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Properties of positive solutions for the operator equation $Ax = \lambda x$ and applications to fractional differential equations with integral boundary conditions

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Abstract
In this article we present a new fixed point theorem for a class of generalized concave operators and we establish some properties of positive solutions for the operator equation $Ax = \lambda x$. Based on them, the existence and uniqueness of positive solutions for a class of fractional differential equations with integral boundary conditions is proved. Moreover, we present some properties of positive solutions to the boundary value problem dependent on the parameter.

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Keywords: positive solution; generalized concave operator; parameter; fractional differential equation; integral boundary condition

1 Introduction

Owing to various applications of integral boundary value problems in applied fields such as blood flow problems, chemical engineering, thermo-elasticity, underground water flow, and population dynamics, the existence of solutions for fractional differential equations with integral boundary conditions has been extensively studied in recent years (see [1–12] and the references therein). In these papers, most of the authors have investigated the existence and multiplicity of positive solutions. For example, by means of the monotone iteration method, Sun and Zhao [7] investigated the existence of positive solutions for a class of Riemann-Liouville fractional differential equations with integral boundary conditions. Zhao *et al.* [8] used Krasnosel'skii's fixed point theorem to obtain the existence and nonexistence of positive solutions for a fractional differential equation with integral boundary conditions. In [9], by using the properties of the Green function, a u_0-bounded function, and fixed point index theory, the authors obtained the existence of positive solutions for a class of nonlinear fractional differential equations with integral boundary conditions and a parameter. But the uniqueness of positive solutions is not treated in [7–9]. As far as we know, there are few papers reported on the integral boundary conditions of fractional differential equations with a parameter. In particular, there are no clear explanations of the relation between positive solutions and the parameter. The reason is that many methods used in the literature are independent of the parameters. So we need some properties of positive solutions for the operator equation

$$Ax = \lambda x, \tag{1.1}$$

where A is a generalized concave operator and $\lambda > 0$ is an eigenvalue. In this article, we first state and prove a new fixed point theorem for a class of generalized concave operators. Further, we establish some properties of positive solutions for the operator equation (1.1). Here we do not assume the existence of upper-lower solutions for the operator A, which is usually done in the literature (for example, see [13, 14]). As applications, we utilize the main results for (1.1) to the following fractional differential equation with integral boundary conditions:

$$\begin{cases} D_{0^+}^{\alpha} u(t) + \lambda f(t, u(t)) = 0, & t \in (0,1), \\ u(0) = u'(0) = 0, & u(1) = \beta \int_0^1 u(s)\,ds, \end{cases} \tag{1.2}$$

where $2 < \alpha \leq 3$, $0 < \beta < \alpha$, $\lambda > 0$ is a parameter. $D_{0^+}^{\alpha}$ is the Riemann-Liouville fractional derivative of order α, which is defined as follows:

$$D_{0^+}^{\alpha} u(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t \frac{u(s)}{(t-s)^{\alpha+1-n}}\,ds, \quad n = [\alpha] + 1,$$

here Γ denotes the Euler gamma function and $[\alpha]$ denotes the integer part of number α provided that the right side is point-wise defined on $(0, +\infty)$; see [15]. We establish the existence and uniqueness of positive solutions for problem (1.2) with any given parameter. Moreover, we present some properties of positive solutions for problem (1.2) dependent on the parameter.

2 Properties of positive solutions for the operator equation $Ax = \lambda x$

For the discussion of this section, we first list some basic notations, concepts in ordered Banach spaces. For the convenience of the reader, we refer to [13, 14, 16] for details.

Let $(E, \|\cdot\|)$ be a real Banach space which is partially ordered by a cone $P \subset E$, *i.e.*, $x \leq y$ if and only if $y - x \in P$. If $x \leq y$ and $x \neq y$, then we denote $x < y$ or $y > x$. By θ we denote the zero element of E. A non-empty closed convex set $P \subset E$ is a cone if it satisfies (i) $x \in P$, $r \geq 0 \Rightarrow rx \in P$; (ii) $x \in P$, $-x \in P \Rightarrow x = \theta$.

P is called normal if there is a constant $N > 0$ such that, for all $x, y \in E$, $\theta \leq x \leq y$ implies $\|x\| \leq N\|y\|$; in this case N is the infimum of such constants, it is called the normality constant of P. We say that an operator $A : E \to E$ is increasing if $x \leq y$ implies $Ax \leq Ay$.

For all $x, y \in E$, the notation $x \sim y$ means that there exist $\lambda > 0$ and $\mu > 0$ such that $\lambda x \leq y \leq \mu x$. Clearly, $\sim$ is an equivalence relation. Given $h > \theta$ (*i.e.*, $h \geq \theta$ and $h \neq \theta$), we denote by P_h the set $P_h = \{x \in E \mid x \sim h\}$. It is easy to see that $P_h \subset P$.

Lemma 2.1 (see Theorem 2.1 in [16]) *Let P be a normal cone in a real Banach space E and $h > \theta$. Assume that*:

(D_1) *$A : P \to P$ is increasing and $Ah + x_0 \in P_h$ with $x_0 \in P$;*
(D_2) *for $x \in P$ and $t \in (0,1)$, there exists $\varphi(t) \in (t,1)$ such that $A(tx) \geq \varphi(t)Ax$.*

Then the operator equation $x = Ax + x_0$ has a unique solution in P_h.

Now we consider the operator equation

$$Ax = x. \tag{2.1}$$

Theorem 2.1 *Let P be a normal cone in a real Banach space E, $h > \theta$, and let $A : P \to P$ be an increasing operator, satisfying*:

(i) *there is $h_0 \in P_h$ such that $Ah_0 \in P_h$*;

(ii) *for any $x \in P$ and $t \in (0,1)$, there exists $\varphi(t) \in (t,1)$ such that $A(tx) \geq \varphi(t)Ax$.*

Then:

(1) *the operator equation* (2.1) *has a unique solution x^* in P_h*;

(2) *for any initial value $x_0 \in P_h$, constructing successively the sequence $x_n = Ax_{n-1}$, $n = 1, 2, \ldots$, we have $x_n \to x^*$ as $n \to \infty$.*

Remark 2.1 We say an operator A is generalized concave if it satisfies the condition (ii) in Theorem 2.1; see [16].

Proof of Theorem 2.1 From the condition (ii), $Ax = A(t \cdot \frac{1}{t}x) \geq \varphi(t)A(\frac{1}{t}x)$, $t \in (0,1)$. Thus we have

$$A\left(\frac{1}{t}x\right) \leq \frac{1}{\varphi(t)}Ax, \quad \forall x \in P, t \in (0,1). \tag{2.2}$$

Since $Ah_0 \in P_h$, there exist constants $\lambda_1, \lambda_2 > 0$ such that $\lambda_1 h \leq Ah_0 \leq \lambda_2 h$. Also from $h_0 \in P_h$, there exists a constant $t_0 \in (0,1)$ such that $t_0 h \leq h_0 \leq \frac{1}{t_0}h$. Then from (2.2) and the monotonicity of operator A, we have

$$Ah \geq A(t_0 h_0) \geq \varphi(t_0)Ah_0 \geq \varphi(t_0)\lambda_1 h,$$
$$Ah \leq A\left(\frac{1}{t_0}h_0\right) \leq \frac{1}{\varphi(t_0)}Ah_0 \leq \frac{\lambda_2}{\varphi(t_0)}h.$$

Note that $\lambda_1\varphi(t_0), \frac{\lambda_2}{\varphi(t_0)} > 0$, we can get $Ah \in P_h$.

Now we show that $A : P_h \to P_h$. For any $x \in P_h$, we can choose a sufficiently small number $t_1 \in (0,1)$ such that $t_1 h \leq x \leq \frac{1}{t_1}h$. Then

$$Ax \geq A(t_1 h) \geq \varphi(t_1)Ah, \qquad Ax \leq A\left(\frac{1}{t_1}h\right) \leq \frac{1}{\varphi(t_1)}Ah.$$

It follows from $Ah \in P_h$ that $Ax \in P_h$. That is, $A : P_h \to P_h$. Letting $x_0 = \theta$ in Lemma 2.1, we can easily get the conclusion (1).

Next we construct successively the sequence $x_n = Ax_{n-1}$, $n = 1, 2, \ldots$ for any initial value $x_0 \in P_h$. Since $x_0 \in P_h$ and $Ax_0 \in P_h$, we can choose a sufficiently small number $t_2 \in (0,1)$ such that $t_2 x_0 \leq Ax_0 \leq \frac{1}{t_2}x_0$. Note that $t_2 < \varphi(t_2) < 1$, we can take a positive integer k such that $(\frac{\varphi(t_2)}{t_2})^k \geq \frac{1}{t_2}$. Put $u_0 = t_2^k x_0$, $v_0 = \frac{1}{t_2^k}x_0$. Then $u_0, v_0 \in P_h$ and $u_0 \leq x_0 \leq v_0$. Take any $r \in (0, t_2^{2k}]$, then $r \in (0,1)$ and $u_0 \geq rv_0$. By the monotonicity of A, we have $Au_0 \leq Ax_0 \leq Av_0$. From the condition (ii),

$$\begin{aligned} Au_0 &= A\bigl(t_2^k x_0\bigr) = A\bigl(t_2 \cdot t_2^{k-1}x_0\bigr) \geq \varphi(t_2)A\bigl(t_2^{k-1}x_0\bigr) \\ &= \varphi(t_2)A\bigl(t_2 \cdot t_2^{k-2}x_0\bigr) \geq \varphi(t_2) \cdot \varphi(t_2)A\bigl(t_2^{k-2}x_0\bigr) \end{aligned}$$

$$\begin{aligned}&\geq \cdots \geq \varphi(t_2)\cdot\varphi(t_2)\cdots\varphi(t_2)\cdot\varphi(t_2)Ax_0\\ &\geq \bigl(\varphi(t_2)\bigr)^k Ax_0 \geq \bigl(\varphi(t_2)\bigr)^k t_2 h \geq t_2^k x_0 = u_0.\end{aligned}$$

From (2.2), we get

$$\begin{aligned}Av_0 &= A\left(\frac{1}{t_2^k}x_0\right) = A\left(\frac{1}{t_2}\cdot\frac{1}{t_2^{k-1}}x_0\right)\\ &\leq \frac{1}{\varphi(t_2)}A\left(\frac{1}{t_2^{k-1}}x_0\right) = \frac{1}{\varphi(t_2)}A\left(\frac{1}{t_2}\cdot\frac{1}{t_2^{k-2}}x_0\right)\\ &\leq \frac{1}{\varphi(t_2)}\cdot\frac{1}{\varphi(t_2)}A\left(\frac{1}{t_2^{k-2}}x_0\right) \leq \cdots\\ &\leq \frac{1}{\varphi(t_2)\cdot\varphi(t_2)\cdots\varphi(t_2)}Ax_0\\ &\leq \frac{1}{\varphi(t_2)^k}\cdot\frac{1}{t_2}x_0 \leq \frac{1}{t_2^k}x_0 = v_0.\end{aligned}$$

Thus we have $u_0 \leq Au_0 \leq Av_0 \leq v_0$. Let $u_n = Au_{n-1}$, $v_n = Av_{n-1}$, $n = 1,2,\ldots$. Evidently, $u_1 \leq x_1 \leq v_1$. In a general way, we obtain $u_n \leq x_n \leq v_n$, $n = 1,2,\ldots$ and then

$$u_0 \leq u_1 \leq \cdots \leq u_n \leq \cdots \leq v_n \leq \cdots \leq v_1 \leq v_0.$$

Because $u_0 \geq rv_0$, we have $u_n \geq u_0 \geq rv_0 \geq rv_n$, $n = 1,2,\ldots$. Let

$$r_n = \sup\{t > 0 \mid u_n \geq tv_n\}, \quad n = 1,2,\ldots.$$

Thus we have $u_n \geq r_n v_n$, $n = 1,2,\ldots$ and then

$$u_{n+1} \geq u_n \geq r_n v_n \geq r_n v_{n+1}, \quad n = 1,2,\ldots.$$

So $r_{n+1} \geq r_n$, *i.e.*, $\{r_n\}$ is increasing with $\{r_n\} \subset (0,1]$. Suppose $r_n \to r^*$ as $n \to \infty$, then $r^* = 1$. Otherwise, $0 < r^* < 1$. Note that $r_n \leq r^*$. We obtain

$$u_{n+1} = Au_n \geq A(r_n v_n) = A\left(\frac{r_n}{r^*}r^* v_n\right) \geq \frac{r_n}{r^*}A\bigl(r^* v_n\bigr) \geq \frac{r_n}{r^*}\cdot\varphi\bigl(r^*\bigr)Av_n.$$

By the definition of $r_n, r_{n+1} \geq \frac{r_n}{r^*}\cdot\varphi(t^*)$. Let $n \to \infty$, we get $r^* \geq \varphi(r^*) > r^*$, which is a contradiction. Thus, $\lim_{n\to\infty} r_n = 1$. For any natural number p, we have

$$\begin{aligned}&\theta \leq u_{n+p} - u_n \leq v_n - u_n \leq v_n - r_n v_n = (1-r_n)v_n \leq (1-r_n)v_0,\\ &\theta \leq v_n - v_{n+p} \leq v_n - u_n \leq (1-r_n)v_0.\end{aligned}$$

Since P is normal, we have

$$\begin{aligned}&\|u_{n+p} - u_n\| \leq N(1-r_n)\|v_0\| \to 0 \quad (\text{as } n \to \infty),\\ &\|v_n - v_{n+p}\| \leq N(1-r_n)\|v_0\| \to 0 \quad (\text{as } n \to \infty),\end{aligned}$$

where N is the normality constant. So $\{u_n\}$ and $\{v_n\}$ are Cauchy sequences in complete space E, and there exist u^*, v^* such that $u_n \to u^*$, $v_n \to v^*$ as $n \to \infty$. It is easy to see that $u_n \leq u^* \leq v^* \leq v_n$ with $u^*, v^* \in P_h$ and $\theta \leq v^* - u^* \leq v_n - u_n \leq (1 - r_n)v_0$. Further, $\|v^* - u^*\| \leq N(1 - t_n)\|v_0\| \to 0$ $(n \to \infty)$, and thus $u^* = v^*$. It follows from conclusion (1) that $u^* = v^* = x^*$ and then $u_n \to x^*$, $v_n \to x^*$ as $n \to \infty$. Thus, from the normality of P, $x_n \to x^*$ as $n \to \infty$. □

Next we state and prove some properties of positive solutions for the operator equation (1.1).

Theorem 2.2 *Assume that all the conditions of Theorem* 2.1 *hold. Let* x_λ $(\lambda > 0)$ *denote the unique solution of operator equation* (1.1). *Then we have the following conclusions*:

(i) x_λ *is strictly decreasing in* λ, *that is*, $0 < \lambda_1 < \lambda_2$ *implies* $x_{\lambda_1} > x_{\lambda_2}$;

(ii) *if there exists* $\gamma \in (0,1)$ *such that* $\varphi(t) \geq t^\gamma$ *for* $t \in (0,1)$, *then* x_λ *is continuous in* λ, *that is*, $\lambda \to \lambda_0$ $(\lambda_0 > 0)$ *implies* $\|x_\lambda - x_{\lambda_0}\| \to 0$;

(iii) $\lim_{\lambda\to\infty} \|x_\lambda\| = 0$, $\lim_{\lambda\to 0^+} \|x_\lambda\| = \infty$.

Proof Fix $\lambda > 0$ and by Theorem 2.1, $\frac{1}{\lambda}A : P_h \to P_h$ is increasing and satisfies

$$\left(\frac{1}{\lambda}A\right)(tx) = \frac{1}{\lambda}A(tx) \geq \frac{1}{\lambda}\varphi(t)Ax = \varphi(t)\left(\frac{1}{\lambda}A\right)(x), \quad x \in P_h, t \in (0,1).$$

So it follows from Theorem 2.1 that $\frac{1}{\lambda}A$ has a unique fixed point x_λ in P_h. That is, $Ax_\lambda = \lambda x_\lambda$.

(i) Suppose $0 < \lambda_1 < \lambda_2$ and let $t_0 = \sup\{t > 0 \mid x_{\lambda_1} \geq tx_{\lambda_2}\}$, then we have $0 < t_0 < \infty$, $x_{\lambda_1} \geq t_0 x_{\lambda_2}$. Next we prove $t_0 \geq 1$. In fact, if $0 < t_0 < 1$, then

$$x_{\lambda_1} = \frac{1}{\lambda_1}Ax_{\lambda_1} \geq \frac{1}{\lambda_1}A(t_0 x_{\lambda_2}) \geq \frac{1}{\lambda_1}\varphi(t_0)Ax_{\lambda_2} = \frac{\lambda_2}{\lambda_1}\varphi(t_0)x_{\lambda_2}.$$

Note that $\frac{\lambda_2}{\lambda_1}\varphi(t_0) > t_0$, which contradicts the definition of t_0. Hence $t_0 \geq 1$ and $x_{\lambda_1} \geq x_{\lambda_2}$,

$$x_{\lambda_1} = \frac{1}{\lambda_1}Ax_{\lambda_1} \geq \frac{1}{\lambda_1}A(x_{\lambda_2}) = \frac{\lambda_2}{\lambda_1}x_{\lambda_2} > x_{\lambda_2}. \tag{2.3}$$

(ii) For given $\lambda_0 > 0$, we know by (2.3),

$$x_\lambda < x_{\lambda_0}, \quad \forall \lambda > \lambda_0. \tag{2.4}$$

On the other hand, let $t_\lambda = \sup\{t > 0 \mid x_\lambda \geq tx_{\lambda_0}\}$ $(\lambda > \lambda_0)$, then we have $0 < t_\lambda < 1$, $x_\lambda \geq t_\lambda x_{\lambda_0}$, for $\lambda > \lambda_0$. Thus $\lambda x_\lambda = Ax_\lambda \geq A(t_\lambda x_{\lambda_0}) \geq \varphi(t_\lambda)Ax_{\lambda_0} = \varphi(t_\lambda)\lambda_0 x_{\lambda_0}$. By the definition of t_λ and the condition (ii), we know $t_\lambda \geq \frac{\lambda_0}{\lambda}\varphi(t_\lambda) \geq \frac{\lambda_0}{\lambda}(t_\lambda)^\gamma$, which in turn yields $t_\lambda \geq (\frac{\lambda_0}{\lambda})^{\frac{1}{1-\gamma}}$, $\forall \lambda > \lambda_0$. Consequently,

$$x_\lambda \geq \left(\frac{\lambda_0}{\lambda}\right)^{\frac{1}{1-\gamma}} x_{\lambda_0}, \quad \forall \lambda > \lambda_0. \tag{2.5}$$

By (2.4), (2.5), and the normality of the cone P,

$$\|x_{\lambda_0} - x_\lambda\| \leq N\left[1 - \left(\frac{\lambda_0}{\lambda}\right)^{\frac{1}{1-\gamma}}\right]\|x_{\lambda_0}\|,$$

where N is the normality constant. Let $\lambda \to \lambda_0 + 0$, we have $\|x_{\lambda_0} - x_\lambda\| \to 0$. Similarly, let $\lambda \to \lambda_0 - 0$, we also have $\|x_\lambda - x_{\lambda_0}\| \to 0$. So the conclusion (ii) holds.

(iii) Let $\lambda_1 = 1, \lambda_2 = \lambda$ in (2.3), we have $x_1 \geq \lambda x_\lambda$, $\forall \lambda > 1$. Thus, we can easily obtain $\|x_\lambda\| \leq \frac{N}{\lambda}\|x_1\|$, where N is the normal constant. Let $\lambda \to \infty$, then $\|x_\lambda\| \to 0$. Similarly, let $\lambda_1 = \lambda$, $\lambda_2 = 1$ in (2.3), then $x_\lambda \geq \frac{1}{\lambda}x_1$, $\forall 0 < \lambda < 1$. Thus $\|x_\lambda\| \geq \frac{1}{N\lambda}\|x_1\|$, where N is the normality constant. Let $\lambda \to 0 + 0$, we have $\|x_\lambda\| \to \infty$. □

Remark 2.2 (1) We do not suppose the condition of upper-lower solutions which is common in many known results and is difficult to verify. Moreover, we give the iterative forms. The existence of a unique positive solution is proved only in the case where the cone P is normal and the operators A is generalized concave.

(2) The eigenvalue problem for generalized concave operators has not been studied in the literature, so Theorem 2.2 complements the eigenvalue results for generalized concave operators.

3 Properties of positive solutions for problem (1.2)

When $\lambda = 1$ in (1.2), Cabada and Hamdi [10] established the existence of one positive solution for problem (1.2) under sublinear case or superlinear case. The method used there is by Guo-Krasnosel'skii fixed point theorem. Different from the main result and the method, we will apply Theorem 2.2 and present some properties of positive solutions for problem (1.2) dependent on the parameter, and the method is also different from those in previous works.

Lemma 3.1 (see [10]) *Let* $2 < \alpha \leq 3$ *and* $\alpha \neq \beta$. *Assume* $y \in C[0,1]$, *then the following fractional differential equation with integral boundary conditions*:

$$\begin{cases} D_{0^+}^\alpha u(t) + y(t) = 0, & t \in (0,1), \\ u(0) = u'(0) = 0, & u(1) = \beta \int_0^1 u(s)\,ds, \end{cases}$$

has a unique solution $u \in C^1[0,1]$, *given by the expression*

$$u(t) = \int_0^1 G(t,s)y(s)\,ds, \tag{3.1}$$

where

$$G(t,s) = \begin{cases} \frac{t^{\alpha-1}(1-s)^{\alpha-1}(\alpha-\beta+\beta s)-(\alpha-\beta)(t-s)^{\alpha-1}}{(\alpha-\beta)\Gamma(\alpha)}, & 0 \leq s \leq t \leq 1, \\ \frac{t^{\alpha-1}(1-s)^{\alpha-1}(\alpha-\beta+\beta s)}{(\alpha-\beta)\Gamma(\alpha)}, & 0 \leq t \leq s \leq 1. \end{cases} \tag{3.2}$$

Lemma 3.2 *Let* $2 < \alpha \leq 3$ *and* $0 < \beta < \alpha$. *The function* $G(t,s)$ *defined by* (3.2) *has the following properties*:

$$\frac{(1-s)^{\alpha-1}\beta s}{(\alpha-\beta)\Gamma(\alpha)}t^{\alpha-1} \leq G(t,s) \leq \frac{(1-s)^{\alpha-1}(\alpha-\beta+\beta s)}{(\alpha-\beta)\Gamma(\alpha)}t^{\alpha-1}, \quad t,s \in [0,1].$$

Proof Evidently, the right inequality holds. So we only need to prove the left inequality. If $0 \leq s \leq t \leq 1$, then we have $0 \leq t - s \leq t - ts = (1-s)t$, and thus

$$(t-s)^{\alpha-1} \leq (1-s)^{\alpha-1}t^{\alpha-1}.$$

Since $\alpha - \beta > 0$, we obtain

$$\begin{aligned} G(t,s) &= \frac{t^{\alpha-1}(1-s)^{\alpha-1}(\alpha-\beta+\beta s) - (\alpha-\beta)(t-s)^{\alpha-1}}{(\alpha-\beta)\Gamma(\alpha)} \\ &\geq \frac{1}{(\alpha-\beta)\Gamma(\alpha)}\bigl[t^{\alpha-1}(1-s)^{\alpha-1}(\alpha-\beta+\beta s) - (\alpha-\beta)(1-s)^{\alpha-1}t^{\alpha-1}\bigr] \\ &= \frac{(1-s)^{\alpha-1}\beta s}{(\alpha-\beta)\Gamma(\alpha)}t^{\alpha-1}. \end{aligned}$$

When $0 \leq t \leq s \leq 1$, from $\alpha - \beta > 0$, we have

$$\begin{aligned} G(t,s) &= \frac{t^{\alpha-1}(1-s)^{\alpha-1}(\alpha-\beta+\beta s)}{(\alpha-\beta)\Gamma(\alpha)} \\ &\geq \frac{(1-s)^{\alpha-1}\beta s}{(\alpha-\beta)\Gamma(\alpha)}t^{\alpha-1}. \end{aligned}$$

So the left inequality also holds. □

In the following considerations we will work in the Banach space $C[0,1]$, the space of all continuous functions on $[0,1]$ with the standard norm $\|x\| = \sup\{|x(t)| : t \in [0,1]\}$. Evidently, this space can be equipped with a partial order given by

$$x, y \in C[0,1], \quad x \leq y \quad \Leftrightarrow \quad x(t) \leq y(t) \quad \text{for } t \in [0,1].$$

Set $P = \{x \in C[0,1] \mid x(t) \geq 0, t \in [0,1]\}$, the standard cone. We know that P is a normal cone in $C[0,1]$ and the normality constant is 1.

Theorem 3.1 *Assume that*:

(H_1) $f : [0,1] \times [0,+\infty) \to [0,+\infty)$ *is continuous with* $f(t,0) \not\equiv 0$;
(H_2) $f(t,x)$ *is increasing in* x *for each* $t \in [0,1]$;
(H_3) *for any* $r \in (0,1)$, *there exists* $\varphi(r) \in (r,1)$ *such that*

$$f(t,rx) \geq \varphi(r)f(t,x), \quad \forall t \in [0,1], x \in [0,+\infty).$$

Then the following conclusions hold:

(1) *For any given* $\lambda > 0$, *problem* (1.2) *has a unique positive solution* u_λ^* *in* P_h, *where* $h(t) = t^{\alpha-1}$, $t \in [0,1]$. *Moreover, for any initial value* $u_0 \in P_h$, *constructing successively the sequence*

$$u_n(t) = \lambda \int_0^1 G(t,s)f\bigl(s, u_{n-1}(s)\bigr)\,ds, \quad n = 1, 2, \ldots,$$

we have $u_n(t) \to u_\lambda^*(t)$ *as* $n \to +\infty$, *where* $G(t,s)$ *is given as in Lemma* 3.1.

(2) u_λ^* *is strictly increasing in* λ, *that is*, $0 < \lambda_1 < \lambda_2$ *implies* $u_{\lambda_1}^* < u_{\lambda_2}^*$.

(3) *If there exists* $\gamma \in (0,1)$ *such that* $\varphi(t) \geq t^\gamma$ *for* $t \in (0,1)$, *then* u_λ^* *is continuous in* λ, *that is*, $\lambda \to \lambda_0$ ($\lambda_0 > 0$) *implies* $\|u_\lambda^* - u_{\lambda_0}^*\| \to 0$.

(4) $\lim_{\lambda \to 0^+} \|u_\lambda^*\| = 0$, $\lim_{\lambda \to +\infty} \|u_\lambda^*\| = +\infty$.

Proof For any $u \in P$, we define an operator A by

$$Au(t) = \int_0^1 G(t,s)f\big(s,u(s)\big)\,ds, \quad t \in [0,1],$$

where $G(t,s)$ is given as in Lemma 3.1. From Lemma 3.1, $u(t)$ is the solution of problem (1.2) if and only if $u(t) = \lambda Au(t)$. Noting that $f(t,x) \geq 0$ and $G(t,s) \geq 0$, it is easy to check that $A : P \to P$. From (H_1), (H_2), we can easily prove that $A : P \to P$ is increasing. In the sequel we check that A satisfies all assumptions of Theorem 2.1.

First of all, we show that A satisfies the second condition of Theorem 2.1. From (H_3), for any $r \in (0,1)$ and $u \in P$, we obtain

$$A(ru)(t) = \int_0^1 G(t,s)f\big(s,ru(s)\big)\,ds \geq \varphi(r)\int_0^1 G(t,s)f\big(s,u(s)\big)\,ds = \varphi(r)Au(t), \quad t \in [0,1].$$

Hence, $A(ru) \geq \varphi(r)Au$, $\forall u \in P$, $r \in (0,1)$.

Next we show that the first condition of Theorem 2.1 also holds. That is, take $h_0 = h$, we prove $Ah \in P_h$. On one hand, it follows from (H_2) and Lemma 3.2 that

$$\begin{aligned} Ah(t) &= \int_0^1 G(t,s)f\big(s,h(s)\big)\,ds = \int_0^1 G(t,s)f\big(s,s^{\alpha-1}\big)\,ds \\ &\geq \int_0^1 \frac{(1-s)^{\alpha-1}\beta s}{(\alpha-\beta)\Gamma(\alpha)}t^{\alpha-1}f(s,0)\,ds \\ &= \frac{\beta}{(\alpha-\beta)\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}sf(s,0)\,ds\cdot t^{\alpha-1}, \quad t \in [0,1]. \end{aligned}$$

On the other hand, also from (H_2) and Lemma 3.2, we obtain

$$\begin{aligned} Ah(t) &\leq \int_0^1 \frac{(1-s)^{\alpha-1}(\alpha-\beta+\beta s)}{(\alpha-\beta)\Gamma(\alpha)}t^{\alpha-1}f(s,1)\,ds \\ &= \frac{1}{(\alpha-\beta)\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}(\alpha-\beta+\beta s)f(s,1)\,ds\cdot t^{\alpha-1}, \quad t \in [0,1]. \end{aligned}$$

Let

$$\begin{aligned} r_1 &= \frac{\beta}{(\alpha-\beta)\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}sf(s,0)\,ds, \\ r_2 &= \frac{1}{(\alpha-\beta)\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}(\alpha-\beta+\beta s)f(s,1)\,ds. \end{aligned}$$

Since $\alpha-\beta > 0$, f is continuous and $f(t,0) \not\equiv 0$, we can get $0 < r_1 \leq r_2$. Consequently,

$$Ah(t) \geq r_1 h(t), \qquad Ah(t) \leq r_2 h(t), \quad t \in [0,1].$$

So we have $r_1 h \leq Ah \leq r_2 h$. Hence $Ah \in P_h$. Therefore, by Theorem 2.2, there exists a unique $u_\lambda^* \in P_h$ such that $A(u_\lambda^*, u_\lambda^*) = \frac{1}{\lambda}u_\lambda^*$. That is, $u_\lambda^* = \lambda A(u_\lambda^*, u_\lambda^*)$, and then

$$u_\lambda^*(t) = \lambda\int_0^1 G(t,s)f\big(s,u_\lambda^*(s)\big)\,ds, \quad t \in [0,1].$$

It is easy to check that u_λ^* is a unique positive solution of problem (1.2) for given $\lambda > 0$. From Theorem 2.2, u_λ^* is strictly increasing in λ, that is, $0 < \lambda_1 < \lambda_2$ implies $u_{\lambda_1}^* \leq u_{\lambda_2}^*$, $u_{\lambda_1}^* \neq u_{\lambda_2}^*$. Further, $\lim_{\lambda\to 0^+} \|u_\lambda^*\| = 0$, $\lim_{\lambda\to\infty} \|u_\lambda^*\| = \infty$. Moreover, if there exists $\gamma \in (0,1)$ such that $\varphi(t) \geq t^\gamma$ for $t \in (0,1)$, Theorem 2.2 means that u_λ^* is continuous in λ, that is, $\lambda \to \lambda_0$ ($\lambda_0 > 0$) implies $\|u_\lambda^* - u_{\lambda_0}^*\| \to 0$.

Let $A_\lambda = \lambda A$, then A_λ also satisfies all the conditions of Theorem 2.1. By Theorem 2.1, for any initial value $u_0 \in P_h$, constructing successively the sequence $u_n = A_\lambda u_{n-1}$, $n = 1, 2, \ldots$, we have $u_n \to u_\lambda^*$ as $n \to \infty$. That is,

$$u_n(t) = \lambda \int_0^1 G(t,s) f\bigl(s, u_{n-1}(s)\bigr)\, ds, \quad n = 1, 2, \ldots, t \in [0,1],$$

and $u_n(t) \to u_\lambda^*(t)$ as $n \to +\infty$. □

In Theorem 3.1, let $\lambda = 1$, we can easily obtain the following conclusions.

Corollary 3.2 *Assume* (H_1)-(H_3) *hold. Then the following Riemann-Liouville fractional differential equation with integral boundary conditions*

$$\begin{cases} D_{0^+}^\alpha u(t) + f(t, u(t)) = 0, & t \in (0,1), \\ u(0) = u'(0) = 0, & u(1) = \beta \int_0^1 u(s)\, ds, \end{cases}$$

where $2 < \alpha \leq 3$, $0 < \beta < \alpha$, *has a unique positive solution* u^* *in* P_h, *where* $h(t) = t^{\alpha-1}$, $t \in [0,1]$. *Moreover, for any initial value* $u_0 \in P_h$, *constructing successively the sequence*

$$u_n(t) = \int_0^1 G(t,s) f\bigl(s, u_{n-1}(s)\bigr)\, ds, \quad n = 1, 2, \ldots,$$

we have $u_n(t) \to u^*(t)$ *as* $n \to +\infty$, *where* $G(t,s)$ *is given as in Lemma* 3.1.

Remark 3.1 Comparing Theorem 3.1 and Corollary 3.2 with many main results in the literature, here we present an alternative approach to the study of similar problems under different conditions. Our results cannot only guarantee the existence of a unique positive solution for any given parameter, but they also help to construct an iterative scheme for approximating it. Moreover, we can show that the positive solution with respect to the parameter has some definite properties. So our results are seldom seen in the literature.

Remark 3.2 (1) We can see that there is a large number of functions which satisfy the conditions of Theorem 3.1 or Corollary 3.2. For example, let $f(t,x) = a(t)[x^{\frac{1}{3}} + b]$, where $a : [0,1] \to [0,+\infty)$ is continuous with $a(t) \not\equiv 0$, $b > 0$. Take $\varphi(r) = r^{\frac{1}{3}}$. Also let $f(t,x) = g(t) + x^{\frac{1}{2}} + x^{\frac{1}{3}} + \cdots + x^{\frac{1}{n}} + b$, where $g : [0,1] \to [0,+\infty)$ is continuous, $n \geq 2$, $b > 0$. Take $\varphi(r) = r^{\frac{1}{2}}$. We can easily prove that (H_1)-(H_3) in Theorem 3.1 hold.

(2) If the Green functions satisfy some properties similar to Lemma 3.2, then Theorems 2.1 and 2.2 can be applied to many fractional differential equations boundary value problems.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors declare that the study was realized in collaboration with the same responsibilities. All authors read and approved the final manuscript.

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