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Remark on global attractor for damped wave equation on $\mathbb{R}^3$

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Abstract
This paper is concerned with the long-time behavior of solutions to the weakly damped wave equation with lower regular forcing defined on the entire space $\mathbb{R}^3$. The existence of a global attractor in $H^1(\mathbb{R}^3)\times L^2(\mathbb{R}^3)$ is proven. Moreover, under some additional condition, the translational regularity of the attractor is established.

MSC: 35B40; 35B41

Keywords: global attractor; damped wave equation; unbounded domain; lower regularity; translational regularity

1 Introduction

In this paper, we study the following weakly damped wave equation defined on $\mathbb{R}^3$:

$$\begin{cases} u_{tt}+\gamma u_t-\Delta u+\lambda u+f(x,u)=g(x), & (t,x)\in\mathbb{R}_+\times\mathbb{R}^3, \\ u(x,0)=u_0, \qquad u_t(x,0)=u_1, & x\in\mathbb{R}^3. \end{cases} \tag{1.1}$$

Here $\gamma>0$, $\lambda>0$, and the initial data (u_0,u_1) belong to the energy space $H^1(\mathbb{R}^3)\times L^2(\mathbb{R}^3)$, $g\in H^{-1}(\mathbb{R}^3)$ is independent of time, $f\in C^1(\mathbb{R}^4)$, and the following conditions are satisfied:

$$\left|\partial_s f(x,s)\right|\le c_1\left(1+s^2\right), \tag{1.2}$$

$$\liminf_{|x|+|s|\to\infty}\partial_s f(x,s)\ge 0, \tag{1.3}$$

$$\left|\nabla_x f(x,s)\right|\le c_2\left(1+|s|^3\right). \tag{1.4}$$

In the case where the spatial variable x belongs to a bounded domain, the existence of global attractors for the weakly damped wave equations (1.1) has been widely investigated; see [1–11] and the references therein. In the case where the underlying domain is unbounded, due to the embedding

$$H^1\left(\mathbb{R}^3\right)\hookrightarrow L^p\left(\mathbb{R}^3\right) \quad \text{for all } p\in[2,6]$$

being continuous but no longer compact, there is a substantial difficulty in dealing with the compactness of the corresponding operator semigroup, however, the compactness is a key point to obtain the existence of global attractor. To avoid such difficulty, some researchers used weighted spaces (or special Banach spaces) instead of the usual Hilbert ones. For example, Babin and Vishik [12] for the first time investigated the corresponding problem for the equations of parabolic type in weighted Sobolev spaces as phase space. Karachalios and Stavrakakis [13] have also employed weighted Sobolev spaces to deal with the wave equation. However, when studying in weighted spaces we have to impose the additional condition that the initial data and forcing term also belong to the corresponding spaces. Based on an approximation of $\mathbb{R}^n$ by a sufficiently large bounded domain $B(0,R)$ and then showing that there is a null convergence of the solutions on $\mathbb{R}^n \setminus B(0,R)$, Wang [14] came up with a new idea of 'tail estimate' to prove the asymptotic compactness of the semigroup, complied with the same idea, Khanmamedov [15, 16] dealt with plate equations, Djiby and You [17] tackled the weakly damped wave equation. By using a finite speed of propagation and specific estimates for the linear wave equations in $\mathbb{R}^n$, Feireisl obtained the existence of a global attractor for the semilinear wave equations with critical and supercritical exponents in an unbounded domain; see [18, 19].

On the other hand, in terms of the forcing term g, all papers we mentioned above require $g \in L^2$ or specially $g = 0$. In the case of lower regular forcing, we refer the reader to [20–22], in which the existence and asymptotic regularity of global attractor have been discussed for the strongly damped wave equations. As mentioned in [22], a strongly damped wave equation contains the strong damping term $-\Delta u_t$, which brings about many advantages for considering the long-time behavior, especially in considering the attractor. However, for the weakly damped wave equation (1.1), it seems difficult to apply the corresponding method to verify asymptotic compactness of the solution semigroup. Just recently, in [23], we presented a new method for the decomposition of the solution to deal with the case of $g \in H^{-1}(\Omega)$ in the bounded domain. Little seems to be known about the weakly damped wave equation in an unbounded domain with lower regular forcing.

The aim of this note is to study the existence and translational regularity of a global attractor for the weakly damped wave equation with lower regular forcing in unbounded domain. It is worth mentioning that the concept of translational regularity of global attractor, in other words, the global attractor being regular after a translation transformation, was introduced in our previous work [23].

Note that the number 2 in (1.2) corresponds to the 'critical exponent' case in bounded domains (see [1]), however, in an unbounded domain, it is possible to relax the growth restriction; see [24–30] for the well-posedness of the wave equation without damping and [19, 31] for the existence of global attractor. Due to the lower regular forcing considered in this paper, the well-posedness of problem (1.1) with faster growth nonlinearity is less clear, we shall discuss the weakly damped wave equation with lower regular forcing in unbounded domain when the growth exponent is larger that 3 in a future paper.

The rest of the paper is organized as follows. In Section 2, well-posedness and dissipativity for (1.1) are given. Section 3 is devoted to the compactness of the semigroup and the existence of global attractor, where the proof of compactness for the semigroup is with the help of a decomposition of the solution which inspired by [18, 19, 23]. Finally, the translational regularity of the global attractor in $\mathcal{H}$ is established in Section 4.

As regards the notations, denote $H^\sigma = \mathrm{dom}((-\Delta)^{\frac{\sigma}{2}})$, then we define the energy product spaces $\mathcal{H}^\sigma = H^{1+\sigma} \times H^\sigma$ with $\mathcal{H} = \mathcal{H}^0$ for short, and we write $\|\cdot\|$ for the norm of L^2. Denote by C any positive constant which may be different from line to line or even on the same line, and we also denote the different positive constants by C_i, $i \in \mathbb{N}$, for distinguishing.

2 Well-posedness and dissipativity

Applying the standard Galerkin method and smooth approximation on g (see, *e.g.*, [32]), under the conditions (1.2) and (1.3), we can verify that problem (1.1) is well-posed, that is, for any $T > 0$ and initial data $(u_0, u_1) \in \mathcal{H}$, problem (1.1) admits a unique weak solution $u \in C(0, T; H^1(\mathbb{R}^3)) \cap C^1(0, T; L^2(\mathbb{R}^3))$. Therefore, we can define an operator semigroup $\{S(t)\}_{t\geq 0}$ in $\mathcal{H}$ as follows:

$$S(t)(u_0, u_1) = \bigl(u(t), u_t(t)\bigr),$$

which is continuous in $\mathcal{H}$. In addition, the weak solution $u(t)$ satisfies the following energy estimate:

$$\begin{aligned} &\bigl\|u_t(t)\bigr\|^2 + \bigl\|\nabla u(t)\bigr\|^2 + \lambda\bigl\|u(t)\bigr\|^2 + 2\int_\Omega F\bigl(x, u(t)\bigr)\,dx - 2\bigl(g, u(t)\bigr) \\ &\quad\leq \|u_1\|^2 + \|\nabla u_0\|^2 + \lambda\|u_0\|^2 + 2\int_\Omega F(x, u_0)\,dx - 2(g, u_0), \end{aligned} \tag{2.1}$$

here and below, (u, v) stands for the classical inner product in L^2 and $F(x, s) = \int_0^s f(x, \tau)\,d\tau$.

Note that $u_{tt} + \gamma u_t - \Delta u - g = -f(x, u) \in L^2$, it allows us to multiply $u_t + \alpha u$ over the equation (1.1), and by proper energy estimates (see, *e.g.*, [18, 20]), we can obtain the existence of an absorbing set. We only state the result.

Lemma 2.1 *Suppose $g \in H^{-1}(\mathbb{R}^3)$ and f satisfies* (1.2), (1.3), *then $\{S(t)\}_{t\geq 0}$ has a bounded absorbing set $\mathscr{B}_0$ in $\mathcal{H}$, that is, for any bounded subset $B \subset \mathcal{H}$, there exists T_0 which depends on $\mathcal{H}$-bounds of B such that*

$$S(t)B \subset \mathscr{B}_0 \quad \textit{for all } t \geq T_0. \tag{2.2}$$

3 Compactness and existence of global attractor

To prove the existence of global attractor by means of well-known results of the theory of dynamical systems, we also need to verify the asymptotic compactness of the solution semigroup $\{S(t)\}_{t\geq 0}$.

To this end, we decompose the solution $u(t)$ as follows:

$$u(t) = h + v(t) + w(t), \tag{3.1}$$

where h is a solution of the elliptic equation

$$\begin{cases} -\Delta h + \lambda h + f(x, h) = g, \\ h \in H^1(\mathbb{R}^3), \end{cases} \tag{3.2}$$

$v(t)$ solves the cutting-off problem

$$\begin{cases} v_{tt} + \gamma v_t - \Delta v + \lambda v + \mu(|x| - \delta)(f(x, v + h) - f(x, h)) = 0, \\ v(0) = u_0 - h, \qquad v_t(0) = u_1, \end{cases} \tag{3.3}$$

with the cutting-off function $\mu \in \mathcal{C}^\infty(\mathbb{R}^1)$ defined by

$$\begin{cases} \mu(\theta) = 0, & \theta < 0, \\ \mu(\theta) \in [0,1], & \theta \in [0,1], \\ \mu(\theta) = 1, & \theta > 1, \end{cases} \tag{3.4}$$

and the remainder $w(t)$ satisfies

$$\begin{cases} w_{tt} + \gamma w_t - \Delta w + \lambda w + f(x,u) - f(x,h) - \mu(|x| - \delta)(f(x,v+h) - f(x,h)) = 0, \\ w(0) = 0, \qquad w_t(0) = 0. \end{cases} \tag{3.5}$$

Lemma 3.1 *Suppose $g \in H^{-1}(\mathbb{R}^3)$ and f satisfies* (1.2), (1.3), *then equation* (3.2) *has a solution $h \in H^1(\mathbb{R}^3)$.*

Proof According to assumptions (1.2) and (1.3), the energy functional corresponding to elliptic equation (3.2) is weakly lower semi-continuous and bounded from below on $H^1(\mathbb{R}^3)$, thus the existence of h can be guaranteed. □

Lemma 3.2 *Under the hypotheses of Lemma* 3.1, *we can find a $\delta_0 > 0$ such that, for any $\varepsilon > 0$, there exists $T_1 = T_1(\varepsilon) > 0$ such that*

$$\|v_t(t)\|_{L^2} + \|v(t)\|_{H^1} \le \varepsilon, \quad \text{for all } t \ge T_1, \delta > \delta_0 \text{ and } (u_0, u_1) \in \mathscr{B}_0. \tag{3.6}$$

Proof Multiplying (3.3) by $v_t + \alpha v$ and integrating in $\mathbb{R}^3$ we infer that

$$\begin{aligned} &\frac{d}{dt}E(t) + (\gamma - \alpha)\|v_t\|_{L^2}^2 + \lambda\alpha\|v\|_{L^2}^2 + \alpha\|\nabla v\|_{L^2}^2 \\ &\quad + \alpha \int_{\mathbb{R}^3} \mu\big(|x| - \delta\big)\big(f(x,v+h) - f(x,h)\big)v\,dx = 0, \end{aligned} \tag{3.7}$$

where

$$\begin{aligned} E(t) = {}& \frac{1}{2}\|v_t\|_{L^2}^2 + \frac{1}{2}\|\nabla v\|_{L^2}^2 + \frac{1}{2}(2\alpha\gamma + \lambda)\|v\|_{L^2}^2 + \alpha(v_t, v) \\ &+ \int_{\mathbb{R}^3} \mu\big(|x| - \delta\big)\big(F(x,v+h) - F(x,h) - f(x,h)v\big)\,dx. \end{aligned}$$

By the mean value theorem and (1.3), there exists a constant $R_1 > 0$, such that, for all $|x| > R_1$, we get

$$\begin{aligned} F(x,v+h) - F(x,h) - f(x,h)v &= \int_0^1 f(x,h+\tau v)v\,d\tau - f(x,h)v \\ &= \int_0^1 \big(f(x,h+\tau v) - f(x,h)\big)v\,d\tau \\ &= \int_0^1 \int_0^\tau \partial_s f(x,h+\eta v)v^2\,d\eta\,d\tau \ge -\frac{\lambda}{3}v^2 \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} & f(x,v+h)v - f(x,h)v - \left[F(x,v+h) - F(x,h) - f(x,h)v\right] \\ & \quad = f(x,v+h)v - F(x,v+h) + F(x,h) = f(x,v+h) - \int_0^1 f(x,h+\tau v)v\,d\tau \\ & \quad = \int_0^1 \bigl(f(x,v+h) - f(x,h+\tau v)\bigr)v d\tau = \int_0^1 \int_\tau^1 \partial_s f(x,h+\eta v)v^2\,d\eta\,d\tau \\ & \quad \geq -\frac{\lambda}{3}v^2. \end{aligned} \tag{3.9}$$

Hence, for $\delta > R_1$, we have

$$\begin{aligned} & \frac{2\lambda}{3}\|v\|_{L^2}^2 + \int_{\mathbb{R}^3} \mu\bigl(|x|-\delta\bigr)\bigl(f(x,v+h) - f(x,h)\bigr)v\,dx \\ & \quad \geq \frac{\lambda}{3}\|v\|_{L^2}^2 + \int_{\mathbb{R}^3} \mu\bigl(|x|-\delta\bigr)\bigl(F(x,v+h) - F(x,h) - f(x,h)v\bigr)v\,dx \\ & \quad \geq \frac{1}{2+2\gamma}\int_{\mathbb{R}^3} \mu\bigl(|x|-\delta\bigr)\bigl(F(x,v+h) - F(x,h) - f(x,h)v\bigr)v\,dx. \end{aligned} \tag{3.10}$$

Choosing $\alpha \leq \min\{\frac{\gamma}{3}, \frac{\lambda}{3}, \frac{1}{2}\}$, we obtain

$$\begin{aligned} & (\gamma - \alpha)\|v_t\|_{L^2}^2 + \frac{\lambda\alpha}{3}\|v\|_{L^2}^2 + \alpha\|\nabla v\|_{L^2}^2 \\ & \quad \geq \kappa\left(\frac{1}{2}\|v_t\|_{L^2}^2 + \frac{1}{2}\|\nabla v\|_{L^2}^2 + \frac{1}{2}(2\alpha\gamma + \lambda)\|v\|_{L^2}^2 + \alpha(v_t, v)\right) \end{aligned} \tag{3.11}$$

and

$$E(t) \geq \frac{1}{1+\gamma}\bigl(\|v(t)\|_{H^1}^2 + \|v_t(t)\|_{L^2}^2\bigr), \tag{3.12}$$

with $\kappa = \frac{\alpha}{2+2\gamma}$.

By substituting (3.10) and (3.11) into (3.7), we arrive at

$$\frac{d}{dt}E(t) + \kappa E(t) \leq 0.$$

Then the Gronwall inequality together with (3.12) will imply the conclusion. □

Before dealing with the w-component, we first introduce a technical tool which will be used in our proofs, that is, the Strichartz estimate for the following linear damped wave equation:

$$v_{tt} + v_t - \Delta v = G(t), \qquad v(0) = v_0, \qquad v_t(0) = v_1.$$

Proposition 3.3 ([28, 29]) *Assume that* $(v_0, v_1) \in \mathcal{H}$ *and* $G \in L^1(0,T;L^2(\mathbb{R}^3))$. *Let* $v(t)$ *be a solution of above equation such that* $(v, v_t) \in C(0,T;\mathcal{H})$. *Then* $v \in L^4(0,T;L^{12}(\mathbb{R}^3))$ *and the* *following estimate holds*:

$$\|v\|_{L^4(0,T;L^{12}(\mathbb{R}^3))} \le C_T\big(\|v_0\|_{H^1} + \|v_1\| + \|G\|_{L^1(0,T;L^2(\mathbb{R}^3))}\big), \tag{3.13}$$

where C_T *may depend on* T, *but it is independent of* v_0, v_1, *and* G.

Lemma 3.4 *Given* $T > 0$, *there is a compact set* $\mathfrak{C}_T \subset \mathcal{H}$ *such that for any* $0 \le t \le T$, *we have*

$$\big(w(t), w_t(t)\big) \in \mathfrak{C}_T,$$

where $w = u - h - v$ *and* h, v *solve* (3.2) *and* (3.3), *respectively, with the initial data* $(u_0, u_1) \in \mathscr{B}_0$.

Proof Based on the finite speed of propagation property, observe that

$$v = u - h \quad \text{on } \big(\mathbb{R}^3 \backslash B_0(\delta + T + 1)\big) \times [0, T]. \tag{3.14}$$

Consequently, the difference $w = u - h - v$ satisfies

$$w \equiv 0 \quad \text{on } \big(\mathbb{R}^3 \backslash B_0(\delta + T + 1)\big) \times [0, T]. \tag{3.15}$$

According to the continuous dependence of the solution on the right-hand side for equation (3.5), it suffices to prove $f(\cdot, u)$ and $f(\cdot, v)$ belong to a compact subset of $L^1(0, T; L^2(B_0(\delta + T + 1)))$, which, in turn, would follow from

$$u, v \text{ are compact in } L^3\big(0, T; L^6\big(B_0(\delta + T + 1)\big)\big). \tag{3.16}$$

Combining with (2.1), Lemma 3.1, and Lemma 3.2, it follows that u, v are bounded both in $L^\infty(0, T; H^1(\mathbb{R}^3))$ and $H^1([0, T] \times \mathbb{R}^3)$. Then u, v belong to a compact subset of $C([0, T]; L^4(B_0(\delta + T + 1)))$ via the Aubin-Lions lemma [33]. On the other hand, according to (3.13), u, v are bounded in $L^4(0, T, L^{12}(\mathbb{R}^3))$. Now as

$$L^\infty\big(0, T; L^4\big) \cap L^4\big(0, T; L^{12}\big) \hookrightarrow L^3\big(0, T; L^6\big), \tag{3.17}$$

(3.16) holds. The proof is completed. □

Lemmas 3.1, 3.2 and 3.4 show that the solution $(u(t), u_t(t))$ with initial data $(u(0), u_t(0)) \in \mathcal{H}$ decomposes into the sum of a fixed point $(h, 0)$ and a uniform decaying term $(v(t), v_t(t))$ and a term $(w(t), w_t(t))$ belonging to a compact set $\mathfrak{C} \subset \mathcal{H}$. Therefore, we can derive the asymptotic compactness of the semigroup $\{S(t)\}_{t\ge 0}$ immediately. Combining with the existence of a bounded absorbing set stated in Lemma 2.1, the existence of a global attractor is obtained by the standard methods of the theory of attractors (see, *e.g.*, [7, 11, 34]). We state the result.

Theorem 3.5 *Let* $g \in H^{-1}(\mathbb{R}^3)$ *and* $f \in C^1(\mathbb{R}^4)$ *satisfy* (1.2) *and* (1.3), $\{S(t)\}_{t\ge 0}$ *be the semigroup generated by the solution of problem* (1.1) *in* $\mathcal{H}$. *Then* $\{S(t)\}_{t\ge 0}$ *possesses a global attractor* $\mathscr{A}$ *in* $\mathcal{H}$, *which is compact and invariant and attracts the bounded sets of* $\mathcal{H}$.

4 Translational regularity of global attractor

Due to the solution of the corresponding stationary equation of (1.1) $-\Delta h + \lambda h + f(h) = g$ only belonging to H^1, we cannot expect any higher regularity of the global attractor than $\mathcal{H}$. In [23], we obtained the translational regularity of a global attractor for the bounded domain. For the unbounded domain in the present text, if we add some condition on f other than (1.2) and (1.3), we can also obtain the translational regularity of the global attractor.

Decompose the solution $u(t)$ again which is similar to the bounded domain [23] as follows:

$$u(t) = h + \tilde{v}(t) + \tilde{w}(t), \tag{4.1}$$

where h is a solution of equation (3.2), $\tilde{v}(t)$ solves the linear problem

$$\begin{cases} \tilde{v}_{tt} + \gamma \tilde{v}_t - \Delta \tilde{v} + \lambda v = 0, \\ \tilde{v}(0) = u_0 - h, \qquad \tilde{v}_t(0) = u_1, \end{cases} \tag{4.2}$$

and the remainder $\tilde{w}(t)$ satisfies

$$\begin{cases} \tilde{w}_{tt} + \gamma \tilde{w}_t - \Delta \tilde{w} + \lambda \tilde{w} = f(x,h) - f(x,u), \\ \tilde{w}(0) = 0, \qquad \tilde{w}_t(0) = 0, \end{cases} \tag{4.3}$$

we denote $S_{\tilde{w}}(t)(u_0,u_1) = (\tilde{w}(t), \tilde{w}_t(t))$.

Since equation (4.2) is linear, it is easy to check that $\tilde{v}(t)$ is exponentially decaying. For the $\tilde{w}$-component, we have the following regularity lifting lemma.

Lemma 4.1 *Let* $g \in H^{-1}(\mathbb{R}^3)$ *and* $f \in C^1(\mathbb{R}^4)$ *satisfy* (1.2), (1.3), *and* (1.4). *Assume that* $\sup_{t\geq 0} \|(u(t) - h, u_t(t))\|_{\mathcal{H}^\sigma}$ *and* $\sup_{t\geq 0} \|f(x,u) - f(x,h)\|_{L^1(t,t+1;H^\sigma)}$ *are both finite for some* $0 \leq \sigma \leq \frac{1}{3}$. *Then* $\{(\tilde{w}(t), \tilde{w}_t(t))\}_{t\geq 0}$ *is bounded in* $\mathcal{H}^{\rho(\sigma)}$ *and*

$$\sup_{t\geq 0} \left\| f(x,u) - f(x,h) \right\|_{L^1(t,t+1;H^{\rho(\sigma)})}$$

is finite, with

$$\rho(\sigma) = \begin{cases} (1+4\sigma)/(5+4\sigma), & 0 \leq \sigma < \frac{1}{4}, \\ \frac{1}{3}, & \sigma > \frac{1}{4}, \\ \frac{1}{3} - \varepsilon, \forall \varepsilon > 0, & \sigma = \frac{1}{4}. \end{cases}$$

For the proof of Lemma 4.1, one of the key points is the following estimate:

$$\begin{aligned} \left\| f(x,u) \right\|_{W^{1,\frac{6}{5}}} &= \|\partial_u f \nabla u + \nabla_x f\|_{L^{\frac{6}{5}}(\mathbb{R}^3)} \leq \|\partial_u f \nabla u\|_{L^{\frac{6}{5}}} + \|\nabla_x f\|_{L^{\frac{6}{5}}} \\ &\leq C_1\left(1 + \|u\|^2_{L^6(\mathbb{R}^3)}\right) \|u\|_{H^1(\mathbb{R}^3)} + C\left(1 + \|u\|^3_{L^{\frac{18}{5}}}\right), \end{aligned} \tag{4.4}$$

which is according to the growth restriction (1.2). Combining with (2.1) we have $f(x,u) \in L^\infty(0,\infty; W^{1,\frac{6}{5}})$, the same as $f(x,h)$. The rest of the proof of Lemma 4.1 could be a repeat nearly word by word from our previous work [23], hence we omit it here.

Note that $\mathscr{A} = \bigcap_{t\geq 0} \overline{\bigcup_{s\geq t} S(s)\mathscr{B}_0}$. The decaying property of $\tilde{v}$ implies that $\mathscr{A} \subset (h, 0) + \bigcap_{t\geq 0} \overline{\bigcup_{s\geq t} S_{\tilde{w}}(s)\mathscr{B}_0}$. Applying Lemma 4.1, we have the following translational regularity of the global attractor.

Theorem 4.2 *Under the conditions of Lemma* 4.1, $\mathscr{A} - (h, 0)$ *is bounded in* $\mathcal{H}^{\frac{1}{3}}$, *where* h *is a solution of the stationary equation* (3.2).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
MFJ completed the main part of this article, LCC corrected the main theorems. All authors read and approved the final manuscript.

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References
1. Arrieta, J, Carvalho, AN, Hale, JK: A damped hyperbolic equation with critical exponent. Commun. Partial Differ. Equ. **17**, 841-866 (1992)
2. Babin, AV, Vishik, MI: Regular attractors of semigroups and evolution equation. J. Math. Pures Appl. **62**, 441-491 (1983)
3. Ball, JM: Global attractors for damped semilinear wave equations. Discrete Contin. Dyn. Syst. **10**, 31-52 (2004)
4. Chepyzhov, V, Vishik, MI: Attractors for Equations of Mathematical Physics. American Mathematical Society Colloquium Publications, vol. 49. Am. Math. Soc., Providence (2002)
5. Chueshov, I, Lasiecka, I: Attractors for second-order evolution equations with a nonlinear damping. J. Dyn. Differ. Equ. **16**, 469-512 (2004)
6. Chueshov, I, Lasiecka, I: Long-time behavior of second order evolution equations with nonlinear damping. Mem. Am. Math. Soc. **195**(912), viii+183 pp. (2008)
7. Hale, JK: Asymptotic Behavior of Dissipative Systems. Am. Math. Soc., Providence (1988)
8. Haraux, A: Semi-linear Hyperbolic Problems in Bounded Domains. Math. Rep., vol. 3. Gordon & Breach, New York (1987)
9. Ladyzhenskaya, OA: Attractors of nonlinear evolution problems with dissipation. J. Sov. Math. **40**, 632-640 (1988)
10. Pata, V, Zelik, S: A remark on the damped wave equation. Commun. Pure Appl. Anal. **5**(3), 611-616 (2006)
11. Temam, R: Infinite-Dimensional Dynamical Systems in Mechanics and Physics, 2nd edn. Applied Mathematical Sciences, vol. 68. Springer, New York (1997)
12. Babin, AV, Vishik, MI: Attractors of partial differential evolution equations in an unbounded domain. Proc. R. Soc. Edinb., Sect. A **116**, 221-243 (1990)
13. Karachalios, NI, Stavrakakis, NM: Existence of a global attractor for semilinear dissipative wave equations on R^N. J. Differ. Equ. **157**, 183-205 (1999)
14. Wang, B: Attractors for reaction-diffusion equation in unbounded domains. Physica D **128**, 41-52 (1999)
15. Khanmamedov, AK: Existence of a global attractor for the plate equation with the critical exponent in an unbounded domain. Appl. Math. Lett. **18**, 827-832 (2005)
16. Khanmamedov, AK: Global attractors for the plate equation with a localized damping and a critical exponent in an unbounded domain. J. Differ. Equ. **225**, 528-548 (2006)
17. Djiby, F, You, Y: Global attractors for the damped nonlinear wave equation in unbounded domain. In: Proceedings of the Fourth World Congress of Nonlinear Analysis (2004)
18. Feireisl, E: Attractors for semilinear damped wave equations on $\mathbb{R}^3$. Nonlinear Anal. **23**, 187-195 (1994)
19. Feireisl, E: Asymptotic behavior and attractors for a semilinear damped wave equation with supercritical exponent. Proc. R. Soc. Edinb., Sect. A **125**, 1051-1062 (1995)
20. Pata, V, Squassina, M: On the strongly damped wave equation. Commun. Math. Phys. **253**, 511-533 (2005)
21. Sun, C: Asymptotic regularity for some dissipative equations. J. Differ. Equ. **248**, 342-362 (2010)
22. Yang, M, Sun, C: Exponential attractors for the strongly damped wave equations. Nonlinear Anal., Real World Appl. **11**, 913-919 (2010)
23. Liu, CC, Meng, FJ, Zhong, CK: A note on the global attractor for the weakly wave equation. Appl. Math. Lett. **41**, 12-16 (2015)
24. Grillakis, M: Regularity and asymptotic behaviour of the wave equation with a critical nonlinearity. Ann. Math. **132**(3), 485-509 (1990)
25. Kapitanski, L: The Cauchy problem for the semilinear wave equation, I. Zap. Nauč. Semin. POMI **163**, Kraev. Zadachi Mat. Fiz. i Smezhn. Vopr. Teor. Funktsii 19, 76-104, 188 (1987) (in Russian); translation in J. Sov. Math. **49**(5), 1166-1186 (1990)
26. Kapitanski, L: The Cauchy problem for the semilinear wave equation, II. Zap. Nauč. Semin. POMI **182**, Kraev. Zadachi Mat. Fiz. i Smezh. Voprosy Teor. Funktsii. 21, 38-85, 171 (1990) (in Russian); translation in J. Sov. Math. **62**(3), 2746-2777 (1992)
27. Kapitanski, L: Global and unique weak solutions of nonlinear wave equations. Math. Res. Lett. **1**(2), 211-223 (1994)

28. Shatah, J, Struwe, M: Regularity results for nonlinear wave equations. Ann. Math. **138**(3), 503-518 (1993)
29. Shatah, J, Struwe, M: Well-posedness in the energy space for semilinear wave equations with critical growth. Int. Math. Res. Not. **7**, 1-7 (1994)
30. Struwe, M: Globally regular solutions to the u^5 Klein-Gordon equation. Ann. Sc. Norm. Super. Pisa, Cl. Sci. **15**, 495-513 (1988)
31. Kapitanski, L: Minimal compact global attractor for a damped semilinear wave equation. Commun. Partial Differ. Equ. **20**(7-8), 1303-1323 (1995)
32. Lions, JL: Quelques méthodes de résolution des problèmes aux limites *n*-linéaires. Dunod, Paris (1969)
33. Simon, J: Compact sets in the space $L^p(0,T;B)$. Ann. Mat. Pura Appl. **CXLVI**, 65-96 (1987)
34. Babin, AV, Vishik, MI: Attractors of Evolution Equations. North-Holland, Amsterdam (1992)