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Dynamics of a stochastic cooperative predator-prey system with Beddington-DeAngelis functional response

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Abstract

In this paper, a stochastic cooperative predator-prey system with Beddington-DeAngelis functional response is proposed. By constructing Lyapunov functions, we demonstrate the existence of global positive solution, which is stochastically bounded and permanent. In addition, under some specific conditions, the solution of this stochastic system is globally asymptotically stable, which means that the properties of the solution will not be changed when the stochastic perturbation is small. Finally, the influence of the stochastic perturbation is demonstrated by simulation.

Keywords: stochastic differential equation; Beddington-DeAngelis functional response; stochastic perturbation; global asymptotic stability

1 Introduction

As an important branch of ecology science, population ecology has become a systematic discipline where mathematics is thoroughly applied. The relationship between predator and prey is one of the basic relationships among species. Many significant functional responses are constructed to model various situations. Most of the functional response are prey-dependent, which fail to model the interference among predators. In fact, when predators have to search, share, and compete for food, the functional response should be predator dependent; such a scenario occurs more frequently in nature and laboratory (see [1] and references therein).

In predator-prey systems, there are three classical predator-dependent functional responses: the Hassell-Varley, Beddington-DeAngelis, and Crowley-Martin responses (see [1] and references therein). By comparing statistical evidence from 19 predator-prey systems, Skalski and Gilliam claimed that the above three predator-dependent functional responses provided a better description of a predator feeding over a range of predator-prey abundances [2]. Amongst the Beddington-DeAngelis type functional responses some cases fitted better [2]. Beddington and DeAngelis *et al.* in 1975 first introduced the Beddington-DeAngelis type predator-prey model (see [3] and references therein), which is

$$\begin{cases} \frac{dx}{dt} = x(r_1 - a_{11}x - \frac{a_{12}y}{1+\beta x+\gamma y}), \\ \frac{dy}{dt} = y(-r_2 + \frac{a_{21}x}{1+\beta x+\gamma y}), \end{cases} \tag{1}$$

where $x = x(t)$ and $y = y(t)$ represent prey and predator densities, respectively; r_1 stands for the average growth rate of prey, r_2 is the death rate of predator, and a_{11} is the predator density-dependence rate. r_i, a_{ij}, β, and γ are positive constants for $i,j = 1,2$. Because of its practical significance, many scholars have studied the system in recent years [4–12].

In the study of predator-prey systems, the cooperation element is always neglected. Nevertheless, the cooperative system is a sometimes rudimentary and sometimes important ecological system in mathematical biology [13, 14]. To describe the mutual cooperation between two species, May [15] proposed the following equations:

$$\begin{cases} \frac{dx}{dt} = r_1 x(1 - \frac{x}{a_1+b_1 y} - c_1 x), \\ \frac{dy}{dt} = r_2 y(1 - \frac{y}{a_2+b_2 y} - c_2 y), \end{cases} \tag{2}$$

where r_i represents the growth rate, and $c_i = \frac{1}{K_i}$ (K_i is the carrying capacity). r_i, a_i, b_i, and c_i are positive constants for $i,j = 1,2$.

For each predator, various species of prey coexist in nature, so the multiple species predator-prey system simulates the actual situation better. However, this has been neglected in many existing studies [2, 16]. In this paper, we investigate the following cooperative predator-prey system of one predator feeding on two prey species with Beddington-DeAngelis functional response:

$$\begin{cases} dx = x(a_1 - b_1 x - \frac{h_1 x}{f_1+g_1 y} - \frac{c_1 z}{1+\alpha_1 x+\beta_1 z})\,dt, \\ dy = y(a_2 - b_2 y - \frac{h_2 y}{f_2+g_2 x} - \frac{c_2 z}{1+\alpha_2 y+\beta_2 z})\,dt, \\ dz = z(a_3 - b_3 z + \frac{d_1 x}{1+\alpha_1 x+\beta_1 z} + \frac{d_2 y}{1+\alpha_2 y+\beta_2 z})\,dt, \end{cases} \tag{3}$$

where the species x, y are the preys of z; x and y are cooperative species. a_i stands for the average growth rate, and b_i is the density-dependence rate. All the parameters in system (3) are positive constants.

In reality, a population system is inevitably affected by environmental perturbation. May [15] have claimed that owing to the environmental perturbation, the birth rates in the system should be stochastic. At the same time, the natural growth of many species vary with t, *e.g.* owing to the seasonality. However, environmental fluctuations have been neglected in many existing studies. References [1, 3, 17] consider the effect of environmental noise. We introduce stochastic perturbation into the intrinsic growth rate. The intrinsic growth rate can be written as an average growth rate with some small random perturbed terms. In general, by the central limit theorem, the small terms follow some normal distributions, thus we can approximate the error term by a white noise $\sigma_i \dot{B}_i(t)$, where σ_i^2 is the intensity of the noise and $\dot{B}_i(t)$ is a standard white noise. $B_i(t)$ is a Brownian motion defined on a complete probability space (Ω, F, P). The growth rates a_1, a_2, and a_3 are disturbed to $a_1 + \sigma_1 \dot{B}_1(t)$, $a_2 + \sigma_2 \dot{B}_2(t)$, $a_3 + \sigma_3 \dot{B}_3(t)$, respectively. Finally we obtain the following stochastic system:

$$\begin{cases} dx = x(a_1 - b_1 x - \frac{h_1 x}{f_1+g_1 y} - \frac{c_1 z}{1+\alpha_1 x+\beta_1 z})\,dt + \sigma_1 x\,dB_1(t), \\ dy = y(a_2 - b_2 y - \frac{h_2 y}{f_2+g_2 x} - \frac{c_2 z}{1+\alpha_2 y+\beta_2 z})\,dt + \sigma_2 y\,dB_2(t), \\ dz = z(a_3 - b_3 z + \frac{d_1 x}{1+\alpha_1 x+\beta_1 z} + \frac{d_2 y}{1+\alpha_2 y+\beta_2 z})\,dt + \sigma_3 z\,dB_3(t), \end{cases} \tag{4}$$

where b_3z denotes the density dependence of the predator population. All the parameters in system (4) are positive constants.

In this paper, we will study the stochastic predator-prey system of one predator feeding on two prey species with Beddington-DeAngelis functional response. The rest of this paper is organized as follows. Section 2, we show the existence of global positive solution. Section 3, the stochastic boundedness of solution is studied. In Section 4, we prove that the system is stochastically permanent. In Section 5, we investigated the global attractivity of system (4). Finally, in Section 6, we present numerical simulation to verify our analytical results.

2 Global positive solutions

In model (4), as $x(t), y(t), z(t)$ represent predator and prey densities, the solutions of model (4) should be non-negative. In order for a stochastic differential equation to have a unique global solution for any given initial value, the coefficients of the equation are generally required to satisfy the linear growth condition and the local Lipschitz condition (see [18]), whereas the coefficients of (4) neither comply with the linear growth condition nor with the local Lipschitz condition. In the following sections, we will construct Lyapunov functions to demonstrate the existence and uniqueness of positive solutions.

Lemma 1 *For any initial value $x_0 > 0, y_0 > 0, z_0 > 0$, system* (4) *has a unique positive local solution $(x(t), y(t), z(t))$ for $t \in [0, \tau_e)$ almost surely* (*a.s.*), *where τ_e is the explosion time.*

Proof Consider the following equations:

$$\begin{cases} du = (a_1 - \frac{\sigma_1^2}{2} - b_1e^u - \frac{h_1e^u}{f_1+g_1e^v} - \frac{c_1e^w}{1+\alpha_1e^u+\beta_1e^w})\,dt + \sigma_1\,dB_1(t), \\ dv = (a_2 - \frac{\sigma_2^2}{2} - b_2e^v - \frac{h_2e^v}{f_2+g_2e^u} - \frac{c_2e^w}{1+\alpha_2e^v+\beta_2e^w})\,dt + \sigma_2\,dB_2(t), \\ dw = (a_3 - \frac{\sigma_3^2}{2} - b_3e^w + \frac{d_1e^u}{1+\alpha_1e^u+\beta_1e^w} + \frac{d_2e^v}{1+\alpha_2e^v+\beta_2e^w})\,dt + \sigma_3\,dB_3(t), \end{cases} \tag{5}$$

on $t \geq 0$ with initial value $u(0) = \ln x_0$, $v(0) = \ln y_0$, $w(0) = \ln z_0$. The coefficients of (5) satisfy the local Lipschitz condition, so there is a unique local solution $(u(t), v(t), z(t))$ on $t \in [0, \tau_e)$. By Itô's formula, we can see that $x(t) = e^{u(t)}$, $y(t) = e^{v(t)}$, $z(t) = e^{w(t)}$ is the unique positive local solution to (5) with initial value $x_0 > 0, y_0 > 0, z_0 > 0$. □

Theorem 1 *Consider system* (4), *for any given initial value $(x_0, y_0, z_0) \in R^3_+$, there is a unique solution $(x(t), y(t), z(t))$ for $t \geq 0$ and the solution will remain in R^3_+ with probability* 1.

Proof Based on Lemma 1, we only need to show the $\tau_e = \infty$. Let $m_0 > 0$ be sufficiently large for x_0, y_0, z_0 lying within the interval $[\frac{1}{m_0}, m_0]$. For each integer $m > m_0$, define the stopping times

$$\tau_m = \inf\left\{t \in [0, \tau_e] : \min\{x(t), y(t), z(t)\} \leq \frac{1}{m} \text{ or } \max\{x(t), y(t), z(t)\} \geq m\right\}. \tag{6}$$

Clearly, τ_m is increasing as $m \to \infty$. Set $\tau_\infty = \lim_{m\to\infty} \tau_m$, whence $\tau_\infty \leq \tau_m$. If we can prove $\tau_\infty = \infty$, then $\tau_e = \infty$ and $(x_0, y_0, z_0) \in R^3_+$ a.s. for all $t \geq 0$. If this statement is false, there is a pair of constants $T > 0$ and $\epsilon \in (0,1)$ such that $P\{\tau_\infty \leq T\} > \epsilon$. So there exists an integer $m_1 \geq m_0$ such that

$$P\{\tau_m \leq T\} \geq \epsilon, \quad m \geq m_1. \tag{7}$$

Define $V(x,y,z) = (x-1-\ln x) + (y-1-\ln y) + (z-1-\ln z)$. Since $u-1-\ln u \geq 0$ for all $u \geq 0$, we see that $V(x,y,z)$ is non-negative. Applying Itô's formula, we have

$$\begin{aligned} dV = {} & (x-1)\left(a_1 - b_1 x - \frac{h_1 x}{f_1 + g_1 y} - \frac{c_1 z}{1+\alpha_1 x + \beta_1 z}\right) dt + \frac{\sigma_1^2}{2}\, dt \\ & + (y-1)\left(a_2 - b_2 y - \frac{h_2 y}{f_2 + g_2 x} - \frac{c_2 z}{1+\alpha_2 y + \beta_2 z}\right) dt + \frac{\sigma_2^2}{2}\, dt \\ & + (z-1)\left(a_3 - b_3 z + \frac{d_1 x}{1+\alpha_1 x + \beta_1 z} + \frac{d_2 y}{1+\alpha_2 y + \beta_2 z}\right) dt + \frac{\sigma_3^2}{2}\, dt \\ & + \sigma_1(x-1)\, dB_1(t) + \sigma_2(y-1)\, dB_2(t) + \sigma_3(z-1)\, dB_3(t), \end{aligned} \tag{8}$$

then

$$\begin{aligned} LV = {} & (x-1)\left(a_1 - b_1 x - \frac{h_1 x}{f_1 + g_1 y} - \frac{c_1 z}{1+\alpha_1 x + \beta_1 z}\right) \\ & + (y-1)\left(a_2 - b_2 y - \frac{h_2 y}{f_2 + g_2 x} - \frac{c_2 z}{1+\alpha_2 y + \beta_2 z}\right) \\ & + (z-1)\left(a_3 - b_3 z + \frac{d_1 x}{1+\alpha_1 x + \beta_1 z} + \frac{d_2 y}{1+\alpha_2 y + \beta_2 z}\right) + \frac{\sigma_1^2 + \sigma_2^2 + \sigma_3^2}{2} \\ \leq {} & x(a_1 - b_1 x) + y(a_2 - b_2 y) + z\left(a_3 - b_3 z + \frac{d_1 x}{1+\alpha_1 x + \beta_1 z} + \frac{d_2 y}{1+\alpha_2 y + \beta_2 z}\right) \\ & + \frac{\sigma_1^2 + \sigma_2^2 + \sigma_3^2}{2} \\ \leq {} & x(a_1 - b_1 x) + y(a_2 - b_2 y) + z\left(a_3 - b_3 z + \frac{d_1}{\alpha_1} + \frac{d_2}{\alpha_2}\right) + \frac{\sigma_1^2 + \sigma_2^2 + \sigma_3^2}{2} \\ \leq {} & K, \end{aligned} \tag{9}$$

where K is a positive number. Consequently,

$$dV \leq K\, dt + (x-1)\sigma_1\, dB_1(t) + (y-1)\sigma_2\, dB_2(t) + (z-1)\sigma_3\, dB_3(t). \tag{10}$$

Integrating both sides of the inequality from 0 to $\tau_m \wedge T$ and then taking the expectations, we get

$$EV\big(x(\tau_m \wedge T), y(\tau_m \wedge T), z(\tau_m \wedge T)\big) \leq V(x_0, y_0, z_0) + KT. \tag{11}$$

Set $\Omega_m = \{\tau_m \leq T\}$ for $m \geq m_1$, then we get $P(\Omega_m) \geq \epsilon$ by (7). For every $\omega \in \Omega_m$, there is at least one of $x(\tau_m, \omega)$, $y(\tau_m, \omega)$, $z(\tau_m, \omega)$, which equals either m or $\frac{1}{m}$, therefore $V(x(\tau_m \wedge T), y(\tau_m \wedge T), z(\tau_m \wedge T))$ is no less than $m - 1 - \ln m$ or $\frac{1}{m} - 1 - \ln\frac{1}{m}$. Consequently, we have

$$V\big(x(\tau_m \wedge T), y(\tau_m \wedge T), z(\tau_m \wedge T)\big) \geq (m-1-\ln m) \wedge \left(\frac{1}{m} - 1 - \ln\frac{1}{m}\right). \tag{12}$$

It follows from (11) that

$$V(x_0,y_0,z_0)+KT \geq E\big[I_{\Omega_m}V\big(x(\tau_m\wedge T),y(\tau_m\wedge T),z(\tau_m\wedge T)\big)\big] \\ \geq \epsilon(m-1-\ln m)\wedge\left(\frac{1}{m}-1-\ln\frac{1}{m}\right), \tag{13}$$

where I_{Ω_m} is the indicator function of Ω_m. Letting $m\to\infty$, it results in the contradiction that $\infty > V(x_0,y_0,z_0)+KT=\infty$, thus we draw a conclusion that $\tau_\infty=\infty$ a.s. □

Theorem 1 only tells us the solution of model (4) will remain in R^3_+ with probability 1. Next, we will discuss how the solution varies in R^3_+ more detail.

3 Stochastic boundedness

Definition 1 (see [3] and references therein) The solution $(x(t),y(t),z(t))$ of system (4) is said to be stochastically ultimately bounded, if for any $\varepsilon\in(0,1)$, there is a positive constant $\delta=\delta(\varepsilon)$. Such that for any initial value $(x_0,y_0,z_0)\in R^3_+$, the solution $(x(t),y(t),z(t))$ of system (4) has the property that

$$\limsup_{t\to\infty}P\Big\{\big|x(t),y(t),z(t)\big|=\sqrt{x^2(t)+y^2(t)+z^2(t)}>\delta\Big\}<\varepsilon. \tag{14}$$

Assumption 1 For any initial value $(x_0,y_0,z_0)\in R^3_+$, there exists $p\geq 1$ such that

$$x_0<\frac{a_1+\frac{p-1}{2}\sigma_1^2}{b_1}, \tag{15}$$

$$y_0<\frac{a_2+\frac{p-1}{2}\sigma_2^2}{b_2}, \tag{16}$$

$$z_0<\frac{a_3+\frac{d_1}{\alpha_1}+\frac{d_2}{\alpha_2}+\frac{p-1}{2}\sigma_3^2}{b_3}. \tag{17}$$

Lemma 2 *If Assumption* 1 *holds, let* $(x(t),y(t),z(t))$ *be a solution to system* (4) *with initial value* $(x_0,y_0,z_0)\in R^3_+$. *Then, for all* $p>1$,

$$E\big[x^p(t)\big]\leq K_1(p), \tag{18}$$

$$E\big[y^p(t)\big]\leq K_2(p), \tag{19}$$

$$E\big[z^p(t)\big]\leq K_3(p), \tag{20}$$

where

$$K_1(p):=\left(\frac{a_1+\frac{p-1}{2}\sigma_1^2}{b_1}\right)^p, \tag{21}$$

$$K_2(p):=\left(\frac{a_2+\frac{p-1}{2}\sigma_2^2}{b_2}\right)^p, \tag{22}$$

$$K_3(p):=\left(\frac{a_3+\frac{d_1}{\alpha_1}+\frac{d_2}{\alpha_2}+\frac{p-1}{2}\sigma_3^2}{b_3}\right)^p. \tag{23}$$

Proof Define $V_1 = x^p$ for $x \in R_+$ and $p > 0$. By Itô's formula, we have

$$\begin{aligned} dV_1 &= px^{p-1}\,dx + \frac{1}{2}p(p-1)x^{p-2}(dx)^2 \\ &= px^{p-1}\left[x\left(a_1 - b_1x - \frac{h_1x}{f_1+g_1y} - \frac{c_1z}{1+\alpha_1x+\beta_1z}\right)dt + \sigma_1x\,dB_1(t)\right] \\ &\quad + \frac{1}{2}p(p-1)x^p\sigma_1^2\,dt \\ &= px^p\left\{\left[a_1 - b_1x - \frac{h_1x}{f_1+g_1y} - \frac{c_1z}{1+\alpha_1x+\beta_1z} + \frac{1}{2}(p-1)\sigma_1^2\right]dt + \sigma_1\,dB_1(t)\right\}. \end{aligned} \tag{24}$$

Integrating both sides of the equality from 0 to t and then taking the expectations, we get

$$\begin{aligned} &E\big[x^p(t)\big] - E\big[x^p(0)\big] \\ &\quad = \int_0^t pE\left\{x^p\left[a_1 - b_1x - \frac{h_1x}{f_1+g_1y} - \frac{c_1z}{1+\alpha_1x+\beta_1z} + \frac{1}{2}(p-1)\sigma_1^2\right]\right\}ds; \end{aligned} \tag{25}$$

thus

$$\begin{aligned} \frac{dE[x^p(t)]}{dt} &= pE\left\{x^p\left[a_1 - b_1x - \frac{h_1x}{f_1+g_1y} - \frac{c_1z}{1+\alpha_1x+\beta_1z} + \frac{1}{2}(p-1)\sigma_1^2\right]\right\} \\ &\le pE\left[a_1x^p - b_1x^{p+1} + \frac{1}{2}(p-1)\sigma_1^2x^p\right] \\ &= p\left\{\left[a_1 + \frac{1}{2}(p-1)\sigma_1^2\right]E\big[x^p(t)\big] - b_1E\big[x^{p+1}(t)\big]\right\} \\ &= pE\big[x^p(t)\big]\left\{a_1 + \frac{1}{2}(p-1)\sigma_1^2 - b_1E\big[x^p(t)\big]^{\frac{1}{p}}\right\}. \end{aligned} \tag{26}$$

Let $X(t) = E[x^p(t)]$, then

$$\frac{dX(t)}{dt} = pX(t)\left\{a_1 + \frac{1}{2}(p-1)\sigma_1^2 - b_1X^{\frac{1}{p}}(t)\right\}. \tag{27}$$

By Assumption 1, we know $0 < b_1X^{\frac{1}{p}}(0) = b_1x(0) < a_1 + \frac{p-1}{2}\sigma_1^2$. Applying the standard comparison argument,

$$E\big[x^p(t)\big]^{\frac{1}{p}} < \frac{a_1 + \frac{p-1}{2}\sigma_1^2}{b_1}; \tag{28}$$

thus

$$E\big[x^p(t)\big] \le K_1(p) = \left(\frac{a_1 + \frac{p-1}{2}\sigma_1^2}{b_1}\right)^p. \tag{29}$$

Similarly, we can prove that

$$E\big[y^p(t)\big] \le \left(\frac{a_2 + \frac{p-1}{2}\sigma_2^2}{b_2}\right)^p, \tag{30}$$

$$E\left[z^p(t)\right] \le \left(\frac{a_3 + \frac{d_1}{\alpha_1} + \frac{d_2}{\alpha_2} + \frac{p-1}{2}\sigma_3^2}{b_3}\right)^p. \tag{31}$$

□

Theorem 2 *Assume that Assumption* 1 *holds, the solutions of system* (4) *with initial value* $(x_0, y_0, z_0) \in R_+^3$ *are stochastically ultimately bounded.*

Proof If $(x(t), y(t), z(t)) \in R_+^3$, its norm here is denoted by

$$\left|x(t), y(t), z(t)\right| = \sqrt{x^2(t) + y^2(t) + z^2(t)}.$$

Then $|x(t), y(t), z(t)|^p = 3^{\frac{p}{2}}(|x^p(t)| + |y^p(t)| + |z^p(t)|)$. By Lemma 2, $E[|x(t), y(t), z(t)|^p] \le L(p)$, $t > 0$. $L(p)$ is dependent on $(x_0, y_0, z_0) \in R_+^3$ and defined by $L(p) = 3^{\frac{p}{2}}[K_1(p) + K_2(p) + K_3(p)]$. By Chebyshev's inequality, the above result is straightforward. □

4 Stochastic permanence

Definition 2 (see [3] and references therein) The solution $(x(t), y(t), z(t))$ of system (4) is said to be stochastically permanent, if for any $\varepsilon \in (0,1)$, there is a positive constant $\delta = \delta(\varepsilon)$ and $\chi = \chi(\varepsilon)$, such that for any initial value $(x_0, y_0, z_0) \in R_+^3$, system (4) has the properties that

$$\liminf_{t\to\infty} P\left\{\left|x(t), y(t), z(t)\right| \ge \delta\right\} \ge 1 - \varepsilon, \tag{32}$$

$$\liminf_{t\to\infty} P\left\{\left|x(t), y(t), z(t)\right| \le \chi\right\} \ge 1 - \varepsilon. \tag{33}$$

Assumption 2

$$\frac{1}{2}\max\left\{\sigma_1^2, \sigma_2^2, \sigma_3^2\right\} < \min\left\{a_1, a_2, a_3 + \frac{c_1}{\alpha_1} + \frac{c_2}{\alpha_2}\right\}. \tag{34}$$

Theorem 3 *If Assumption* 2 *holds, for any initial value* $(x_0, y_0, z_0) \in R_+^3$, *the solution* $(x(t), y(t), z(t))$ *satisfies* $\limsup_{t\to\infty} E[\frac{1}{|x(t),y(t),z(t)|^\theta}] \le H$, *where* θ *is an arbitrary positive constant satisfying*

$$\frac{\theta+3}{2}\max\left\{\sigma_1^2, \sigma_2^2, \sigma_3^2\right\} < \min\left\{a_1, a_2, a_3 + \frac{c_1}{\alpha_1} + \frac{c_2}{\alpha_2}\right\}, \tag{35}$$

where k *is an arbitrary positive constant satisfying*

$$k + \frac{\theta(\theta+3)}{2}\max\left\{\sigma_1^2, \sigma_2^2, \sigma_3^2\right\} < \theta\min\left\{a_1, a_2, a_3 + \frac{c_1}{\alpha_1} + \frac{c_2}{\alpha_2}\right\}. \tag{36}$$

Proof Define $U(x, y, z) = x + y + z$. Applying Itô's formula, we have

$$\begin{aligned} dU(x,y,z) = {}& x\left(a_1 - b_1 x - \frac{h_1 x}{f_1 + g_1 y} - \frac{c_1 z}{1 + \alpha_1 x + \beta_1 z}\right)dt + \sigma_1 x\,dB_1(t) \\ &+ y\left(a_2 - b_2 y - \frac{h_2 y}{f_2 + g_2 x} - \frac{c_2 z}{1 + \alpha_2 y + \beta_2 z}\right)dt + \sigma_2 y\,dB_2(t) \\ &+ z\left(a_3 - b_3 z + \frac{d_1 x}{1 + \alpha_1 x + \beta_1 z} + \frac{d_2 y}{1 + \alpha_2 y + \beta_2 z}\right)dt + \sigma_3 z\,dB_3(t). \end{aligned} \tag{37}$$

Define $V(x,y,z) = \frac{1}{U(x,y,z)}$. Using Itô's formula, we obtain

$$\begin{aligned} dV(x,y,z) = -V^2\bigg[& x\left(a_1 - b_1 x - \frac{h_1 x}{f_1 + g_1 y} - \frac{c_1 z}{1 + \alpha_1 x + \beta_1 z}\right) \\ & + y\left(a_2 - b_2 y - \frac{h_2 y}{f_2 + g_1 x} - \frac{c_2 z}{1 + \alpha_2 y + \beta_2 z}\right) \\ & + z\left(a_3 - b_3 z + \frac{d_1 x}{1 + \alpha_1 x + \beta_1 z} + \frac{d_2 y}{1 + \alpha_2 y + \beta_2 z}\right)\bigg] dt \\ & + V^3\left(\sigma_1^2 x^2 + \sigma_2^2 y^2 + \sigma_3^2 z^2\right) dt \\ & - V^2\left(\sigma_1 x\, dB_1(t) + \sigma_2 y\, dB_2(t) + \sigma_3 z\, dB_3(t)\right), \end{aligned} \tag{38}$$

where

$$\begin{aligned} LV = -V^2\bigg[& x\left(a_1 - b_1 x - \frac{h_1 x}{f_1 + g_1 y} - \frac{c_1 z}{1 + \alpha_1 x + \beta_1 z}\right) \\ & + y\left(a_2 - b_2 y - \frac{h_2 y}{f_2 + g_1 x} - \frac{c_2 z}{1 + \alpha_2 y + \beta_2 z}\right) \\ & + z\left(a_3 - b_3 z + \frac{d_1 x}{1 + \alpha_1 x + \beta_1 z} + \frac{d_2 y}{1 + \alpha_2 y + \beta_2 z}\right)\bigg] \\ & + V^3\left(\sigma_1^2 x^2 + \sigma_2^2 y^2 + \sigma_3^2 z^2\right). \end{aligned} \tag{39}$$

Under Assumption 2, choosing a positive constant θ such that it satisfies (35), then by Itô's formula, we have

$$\begin{aligned} d(1+V)^\theta &= \theta(1+V)^{\theta-1}\, dV + \frac{1}{2}\theta(\theta-1)(1+V)^{\theta-2}(dV)^2 \\ &= \left\{\theta(1+V)^{\theta-1} LV + \frac{1}{2}\theta(\theta-1)(1+V)^{\theta-2} V^4\left(\sigma_1^2 x^2 + \sigma_2^2 y^2 + \sigma_3^2 z^2\right)\right\} dt \\ &\quad - \theta V^2 (1+V)^{\theta-1}\left(\sigma_1 x\, dB_1(t) + \sigma_2 y\, dB_2(t) + \sigma_3 z\, dB_3(t)\right). \end{aligned} \tag{40}$$

Thus

$$L(1+V)^\theta = \theta(1+V)^{\theta-1} LV + \frac{1}{2}\theta(\theta-1)(1+V)^{\theta-2} V^4\left(\sigma_1^2 x^2 + \sigma_2^2 y^2 + \sigma_3^2 z^2\right). \tag{41}$$

Then choosing a positive constant k such that it satisfies (36). By Itô's formula, we get

$$\begin{aligned} de^{kt}(1+V)^\theta &= e^{kt} d(1+V)^\theta + ke^{kt}(1+V)^\theta\, dt \\ &= e^{kt} L(1+V)^\theta\, dt + ke^{kt}(1+V)^\theta\, dt \\ &\quad - e^{kt}\theta V^2(1+V)^{\theta-1}\left[\sigma_1 x\, dB_1(t) + \sigma_2 y\, dB_2(t) + \sigma_3 z\, dB_3(t)\right], \end{aligned}$$

$$\begin{aligned} & Le^{kt}(1+V)^\theta \\ &\quad = e^{kt}(1+V)^{\theta-2}\bigg\{k(1+V)^2 + \frac{1}{2}\theta(\theta-1)V^4\left(\sigma_1^2 x^2 + \sigma_2^2 y^2 + \sigma_3^2 z^2\right) \\ &\qquad + \theta(1+V)V^3\left(\sigma_1^2 x^2 + \sigma_2^2 y^2 + \sigma_3^2 z^2\right) \end{aligned}$$

$$
\begin{aligned}
&-\theta(1+V)V^2\left[x\left(a_1-b_1x-\frac{h_1x}{f_1+g_1y}-\frac{c_1z}{1+\alpha_1x+\beta_1z}\right)\right.\\
&+y\left(a_2-b_2y-\frac{h_2y}{f_2+g_2x}-\frac{c_2z}{1+\alpha_2y+\beta_2z}\right)\\
&\left.\left.+z\left(a_3-b_3z+\frac{d_1x}{1+\alpha_1x+\beta_1z}+\frac{d_2y}{1+\alpha_2y+\beta_2z}\right)\right]\right\}\\
\le{}& e^{kt}(1+V)^{\theta-2}\left\{k(1+V)^2+\frac{1}{2}\theta(\theta+1)V^4\max\{\sigma_1^2,\sigma_2^2,\sigma_3^2\}(x^2+y^2+z^2)\right.\\
&+\theta(1+V)V^3\max\{\sigma_1^2,\sigma_2^2,\sigma_3^2\}(x^2+y^2+z^2)\\
&-\theta(1+V)V^2\left[x\left(a_1-b_1x-\frac{h_1x}{f_1}-\frac{c_1z}{\alpha_1x}\right)\right.\\
&\left.\left.+y\left(a_2-b_2y-\frac{h_2y}{f_2}-\frac{c_2z}{\alpha_2y}\right)+z(a_3-b_3z)\right]\right\}\\
\le{}& e^{kt}(1+V)^{\theta-2}\left\{k(1+V)^2+\frac{1}{2}\theta(\theta+1)V^4\max\{\sigma_1^2,\sigma_2^2,\sigma_3^2\}(x^2+y^2+z^2)\right.\\
&+\theta(1+V)V^3\max\{\sigma_1^2,\sigma_2^2,\sigma_3^2\}(x^2+y^2+z^2)\\
&-\theta(1+V)V^2\left[a_1x+a_2y+\left[a_3-\frac{c_1}{\alpha_1}-\frac{c_2}{\alpha_2}\right]z\right.\\
&\left.\left.-\left(b_1+\frac{h_1}{f_1}\right)x^2-\left(b_2+\frac{h_2}{f_2}\right)y^2-b_3z^2\right]\right\}\\
\le{}& e^{kt}(1+V)^{\theta-2}\left\{k(1+V)^2+\frac{1}{2}\theta(\theta+1)V^4\max\{\sigma_1^2,\sigma_2^2,\sigma_3^2\}\frac{1}{V^2}\right.\\
&+\theta(1+V)V^3\max\{\sigma_1^2,\sigma_2^2,\sigma_3^2\}\frac{1}{V^2}-\theta(1+V)V^2\left[\min\left\{a_1,a_2,a_3-\frac{c_1}{\alpha_1}-\frac{c_2}{\alpha_2}\right\}\frac{1}{V}\right.\\
&\left.\left.-\max\left\{b_1+\frac{h_1}{f_1},b_2+\frac{h_2}{f_2},b_3\right\}\frac{1}{V^2}\right]\right\}\\
\le{}& e^{kt}(1+V)^{\theta-2}\left\{\left[k+\theta\max\left\{b_1+\frac{h_1}{f_1},b_2+\frac{h_2}{f_2},b_3\right\}\right]\right.\\
&+\left[2k+\theta\max\{\sigma_1^2,\sigma_2^2,\sigma_3^2\}-\theta\min\left\{a_1,a_2,a_3-\frac{c_1}{\alpha_1}-\frac{c_2}{\alpha_2}\right\}\right.\\
&\left.+\theta\max\left\{b_1+\frac{h_1}{f_1},b_2+\frac{h_2}{f_2},b_3\right\}\right]V\\
&\left.+\left[k+\frac{1}{2}\theta(\theta+3)\max\{\sigma_1^2,\sigma_2^2,\sigma_3^2\}-\theta\min\left\{a_1,a_2,a_3-\frac{c_1}{\alpha_1}-\frac{c_2}{\alpha_2}\right\}\right]V^2\right\},
\end{aligned}
\tag{42}
$$

where

$$V^4(\sigma_1^2x^2+\sigma_2^2y^2+\sigma_3^2z^2)\le V^2\max\{\sigma_1^2,\sigma_2^2,\sigma_3^2\}, \tag{43}$$

$$a_1x+a_2y+\left(a_3-\frac{c_1}{\alpha_1}-\frac{c_2}{\alpha_2}\right)z\ge\min\left\{a_1,a_2,a_3-\frac{c_1}{\alpha_1}-\frac{c_2}{\alpha_2}\right\}\frac{1}{V}. \tag{44}$$

Hence, we get $Le^{kt}(1+V)^\theta\le K_4e^{kt}$. Then

$$de^{kt}(1+V)^\theta\le K_4e^{kt}\,dt-\theta e^{kt}V^2(1+V)^{\theta-1}\big[\sigma_1x\,dB_1(t)+\sigma_2y\,dB_2(t)+\sigma_3z\,dB_3(t)\big]. \tag{45}$$

Integrating both sides of the inequality from 0 to t and then taking the expectations, we can see that

$$Ee^{kt}(1+V)^{\theta} \leq \bigl(1+V(0)\bigr)^{\theta} + \frac{K_4}{k}e^{kt} = \bigl(1+V(0)\bigr)^{\theta} + K_5 e^{kt}, \tag{46}$$

where $K_5 = \frac{K_4}{k}$, so

$$\limsup_{t\to\infty} E\bigl[V(t)\bigr]^{\theta} \leq \limsup_{t\to\infty} E\bigl[1+V(t)\bigr]^{\theta} \leq K_5. \tag{47}$$

Since $(x+y+z)^{\theta} \leq 3^{\theta}(x^2+y^2+z^2)^{\frac{\theta}{2}} = 3^{\theta}|x+y+z|^{\theta}$ and $V(x,y,z) = \frac{1}{x+y+z}$,

$$\frac{1}{|x+y+z|^{\theta}} \leq 3^{\theta}\frac{1}{(x+y+z)^{\theta}} = 3^{\theta}V^{\theta}(t). \tag{48}$$

Clearly,

$$\limsup_{t\to\infty} E\biggl[\frac{1}{|x+y+z|^{\theta}}\biggr] \leq 3^{\theta}\limsup_{t\to\infty} E\bigl[V^{\theta}(t)\bigr] \leq 3^{\theta}K_5 = K_6, \tag{49}$$

which is the desired assertion. □

Theorem 4 *Under Assumption* 2, *system* (4) *is stochastically permanent.*

Proof By Theorem 2, we know that $\limsup_{t\to\infty} E|x+y+z|^p \leq L(p)$. Let $\chi = (\frac{L(p)}{\varepsilon})^{\frac{1}{p}}$, for all $\varepsilon > 0$. By Chebyshev's inequality, we can obtain the required assertion. □

5 Global asymptotic stability

Definition 3 (see [3] and references therein) Let $(x_1(t), y_1(t), z_1(t))$ be a positive solution of system (4). If we say that $(x_1(t), y_1(t), z_1(t))$ is globally asymptotically stable in expectation, it means that any other solution $(x_2(t), y_2(t), z_2(t))$ of system (4) has $t \geq 0$ and that we have initial value $(x_0, y_0, z_0) \in R^3_+$. That is,

$$P\Bigl\{\lim_{t\to\infty} E\bigl[\bigl|\bigl(x_1(t), y_1(t), z_1(t)\bigr) - \bigl(x_2(t), y_2(t), z_2(t)\bigr)\bigr|\bigr] = 0\Bigr\} = 1. \tag{50}$$

Lemma 3 [19] *Suppose that an n-dimensional stochastic process $X(t)$ on $t \geq 0$ satisfies the condition*

$$E\bigl|X(t) - X(s)\bigr|^{\alpha} \leq c|t-s|^{1+\beta}, \quad 0 \leq s, t < \infty, \tag{51}$$

for some positive constants α, β, and c. There exists a continuous modification $\widetilde{X}(t)$ of $X(t)$ which has the property that for every $\vartheta \in (0, \frac{\beta}{\alpha})$ there is a positive random variable $h(\omega)$ such that

$$P\biggl\{\omega : \sup_{0<|t-s|<h(\omega), 0\leq s,t<\infty} \frac{\widetilde{X}(t,\omega) - \widetilde{X}(s,\omega)}{|t-s|^{\vartheta}} \leq \frac{2}{1-2^{-\vartheta}}\biggr\} = 1. \tag{52}$$

In other words, almost every sample path of $\widetilde{X}(t)$ is locally but uniformly Hölder continuous with exponent ϑ.

Lemma 4 *Let $(x(t),y(t),z(t))$ be a positive solution of system* (4) *on $t \geq 0$ with initial value $(x_0,y_0,z_0) \in R_+^3$. Then almost every sample path of $(x(t),y(t),z(t))$ is uniformly continuous on $t \geq 0$.*

Proof The first equation of system (4) is equivalent to the following stochastic integral equation:

$$\begin{aligned} x(t) = x(0) + \int_0^t x(s)\left(a_1 - b_1x(s) - \frac{h_1x(s)}{f_1 + g_1y(s)} - \frac{c_1z(s)}{1 + \alpha_1x(s) + \beta_1z(s)}\right) ds \\ + \int_0^t \sigma_1x(s)\,dB_1(s). \end{aligned} \tag{53}$$

We estimate

$$\begin{aligned} &E\left|x\left(a_1 - b_1x - \frac{h_1x}{f_1 + g_1y} - \frac{c_1z}{1 + \alpha_1x + \beta_1z}\right)\right|^p \\ &\quad\leq \frac{1}{2}E|x|^{2p} + \frac{1}{2}E\left|a_1 - b_1x - \frac{h_1x}{f_1 + g_1y} - \frac{c_1z}{1 + \alpha_1x + \beta_1z}\right|^{2p} \\ &\quad\leq \frac{1}{2}E|x|^{2p} + \frac{1}{2}E\left|a_1 + b_1x + \frac{h_1x}{f_1} + \frac{c_1}{\beta_1}\right|^{2p} \\ &\quad\leq \frac{1}{2}E|x|^{2p} + \frac{3^{2p-1}}{2}\left\{a_1^{2p} + \left(b_1 + \frac{h_1}{f_1}\right)^{2p}E|x|^{2p} + \left(\frac{c_1}{\beta_1}\right)^{2p}\right\} \\ &\quad\leq \frac{1}{2}K_1(2p) + \frac{3^{2p-1}}{2}\left\{a_1^{2p} + \left(b_1 + \frac{h_1}{f_1}\right)^{2p}K_1(2p) + \left(\frac{c_1}{\beta_1}\right)^{2p}\right\} \\ &\quad=: K_5(p). \end{aligned} \tag{54}$$

In addition, applying the moment inequality for stochastic integrals [18], we have, for $0 < t_1 \leq t_2 < \infty$ and $p > 2$,

$$\begin{aligned} E\left[\left|\int_{t_1}^{t_2} \sigma_1x(s)\,dB_1(s)\right|^p\right] &\leq \left(\sigma_1^2\right)^p\left[\frac{p(p-1)}{2}\right]^{\frac{p}{2}}(t_2 - t_1)^{\frac{p-2}{2}}\int_{t_1}^{t_2} E\left|x(s)\right|^p ds \\ &\leq \left(\sigma_1^2\right)^p\left[\frac{p(p-1)}{2}\right]^{\frac{p}{2}}(t_2 - t_1)^{\frac{p}{2}}K_1(p). \end{aligned} \tag{55}$$

Let $t_2 - t_1 \leq 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, we obtain

$$\begin{aligned} &E\left|x(t_2) - x(t_1)\right|^p \\ &\quad= E\left|\int_{t_1}^{t_2} x(s)\left(a_1 - b_1x(s) - \frac{h_1x(s)}{f_1 + g_1y(s)} - \frac{c_1z(s)}{1 + \alpha_1x(s) + \beta_1z(s)}\right) ds + \int_{t_1}^{t_2} \sigma_1x(s)\,dB_1(s)\right|^p \\ &\quad\leq 2^{p-1}E\left|\int_{t_1}^{t_2} x(s)\left(a_1 - b_1x(s) - \frac{h_1x(s)}{f_1 + g_1y(s)} - \frac{c_1z(s)}{1 + \alpha_1x(s) + \beta_1z(s)}\right) ds\right|^p \\ &\qquad + 2^{p-1}E\left[\left|\int_{t_1}^{t_2} \sigma_1x(s)\,dB_1(s)\right|^p\right] \\ &\quad\leq 2^{p-1}(t_2 - t_1)^{\frac{p}{q}}\int_{t_1}^{t_2} E\left|x(s)\left(a_1 - b_1x(s) - \frac{h_1x(s)}{f_1 + g_1y(s)} - \frac{c_1z(s)}{1 + \alpha_1x(s) + \beta_1z(s)}\right)\right|^p ds \end{aligned}$$

$$
\begin{aligned}
&+ 2^{p-1}(\sigma_1^2)^p\left[\frac{p(p-1)}{2}\right]^{\frac{p}{2}}(t_2-t_1)^{\frac{p}{2}}K_1(p) \\
&\le 2^{p-1}(t_2-t_1)^p K_5(p) + 2^{p-1}(t_2-t_1)^{\frac{p}{2}}(\sigma_1^2)^p\left[\frac{p(p-1)}{2}\right]^{\frac{p}{2}}K_1(p) \\
&\le 2^{p-1}(t_2-t_1)^{\frac{p}{2}}\left\{1+\left[\frac{p(p-1)}{2}\right]^{\frac{p}{2}}\right\}\max\left\{K_5(p),(\sigma_1^2)^p K_1(p)\right\} \\
&\le 2^{p-1}(t_2-t_1)^{\frac{p}{2}}\left\{1+\left[\frac{p(p-1)}{2}\right]^{\frac{p}{2}}\right\}K_6(p), \qquad (56)
\end{aligned}
$$

where $K_6(p) = \max\{K_5(p), (\sigma_1^2)^p K_1(p)\}$. Then according to Lemma 3, we know that almost every sample path of $x(t)$ is locally but uniformly Hölder continuous with exponent ϑ for every $\vartheta \in (0, \frac{p-2}{2p})$. Therefore almost every sample path of $x(t)$ is uniformly continuous on $t \ge 0$. In the same way, we can demonstrate that almost every sample path of $y(t)$, $z(t)$ is uniformly continuous on $t \ge 0$. □

Lemma 5 [1] *Let f be a non-negative function defined on R_+ such that f is integrable on R_+ and is uniformly continuous on $t \ge 0$. Then* $\lim_{t\to\infty} f(t) = 0$.

Theorem 5 *If*

$$
\begin{cases}
A = b_1 - h_1 - \frac{h_1}{f_1} - \frac{c_1\alpha_1}{\beta_1} - 2d_1 - \frac{h_2 g_2[a_2+\frac{p-1}{2}\sigma_2^2]}{b_2 f_2^2} > 0, \\
B = b_2 - h_2 - \frac{h_2}{f_2} - \frac{c_2\alpha_2}{\beta_2} - 2d_2 - \frac{h_1 g_1[a_1+\frac{p-1}{2}\sigma_1^2]}{b_1 f_1^2} > 0, \\
C = b_3 - 2c_1 - 2c_2 - \frac{\beta_1 d_1}{\alpha_1} - \frac{\beta_2 d_2}{\alpha_2} > 0,
\end{cases} \qquad (57)
$$

then system (4) *is globally attractive.*

Proof Define $W(t) = |\ln x_1(t) - \ln x_2(t)| + |\ln y_1(t) - \ln y_2(t)| + |\ln z_1(t) - \ln z_2(t)|$. We can prove $W(t)$ is continuous positive function on $t \ge 0$. By a direct calculation of the right differential $d^+W(t)$ of $W(t)$, and then applying Itô's formula, we get

$$
\begin{aligned}
d^+W(t) &= \operatorname{sgn}(x_1-x_2)\left\{\left[\frac{dx_1}{x_1} - \frac{(dx_1)^2}{2x_1^2}\right] - \left[\frac{dx_2}{x_2} - \frac{(dx_2)^2}{2x_2^2}\right]\right\} \\
&\quad + \operatorname{sgn}(y_1-y_2)\left\{\left[\frac{dy_1}{y_1} - \frac{(dy_1)^2}{2y_1^2}\right] - \left[\frac{dy_2}{y_2} - \frac{(dy_2)^2}{2y_2^2}\right]\right\} \\
&\quad + \operatorname{sgn}(z_1-z_2)\left\{\left[\frac{dz_1}{z_1} - \frac{(dz_1)^2}{2z_1^2}\right] - \left[\frac{dz_2}{z_2} - \frac{(dz_2)^2}{2z_2^2}\right]\right\} \\
&= \operatorname{sgn}(x_1-x_2)\left\{-b_1(x_1-x_2) - \left(\frac{h_1x_1}{f_1+g_1y_1} - \frac{h_1x_2}{f_1+g_1y_2}\right)\right. \\
&\quad \left. - \left(\frac{c_1z_1}{1+\alpha_1x_1+\beta_1z_1} - \frac{c_1z_2}{1+\alpha_1x_2+\beta_1z_2}\right)\right\}dt \\
&\quad + \operatorname{sgn}(y_1-y_2)\left\{-b_1(y_1-y_2) - \left(\frac{h_2y_1}{f_2+g_2x_1} - \frac{h_2y_2}{f_2+g_2x_2}\right)\right. \\
&\quad \left. - \left(\frac{c_2z_1}{1+\alpha_2y_1+\beta_2z_1} - \frac{c_2z_2}{1+\alpha_2y_2+\beta_2z_2}\right)\right\}dt
\end{aligned}
$$

$$
\begin{aligned}
&+\operatorname{sgn}(z_1-z_2)\biggl\{-b_3(z_1-z_2)+\biggl(\frac{d_1x_1}{1+\alpha_1x_1+\beta_1z_1}-\frac{d_1x_2}{1+\alpha_1x_2+\beta_1z_2}\biggr)\\
&+\biggl(\frac{d_2y_1}{1+\alpha_2y_1+\beta_2z_1}-\frac{d_2y_2}{1+\alpha_2y_2+\beta_2z_2}\biggr)\biggr\}\,dt. \qquad (58)
\end{aligned}
$$

Integrating both sides of the equality from 0 to t and then taking expectations

$$
\begin{aligned}
&E\bigl[W(t)-W(0)\bigr]\\
&=E\biggl\{\int_0^t\biggl[\operatorname{sgn}\bigl(x_1(s)-x_2(s)\bigr)\biggl\{-b_1\bigl(x_1(s)-x_2(s)\bigr)-\biggl(\frac{h_1x_1(s)}{f_1+g_1y_1(s)}-\frac{h_1x_2(s)}{f_1+g_1y_2(s)}\biggr)\\
&\quad-\biggl(\frac{c_1z_1(s)}{1+\alpha_1x_1(s)+\beta_1z_1(s)}-\frac{c_1z_2(s)}{1+\alpha_1x_2(s)+\beta_1z_2(s)}\biggr)\biggr\}\\
&\quad+\operatorname{sgn}\bigl(y_1(s)-y_2(s)\bigr)\biggl\{-b_1\bigl(y_1(s)-y_2(s)\bigr)-\biggl(\frac{h_2y_1(s)}{f_2+g_2x_1(s)}-\frac{h_2y_2(s)}{f_2+g_2x_2(s)}\biggr)\\
&\quad-\biggl(\frac{c_2z_1(s)}{1+\alpha_2y_1(s)+\beta_2z_1(s)}-\frac{c_2z_2(s)}{1+\alpha_2y_2(s)+\beta_2z_2(s)}\biggr)\biggr\}\\
&\quad+\operatorname{sgn}\bigl(z_1(s)-z_2(s)\bigr)\biggl\{-b_3\bigl(z_1(s)-z_2(s)\bigr)\\
&\quad+\biggl(\frac{d_1x_1(s)}{1+\alpha_1x_1(s)+\beta_1z_1(s)}-\frac{d_1x_2(s)}{1+\alpha_1x_2(s)+\beta_1z_2(s)}\biggr)\\
&\quad+\biggl(\frac{d_2y_1(s)}{1+\alpha_2y_1(s)+\beta_2z_1(s)}-\frac{d_2y_2(s)}{1+\alpha_2y_2(s)+\beta_2z_2(s)}\biggr)\biggr\}\,ds\biggr]\biggr\}. \qquad (59)
\end{aligned}
$$

So

$$
\begin{aligned}
\frac{dE[W(t)]}{dt}&=E\biggl[\operatorname{sgn}\bigl(x_1(t)-x_2(t)\bigr)\biggl\{-b_1\bigl(x_1(t)-x_2(t)\bigr)-\biggl(\frac{h_1x_1(t)}{f_1+g_1y_1(t)}-\frac{h_1x_2(t)}{f_1+g_1y_2(t)}\biggr)\\
&\quad-\biggl(\frac{c_1z_1(t)}{1+\alpha_1x_1(t)+\beta_1z_1(t)}-\frac{c_1z_2(t)}{1+\alpha_1x_2(t)+\beta_1z_2(t)}\biggr)\biggr\}\\
&\quad+\operatorname{sgn}\bigl(y_1(t)-y_2(t)\bigr)\biggl\{-b_2\bigl(y_1(t)-y_2(t)\bigr)-\biggl(\frac{h_2y_1(t)}{f_2+g_2x_1(t)}-\frac{h_2y_2(t)}{f_2+g_2x_2(t)}\biggr)\\
&\quad-\biggl(\frac{c_2z_1(t)}{1+\alpha_2y_1(t)+\beta_2z_1(t)}-\frac{c_2z_2(t)}{1+\alpha_2y_2(t)+\beta_2z_2(t)}\biggr)\biggr\}\\
&\quad+\operatorname{sgn}\bigl(z_1(t)-z_2(t)\bigr)\biggl\{-b_3\bigl(z_1(t)-z_2(t)\bigr)\\
&\quad+\biggl(\frac{d_1x_1(t)}{1+\alpha_1x_1(t)+\beta_1z_1(t)}-\frac{d_1x_2(t)}{1+\alpha_1x_2(t)+\beta_1z_2(t)}\biggr)\\
&\quad+\biggl(\frac{d_2y_1(t)}{1+\alpha_2y_1(t)+\beta_2z_1(t)}-\frac{d_2y_2(t)}{1+\alpha_2y_2(t)+\beta_2z_2(t)}\biggr)\biggr\}\biggr]\\
&\le-b_1E\bigl|x_1(t)-x_2(t)\bigr|+E\biggl|\frac{h_1x_1(t)}{f_1+g_1y_1(t)}-\frac{h_1x_2(t)}{f_1+g_1y_2(t)}\biggr|\\
&\quad+E\biggl|\frac{c_1z_1(t)}{1+\alpha_1x_1(t)+\beta_1z_1(t)}-\frac{c_1z_2(t)}{1+\alpha_1x_2(t)+\beta_1z_2(t)}\biggr|\\
&\quad-b_2E\bigl|y_1(t)-y_2(t)\bigr|-E\biggl|\frac{h_2y_1(t)}{f_2+g_2x_1(t)}-\frac{h_2y_2(t)}{f_2+g_2x_2(t)}\biggr|
\end{aligned}
$$

$$\begin{aligned}
&- E\left|\frac{c_2 z_1(t)}{1+\alpha_2 y_1(t)+\beta_2 z_1(t)} - \frac{c_2 z_2(t)}{1+\alpha_2 y_2(t)+\beta_2 z_2(t)}\right| \\
&- b_3 E\left|z_1(t)-z_2(t)\right| + E\left|\frac{d_1 x_1(t)}{1+\alpha_1 x_1(t)+\beta_1 z_1(t)} - \frac{d_1 x_2(t)}{1+\alpha_1 x_2(t)+\beta_1 z_2(t)}\right| \\
&+ E\left|\frac{d_2 y_1(t)}{1+\alpha_2 y_1(t)+\beta_2 z_1(t)} - \frac{d_2 y_2(t)}{1+\alpha_2 y_2(t)+\beta_2 z_2(t)}\right| \\
\le{}& -b_1 E\left|x_1(t)-x_2(t)\right| + h_1\left(1+\frac{1}{f_1}\right)E\left|x_1(t)-x_2(t)\right| \\
&+ \frac{h_1 g_1}{f_1^2}E\left|x_1(t)\right|E\left|y_1(t)-y_2(t)\right| + 2c_1 E\left|z_1(t)-z_2(t)\right| \\
&+ \frac{c_1\alpha_1}{\beta_1}E\left|x_1(t)-x_2(t)\right| - b_2 E\left|y_1(t)-y_2(t)\right| \\
&+ h_2\left(1+\frac{1}{f_2}\right)E\left|y_1(t)-y_2(t)\right| + \frac{h_2 g_2}{f_2^2}E\left|y_1(t)\right|E\left|x_1(t)-x_2(t)\right| \\
&+ 2c_2 E\left|z_1(t)-z_2(t)\right| + \frac{c_2\alpha_2}{\beta_2}E\left|y_1(t)-y_2(t)\right| \\
&- b_3 E\left|z_1(t)-z_2(t)\right| + 2d_1 E\left|x_1(t)-x_2(t)\right| \\
&+ \frac{\beta_1 d_1}{\alpha_1}E\left|z_1(t)-z_2(t)\right| + 2d_2 E\left|y_1(t)-y_2(t)\right| + \frac{\beta_2 d_2}{\alpha_2}E\left|z_1(t)-z_2(t)\right|, \qquad (60)
\end{aligned}$$

$$\begin{aligned}
\frac{dE[W(t)]}{dt} ={}& -\left[b_1 - h_1 - \frac{h_1}{f_1} - \frac{c_1\alpha_1}{\beta_1} - 2d_1 - \frac{h_2 g_2}{f_2^2}E\left|y_1(t)\right|\right]E\left|x_1(t)-x_2(t)\right| \\
&- \left[b_2 - h_2 - \frac{h_2}{f_2} - \frac{c_2\alpha_2}{\beta_2} - 2d_2 - \frac{h_1 g_1}{f_1^2}E\left|x_1(t)\right|\right]E\left|y_1(t)-y_2(t)\right| \\
&- \left[b_3 - 2c_1 - 2c_2 - \frac{\beta_1 d_1}{\alpha_1} - \frac{\beta_2 d_2}{\alpha_2}\right]E\left|z_1(t)-z_2(t)\right|. \qquad (61)
\end{aligned}$$

By Lemma 2,

$$E\left|x_1(t)\right| = E\left|x_1^3(t)\right|^{\frac{1}{3}} = \left\{E\left[x_1^3(t)\right]\right\}^{\frac{1}{3}} \le \frac{a_1 + \frac{p-1}{2}\sigma_1^2}{b_1}, \qquad (62)$$

$$E\left|y_1(t)\right| = E\left|y_1^3(t)\right|^{\frac{1}{3}} = \left\{E\left[y_1^3(t)\right]\right\}^{\frac{1}{3}} \le \frac{a_2 + \frac{p-1}{2}\sigma_2^2}{b_2}, \qquad (63)$$

$$E\left|z_1(t)\right| = E\left|z_1^3(t)\right|^{\frac{1}{3}} = \left\{E\left[z_1^3(t)\right]\right\}^{\frac{1}{3}} \le \frac{a_3 + \frac{d_1}{\alpha_1} + \frac{d_2}{\alpha_2} + \frac{p-1}{2}\sigma_3^2}{b_3}. \qquad (64)$$

Thus

$$\begin{aligned}
\frac{dE[W(t)]}{dt} \le{}& -\left[b_1 - h_1 - \frac{h_1}{f_1} - \frac{c_1\alpha_1}{\beta_1} - 2d_1 - \frac{h_2 g_2(a_2 + \frac{p-1}{2}\sigma_2^2)}{b_2 f_2^2}\right]E\left|x_1(t)-x_2(t)\right| \\
&- \left[b_2 - h_2 - \frac{h_2}{f_2} - \frac{c_2\alpha_2}{\beta_2} - 2d_2 - \frac{h_1 g_1(a_1 + \frac{p-1}{2}\sigma_1^2)}{b_1 f_1^2}\right]E\left|y_1(t)-y_2(t)\right| \\
&- \left[b_3 - 2c_1 - 2c_2 - \frac{\beta_1 d_1}{\alpha_1} - \frac{\beta_2 d_2}{\alpha_2}\right]E\left|z_1(t)-z_2(t)\right| \\
\le{}& -AE\left|x_1(t)-x_2(t)\right| - BE\left|y_1(t)-y_2(t)\right| - CE\left|z_1(t)-z_2(t)\right|, \qquad (65)
\end{aligned}$$

where

$$A = b_1 - h_1 - \frac{h_1}{f_1} - \frac{c_1\alpha_1}{\beta_1} - 2d_1 - \frac{h_2 g_2(a_2 + \frac{p-1}{2}\sigma_2^2)}{b_2 f_2^2}, \tag{66}$$

$$B = b_2 - h_2 - \frac{h_2}{f_2} - \frac{c_2\alpha_2}{\beta_2} - 2d_2 - \frac{h_1 g_1(a_1 + \frac{p-1}{2}\sigma_1^2)}{b_1 f_1^2}, \tag{67}$$

$$C = b_3 - 2c_1 - 2c_2 - \frac{\beta_1 d_1}{\alpha_1} - \frac{\beta_2 d_2}{\alpha_2}. \tag{68}$$

Integrating both sides, we get

$$E\big[W(t)\big] \le E\big[W(0)\big] - \int_0^t \big[AE\big|x_1(s) - x_2(s)\big| + BE\big|y_1(s) - y_2(s)\big| + CE\big|z_1(s) - z_2(s)\big|\big]\,ds. \tag{69}$$

Consequently,

$$W(t) + \int_0^t \big[AE\big|x_1(s) - x_2(s)\big| + BE\big|y_1(s) - y_2(s)\big| + CE\big|z_1(s) - z_2(s)\big|\big]\,ds \le W(0) < \infty. \tag{70}$$

Recall that $W(t) \ge 0$ and $A > 0$, $B > 0$, $C > 0$, yielding

$$\begin{aligned} &\big|x_1(t) - x_2(t)\big| \in L^1[0, +\infty), \qquad \big|y_1(t) - y_2(t)\big| \in L^1[0, +\infty), \\ &\big|z_1(t) - z_2(t)\big| \in L^1[0, +\infty). \end{aligned} \tag{71}$$

Then from Lemmas 4 and 5, we get the desired assertion. □

6 Numerical simulations

In this section, to substantiate the analytical results; the dynamics of system (4) with and without environmental noises are illustrated by the Milstein method [20].

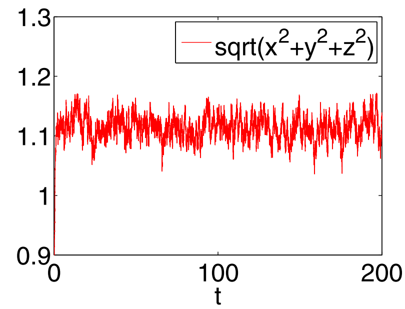
Consider the discrete equations:

$$\begin{aligned} x_{k+1} &= x_k + x_k\left(a_1 - b_1 x_k - \frac{h_1 x_k}{f_1 + g_1 y_k} - \frac{c_1 z_k}{1 + \alpha_1 x_k + \beta_1 z_k}\right)\Delta t \\ &\quad + \sigma_1 x_k \xi_k \sqrt{\Delta t} + \frac{1}{2}\sigma_1^2 x_k\big(\xi_k^2 - 1\big)\Delta t, \\ y_{k+1} &= y_k + y_k\left(a_2 - b_2 y_k - \frac{h_2 y_k}{f_2 + g_2 x_k} - \frac{c_2 z_k}{1 + \alpha_2 x_k + \beta_2 z_k}\right)\Delta t \\ &\quad + \sigma_2 y_k \eta_k \sqrt{\Delta t} + \frac{1}{2}\sigma_2^2 y_k\big(\eta_k^2 - 1\big)\Delta t, \\ z_{k+1} &= z_k + z_k\left(a_3 - b_3 z_k + \frac{d_1 x_k}{1 + \alpha_1 x_k + \beta_1 z_k} + \frac{d_2 y_k}{1 + \alpha_2 x_k + \beta_2 z_k}\right)\Delta t \\ &\quad + \sigma_3 z_k \zeta_k \sqrt{\Delta t} + \frac{1}{2}\sigma_3^2 z_k\big(\zeta_k^2 - 1\big)\Delta t, \end{aligned} \tag{72}$$

where ξ_k, η_k, and ζ_k are Gaussian random variables that follow $N(0,1)$.

In Figure 1, we choose the initial value $(x_0, y_0, z_0) = (0.5, 0.5, 0.5)$ and all the parameters satisfying the conditions of Theorem 3, the system (4) is stochastically permanent.

Figure 1 Solutions of system (4) for $(x_0, y_0, z_0) = (0.5, 0.5, 0.5)$, $a_1 = 2$, $a_2 = 1$, $a_3 = 1.5$, $b_1 = 2$, $b_2 = 2$, $b_3 = 3$, $c_1 = 0.2$, $c_2 = 0.1$, $d_1 = 1$, $d_2 = 1$, $\alpha_1 = 1$, $\beta_1 = 0.5$, $\alpha_2 = 0.6$, $\beta_2 = 0.8$, $h_1 = 0.5$, $h_2 = 0.8$, $f_1 = 1$, $f_2 = 0.3$, $g_1 = 0.5$, $g_2 = 1$, $\sigma_1 = 0.05$, $\sigma_2 = 0.05$, $\sigma_2 = 0.05$.



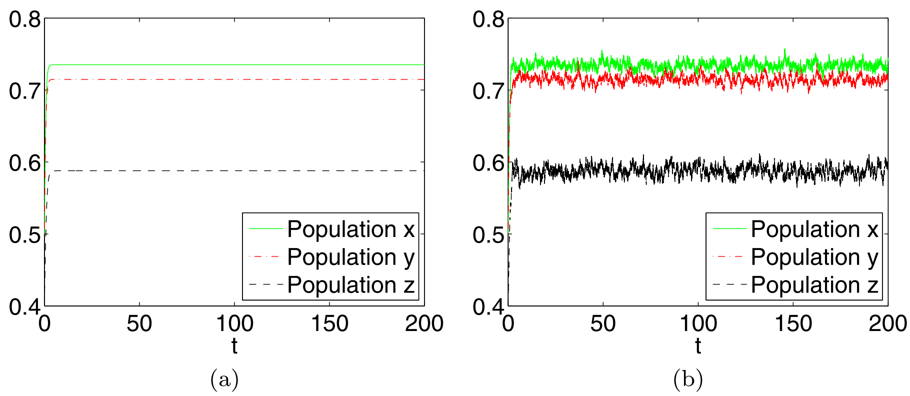


Figure 2 Solutions of system (4) for $(x_0, y_0, z_0) = (0.5, 0.5, 0.4)$, $a_1 = 2$, $a_2 = 2$, $a_3 = 1$, $b_1 = 2.5$, $b_2 = 2.5$, $b_3 = 2.6$, $c_1 = 0.1$, $c_2 = 0.15$, $d_1 = 0.8$, $d_2 = 0.8$, $\alpha_1 = 1$, $\beta_1 = 0.8$, $\alpha_2 = 1$, $\beta_2 = 0.8$, $h_1 = 0.25$, $h_2 = 0.3$, $f_1 = 1$, $f_2 = 0.8$, $g_1 = 0.5$, $g_2 = 0.6$, $\sigma_1 = 0.05$, $\sigma_2 = 0.05$, $\sigma_2 = 0.05$.

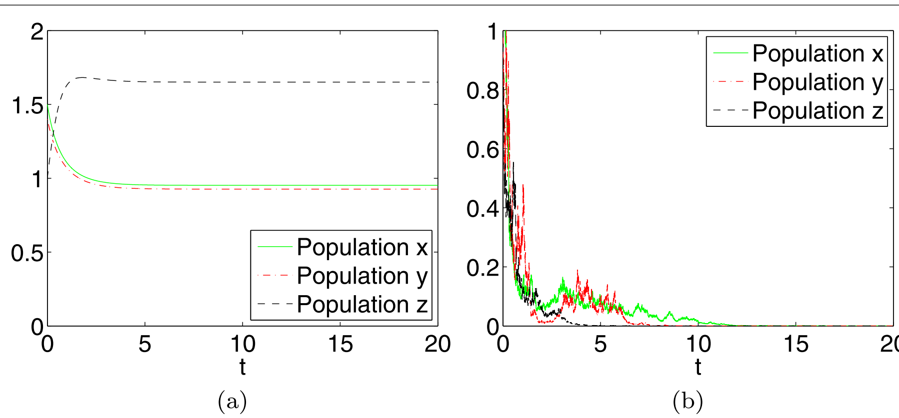


Figure 3 Solutions of system (4) for $(x_0, y_0, z_0) = (1.5, 1.2, 1.0)$, $a_1 = 1.1$, $a_2 = 1.1$, $a_3 = 1$, $b_1 = 0.8$, $b_2 = 0.9$, $b_3 = 1.1$, $c_1 = 0.02$, $c_2 = 0.01$, $d_1 = 1.2$, $d_2 = 1$, $\alpha_1 = 0.8$, $\beta_1 = 0.5$, $\alpha_2 = 0.7$, $\beta_2 = 0.5$, $h_1 = 0.5$, $h_2 = 0.3$, $f_1 = 1$, $f_2 = 0.5$, $g_1 = 0.5$, $g_2 = 0.6$, $\sigma_1 = 1.8$, $\sigma_2 = 1.8$, $\sigma_2 = 2$.

In Figure 2, we choose the initial value $(x_0, y_0, z_0) = (0.5, 0.5, 0.5)$ and all the parameters satisfying conditions of Theorem 5. Using Matlab, we see that, under a small perturbation, the solution of system (4) is fluctuating in a small neighborhood, at this time, the stochastic system is getting more similar to the deterministic.

In Figure 3, suffering sufficiently large white noise, system (4) gets extinct, while none of the species in the deterministic system die out.

7 Conclusion

In this paper, we studied a stochastic cooperative predator-prey system with Beddington-DeAngelis functional response. We first demonstrate the existence and uniqueness of global positive solution. We then investigate that the solution is stochastically bounded and permanent. Under some specific conditions, the solution of this stochastic system is globally asymptotically stable, which is useful to estimate the risk of extinction of species in the system. Some interesting topics deserve further study. For example, we can try to study three or more trophic levels, which simulates the actual situation better. Moreover, we can consider the extinction of system (4).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
XL proposed the model and drafted the main part of the manuscript. YW contributed to the revision of the manuscript and polished the language. All authors read and approved the final manuscript.

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References

1. Qiu, H, Liu, M, Wang, K, Wang, Y: Dynamics of a stochastic predator-prey system with Beddington-DeAngelis functional response. Appl. Math. Comput. **219**, 2303-2312 (2012)
2. Skalski, GT, Gilliam, JF: Functional responses with predator interference: viable alternatives to the Holling type II model. Ecology **82**, 3083-3092 (2001)
3. Huang, D, Li, Y, Guo, Y: Dynamics of a stochastic cooperative predator-prey system. Abstr. Appl. Anal. **2014**, Article ID 927217 (2014)
4. Cosner, C, DeAngelis, DL, Ault, JS, Olson, DB: Effects of spatial grouping on the functional response of predators. Theor. Popul. Biol. **56**, 65-75 (1999)
5. Cantrell, RS, Cosner, C: On the dynamics of predator-prey models with the Beddington-DeAngelis functional response. J. Math. Anal. Appl. **257**, 206-222 (2001)
6. Hwang, TW: Uniqueness of limit cycles of the predator-prey system with Beddington-DeAngelis functional response. J. Math. Anal. Appl. **290**, 113-122 (2004)
7. Dimitrov, D, Kojouharov, H: Complete mathematical analysis of predator-prey models with linear prey growth and Beddington-DeAngelis functional response. Appl. Math. Comput. **162**, 523-538 (2005)
8. Cui, J, Takeuchi, Y: Permanence, extinction and periodic solution of predator-prey system with Beddington-DeAngelis functional response. J. Math. Anal. Appl. **317**, 464-474 (2006)
9. Lin, G, Hong, Y: Delay induced oscillation in predator-prey system with Beddington-DeAngelis functional response. Appl. Math. Comput. **190**, 1296-1311 (2007)
10. Ko, W, Ryu, K: Analysis of diffusive two-competing-prey and one-predator systems with Beddington-DeAngelis functional response. Nonlinear Anal. **71**, 4185-4202 (2009)
11. Li, H, Takeuchi, Y: Dynamics of the density dependent predator-prey system with Beddington-DeAngelis functional response. J. Math. Anal. Appl. **374**, 644-654 (2011)
12. Baek, H: Qualitative analysis of Beddington-DeAngelis type impulsive predator-prey models. Nonlinear Anal., Real World Appl. **11**, 1312-1322 (2010)
13. Goh, BS: Stability in models of mutualism. Am. Nat. **113**, 261-275 (1979)
14. Yang, PH, Xu, R: Global asymptotic stability of periodic solution in n-species cooperative system with time delays. J. Biomath. **13**, 841-862 (1998)
15. May, RM: Stability and Complexity in Model Ecosystems. Princeton University Press, Princeton (2001)
16. Jost, C, Arditi, R: From pattern to process: identifying predator-prey interactions. Popul. Ecol. **43**, 229-243 (2001)
17. Li, X, Mao, X: Population dynamical behavior of nonautonomous Lotka-Volterra competitive system with random perturbation. Discrete Contin. Dyn. Syst., Ser. A **24**, 523-545 (2009)
18. Mao, X: Stochastic Differential Equations and Applications. Horwood, Chichester (2007)

19. Karatzas, I, Shreve, SE: Brownian Motion and Stochastic Calculus. Springer, Berlin (1991)
20. Higham, DJ: An algorithmic introduction to numerical simulation of stochastic differential equations. SIAM Rev. **43**, 525-546 (2001)