_{hn}^+, \\
u^h(0,x) = u^h(T,x), \qquad u_t^h(0,x) = u_t^h(T,x), \quad x \in \mathbb{R}_{hn}^+,
\end{cases}
\tag{68}
$$

for an infinite system of ordinary differential equations.

In the second step, we replace problem (68) by the first order of approximation in the $t$ difference scheme

$$
\begin{cases}
-\frac{1}{\tau^2}(u_{k+1}^h(x) - 2u_k^h(x) + u_{k-1}^h(x)) + A_h^x u_k^h(x) = f_k^h(x), \\
f_k^h(x) = f^h(t_k,x), \quad t_k = k\tau, 1 \le k \le N-1, N\tau = T, x \in \mathbb{R}_{hn}^+, \\
u_0^h(x) = u_N^h(x), \qquad u_1^h(x) - u_0^h(x) = u_N^h(x) - u_{N-1}^h(x), \quad x \in \mathbb{R}_{hn}^+,
\end{cases}
\tag{69}
$$

and the second order of approximation in the $t$ difference scheme

$$
\begin{cases}
-\frac{1}{\tau^2}(u_{k+1}^h(x) - 2u_k^h(x) + u_{k-1}^h(x)) + A_h^x u_k^h(x) = f_k^h(x), \\
f_k^h(x) = f^h(t_k,x), \quad t_k = k\tau, 1 \le k \le N-1, N\tau = T, x \in \mathbb{R}_{hn}^+, u_0^h(x) = u_N^h(x), \\
-u_2^h(x) + 4u_1^h(x) - 3u_0^h(x) = u_{N-2}^h(x) - 4u_{N-1}^h(x) + 3u_N^h(x), \quad x \in \mathbb{R}_{hn}^+,
\end{cases}
\tag{70}
$$

for the approximate solution of boundary value problem (67).

**Theorem 14** *Let $0 \le \mu < 1$. Then the solution of the difference schemes* (69) *and* (70) *satisfies the following stability estimate*:

$$
\max_{0 \le k \le N} \|u_k^h\|_{\dot{C}_h^\mu} \le M(\mu)\left[ \|\varphi^h\|_{\dot{C}_h^\mu} + \|\varphi^h\|_{\dot{C}_h^\mu} + \max_{1 \le k \le N-1} \|f_k^h\|_{\dot{C}_h^\mu} \right].
$$

The proof of Theorem 14 is based on Theorems 2 and 4, on the positivity of the difference operator $A_h^x$ in the Banach space $\dot{C}_h^\mu$, $\mu \in [0,1)$, and on the following abstract theorem, and on the stability of the difference schemes (69) and (70).

**Theorem 15** [3] *Let A be a positive operator in a Banach space E. Then, for the solution of difference schemes* (69) *and* (70), *the following stability inequality holds*:

$$\max_{0 \le k \le N} \|u_k\|_{E_\alpha} \le M(\alpha) \Big[ \|\varphi\|_{E_\alpha} + \|\psi\|_{E_\alpha} + \max_{1 \le k \le N-1} \|f_k\|_{E_\alpha} \Big].$$

**Theorem 16** *For the solution of difference schemes* (69) *and* (70), *the following almost coercive stability estimate holds*:

$$\max_{1 \le k \le N-1} \left\| \frac{1}{\tau^2} \big( u_{k+1}^h - 2u_k^h + u_{k-1}^h \big) \right\|_{\dot{C}_h}$$

$$+ h^{-2m} \sum_{2m \le |p| \le S} \big\| \Delta_{1-}^{p_1} \Delta_{1+}^{p_2} \cdots \Delta_{(n-1)-}^{p_{2n-3}} \Delta_{(n-1)+}^{p_{2n-2}} u^h \big\|_{\dot{C}_h} + h^{-2m} \sum_{\substack{2m \le r+s \le S \\ s \le m}} \big\| \Delta_+^r \Delta_-^s u^h \big\|_{\dot{C}_h}$$

$$\le M \ln \frac{1}{h} \Big[ h^{-2m} \sum_{2m \le |p| \le S} \big\| \Delta_{1-}^{p_1} \Delta_{1+}^{p_2} \cdots \Delta_{(n-1)-}^{p_{2n-3}} \Delta_{(n-1)+}^{p_{2n-2}} \varphi^h \big\|_{\dot{C}_h}$$

$$+ h^{-2m} \sum_{\substack{2m \le r+s \le S \\ s \le m}} \big\| \Delta_+^r \Delta_-^s \varphi^h \big\|_{\dot{C}_h} + h^{-2m} \sum_{2m \le |p| \le S} \big\| \Delta_{1-}^{p_1} \Delta_{1+}^{p_2} \cdots \Delta_{(n-1)-}^{p_{2n-3}} \Delta_{(n-1)+}^{p_{2n-2}} \psi^h \big\|_{\dot{C}_h}$$

$$+ h^{-2m} \sum_{\substack{2m \le r+s \le S \\ s \le m}} \big\| \Delta_+^r \Delta_-^s \psi^h \big\|_{\dot{C}_h} \Big] + M \ln \frac{1}{\tau + h} \max_{1 \le k \le N-1} \big\| f_k^h \big\|_{\dot{C}_h}.$$

The proof of Theorem 16 is based on Theorem 2, on the positivity of an elliptic difference operator $A_h^x$ in the Banach space $\dot{C}_h$, on the estimate

$$\min \left\{ \ln \frac{1}{\tau}, 1 + \big| \ln \big\| A_h^x \big\|_{\dot{C}_h \to \dot{C}_h} \big| \right\} \le M \ln \frac{1}{\tau + h}, \tag{71}$$

and on the following theorem on the almost coercive stability of difference schemes (69) and (70) and on Theorem 7 on the almost coercive stability of elliptic difference problem (66).

**Theorem 17** [3] *Let A be a positive operator in a Banach space E. Then, for the solution of difference schemes* (69) *and* (70), *the following almost coercive stability inequality holds*:

$$\max_{1 \le k \le N-1} \left\| \frac{1}{\tau^2} (u_{k+1} - 2u_k + u_{k-1}) \right\|_E + \max_{0 \le k \le N} \|Au_k\|_E$$

$$\le M \min \left\{ \ln \frac{1}{\tau}, 1 + \big| \ln \|A\|_{E \to E} \big| \right\} \max_{1 \le k \le N-1} \|f_k\|_E.$$

**Theorem 18** *Let* $0 < 2m\alpha < 1$. *Then, the solution of the difference schemes* (69) *and* (70) *satisfies the following coercive stability estimate*:

$$\max_{1 \le k \le N-1} \left\| \frac{1}{\tau^2} \big( u_{k+1}^h - 2u_k^h + u_{k-1}^h \big) \right\|_{\dot{C}_h^{2m\alpha}}$$

$$+ h^{-2m} \sum_{2m \le |p| \le S} \big\| \Delta_{1-}^{p_1} \Delta_{1+}^{p_2} \cdots \Delta_{(n-1)-}^{p_{2n-3}} \Delta_{(n-1)+}^{p_{2n-2}} u^h \big\|_{\dot{C}_h^{2m\alpha}}$$

$$+ h^{-2m} \sum_{\substack{2m \le r+s \le S \\ s \le m}} \left\| \Delta_+^r \Delta_-^s u^h \right\|_{\dot{C}_h^{2m\alpha}}$$

$$\le M(\alpha) \Bigg[ h^{-2m} \sum_{2m \le |p| \le S} \left\| \Delta_{1-}^{p_1} \Delta_{1+}^{p_2} \cdots \Delta_{(n-1)-}^{p_{2n-3}} \Delta_{(n-1)+}^{p_{2n-2}} \varphi^h \right\|_{\dot{C}_h^{2m\alpha}}$$

$$+ h^{-2m} \sum_{\substack{2m \le r+s \le S \\ s \le m}} \left\| \Delta_+^r \Delta_-^s \varphi^h \right\|_{\dot{C}_h^{2m\alpha}}$$

$$+ h^{-2m} \sum_{2m \le |p| \le S} \left\| \Delta_{1-}^{p_1} \Delta_{1+}^{p_2} \cdots \Delta_{(n-1)-}^{p_{2n-3}} \Delta_{(n-1)+}^{p_{2n-2}} \psi^h \right\|_{\dot{C}_h^{2m\alpha}}$$

$$+ h^{-2m} \sum_{\substack{2m \le r+s \le S \\ s \le m}} \left\| \Delta_+^r \Delta_-^s \psi^h \right\|_{\dot{C}_h^{2m\alpha}} \Bigg] + M(\alpha) \max_{1 \le k \le N-1} \left\| f_k^h \right\|_{\dot{C}_h^{2m\alpha}}.$$

The proof of Theorem 18 is based on Theorem 3, on the structure of the fractional spaces $E_\alpha(\dot{C}_h, A_h^x)$, on Theorem 11, on the structure of the fractional space $E'_\alpha = E_\alpha(E, A^{1/2})$, on Theorem 2, on the positivity of an elliptic difference operator $A_h^x$ in the Banach space $\dot{C}_h$, and on the following theorem on the coercive stability of difference schemes (69) and (70) and on Theorem 13 on the coercive stability of elliptic difference problem (66).

**Theorem 19** [2] *Let $A$ be a strongly positive operator in a Banach space $E$. Then, for the solution of difference schemes* (69) *and* (70), *the following coercive stability inequality*:

$$\max_{1 \le k \le N-1} \left\| \frac{1}{\tau^2} (u_{k+1} - 2u_k + u_{k-1}) \right\|_{E_\alpha} + \max_{0 \le k \le N} \| A u_k \|_{E_\alpha} \le M(\alpha) \max_{1 \le k \le N-1} \| f_k \|_{E_\alpha}$$

*is valid.*

## 5 Conclusion

In the present article, the structure of the fractional spaces $E_\alpha(\dot{C}(\mathbb{R}_{hn}^+), A_h^x)$ generated by the multi-dimensional elliptic difference operator $A_h^x$ is investigated.

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