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Multiple solutions of ordinary differential systems with min-max terms and applications to the fuzzy differential equations

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Abstract

In this paper, we investigate the existence of multiple solutions for a class of ordinary differential systems with min-max terms. We present two fundamental results for the existence of solutions. An illustrative example shows that the uniqueness of solution does not hold although the Lipschitz condition is added. Finally, there are some nontrivial applications of the considered theory to fuzzy differential equations with the generalized Hukuhara derivative.

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Keywords: nonlinear fuzzy differential equations; generalized Hukuhara derivative; multiple solutions

1 Introduction

Crisp differential equations are popular models to approach various phenomena in the real world when the conditions/hypotheses are clear. If the occurrence of the phenomenon or the initial data is not precisely known, fuzzy differential equations [1–3] and stochastic fuzzy differential equations [4–8] appear to be a natural way to model the aleatory and epistemic uncertainty. For example these equations are used to model cell growth and the dynamics of populations [4, 6–9], dry friction [10], tumor growth [11], and the phenomenon of nuclear disintegration [12] and the transition from HIV to AIDS [13] under uncertainty. It would have better application prospect to investigate the foundational theory of fuzzy differential equation deeply. Especially, there are many approaches to interpret a solution for various fuzzy differential equations. As for most differential equations, the Lipschitz condition is a popular assumption as regards the uniqueness of the local solution. Also, the relationship between crisp differential equations defined in terms of the Hukuhara derivative and set differential equations was investigated by Lakshmikantham *et al.* [14]. As we observe, the uniqueness of the solution does not hold although the Lipschitz condition is added to the ordinary differential equation with min-max terms.

One of the earliest suggestions to define the concept of differentiability for fuzzy functions and, in consequence, to study fuzzy differential equations is the Hukuhara derivative [2, 15]. Naturally, the solution of fuzzy differential equations interpreted by the Hukuhara derivative became fuzzier as time went by [16]. Recently, there were two extensions of the concept of Hukuhara derivatives with different differences [1, 17]. The different definition of the Hukuhara derivative will lead to a different level-cut system. The new generalization of the Hukuhara difference, due to Stefanini [17], raising the fuzzy differential equations associated with the level-cut system involving the min-max terms, is as follows:

$$\begin{cases} \min\{u'(t), v'(t)\} = F(t, u(t), v(t)), \\ \max\{u'(t), v'(t)\} = G(t, u(t), v(t)), \quad t \in I. \end{cases} \tag{1}$$

There are more results on the Hukuhara derivative in the references, for example in [18–23].

In this paper, we consider the differential system (1) with min-max terms subject to the initial value conditions $u(0) = u_0$, $v(0) = v_0$, where $I = [0,1]$ is an interval, $F, G \in C(I \times R \times R, R)$. We say the pair $(u(t), v(t))$ is a solution of system (1) if $u, v \in C(I)$ and $u(t)$, $v(t)$ satisfy equations (1) and the initial value conditions.

2 Existence results

Theorem 2.1 *Let $F, G : I \times R \times R \to R$ be two continuous functions, F and G being Lipschitz continuous relative to their last two arguments, i.e. there exist real numbers $L_1, L_2, L_3, L_4 > 0$ such that, for $(t, u_1, v_1), (t, u_2, v_2) \in I \times R \times R$,*

$$\bigl|F(t, u_1, v_1) - F(t, u_2, v_2)\bigr| \le L_1|u_1 - u_2| + L_2|v_1 - v_2|,$$
$$\bigl|G(t, u_1, v_1) - G(t, u_2, v_2)\bigr| \le L_3|u_1 - u_2| + L_3|v_1 - v_2|.$$

Moreover, if $f, g \in C(I)$ and $f(t) \le g(t)$ for all $t \in I$ implies that $F(t, f(t), g(t)) \le G(t, f(t), g(t))$, then the problem (1) *has at least one solution on I provided $u_0 \le v_0$.*

Proof Let $u(t) = \int_0^t f(s)\,ds + u_0$ and $v(t) = \int_0^t g(s)\,ds + v_0$, where $f, g \in C(I)$. It is easy to see that the pair $(u(t), v(t))$ is a solution of (1) if and only if the pair $(f(t), g(t))$ is a solution of following group integral equations with min-max terms:

$$\min\bigl\{f(t), g(t)\bigr\} = F\left(t, \int_0^t f(s)\,ds + u_0, \int_0^t g(s)\,ds + v_0\right),$$
$$\max\bigl\{f(t), g(t)\bigr\} = G\left(t, \int_0^t f(s)\,ds + u_0, \int_0^t g(s)\,ds + v_0\right), \quad t \in I.$$

Define the successive iterations $\{f_n(t)\}$ and $\{g_n(t)\}$ by

$$f_0(t) = 0, \qquad g_0(t) = 0,$$
$$f_{n+1}(t) = F\left(t, \int_0^t f_n(s)\,ds + u_0, \int_0^t g_n(s)\,ds + v_0\right), \tag{2}$$
$$g_{n+1}(t) = G\left(t, \int_0^t f_n(s)\,ds + u_0, \int_0^t g_n(s)\,ds + v_0\right), \quad t \in I, n = 0, 1, \ldots. \tag{3}$$

Since $u_0 \le v_0$, we have $F(t, u_0, v_0) \le G(t, u_0, v_0)$. Thus $f_1(t) \le g_1(t)$ for all $t \in I$. Also, for $t \in I$, by $\int_0^t f_1(s)\,ds + u_0 \le \int_0^t g_1(s)\,ds + v_0$ and the assumptions, we see that $f_2(t) \le g_2(t)$ for all $t \in I$. By mathematical induction, we conclude that $f_n(t) \le g_n(t)$ for all $t \in I$ and $n \in \mathbb{N}$.

Next, we would prove that the successive iterations $\{f_n(t)\}$ and $\{g_n(t)\}$ are uniformly convergent on I. To this end, we choose a constant $c > 0$ satisfying $\frac{L_1+L_2+L_3+L_4}{c} < 1$ and introduce the norm $\|\cdot\|$ in $C(I)$ by

$$\|x\| = \sup\{e^{-ct}|x(t)| : t \in I\} \quad \text{for } x \in C(I).$$

Then $|x(t)| \le e^{ct}\|x\|$ for all $t \in I$.

On the other hand, from the successive iterations, we see that

$$\begin{aligned}
|f_{n+1}(t) - f_n(t)| &= \left| F\left(t, \int_0^t f_n(s)\,ds + u_0, \int_0^t g_n(s)\,ds + v_0\right) \right. \\
&\quad \left. - F\left(t, \int_0^t f_{n-1}(s)\,ds + u_0, \int_0^t g_{n-1}(s)\,ds + v_0\right) \right| \\
&\le L_1 \int_0^t |f_n(s) - f_{n-1}(s)|\,ds + L_2 \int_0^t |g_n(s) - g_{n-1}(s)|\,ds \\
&\le L_1 \int_0^t e^{cs}\,ds \|f_n - f_{n-1}\| + L_2 \int_0^t e^{cs}\,ds \|g_n - g_{n-1}\| \\
&\le \frac{L_1}{c} e^{ct} \|f_n - f_{n-1}\| + \frac{L_2}{c} e^{ct} \|g_n - g_{n-1}\|.
\end{aligned}$$

Thus

$$e^{-ct}|f_{n+1}(t) - f_n(t)| \le \frac{L_1}{c}\|f_n - f_{n-1}\| + \frac{L_2}{c}\|g_n - g_{n-1}\|.$$

This implies that

$$\|f_{n+1} - f_n\| \le \frac{L_1}{c}\|f_n - f_{n-1}\| + \frac{L_2}{c}\|g_n - g_{n-1}\|.$$

Similarly, we have

$$\|g_{n+1} - g_n\| \le \frac{L_3}{c}\|f_n - f_{n-1}\| + \frac{L_4}{c}\|g_n - g_{n-1}\|.$$

From the above two inequalities, we see that

$$\begin{aligned}
\|f_{n+1} - f_n\| + \|g_{n+1} - g_n\| &\le \frac{L_1 + L_3}{c}\|f_n - f_{n-1}\| + \frac{L_2 + L_4}{c}\|g_n - g_{n-1}\| \\
&\le \frac{L_1 + L_3 + L_2 + L_4}{c}\left(\|f_n - f_{n-1}\| + \|g_n - g_{n-1}\|\right).
\end{aligned}$$

Let $\gamma = \frac{L_1+L_3+L_2+L_4}{c}$, then $\gamma < 1$ and

$$\|f_{n+1} - f_n\| + \|g_{n+1} - g_n\| \le \gamma^n\left(\|f_1\| + \|g_1\|\right).$$

Then, for any $k \in \mathbb{N}$, we have

$$\|f_{n+k} - f_n\| + \|g_{n+k} - g_n\| \le \sum_{i=1}^{k} \gamma^{n+i-1}\left(\|f_1\| + \|g_1\|\right) \le \frac{\gamma^n}{1-\gamma}\left(\|f_1\| + \|g_1\|\right).$$

This means that both successive iterations $\{f_n(t)\}$ and $\{g_n(t)\}$ are uniformly convergent on I. Let $\lim_{n\to\infty} f_n(t) = \bar{f}(t)$, $\lim_{n\to\infty} g_n(t) = \bar{g}(t)$, then $\bar{f}(t) \le \bar{g}(t)$ for $t \in I$. Also, we have

$$\begin{aligned} \bar{f}(t) &= \min\{\bar{f}(t), \bar{g}(t)\} = F\left(t, \int_0^t \bar{f}(s)\,ds + u_0, \int_0^t \bar{g}(s)\,ds + v_0\right), \\ \bar{g}(t) &= \max\{\bar{f}(t), \bar{g}(t)\} = G\left(t, \int_0^t \bar{f}(s)\,ds + u_0, \int_0^t \bar{g}(s)\,ds + v_0\right), \quad t \in I. \end{aligned}$$

Then $u(t) = \int_0^t \bar{f}(s)\,ds + u_0$, $v(t) = \int_0^t \bar{g}(s)\,ds + v_0$ is a solution of problem (1) on I. The proof of Theorem 2.1 is complete. □

Define the successive iteration

$$\begin{aligned} &u_0(t) = 0, \qquad v_0(t) = 0, \\ &v_{n+1}(t) = F\left(t, \int_0^t u_n(s)\,ds + u_0, \int_0^t v_n(s)\,ds + v_0\right), \\ &u_{n+1}(t) = G\left(t, \int_0^t u_n(s)\,ds + u_0, \int_0^t v_n(s)\,ds + v_0\right), \quad t \in I, n = 0, 1, \ldots, \end{aligned}$$

then we have the following theorem.

Theorem 2.2 *Suppose that $F, G : I \times R \times R \to R$ are two continuous functions, Lipschitz continuous relative to their last two arguments. Moreover, if $F(t, f(t), g(t)) \le G(t, f(t), g(t))$ for all $f, g \in C(I)$ and $t \in I$, then both the successive iteration pairs $(P_n(t), Q_n(t))$ and $(U_n(t), V_n(t))$ converge to the solutions of problem* (1) *on I, where*

$$\begin{aligned} \bigl(P_n(t), Q_n(t)\bigr) &= \left(\int_0^t f_n(s)\,ds + u_0, \int_0^t g_n(s)\,ds + v_0\right), \\ \bigl(U_n(t), V_n(t)\bigr) &= \left(\int_0^t u_n(s)\,ds + u_0, \int_0^t v_n(s)\,ds + v_0\right), \quad n = 0, 1, \ldots. \end{aligned}$$

Proof It follows from the proof of Theorem 2.1 that the successive iteration pair $\{(P_n(t), Q_n(t))\}$ converges to the solution of problem (1) on I. Thus it is sufficient to prove that the pair $\{(U_n(t), V_n(t))\}$ converges to the solution of problem (1) on I. Since $F(t, f(t), g(t)) \le G(t, f(t), g(t))$ for all $f, g \in C(I)$ and $t \in I$, we have

$$v_n(t) = \min\{u_n(t), v_n(t)\} \quad \text{and} \quad u_n(t) = \max\{u_n(t), v_n(t)\}.$$

With similar arguments to the proof of Theorem 2.1, we see that $(u_n(t), v_n(t))$ uniformly converge to the continuous functions $u(t)$ and $v(t)$ on I. Thus the pair $(U_n(t), V_n(t))$ converges to the pair $(\int_0^t u(s)\,ds + u_0, \int_0^t v(s)\,ds + v_0)$, which is a solution of problem (1) on I. The proof is complete. □

Remark 2.1 The popular examples for the functions F and G in Theorem 2.2 are $F(t, x, y) = \min\{f_1(t, x, y), \ldots, f_k(t, x, y)\}$ and $G(t, x, y) = \max\{g_1(t, x, y), \ldots, g_k(t, x, y)\}$, where f_i and g_i are Lipschitz continuous relative to their last two arguments. Following the Appendix, we see that both min function and max function are also Lipschitz continuous.

Next, we show an example to illustrate that the initial value problem (1) admits a solution on I although some assumptions in Theorem 2.1 and Theorem 2.2 do not hold.

Example 2.1 Consider the following system with min-max terms:

$$\begin{cases} \min\{x'(t), y'(t)\} = ax(t) + 3, & x(0) = x_0, \\ \max\{x'(t), y'(t)\} = ay(t) + 1, & y(0) = y_0, \quad t \in [0,1], \end{cases} \tag{4}$$

where $a \neq 0$ is a constant.

By direct computations, the corresponding solutions of the subsystems

$$\begin{cases} x'(t) = ax(t) + 3, & x(0) = x_0, \\ y'(t) = ay(t) + 1, & y(0) = y_0 \end{cases}$$

and

$$\begin{cases} y'(t) = ax(t) + 3, & x(0) = x_0, \\ x'(t) = ay(t) + 1, & y(0) = y_0, \end{cases}$$

are given as follows:

$$\begin{cases} x(t) = (x_0 + \frac{3}{a})e^{at} - \frac{3}{a}, \\ y(t) = (y_0 + \frac{1}{a})e^{at} - \frac{1}{a} \end{cases} \tag{5}$$

and

$$\begin{cases} x(t) = (\frac{x_0+y_0}{2} + \frac{2}{a})e^{at} + (\frac{x_0-y_0}{2} + \frac{1}{a})e^{-at} - \frac{3}{a}, \\ y(t) = (\frac{x_0+y_0}{2} + \frac{2}{a})e^{at} - (\frac{x_0-y_0}{2} + \frac{1}{a})e^{-at} - \frac{1}{a}, \end{cases} \tag{6}$$

respectively.

For equations (5) and (6), it is sufficient for the pair $(x(t), y(t))$ to be a solution of (4) that $ax(t) + 3 \leq ay(t) + 1$ for all $t \in I$. Thus, if $a(x_0 - y_0) + 2 \leq 0$ then (5) is a solution of problem (4) in $[0,1]$. Similarly, if $a(x_0 - y_0) \leq 0$ then (6) is a solution of problem (4) in $[0,1]$. Thus for the problem (4), we can conclude the following.

Conclusion 2.1 If $a(x_0 - y_0) \leq -2$, then the initial value problem (4) has two solutions on $[0,1]$, which are given by (5) and (6), respectively. If $-2 < a(x_0 - y_0) \leq 0$, then the initial value problem (4) has a unique solution on $[0,1]$, which is given by (6). If $a(x_0 - y_0) > 0$, then the initial value problem (4) has no solution on $[0,1]$.

3 Application to the fuzzy differential equation

Let us denote by $\mathbb{R}_F$ the class of fuzzy subsets of the real axis (*i.e.* $u : R \to [0,1]$) satisfying the following properties [16, 24, 25]:

(i) u is normal, *i.e.* there exists $s_0 \in R$ such that $u(s_0) = 1$,
(ii) u is a convex fuzzy set (*i.e.* $u(ts + (1-t)r) \geq \min\{u(s), u(r)\}$, for $t \in [0,1]$, $s, r \in R$),
(iii) u is upper semicontinuous on R,
(iv) $\mathrm{cl}\{s \in R | u(s) > 0\}$ is compact where cl denotes the closure of a subset.

Then $\mathbb{R}_F$ is called the space of fuzzy numbers. For $x, y \in \mathbb{R}_F$, if there exists a fuzzy number $z \in \mathbb{R}_F$ such that $y + z = x$, then z is called the H-difference of x, y and is denoted by $x \ominus y$ (see *e.g.* [26]). If there exists a fuzzy number $z \in \mathbb{R}_F$ such that $y + z = x$ or $y = x + (-1)z$, then z is called the gH-difference of x and y and is denoted by $x \ominus_g y$ (see *e.g.* [17]).

If $v \in \mathbb{R}_f$ then the α-level set

$$[v]^\alpha = \{s|v(s) \geq \alpha\}, \quad \alpha \in (0,1] \quad \text{and} \quad [v]^0 = \bigcup_{\alpha \in (0,1]} [v]^\alpha$$

are closed bounded intervals, which is denoted $[v]^\alpha = [\underline{v}_\alpha, \bar{v}_\alpha]$.

Lemma 3.1 *For $x, y \in \mathbb{R}_f$ and k be scalar. For $\alpha \in (0,1]$,*

$$[x+y]^\alpha = [\underline{x}_\alpha + \underline{y}_\alpha, \bar{x}_\alpha + \bar{y}_\alpha],$$
$$[x-y]^\alpha = [\underline{x}_\alpha - \underline{y}_\alpha, \bar{x}_\alpha - \bar{y}_\alpha],$$
$$[x \odot y]^\alpha = [\min\{\underline{x}_\alpha\underline{y}_\alpha, \underline{x}_\alpha\bar{y}_\alpha, \bar{x}_\alpha\underline{y}_\alpha, \bar{x}_\alpha\bar{y}_\alpha\}, \max\{\underline{x}_\alpha\underline{y}_\alpha, \underline{x}_\alpha\bar{y}_\alpha, \bar{x}_\alpha\underline{y}_\alpha, \bar{x}_\alpha\bar{y}_\alpha\}].$$

Definition 3.1 [17] For fuzzy value functions $f : [a,b] \to \mathbb{R}_f$. If $x \in (a,b)$ and $x + h \in (a,b)$, the gH-derivative of f at x is defined as the limit

$$\lim_{h \to 0} \frac{f(x+h) \ominus_g f(x)}{h} = f'(x)$$

provided that the gH-differences exist for small h.

Also, the fuzzy derivative, in terms of level-cuts, is

$$[f'(x)]^\alpha = [\min\{\underline{f}'_\alpha(x), \bar{f}'_\alpha(x)\}, \max\{\underline{f}'_\alpha(x), \bar{f}'_\alpha(x)\}].$$

Example 3.1 Consider the following fuzzy differential equation:

$$u'(t) = au(t) + \gamma, \quad t \in [0,1], \qquad u(0) = u_0, \tag{7}$$

where $a \in R$, $u_0, \gamma \in \mathbb{R}_F$ are given by $[\gamma]^\alpha = [\alpha - 1, 1 - \alpha]$ and $[u_0]^\alpha = [\underline{u}_{0\alpha}, \bar{u}_{0\alpha}]$ and $\underline{u}_{0\alpha} < \bar{u}_{0\alpha}$ for some $\alpha \in [0,1]$, $'$ denotes the gH-derivative.

By the definition of the gH-derivative, the level-cut of a fuzzy derivative for a fuzzy-valued function $u(t)$ should be given by

$$[u'(t)]^\alpha = [\min\{\underline{u}'_\alpha(t), \bar{u}'_\alpha(t)\}, \max\{\underline{u}'_\alpha(t), \bar{u}'_\alpha(t)\}].$$

Thus the α-level-cut system of (7) should be

$$\begin{cases} \min\{\underline{u}'_\alpha(t), \bar{u}'_\alpha(t)\} = a\underline{u}_\alpha(t) + \alpha - 1, & \underline{u}_\alpha(0) = \underline{u}_{0\alpha}, \\ \max\{\underline{u}'_\alpha(t), \bar{u}'_\alpha(t)\} = a\bar{u}_\alpha(t) + 1 - \alpha, & \bar{u}_\alpha(0) = \bar{u}_{0\alpha}, \quad t \in [0,1]. \end{cases}$$

Noting the claims in Conclusion 2.1, we have

$$\begin{cases} \underline{u}_\alpha(t) = (\underline{u}_{0\alpha} + \frac{\alpha-1}{a})e^{at} - \frac{\alpha-1}{a}, \\ \bar{u}_\alpha(t) = (\bar{u}_{0\alpha} + \frac{1-\alpha}{a})e^{at} - \frac{1-\alpha}{a} \end{cases} \tag{8}$$

and

$$\begin{cases} \underline{u}_\alpha(t) = \frac{\underline{u}_{0\alpha}+\bar{u}_{0\alpha}}{2}e^{at} + (\frac{\underline{u}_{0\alpha}-\bar{u}_{0\alpha}}{2} + \frac{\alpha-1}{a})e^{-at} - \frac{\alpha-1}{a}, \\ \bar{u}_\alpha(t) = \frac{\underline{u}_{0\alpha}+\bar{u}_{0\alpha}}{2}e^{at} - (\frac{\underline{u}_{0\alpha}-\bar{u}_{0\alpha}}{2} + \frac{\alpha-1}{a})e^{-at} - \frac{1-\alpha}{a}. \end{cases} \tag{9}$$

Obviously, if both $a\underline{u}_\alpha(t) + \alpha - 1 \le a\bar{u}_\alpha(t) + 1 - \alpha$ and $\underline{u}_\alpha(t) \le \bar{u}_\alpha(t)$ hold on $[0,1]$, then the pair $(\underline{u}_\alpha(t), \bar{u}_\alpha(t))$ would be a solution of (7) on $[0,1]$. Thus, for (8), if

$$a(\underline{u}_{0\alpha} - \bar{u}_{0\alpha}) \le 0 \quad \text{and} \quad \left[(\underline{u}_{0\alpha} - \bar{u}_{0\alpha}) + \frac{2\alpha-2}{a}\right]e^{at} \le \frac{2\alpha-2}{a}$$

for all $t \in [0,1]$, $\alpha \in [0,1]$, then it is a solution of problem (7). Also, if

$$a(\underline{u}_{0\alpha} - \bar{u}_{0\alpha}) + 2(\alpha-1) \le 0 \quad \text{and} \quad \left[(\underline{u}_{0\alpha} - \bar{u}_{0\alpha}) + \frac{2\alpha-2}{a}\right]e^{-at} \le \frac{2\alpha-2}{a}$$

for all $t \in [0,1]$, $\alpha \in [0,1]$, then (9) is a solution of problem (7). Next we would discuss the solutions of problems (7) by three cases: $a > 0$, $a < 0$, and $a = 0$.

Case 1. $a > 0$: If $\underline{u}_{0\alpha} - \bar{u}_{0\alpha} \le \frac{2\alpha-2}{a}(e^a - 1)$, then both (8) and (9) are solutions of problem (7). If $\frac{2\alpha-2}{a}(e^a - 1) < \underline{u}_{0\alpha} - \bar{u}_{0\alpha} \le 0$, then there is a unique solution (8) on $[0,1]$.

Case 2. $a < 0$: If $2(\alpha-1)(e^a - 1) \le a(\underline{u}_{0\alpha} - \bar{u}_{0\alpha}) \le 2(1-\alpha)$, by direct computation, we see that there is a unique solution of problem (7) on $[0,1]$, which is given by (9).

Case 3. $a = 0$: In this case, it is easy to see that, when $2(1-\alpha) \le \bar{u}_{0\alpha} - \underline{u}_{0\alpha}$ for all $\alpha \in [0,1]$, both

$$\begin{cases} \underline{u}_\alpha(t) = (\alpha-1)t + \underline{u}_{0\alpha}, \\ \bar{u}_\alpha(t) = (1-\alpha)t + \bar{u}_{0\alpha} \end{cases}$$

and

$$\begin{cases} \underline{u}_\alpha(t) = (1-\alpha)t + \underline{u}_{0\alpha}, \\ \bar{u}_\alpha(t) = (\alpha-1)t + \bar{u}_{0\alpha}, \end{cases}$$

are solutions of problems (7) on $[0,1]$.

When $2(1-\alpha) > \bar{u}_{0\alpha} - \underline{u}_{0\alpha}$ for all $\alpha \in [0,1]$, then

$$\begin{cases} \underline{u}_\alpha(t) = (\alpha-1)t + \underline{u}_{0\alpha}, \\ \bar{u}_\alpha(t) = (1-\alpha)t + \bar{u}_{0\alpha}, \end{cases}$$

is the unique solution of problems (7) on $[0,1]$.

Example 3.2 Consider the following fuzzy differential equation:

$$u'(t) = a \odot u(t) + \gamma, \quad t \in J, \qquad u(0) = u_0, \tag{10}$$

where $J \subset R$ is an interval, $a, u_0, \gamma \in \mathbb{R}_F$ are given by $[a]^\alpha = [\underline{a}_\alpha, \bar{a}_\alpha]$ with $\underline{a}_\alpha > 0$, $[\gamma]^\alpha = [\alpha - 1, 1-\alpha]$ and $[u_0]^\alpha = [\underline{u}_{0\alpha}, \bar{u}_{0\alpha}]$ and $\underline{u}_{0\alpha} < \bar{u}_{0\alpha}$ for some $\alpha \in [0,1]$, $'$ denotes the gH-derivative, $\odot$ denotes the fuzzy product of two fuzzy number.

First of all, by the definitions of gH-derivative and fuzzy product, we can translate the fuzzy equation (10) into the α-level-cuts systems as follows:

$$\begin{cases} \min\{\underline{u}'_\alpha(t), \bar{u}'_\alpha(t)\} = \min\{\underline{a}_\alpha \underline{u}_\alpha(t), \underline{a}_\alpha \bar{u}_\alpha(t), \bar{a}_\alpha \underline{u}_\alpha(t), \bar{a}_\alpha \bar{u}_\alpha(t)\} + \alpha - 1, \\ \max\{\underline{u}'_\alpha(t), \bar{u}'_\alpha(t)\} = \max\{\underline{a}_\alpha \underline{u}_\alpha(t), \underline{a}_\alpha \bar{u}_\alpha(t), \bar{a}_\alpha \underline{u}_\alpha(t), \bar{a}_\alpha \bar{u}_\alpha(t)\} + 1 - \alpha, \\ \underline{u}_\alpha(0) = \underline{u}_{0\alpha}, \qquad \bar{u}_\alpha(0) = \bar{u}_{0\alpha}, \quad t \in J. \end{cases}$$

Since $\underline{a}_\alpha > 0$, we have $\underline{a}_\alpha \underline{u}_\alpha(t) \le \underline{a}_\alpha \bar{u}_\alpha(t)$ and $\bar{a}_\alpha \underline{u}_\alpha(t) \le \bar{a}_\alpha \bar{u}_\alpha(t)$ for all $t \in J$. Thus

$$\min\bigl\{\underline{a}_\alpha \underline{u}_\alpha(t), \underline{a}_\alpha \bar{u}_\alpha(t), \bar{a}_\alpha \underline{u}_\alpha(t), \bar{a}_\alpha \bar{u}_\alpha(t)\bigr\} = \min\bigl\{\underline{a}_\alpha \underline{u}_\alpha(t), \bar{a}_\alpha \underline{u}_\alpha(t)\bigr\}$$

and

$$\max\bigl\{\underline{a}_\alpha \underline{u}_\alpha(t), \underline{a}_\alpha \bar{u}_\alpha(t), \bar{a}_\alpha \underline{u}_\alpha(t), \bar{a}_\alpha \bar{u}_\alpha(t)\bigr\} = \max\bigl\{\underline{a}_\alpha \bar{u}_\alpha(t), \bar{a}_\alpha \bar{u}_\alpha(t)\bigr\}.$$

Thus the α-level-cuts systems can be rewriten as

$$\begin{cases} \min\{\underline{u}'_\alpha(t), \bar{u}'_\alpha(t)\} = \min\{\underline{a}_\alpha \underline{u}_\alpha(t), \bar{a}_\alpha \underline{u}_\alpha(t)\} + \alpha - 1, \\ \max\{\underline{u}'_\alpha(t), \bar{u}'_\alpha(t)\} = \max\{\underline{a}_\alpha \bar{u}_\alpha(t), \bar{a}_\alpha \bar{u}_\alpha(t)\} + 1 - \alpha, \\ \underline{u}_\alpha(0) = \underline{u}_{0\alpha}, \qquad \bar{u}_\alpha(0) = \bar{u}_{0\alpha}, \quad t \in J. \end{cases} \tag{11}$$

It follows from Theorem 2.2 that both of the pairs $(P_n(t), Q_n(t))$ and $(U_n(t), V_n(t))$ converge to the solutions of the problem (11).

In fact, if $\underline{a}_\alpha \underline{u}_\alpha(t) \le \bar{a}_\alpha \underline{u}_\alpha(t)$ then $\underline{u}_\alpha(t) \ge 0$. Thus $\max\{\underline{a}_\alpha \bar{u}_\alpha(t), \bar{a}_\alpha \bar{u}_\alpha(t)\} = \bar{a}_\alpha \bar{u}_\alpha(t)$. Similarly, if $\underline{a}_\alpha \bar{u}_\alpha(t) \ge \bar{a}_\alpha \bar{u}_\alpha(t)$ then $\bar{u}_\alpha(t) \le 0$. Thus $\max\{\underline{a}_\alpha \bar{u}_\alpha(t), \bar{a}_\alpha \bar{u}_\alpha(t)\} = \underline{a}_\alpha \bar{u}_\alpha(t)$. Then the α-cuts system (11) can be reduced to the following subsystem:

$$\begin{cases} \min\{\underline{u}'_\alpha(t), \bar{u}'_\alpha(t)\} = \underline{a}_\alpha \underline{u}_\alpha(t) + \alpha - 1, \\ \max\{\underline{u}'_\alpha(t), \bar{u}'_\alpha(t)\} = \bar{a}_\alpha \bar{u}_\alpha(t) + 1 - \alpha, \\ \underline{u}_\alpha(0) = \underline{u}_{0\alpha}, \qquad \bar{u}_\alpha(0) = \bar{u}_{0\alpha}, \quad t \in J \end{cases} \tag{12}$$

or

$$\begin{cases} \min\{\underline{u}'_\alpha(t), \bar{u}'_\alpha(t)\} = \bar{a}_\alpha \underline{u}_\alpha(t) + \alpha - 1, \\ \max\{\underline{u}'_\alpha(t), \bar{u}'_\alpha(t)\} = \underline{a}_\alpha \bar{u}_\alpha(t) + 1 - \alpha, \\ \underline{u}_\alpha(0) = \underline{u}_{0\alpha}, \qquad \bar{u}_\alpha(0) = \bar{u}_{0\alpha}, \quad t \in J. \end{cases} \tag{13}$$

Following similar arguments to Example 3.1, we see that there would be two solutions for the problems (12) and (13) subject to a certain additional restriction. The details can be found in Figure 1 and Figure 2.

4 Two counterexamples

Recently, Alikhani *et al.* [27] considered the following initial value problem for the first-order fuzzy integro-differential equation of Volterra type:

$$u'(t) = f\bigl(t, u(t)\bigr) + \int_a^t k\bigl(t, s, u(s)\bigr)\, ds, \quad t \in J, \qquad u(a) = u_0, \tag{14}$$

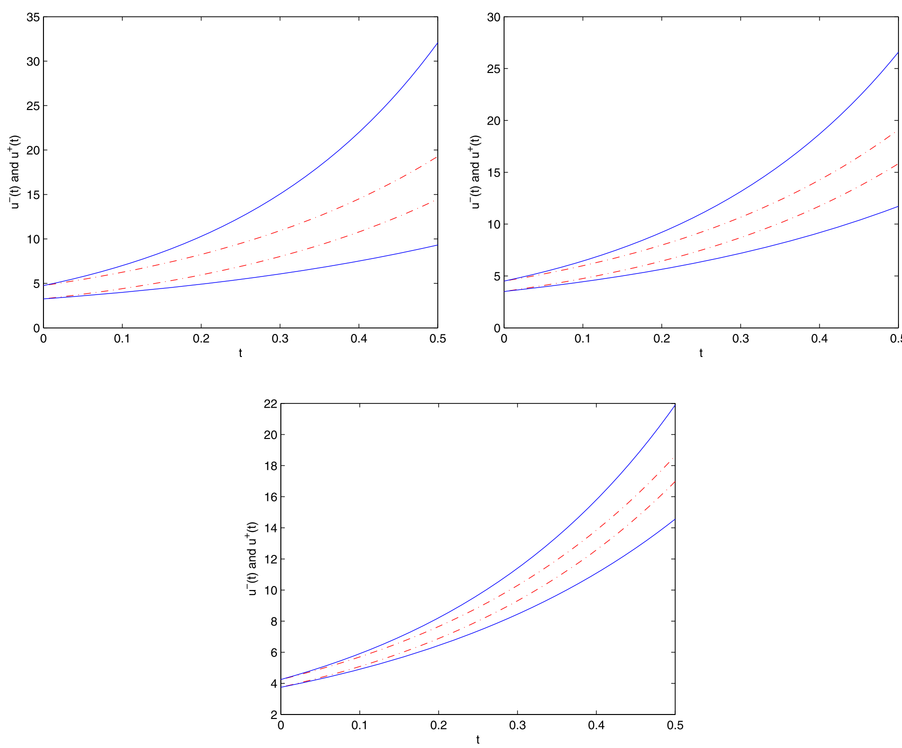

Figure 1 Two level-set solutions of fuzzy differential equation (10) for $J = [0, \frac{1}{2}]$, $[a]^\alpha = [2 + \alpha, 4 - \alpha]$, $[u_0]^\alpha = [3 + \alpha, 5 - \alpha]$. We have $\alpha = \frac{1}{4}$ (left), $\frac{1}{2}$ (right) and $\frac{3}{4}$ (bottom). All these level-set solutions are assigned to the subsystem (12). The dash lines (red) and solid lines (blue) sketch the two level-cut solutions, respectively.

where $J \subset R$ is an interval, $f \in C(J \times \mathbb{R}_F, \mathbb{R}_F)$ and $k \in C(J \times J \times \mathbb{R}_F, \mathbb{R}_F)$. They established the following two results. Unfortunately, both of them are wrong. Two counterexamples are presented in this section.

Theorem 4.1 ([27], Theorem 1) *Let $f : J \times \mathbb{R}_F \to \mathbb{R}_F$ and $k : J \times J \times \mathbb{R}_F \to \mathbb{R}_F$ be bonded continuous functions. Then the problem* (14) *has at least a proper solution which is* (i)*-differentiable on J. Moreover, if f and k are Lipschitz continuous relative to their last argument, i.e. there exist real numbers $L_1, L_2 > 0$ such that*

$$D\bigl(k(t,s,x), k(t,s,y)\bigr) \le L_1 D(x,y) \quad \textit{for } (t,s,x), (t,s,y) \in J \times J \times \mathbb{R}_F,$$
$$D\bigl(f(t,x), f(t,y)\bigr) \le L_2 D(x,y) \quad \textit{for } (t,x), (t,y) \in J \times \mathbb{R}_F.$$

Then the proper solution of the problem (14) *is unique on J.*

Theorem 4.2 ([27], Theorem 2) *Let $f : J \times \mathbb{R}_F \to \mathbb{R}_F$ and $k : J \times J \times \mathbb{R}_F \to \mathbb{R}_F$ be bounded continuous and Lipschitz continuous functions as mentioned in Theorem* 2.1. *Let the sequence $u_n : J \to \mathbb{R}_F$ given by*

$$u_0(t) = u_0, \qquad u_{n+1}(t) = u_0 \ominus (-1) \int_a^t f\bigl(s, u_n(s)\bigr)\, ds \ominus (-1) \int_a^t \int_a^s k\bigl(s, r, u_n(r)\bigr)\, dr\, ds$$

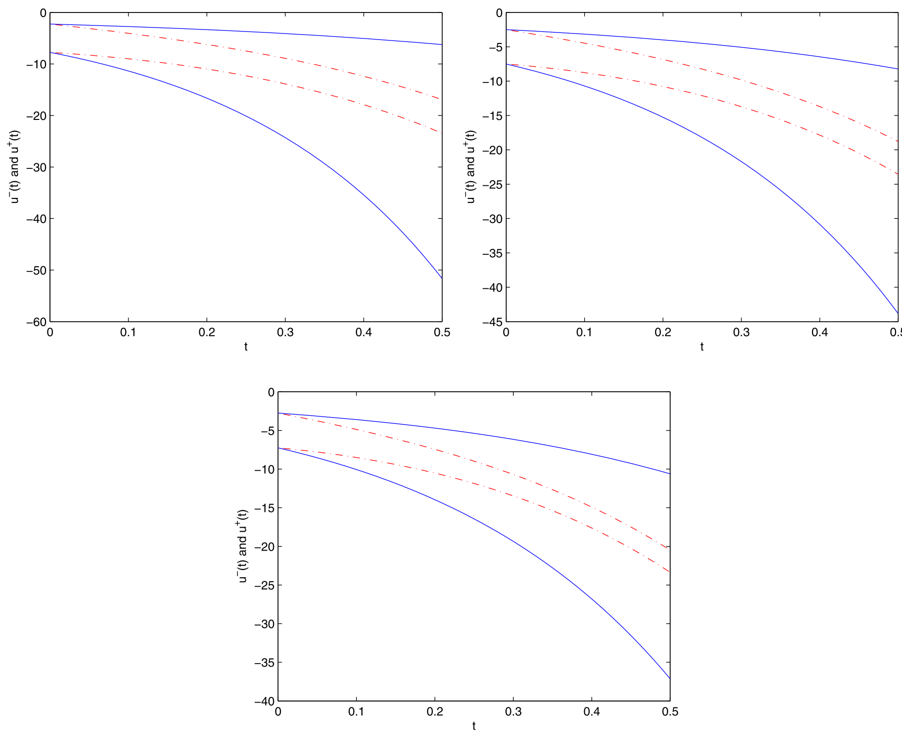

Figure 2 Two level-set solutions of fuzzy differential equation (10) for $J = [0, \frac{1}{2}]$, $[a]^\alpha = [2+\alpha, 4-\alpha]$, $[u_0]^\alpha = [-8+\alpha, -2-\alpha]$. We have $\alpha = \frac{1}{4}$ (left), $\frac{1}{2}$ (right) and $\frac{3}{4}$ (bottom). All these level-set solutions are assigned to the subsystem (13). The dash lines (red) and solid lines (blue) sketch the two level-cut solutions, respectively.

be defined for any $n \in \mathbb{N}$. Then the problem (14) *has a unique proper solution which is* (ii)-*differentiable on J. Furthermore, the successive iteration*

$$u_0(t) = u_0, \qquad u_{n+1}(t) = u_0 \ominus (-1) \int_a^t f\big(s, u_n(s)\big)\, ds \ominus (-1) \int_a^t \int_a^s k\big(s, r, u_n(r)\big)\, dr\, ds$$

converges to this solution.

Example 4.1 Consider the following fuzzy differential equation:

$$u'(t) = 3(t-2)^2 \mu_0, \quad t \in [0,2], \qquad u(0) = \mu_1 \in \mathbb{R}_F, \tag{15}$$

where μ_0, μ_1 are given by

$$\mu_0(x) = \begin{cases} 0, & \text{as } x < 0, \\ 1-(x-1)^2, & \text{as } x \in [0,2], \\ 0, & \text{as } x > 2 \end{cases}$$

and

$$\mu_1(x) = \begin{cases} 0, & \text{as } x > 0, \\ 1-(x+1)^2, & \text{as } x \in [-2,0], \\ 0, & \text{as } x < -2. \end{cases}$$

Let $a = 0, J = [0, 2], k(t, s, u) \equiv 0, f(t, u(t)) = 3(t-2)^2\mu_1$, then all the assumptions in Theorem 4.1 hold. Unfortunately, there are no the proper solutions of the problem (15) on J. More details now follow.

Theorem 4.3 *There is a unique mixed solution and no proper solution for* (15) *on* J.

Proof Since $[\mu_0]^\alpha = [1-\sqrt{1-\alpha}, 1+\sqrt{1-\alpha}]$ and $[\mu_1]^\alpha = [-1-\sqrt{1-\alpha}, -1+\sqrt{1-\alpha}]$, if u is (i)-differentiable, then we transform equation (15) into the α-level system

$$\left[\underline{u}'^\alpha(t), \bar{u}'^\alpha(t)\right] = 3(t-2)^2[\mu_0]^\alpha = 3(t-2)^2[1-\sqrt{1-\alpha}, 1+\sqrt{1-\alpha}]$$

subject to the initial conditions

$$\left[u(0)\right]^\alpha = [\mu_1]^\alpha = [-1-\sqrt{1-\alpha}, -1+\sqrt{1-\alpha}]. \tag{16}$$

Then

$$\begin{aligned}\underline{u}^\alpha(t) &= -1-\sqrt{1-\alpha} + (t-2)^3(1-\sqrt{1-\alpha}),\\ \bar{u}^\alpha(t) &= -1+\sqrt{1-\alpha} + (t-2)^3(1-\sqrt{1+\alpha}).\end{aligned}$$

Noting that $\underline{u}^\alpha(t) \le \bar{u}^\alpha(t)$, we see that $t \in [1, 2]$. This implies that the problem (15) has an (i)-solution on $[1, 2]$. Similarly, if u is (ii)-differentiable, then we transform equation (15) into the α-level system

$$\left[\bar{u}'^\alpha(t), \underline{u}'^\alpha(t)\right] = 3(t-2)^2[1-\sqrt{1-\alpha}, 1+\sqrt{1-\alpha}]$$

subject to the same initial conditions (16). Then for $t \in [0, 1]$,

$$\begin{aligned}\bar{u}^\alpha(t) &= -1-\sqrt{1-\alpha} + (t-2)^3(1-\sqrt{1-\alpha}),\\ \underline{u}^\alpha(t) &= -1+\sqrt{1-\alpha} + (t-2)^3(1-\sqrt{1+\alpha}).\end{aligned}$$

Then the problem (15) has a (ii)-solution on $[0, 1]$. Thus there is a unique mixed solution and no proper solution for (15) on J. □

Example 4.2 Let us consider the linear first-order fuzzy differential equation

$$y'(t) = y(t), \qquad y(0) = \gamma,$$

where $[\gamma]^\alpha = [\alpha - 1, 1-\alpha]$. By direct computations, all the assumptions in Theorem 4.1 and Theorem 4.2 hold for the problem (15); then there would have to be a unique proper solution. But there are two proper solutions of problem (15) ([3], Example 4.1): (i) the solution $y(t) = e^t\gamma$ and (ii) the solution $y(t) = \cosh(t)\gamma \ominus (-1)\sinh(t)\gamma$. Thus the results in both Theorem 4.1 and Theorem 4.2 are invalid.

5 Conclusions

As we know, the crisp differential equations are popular models to approach the various phenomena in the real world when the conditions/hypotheses are clear. Also, the stochastic fuzzy differential equation is a candidate to describe the occurrence of the phenomenon or the unknown initial data. In this paper, we build a relationship between differential system with min-max terms and fuzzy differential equations, and we investigate the existence and multiple solutions for a class of first-order differential system with min-max terms. As applications, the existence results for some linear fuzzy differential equations are obtained. Work in progress uses fixed point theorems for nonlinear operators to study the existence and multiple solutions of interval and fuzzy differential equations (see *e.g.* [28, 29]).

Appendix

Lemma A.1 *If $f_i : R \times R \to R$ $(i = 1, 2, \ldots, m)$ are Lipschitz continuous functions relative to their two arguments, then both the minimum function* $\min\{f_i : i = 1, 2, \ldots, m\}$ *and the maximum function* $\max\{f_i : i = 1, 2, \ldots, m\}$ *are also Lipschitz continuous functions.*

Proof Assume that the Lipschitz constants of f_i are L_{i1} and L_{i2}, that is, for all $(x, y), (u, v) \in R^2$, $|f_i(x, y) - f_i(u, v)| \le L_{i1}|x - u| + L_{i2}|y - v|$. Let $f_{i_0}(x, y) = \min\{f_i(x, y) : i = 1, 2, \ldots, m\}$ and $f_{j_0}(u, v) = \min\{f_i(u, v) : i = 1, 2, \ldots, m\}$, where $1 \le i_0, j_0 \le m$. Noting that $f_{i_0}(u, v) - f_{j_0}(u, v) \ge 0$ and $f_{i_0}(x, y) - f_{j_0}(x, y) \le 0$, we have

$$\begin{aligned} f_{i_0}(x, y) - f_{j_0}(u, v) &= f_{i_0}(x, y) - f_{i_0}(u, v) + f_{i_0}(u, v) - f_{j_0}(u, v) \ge f_{i_0}(x, y) - f_{i_0}(u, v), \\ f_{i_0}(x, y) - f_{j_0}(u, v) &= f_{i_0}(x, y) - f_{j_0}(x, y) + f_{j_0}(x, y) - f_{j_0}(u, v) \le f_{j_0}(x, y) - f_{j_0}(u, v). \end{aligned}$$

Thus

$$f_{i_0}(x, y) - f_{i_0}(u, v) \le f_{i_0}(x, y) - f_{j_0}(u, v) \le f_{j_0}(x, y) - f_{j_0}(u, v).$$

This implies that

$$\begin{aligned} \left|f_{i_0}(x, y) - f_{j_0}(u, v)\right| &\le \max\left\{\left|f_{j_0}(x, y) - f_{j_0}(u, v)\right|, \left|f_{i_0}(x, y) - f_{i_0}(u, v)\right|\right\} \\ &\le L_1|x - u| + L_2|y - v|, \end{aligned}$$

where $L_j = \max\{L_{ij} : i = 1, 2, \ldots, m\}$ $(j = 1, 2)$. Thus, the minimum function $\min\{f_i : i = 1, 2, \ldots, m\}$ is a Lipschitz continuous function. Similarly, the maximum function $\max\{f_i : i = 1, 2, \ldots, m\}$ is also a Lipschitz continuous function. The proof is complete. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All results in this paper have been obtained by all authors' in mutual discussion. The first author's main contributions are the deduction of the theorems and examples; the second author's main contributions are writing and simulation. Both authors read and approved the final manuscript.

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