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Some generalized Hermite-Hadamard type integral inequalities for generalized s-convex functions on fractal sets

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Abstract
In this article, some new integral inequalities of generalized Hermite-Hadamard type for generalized s-convex functions in the second sense on fractal sets have been established.

MSC: 26A51; 26D07; 26D15; 53C22

Keywords: s-convex functions; fractal space; local fractional derivative

1 Introduction

The convexity of functions is an important concept in the class mathematical analysis course, and it plays a significant role in many fields, for example, in biological system, economy, optimization, and so on [1–7]. Furthermore, there are a lot of several inequalities related to the class of convex functions. For example, Hermite-Hadamard's inequality is one of the well-known results in the literature, which can be stated as follows.

Theorem 1.1 (Hermite-Hadamard's inequality) *Let f be a convex function on $[a_1, a_2]$ with $a_1 < a_2$. If f is integral on $[a_1, a_2]$, then*

$$f\left(\frac{a_1 + a_2}{2}\right) \leq \frac{1}{a_2 - a_1} \int_{a_1}^{a_2} f(x)\,dx \leq \frac{f(a_1) + f(a_2)}{2}. \tag{1}$$

In [8], Dragomir and Fitzpatrick demonstrated a variation of Hadamard's inequality which holds for s-convex functions in the second sense.

Theorem 1.2 *Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be an s-convex function in the second sense, $0 < s < 1$ and $a_1, a_2 \in \mathbb{R}_+$, $a_1 < a_2$. If $f \in L^1([a_1, a_2])$, then*

$$2^{s-1} f\left(\frac{a_1 + a_2}{2}\right) \leq \frac{1}{a_2 - a_1} \int_{a_1}^{a_2} f(x)\,dx \leq \frac{f(a_1) + f(a_2)}{s + 1}. \tag{2}$$

In recent years, fractional calculus played an important part in fractal mathematics and engineering. In the sense of Mandelbrot, a fractal set is the one whose Hausdorff dimension strictly exceeds the topological dimension [9–15]. Many researchers studied the properties of functions on fractal space and constructed many kinds of fractional calculus by

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using different approaches [16–18]. Particularly, in [19], Yang stated the analysis of local fractional functions on fractal space systematically, which includes local fractional calculus and the monotonicity of function.

The outline of this article is as follows. In Section 2, we state the operations with real line number fractal sets and some definitions are given. Some integral inequalities of generalized Hermite-Hadamard type for generalized s-convex functions in the second sense are studied in Section 3. Finally, some applications are also illustrated in Section 4. The conclusions are in Section 5.

2 Preliminaries

Let $\mathbb{R}^\alpha$ be the real line numbers on fractal space. Then, by using Gao-Yang-Kang's concept, one can explain the definitions of the local fractional derivative and local fractional integral as in [19–23]. Now, if r_1^α, r_2^α and $r_3^\alpha \in \mathbb{R}^\alpha$ $(0 < \alpha \leq 1)$, then

(1) $r_1^\alpha + r_2^\alpha \in \mathbb{R}^\alpha$, $r_1^\alpha r_2^\alpha \in \mathbb{R}^\alpha$,
(2) $r_1^\alpha + r_2^\alpha = r_2^\alpha + r_1^\alpha = (r_1 + r_2)^\alpha = (r_2 + r_1)^\alpha$,
(3) $r_1^\alpha + (r_2^\alpha + r_3^\alpha) = (r_1^\alpha + r_2^\alpha) + r_3^\alpha$,
(4) $r_1^\alpha r_2^\alpha = r_2^\alpha r_1^\alpha = (r_1 r_2)^\alpha = (r_2 r_1)^\alpha$,
(5) $r_1^\alpha (r_2^\alpha r_3^\alpha) = (r_1^\alpha r_2^\alpha) r_3^\alpha$,
(6) $r_1^\alpha (r_2^\alpha + r_3^\alpha) = (r_1^\alpha r_2^\alpha) + (r_1^\alpha r_3^\alpha)$,
(7) $r_1^\alpha + 0^\alpha = 0^\alpha + r_1^\alpha = r_1^\alpha$ and $r_1^\alpha \cdot 1^\alpha = 1^\alpha \cdot r_1^\alpha = r_1^\alpha$.

Let us state some definitions about the local fractional calculus on $\mathbb{R}^\alpha$.

Definition 2.1 [19] A non-differentiable function $y\colon \mathbb{R} \to \mathbb{R}^\alpha$ is called local fractional continuous at x_0 if, for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\bigl|y(x) - y(x_0)\bigr| < \varepsilon^\alpha$$

holds for $|x - x_0| < \delta$, where $\varepsilon, \delta \in \mathbb{R}$. $y \in C_\alpha(a_1, a_2)$ if it is local fractional continuous on the interval (a_1, a_2).

Definition 2.2 [19] The local fractional derivative of $y(m)$ of order α at $m = m_0$ is defined by

$$y^\alpha(m_0) = \frac{d^\alpha y(m)}{dm^\alpha}\bigg|_{m=m_0} = \lim_{m\to m_0} \frac{\Gamma(1+\alpha)(y(m) - y(m_0))}{(m - m_0)^\alpha},$$

where $\Gamma(m) = \int_0^\infty m^{z-1} e^{-m}\,dm$. If there exists $y^{(n+1)\alpha}(m) = D_m^\alpha \cdots D_m^\alpha y(m)$ ($n + 1$ times) for any $m \in I \subseteq \mathbb{R}$, then $y \in D_{(n+1)\alpha}(I)$, $n = 0, 1, 2, \ldots$.

Definition 2.3 [19] The local fractional integral of function $y(m)$ of order α is defined by, where $y \in C_\alpha[a_1, a_2]$,

$$\begin{aligned} {}_{a_1}I_{a_2}^{(\alpha)} y(m) &= \frac{1}{\Gamma(1+\alpha)} \int_{a_1}^{a_2} y(t)(dt)^\alpha \\ &= \frac{1}{\Gamma(1+\alpha)} \lim_{\triangle t\to 0} \sum_{i=1}^{n} y(t_i)(\triangle t_i)^\alpha \end{aligned}$$

with $\triangle t_i = t_{i+1} - t_i$ and $\triangle t = \max\{\triangle t_i : i = 1, 2, \ldots, n-1\}$, where $[t_i, t_{i+1}]$, $i = 0, 1, \ldots, n-1$ and $t_0 = a_1 < t_1 < \cdots < t_{n-1} < t_n = a_2$ is a partition of the interval $[a_1, a_2]$.

In [24], the authors introduced the generalized convex function and established the generalized Hermite-Hadamard's inequality on fractal space. Let $f : I \subset \mathbb{R} \to \mathbb{R}^\alpha$ for any $x_1, x_2 \in I$ and $\gamma \in [0, 1]$ if the following inequality

$$f\big(\gamma x_1 + (1-\gamma)x_2\big) \leq \gamma^\alpha f(x_1) + (1-\gamma)^\alpha f(x_2)$$

holds, then f is called a generalized convex function on I. In $\alpha = 1$, we have a convex function, convexity is defined only in geometrical terms as being the property of a function whose graph bears tangents only under it [25].

Theorem 2.1 (Generalized Hermite-Hadamard's inequality) *Let $f \in {}_{a_1}I_{a_2}^{(\alpha)}$ be a generalized convex function on $[a_1, a_2]$ with $a_1 < a_2$. Then*

$$f\left(\frac{a_1 + a_2}{2}\right) \leq \frac{\Gamma(1+\alpha)}{(a_2 - a_1)^\alpha} {}_{a_1}I_{a_2}^{(\alpha)} f(x) \leq \frac{f(a_1) + f(a_2)}{2^\alpha}.$$

Note that it will be reduced to the class Hermite-Hadamard's inequality (1) if $\alpha = 1$.

In [23], Mo and Sui introduced the definitions of two kinds of generalized s-convex functions on fractal sets as follows.

Definition 2.4

(i) A function $f : \mathbb{R}_+ \to \mathbb{R}^\alpha$ is called generalized s-convex $(0 < s < 1)$ in the first sense if

$$f(\gamma_1 x_1 + \gamma_2 x_2) \leq \gamma_1^{s\alpha} f(x_1) + \gamma_2^{s\alpha} f(x_2) \tag{3}$$

for all $x_1, x_2 \in \mathbb{R}_+$ and all $\gamma_1, \gamma_2 \geq 0$ with $\gamma_1^s + \gamma_2^s = 1$, we denote this class of functions by GK_s^1.

(ii) A function $f : \mathbb{R}_+ \to \mathbb{R}^\alpha$ is called generalized s-convex $(0 < s < 1)$ in the second sense if inequality (3) holds for all $x_1, x_2 \in \mathbb{R}_+$ and all $\gamma_1, \gamma_2 \geq 0$ with $\gamma_1 + \gamma_2 = 1$, we denote this class of functions by GK_s^2.

In the same paper [23], Mo and Sui proved that all functions from GK_s^2, $s \in (0, 1)$, are non-negative.

3 Main results

In [26], the authors demonstrated a variation of generalized Hadamard's inequality which holds for a generalized s-convex function in the second sense. Now, we will give another proof for generalized s-Hadamard's inequality.

Theorem 3.1 *Let $f : \mathbb{R}_+ \to \mathbb{R}_+^\alpha$ be a generalized s-convex function in the second sense*, $0 < s < 1$ *and $a_1, a_2 \in \mathbb{R}_+$ with $a_1 < a_2$. If $f \in L^1([a_1, a_2])$, then*

$$\begin{aligned} 2^{\alpha(s-1)} f\left(\frac{a_1 + a_2}{2}\right) &\leq \frac{\Gamma(1+\alpha)}{(a_2 - a_1)^\alpha} {}_{a_1}I_{a_2}^{(\alpha)} f(x) \\ &\leq \frac{\Gamma(1+s\alpha)\Gamma(1+\alpha)}{\Gamma(1+(s+1)\alpha)} \big(f(a_1) + f(a_2)\big). \end{aligned} \tag{4}$$

Proof Since f is generalized s-convex in the second sense, then

$$f\big(\gamma a_1 + (1-\gamma)a_2\big) \leq \gamma^{\alpha s} f(a_1) + (1-\gamma)^{\alpha s} f(a_2), \quad \forall \gamma \in [0,1].$$

Integrating the above inequality with respect to γ on $[0,1]$, we have

$$\begin{aligned}\Gamma(1+\alpha)_0 I_1^{(\alpha)} f\big(\gamma a_1 + (1-\gamma)a_2\big) &\leq f(a_1)\Gamma(1+\alpha)_0 I_1^{(\alpha)} \gamma^{\alpha s} \\ &\quad + f(a_2)\Gamma(1+\alpha)_0 I_1^{(\alpha)} (1-\gamma)^{\alpha s} \\ &= \frac{\Gamma(1+s\alpha)\Gamma(1+\alpha)}{\Gamma(1+(s+1)\alpha)}\big(f(a)+f(b)\big).\end{aligned}$$

Let $x = \gamma a_1 + (1-\gamma)a_2$. Then we have

$$\begin{aligned}\Gamma(1+\alpha)_0 I_1^{(\alpha)} f\big(\gamma a_1 + (1-\gamma)a_2\big) &= \frac{\Gamma(1+\alpha)}{(a_1-a_2)^\alpha}\, {}_{a_2}I_{a_1}^{(\alpha)} f(x) \\ &= \frac{\Gamma(1+\alpha)}{(a_2-a_1)^\alpha}\, {}_{a_1}I_{a_2}^{(\alpha)} f(x).\end{aligned}$$

Now, it follows that

$$\frac{\Gamma(1+\alpha)}{(a_2-a_1)^\alpha}\, {}_{a_1}I_{a_2}^{(\alpha)} f(x) \leq \frac{\Gamma(1+s\alpha)\Gamma(1+\alpha)}{\Gamma(1+(s+1)\alpha)}\big(f(a_1)+f(a_2)\big).$$

Then the second inequality in (4) is proved.

In order to prove the first inequality in (4), we use the following inequality:

$$f\left(\frac{x_1+x_2}{2}\right) \leq \frac{f(x_1)+f(x_2)}{2^{\alpha s}}, \quad \forall x_1, x_2 \in I. \tag{5}$$

Now, assume that $x_1 = \gamma a_1 + (1-\gamma)a_2$ and $x_2 = (1-\gamma)a_1 + \gamma a_2$ with $\gamma \in [0,1]$.

Then we get by inequality (5) that

$$f\left(\frac{a_1+a_2}{2}\right) \leq \frac{f(\gamma a_1 + (1-\gamma)a_2) + f((1-\gamma)a_1 + \gamma a_2)}{2^{\alpha s}}, \quad \forall \gamma \in [0,1].$$

By integrating both sides of the above inequalities over $[0,1]$, we have

$$\frac{1}{\Gamma(1+\alpha)} \int_0^1 f\left(\frac{a_1+a_2}{2}\right)(d\gamma)^\alpha \leq \frac{1}{2^{\alpha(s-1)}(a_2-a_1)^\alpha}\, {}_{a_1}I_{a_2}^{(\alpha)} f(x).$$

Then it follows that

$$2^{\alpha(s-1)} f\left(\frac{a_1+a_2}{2}\right) \leq \frac{\Gamma(1+\alpha)}{(a_2-a_1)^\alpha}\, {}_{a_1}I_{a_2}^{(\alpha)} f(x).$$

This completes the proof. □

Remark 3.1 If we set $c = \frac{\Gamma(1+s\alpha)\Gamma(1+\alpha)}{\Gamma(1+(s+1)\alpha)}$ for $s \in (0,1]$, then it is best possible in the second inequality of (4).

As the function $f\colon [0,1] \to [0^\alpha, 1^\alpha]$ given by $f(x) = x^{s\alpha}$ is generalized s-convex in the second sense,

$$\begin{aligned}\frac{\Gamma(1+\alpha)}{(a_2-a_1)^\alpha}{}_{a_1}I_{a_2}^{(\alpha)}f(x) &= \Gamma(1+\alpha)\frac{1}{\Gamma(1+\alpha)}\int_0^1 x^{\alpha s}(dx)^\alpha \\ &= \frac{\Gamma(1+s\alpha)\Gamma(1+\alpha)}{\Gamma(1+(s+1)\alpha)}\end{aligned}$$

and

$$\frac{\Gamma(1+s\alpha)\Gamma(1+\alpha)}{\Gamma(1+(s+1)\alpha)}\bigl(f(0)+f(1)\bigr) = \frac{\Gamma(1+s\alpha)\Gamma(1+\alpha)}{\Gamma(1+(s+1)\alpha)}.$$

Similarly, if $\alpha = 1$, then inequalities (4) reduce to inequalities (2).

Theorem 3.2 *Let $A\colon [0,1] \to \mathbb{R}^\alpha$ be a function such as*

$$A(\gamma) = \frac{\Gamma(1+\alpha)}{(a_2-a_1)^\alpha}{}_{a_1}I_{a_2}^{(\alpha)}f\left(\gamma x + (1-\gamma)\frac{a_1+a_2}{2}\right), \quad \gamma \in [0,1],$$

where $f\colon [a_1,a_2] \to \mathbb{R}^\alpha$ is a generalized s-convex function in the second sense, $s \in (0,1]$, $a_1, a_2 \in \mathbb{R}_+$, $a_1 < a_2$ and $f \in L^1([a_1,a_2])$. Then

(i) *$A \in GK_s^2$ on $[0,1]$,*

(ii) *we have the inequality*

$$A(\gamma) \geq 2^{\alpha(s-1)}f\left(\frac{a_1+a_2}{2}\right), \quad \forall \gamma \in [0,1], \tag{6}$$

(iii) *and the following inequality also holds:*

$$A \leq \min\bigl\{A_1(\gamma), A_2(\gamma)\bigr\}, \quad \gamma \in [0,1], \tag{7}$$

where

$$A_1(\gamma) = \gamma^{\alpha s}\frac{\Gamma(1+\alpha)}{(a_2-a_1)^\alpha}{}_{a_1}I_{a_2}^{(\alpha)}f(x) + (1-\gamma)^{\alpha s}f\left(\frac{a_1+a_2}{2}\right)$$

and

$$\begin{aligned}A_2(\gamma) = \frac{\Gamma(1+\alpha s)\Gamma(1+\alpha)}{\Gamma(1+(s+1)\alpha)}\biggl(&f\left(\gamma a_1 + (1-\gamma)\frac{a_1+a_2}{2}\right) \\ &+ f\left(\gamma a_2 + (1-\gamma)\frac{a_1+a_2}{2}\right)\biggr)\end{aligned}$$

for $\gamma \in (0,1]$.

(iv) *If $\tilde{A} = \max\{A_1(\gamma), A_2(\gamma)\}$, $\gamma \in [0,1]$, then*

$$\tilde{A} \leq \frac{\Gamma(1+\alpha s)\Gamma(1+\alpha)}{\Gamma(1+(s+1)\alpha)}\left\{\gamma^{\alpha s}\bigl(f(a_1)+f(a_2)\bigr) + 2^\alpha(1-\gamma)^{\alpha s}f\left(\frac{a_1+a_2}{2}\right)\right\}.$$

Proof (i) Let $\gamma_1, \gamma_2 \in [0,1]$ and $\mu_1, \mu_2 \geq 0$ with $\mu_1 + \mu_2 = 1$, then

$$\begin{aligned} A(\mu_1\gamma_1 + \mu_2\gamma_2) &= \frac{\Gamma(1+\alpha)}{(a_2-a_1)^\alpha} {}_{a_1}I_{a_2}^{(\alpha)} f\left((\mu_1\gamma_1 + \mu_2\gamma_2)x + \bigl(1 - (\mu_1\gamma_1 + \mu_2\gamma_2)\bigr)\frac{a_1+a_2}{2} \right) \\ &\leq \frac{\Gamma(1+\alpha)}{(a_2-a_1)^\alpha} {}_{a_1}I_{a_2}^{(\alpha)} \left\{ \mu_1^{\alpha s} f\left(\gamma_1 x + (1-\gamma_1)\frac{a_1+a_2}{2} \right) \right. \\ &\quad \left. + \mu_2^{\alpha s} f\left(\gamma_2 x + (1-\gamma_2)\frac{a_1+a_2}{2} \right) \right\} \\ &= \mu_1^{\alpha s} A(\gamma_1) + \mu_2^{\alpha s} A(\gamma_2), \end{aligned}$$

which implies that $A \in GK_s^2$ on $[0,1]$.

(ii) Let $\gamma \in (0,1]$ and by the change of variable $m = \gamma x + (1-\gamma)\frac{a_1+a_2}{2}$, we have

$$A(\gamma) = \frac{\Gamma(1+\alpha)}{\gamma^\alpha (a_2-a_1)^\alpha} {}_{\gamma a_1 + (1-\gamma)\frac{a_1+a_2}{2}}I^{(\alpha)}_{\gamma a_2 + (1-\gamma)\frac{a_1+a_2}{2}} f(m) = \frac{\Gamma(1+\alpha)}{(b_2-b_1)^\alpha} {}_{b_1}I_{b_2}^{(\alpha)} f(m).$$

By using the first generalized Hermite-Hadamard inequality, we have

$$\frac{\Gamma(1+\alpha)}{(b_2-b_1)^\alpha} {}_{b_1}I_{b_2}^{(\alpha)} f(m) \geq 2^{\alpha(s-1)} f\left(\frac{b_1+b_2}{2} \right) = 2^{\alpha(s-1)} f\left(\frac{a_1+a_2}{2} \right),$$

and inequality (6) is obtained.

If $\gamma = 0$, the inequality

$$f\left(\frac{a_1+a_2}{2} \right) \geq 2^{\alpha(s-1)} f\left(\frac{a_1+a_2}{2} \right)$$

also holds.

(iii) By using the second part of generalized Hadamard's inequality, we get

$$\begin{aligned} \frac{\Gamma(1+\alpha)}{(b_2-b_1)^\alpha} {}_{b_1}I_{b_2}^{(\alpha)} f(m) &\leq \frac{\Gamma(1+s\alpha)\Gamma(1+\alpha)}{\Gamma(1+(s+1)\alpha)} \bigl(f(b_1) + f(b_2) \bigr) \\ &= \frac{\Gamma(1+s\alpha)\Gamma(1+\alpha)}{\Gamma(1+(s+1)\alpha)} \left(f\left(\gamma a_1 + (1-\gamma)\frac{a_1+a_2}{2} \right) \right. \\ &\quad \left. + f\left(\gamma a_2 + (1-\gamma)\frac{a_1+a_2}{2} \right) \right) \\ &= A_2(\gamma), \quad \forall \gamma \in [0,1]. \end{aligned}$$

If $\gamma = 0$, then the inequality

$$f\left(\frac{a_1+a_2}{2} \right) = A(0) \leq A_2(0) = \frac{2^\alpha \Gamma(1+s\alpha)\Gamma(1+\alpha)}{\Gamma(1+(s+1)\alpha)} f\left(\frac{a_1+a_2}{2} \right)$$

holds as it is equivalent to

$$\left(\frac{\Gamma(1+(s+1)\alpha)}{\Gamma(1+s\alpha)\Gamma(1+\alpha)} - 2^\alpha \right) f\left(\frac{a_1+a_2}{2} \right) \leq 0^\alpha$$

and we know that for $s \in (0,1)$,

$$f\left(\frac{a_1 + a_2}{2}\right) \geq 0^{\alpha}.$$

Since

$$f\left(\gamma x + (1-\gamma)\frac{a_1 + a_2}{2}\right) \leq \gamma^{\alpha s} f(x) + (1-\gamma)^{\alpha s} f\left(\frac{a_1 + a_2}{2}\right)$$

for $\forall \gamma \in [0,1]$ and $x \in [a_1, a_2]$, then we obtain

$$\begin{aligned} A(\gamma) &= \frac{\Gamma(1+\alpha)}{(a_2 - a_1)^{\alpha}} {}_{a_1}I_{a_2}^{(\alpha)} f\left(\gamma x + (1-\gamma)\frac{a_1 + a_2}{2}\right) \\ &\leq \gamma^{\alpha s} \frac{\Gamma(1+\alpha)}{(a_2 - a_1)^{\alpha}} {}_{a_1}I_{a_2}^{(\alpha)} f(x) + (1-\gamma)^{\alpha s} f\left(\frac{a_1 + a_2}{2}\right) \\ &= A_1(\gamma). \end{aligned}$$

Then, the proof of inequality (7) is complete.

(iv) We have

$$\begin{aligned} A_2(\gamma) &= \frac{\Gamma(1+s\alpha)\Gamma(1+\alpha)}{\Gamma(1+(s+1)\alpha)} \left[f\left(\gamma a_1 + (1-\gamma)\frac{a_1 + a_2}{2}\right) + f\left(\gamma a_2 + (1-\gamma)\frac{a_1 + a_2}{2}\right) \right] \\ &\leq \frac{\Gamma(1+s\alpha)\Gamma(1+\alpha)}{\Gamma(1+(s+1)\alpha)} \left[\gamma^{\alpha s} f(a_1) + (1-\gamma)^{\alpha s} f\left(\frac{a_1 + a_2}{2}\right) \right. \\ &\quad \left. + \gamma^{\alpha s} f(a_2) + (1-\gamma)^{\alpha s} f\left(\frac{a_1 + a_2}{2}\right) \right] \\ &= \frac{\Gamma(1+s\alpha)\Gamma(1+\alpha)}{\Gamma(1+(s+1)\alpha)} \left[\gamma^{\alpha s} \big(f(a_1) + f(a_2)\big) + 2^{\alpha} (1-\gamma)^{\alpha s} f\left(\frac{a_1 + a_2}{2}\right) \right], \\ &\quad \forall \gamma \in [0,1]. \end{aligned}$$

Since

$$\frac{\Gamma(1+\alpha)}{(a_2 - a_1)^{\alpha}} {}_{a_1}I_{a_2}^{(\alpha)} f(x) \leq \frac{\Gamma(1+s\alpha)\Gamma(1+\alpha)}{\Gamma(1+(s+1)\alpha)} \big(f(a_1) + f(a_2)\big)$$

and

$$(1-\gamma)^{\alpha s} f\left(\frac{a_1 + a_2}{2}\right) \leq 2^{\alpha} (1-\gamma)^{\alpha s} \frac{\Gamma(1+s\alpha)\Gamma(1+\alpha)}{\Gamma(1+(s+1)\alpha)} f\left(\frac{a_1 + a_2}{2}\right),$$

then

$$\begin{aligned} A_1(\gamma) &\leq \gamma^{\alpha s} \frac{\Gamma(1+s\alpha)\Gamma(1+\alpha)}{\Gamma(1+(s+1)\alpha)} \big(f(a_1) + f(a_2)\big) \\ &\quad + 2^{\alpha} (1-\gamma)^{\alpha s} \frac{\Gamma(1+s\alpha)\Gamma(1+\alpha)}{\Gamma(1+(s+1)\alpha)} f\left(\frac{a_1 + a_2}{2}\right) \end{aligned}$$

and the proof of Theorem 3.2 is complete. □

Remark 3.2 In particular:

1. If we choose $s = 1$ in Theorem 3.2, then we get:

(a)

$$\begin{aligned}
&\frac{\Gamma(1+\alpha)}{(a_2-a_1)^\alpha}\, {}_{a_1}I_{a_2}^{(\alpha)} f\left(\gamma x + (1-\gamma)\frac{a_1+a_2}{2}\right) \\
&\quad\le \min\left\{\gamma^\alpha \frac{\Gamma(1+\alpha)}{(a_2-a_1)^\alpha}\, {}_{a_1}I_{a_2}^{(\alpha)} f(x) + (1-\gamma)^\alpha f\left(\frac{a_1+a_2}{2}\right),\right. \\
&\qquad \frac{(\Gamma(1+\alpha))^2}{\Gamma(1+2\alpha)}\left(f\left(\gamma a_1 + (1-\gamma)\frac{a_1+a_2}{2}\right)\right. \\
&\qquad \left.\left. + f\left(\gamma a_2 + (1-\gamma)\frac{a_2+a_2}{2}\right)\right)\right\}.
\end{aligned}$$

(b) Since

$$\begin{aligned}
\tilde{A} &= \max\left\{\gamma^\alpha \frac{\Gamma(1+\alpha)}{(a_2-a_1)^\alpha}\, {}_{a_1}I_{a_2}^{(\alpha)} f(x) + (1-\gamma)^\alpha f\left(\frac{a_1+a_2}{2}\right),\right. \\
&\quad \left.\frac{(\Gamma(1+\alpha))^2}{\Gamma(1+2\alpha)}\left(f\left(\gamma a_1 + (1-\gamma)\frac{a_1+a_2}{2}\right) + f\left(\gamma a_2 + (1-\gamma)\frac{a_1+a_2}{2}\right)\right)\right\},
\end{aligned}$$

we have

$$\tilde{A}(\gamma) \le \frac{(\Gamma(1+\alpha))^2}{\Gamma(1+2\alpha)}\left[\gamma^\alpha\big(f(a_1)+f(a_2)\big) + 2^\alpha(1-\gamma)^\alpha f\left(\frac{a_1+a_2}{2}\right)\right].$$

2. Now if one chooses $\alpha = 1$ in Theorem 3.2, then we can easily obtain:

(a)

$$\begin{aligned}
&\frac{1}{(a_2-a_1)}\int_{a_1}^{a_2} f\left(\gamma x + (1-\gamma)\frac{a_1+a_2}{2}\right) dx \\
&\quad\le \min\left\{\gamma^s \frac{1}{(a_2-a_1)}\int_{a_1}^{a_2} f(x)\,dx + (1-\gamma)^s f\left(\frac{a_2+a_2}{2}\right),\right. \\
&\qquad \left.\frac{1}{s+1}\left(f\left(\gamma a_1 + (1-\gamma)\frac{a_1+a_2}{2}\right) + f\left(\gamma a_2 + (1-\gamma)\frac{a_1+a_2}{2}\right)\right)\right\}.
\end{aligned}$$

(b) Similarly we have

$$\begin{aligned}
\tilde{A} &= \max\left\{\gamma^s \frac{1}{(a_2-a_1)}\int_{a_1}^{a_2} f(x)\,dx + (1-\gamma)^s f\left(\frac{a_1+a_2}{2}\right),\right. \\
&\quad \left.\frac{1}{s+1}\left(f\left(\gamma a_1 + (1-\gamma)\frac{a_1+a_2}{2}\right) + f\left(\gamma a_2 + (1-\gamma)\frac{a_1+a_2}{2}\right)\right)\right\}
\end{aligned}$$

and

$$\tilde{A}(\gamma) \le \frac{1}{s+1}\left[\gamma^s\big(f(a_1)+f(a_2)\big) + 2(1-\gamma)^s f\left(\frac{a_2+a_2}{2}\right)\right] \quad \text{for } \forall\gamma \in [0,1].$$

3. If one considers $\alpha = 1$ and $s = 1$ in Theorem 3.2, then we get:

(a)

$$\begin{aligned}&\frac{1}{(a_2-a_1)}\int_{a_1}^{a_2} f\left(\gamma x+(1-\gamma)\frac{a_1+a_2}{2}\right)dx\\&\quad\leq \min\left\{\gamma\frac{1}{(a_2-a_1)}\int_{a_1}^{a_2} f(x)\,dx+(1-\gamma)f\left(\frac{a_2+a_2}{2}\right),\right.\\&\quad\left.\frac{1}{2}\left(f\left(\gamma a_1+(1-\gamma)\frac{a_1+a_2}{2}\right)+f\left(\gamma a_2+(1-\gamma)\frac{a_1+a_2}{2}\right)\right)\right\}.\end{aligned}$$

(b)

$$\begin{aligned}\tilde{A}&=\max\left\{\gamma\frac{1}{(a_2-a_1)}\int_{a_1}^{a_2} f(x)\,dx+(1-\gamma)f\left(\frac{a_1+a_2}{2}\right),\right.\\&\quad\left.\frac{1}{2}\left(f\left(\gamma a_1+(1-\gamma)\frac{a_1+a_2}{2}\right)+f\left(\gamma a_2+(1-\gamma)\frac{a_1+a_2}{2}\right)\right)\right\}\end{aligned}$$

and

$$\tilde{A}(\gamma)\leq\frac{1}{2}\left[\gamma\big(f(a_1)+f(a_2)\big)+2(1-\gamma)f\left(\frac{a_2+a_2}{2}\right)\right]\quad\text{for }\forall\gamma\in[0,1].$$

Theorem 3.3 *Let $g\colon[0,1]\to\mathbb{R}^\alpha$ be a function such as*

$$g(\gamma)=\frac{1}{(\Gamma(1+\alpha))^2}\frac{1}{(a_2-a_1)^{2\alpha}}\int_{a_1}^{a_2}\int_{a_1}^{a_2} f\big(\gamma x_1+(1-\gamma)x_2\big)(dx_1)^\alpha(dx_2)^\alpha,\quad \gamma\in[0,1],$$

where $f\colon[a_1,a_2]\to\mathbb{R}_+^\alpha$ is a generalized s-convex function in the second sense, $s\in(0,1]$, $a_1,a_2\in\mathbb{R}_+$ which $a_1<a_2$ and $f\in L^1([a_1,a_2])$. Then:

(i) *$g\in GK_s^2$ in $[0,1]$. If $f\in GK_s^1$, then $g\in GK_s^1$.*

(ii) *$g(\gamma+\frac{1}{2})=g(\frac{1}{2}-\gamma)$ for all $\gamma=[0,\frac{1}{2}]$ and $g(\gamma)$ is symmetric about $\gamma=\frac{1}{2}$.*

(iii) *We have the inequality*

$$g(\gamma)\geq\frac{2^{\alpha(s-1)}}{(\Gamma(1+\alpha))^2}A(\gamma)\geq\frac{4^{\alpha(s-1)}}{(\Gamma(1+\alpha))^2}f\left(\frac{a_1+a_2}{2}\right)\quad\textit{for }\forall\gamma\in[0,1].\tag{8}$$

(iv) *We have the inequality*

$$g(\gamma)\leq\min\big\{g_1(\gamma),g_2(\gamma)\big\},\tag{9}$$

where

$$g_1(\gamma)=\big[\gamma^{\alpha s}+(1-\gamma)^{\alpha s}\big]\frac{1}{\Gamma(1+\alpha)(a_2-a_1)^\alpha}\,{}_{a_1}I_{a_2}^{(\alpha)}f(x_1)$$

and

$$g_2(\gamma)=\left[\frac{\Gamma(1+s\alpha)}{\Gamma(1+(s+1)\alpha)}\right]^2\big[f(a_1)+f\big(\gamma a_1+(1-\gamma)a_2\big)+f\big((1-\gamma)a_1+\gamma a_2\big)+f(a_2)\big]$$

for $\forall\gamma\in[0,1]$.

Proof (i) Take $\{\gamma_1, \gamma_2\} \subset [0,1]$, $\gamma_1 + \gamma_2 = 1$, $t_1, t_2 \in D$ and $f \in GK_s^2$, then we have

$$\begin{aligned}
g(\gamma_1 t_1 + \gamma_2 t_2) &= \frac{1}{(a_2-a_1)^{2\alpha}(\Gamma(1+\alpha))^2} \\
&\quad \times \int_{a_1}^{a_2}\int_{a_1}^{a_2} f\bigl((\gamma_1 t_1 + \gamma_2 t_2)x_1 + \bigl(1-(\gamma_1 t_1 + \gamma_2 t_2)\bigr)x_2\bigr)(dx_1)^\alpha (dx_2)^\alpha \\
&\le \frac{1}{(a_2-a_1)^{2\alpha}(\Gamma(1+\alpha))^2} \\
&\quad \times \int_{a_1}^{a_2}\int_{a_1}^{a_2} \bigl[\gamma_1^{\alpha s} f(t_1 x_1 + x_2 - t_1 x_2) \\
&\quad + \gamma_2^{\alpha s} f(t_2 x_1 + x_2 - t_2 x_2)\bigr](dx_1)^\alpha (dx_2)^\alpha \\
&= \gamma_1^{\alpha s}\frac{1}{(a_2-a_1)^{2\alpha}(\Gamma(1+\alpha))^2}\int_{a_1}^{a_2}\int_{a_1}^{a_2} f(t_1 x_1 + x_2 - t_1 x_2)(dx_1)^\alpha (dx_2)^\alpha \\
&\quad + \gamma_2^{\alpha s}\frac{1}{(a_2-a_1)^{2\alpha}(\Gamma(1+\alpha))^2}\int_{a_1}^{a_2}\int_{a_1}^{a_2} f(t_2 x_1 + x_2 - t_2 x_2)(dx_1)^\alpha (dx_2)^\alpha \\
&= \gamma_1^{\alpha s} g(t_1) + \gamma_2^{\alpha s} g(t_2),
\end{aligned}$$

which implies that $g \in GK_s^2$ in $[0,1]$.

(ii) Let $\gamma \in [0, \frac{1}{2}]$, then

$$\begin{aligned}
g\left(\gamma + \frac{1}{2}\right) &= \frac{1}{(a_2-a_1)^{2\alpha}(\Gamma(1+\alpha))^2} \\
&\quad \times \int_{a_1}^{a_2}\int_{a_1}^{a_2} f\left(\left(\gamma + \frac{1}{2}\right)x_1 + \left(1-\gamma-\frac{1}{2}\right)x_2\right)(dx_1)^\alpha (dx_2)^\alpha \\
&= \frac{1}{(a_2-a_1)^{2\alpha}(\Gamma(1+\alpha))^2} \\
&\quad \times \int_{a_1}^{a_2}\int_{a_1}^{a_2} f\left(\left(\frac{1}{2}-\gamma\right)x_1 + \left(\frac{1}{2}+\gamma\right)x_2\right)(dx_1)^\alpha (dx_2)^\alpha \\
&= g\left(\frac{1}{2}-\gamma\right).
\end{aligned}$$

$g(\gamma)$ is symmetric about $\gamma = \frac{1}{2}$ because $g(\gamma) = g(1-\gamma)$.

(iii) Let us observe that

$$g(\gamma) = \frac{1}{(\Gamma(1+\alpha))^2}\frac{1}{(a_2-a_1)^\alpha}\int_{a_1}^{a_2}\left(\frac{1}{(a_2-a_1)^\alpha}\int_{a_1}^{a_2} f\bigl(\gamma x_1 + (1-\gamma)x_2\bigr)\right)(dx_1)^\alpha (dx_2)^\alpha.$$

Now, since x_2 is fixed in $[a_1, a_2]$, then the function

$$A_{x_2}: [0,1] \to \mathbb{R}^\alpha$$

can be given by

$$\begin{aligned}
A_{x_2}(\gamma) &= \frac{1}{(a_2-a_1)^\alpha}\int_{a_1}^{a_2} f\bigl(\gamma x_1 + (1-\gamma)x_2\bigr)(dx_1)^\alpha \\
&= \frac{\Gamma(1+\alpha)}{(a_2-a_1)^\alpha}\,{}_{a_1}I_{a_2}^{(\alpha)} f\bigl(\gamma x_1 + (1-\gamma)x_2\bigr).
\end{aligned}$$

As it was shown in the proof of Theorem 3.2, for $\gamma \in [0,1]$, we have equality

$$A_{x_2}(\gamma) = \frac{\Gamma(1+\alpha)}{(b_2-b_1)^\alpha} {}_{b_1}I_{b_2}^{(\alpha)} f(m),$$

where $b_2 = \gamma a_2 + (1-\gamma)x_2$ and $b_1 = \gamma a_1 + (1-\gamma)x_2$. By using the generalized Hermite-Hadamard inequality, we have

$$\begin{aligned}\frac{\Gamma(1+\alpha)}{(b_2-b_1)^\alpha} {}_{b_1}I_{b_2}^{(\alpha)} f(m) &\geq 2^{\alpha(s-1)} f\left(\frac{b_1+b_2}{2}\right) \\ &= 2^{\alpha(s-1)} f\left(\gamma \frac{a_1+a_2}{2} + (1-\gamma)x_2\right)\end{aligned}$$

for all $\gamma \in (0,1)$ and $x_2 \in [a_1,a_2]$. Integrating on $[a_1,a_2]$ over x_2, we have

$$g(\gamma) \geq \frac{2^{\alpha(s-1)}}{(\Gamma(1+\alpha))^2} A(1-\gamma) \quad \text{for } \forall \gamma \in (0,1).$$

Further, since $g(\gamma) = g(1-\gamma)$, then the proof of inequality (8) is done for $\gamma \in (0,1)$. If $\gamma = 0$ or $\gamma = 1$, then inequality (8) also holds.

(iv) Since $f(\gamma x_1 + (1-\gamma)x_2) \leq \gamma^{\alpha s} f(x_1) + (1-\gamma)^{\alpha s} f(x_2)$ for all $x_1, x_2 \in [a_1,a_2]$ and $\gamma \in [0,1]$, integrating the above inequality on $[a_1,a_2]^2$, we have

$$\begin{aligned}&\frac{1}{(a_2-a_1)^{2\alpha}} \int_{a_1}^{a_2}\int_{a_1}^{a_2} f\big(\gamma x_1 + (1-\gamma)x_2\big)(dx_1)^\alpha (dx_2)^\alpha \\ &\quad\leq \gamma^{\alpha s} \frac{(\Gamma(1+\alpha))^2}{(a_2-a_1)^{2\alpha}} \frac{1}{(\Gamma(1+\alpha))^2} \int_{a_1}^{a_2}\int_{a_1}^{a_2} f(x_1)(dx_1)^\alpha (dx_2)^\alpha \\ &\qquad + (1-\gamma)^{\alpha s} \frac{(\Gamma(1+\alpha))^2}{(a_2-a_1)^{2\alpha}} \frac{1}{(\Gamma(1+\alpha))^2} \int_{a_1}^{a_2}\int_{a_1}^{a_2} f(x_2)(dx_1)^\alpha (dx_2)^\alpha \\ &\quad= \gamma^{\alpha s} \frac{\Gamma(1+\alpha)}{(a_2-a_1)^\alpha} {}_{a_1}I_{a_2}^{(\alpha)} f(x_1) + (1-\gamma)^{\alpha s} \frac{\Gamma(1+\alpha)}{(a_2-a_1)^\alpha} {}_{a_1}I_{a_2}^{(\alpha)} f(x_2) \\ &\quad= \big(\gamma^{\alpha s} + (1-\gamma)^{\alpha s}\big) \frac{\Gamma(1+\alpha)}{(a_2-a_1)^\alpha} {}_{a_1}I_{a_2}^{(\alpha)} f(x_1).\end{aligned}$$

The proof of the first part in (9) is done.

By the second part of the generalized Hermite-Hadamard inequality, we obtain

$$\begin{aligned}A_{x_2}(\gamma) &= \frac{\Gamma(1+\alpha)}{(b_2-b_1)^\alpha} {}_{b_1}I_{b_2}^{(\alpha)} f(m) \\ &\leq \frac{\Gamma(1+s\alpha)\Gamma(1+\alpha)}{\Gamma(1+(s+1)\alpha)} \big(f\big(\gamma a_1 + (1-\gamma)x_2\big) + f\big(\gamma a_2 + (1-\gamma)x_2\big)\big),\end{aligned}$$

where $b_2 = \gamma a_2 + (1-\gamma)x_2$ and $b_1 = \gamma a_1 + (1-\gamma)x_2$, $\gamma \in [0,1]$. Integrating this inequality on $[a_1,a_2]$ over x_2, then

$$\begin{aligned}g(\gamma) \leq \frac{\Gamma(1+s\alpha)}{\Gamma(1+\alpha)\Gamma(1+(s+1)\alpha)} \bigg[&\frac{\Gamma(1+\alpha)}{(a_2-a_1)^\alpha} {}_{a_1}I_{a_2}^{(\alpha)} f\big(\gamma a_1 + (1-\gamma)x_2\big) \\ &+ \frac{\Gamma(1+\alpha)}{(a_2-a_1)^\alpha} {}_{a_1}I_{a_2}^{(\alpha)} f\big(\gamma a_2 + (1-\gamma)x_2\big)\bigg].\end{aligned}$$

A simple calculation shows that

$$\begin{aligned}
&\frac{\Gamma(1+\alpha)}{(a_2-a_1)^\alpha}\,{}_{a_1}I_{a_2}^{(\alpha)} f\bigl(\gamma a_2+(1-\gamma)x_2\bigr)\\
&\quad=\frac{\Gamma(1+\alpha)}{(c_2-c_1)^\alpha}\,{}_{c_1}I_{c_2}^{(\alpha)} f(m)\\
&\quad\le\frac{\Gamma(1+s\alpha)\Gamma(1+\alpha)}{\Gamma(1+(s+1)\alpha)}\bigl[f(c_1)+f(c_2)\bigr]\\
&\quad=\frac{\Gamma(1+s\alpha)\Gamma(1+\alpha)}{\Gamma(1+(s+1)\alpha)}\bigl[f(a_2)+f\bigl(\gamma a_2+(1-\gamma)a_1\bigr)\bigr],
\end{aligned}$$

where $c_2=a_2$, $c_1=\gamma a_2+(1-\gamma)a_1$ and $\gamma\in(0,1)$. Similarly, for $\gamma\in(0,1)$,

$$\begin{aligned}
&\frac{\Gamma(1+\alpha)}{(a_2-a_1)^\alpha}\,{}_{a_1}I_{a_2}^{(\alpha)} f\bigl(\gamma a_1+(1-\gamma)x_2\bigr)\\
&\quad\le\frac{\Gamma(1+s\alpha)\Gamma(1+\alpha)}{\Gamma(1+(s+1)\alpha)}\bigl[f(a_1)+f\bigl(\gamma a_1+(1-\gamma)a_2\bigr)\bigr].
\end{aligned}$$

Then

$$g(\gamma)\le\left[\frac{\Gamma(1+s\alpha)}{\Gamma(1+(s+1)\alpha)}\right]^2\bigl[f(a_1)+f\bigl(\gamma a_1+(1-\gamma)a_2\bigr)+f\bigl((1-\gamma)a_1+\gamma a_2\bigr)+f(a_2)\bigr].$$

If $\gamma=0$ or $\gamma=1$, then this inequality also holds. □

Remark 3.3 If $\alpha=1$ in the above theorem, then

$$g(\gamma)=\frac{1}{(a_2-a_1)^2}\int_{a_1}^{a_2}\int_{a_1}^{a_2} f\bigl(\gamma x_1+(1-\gamma)x_2\bigr)(dx_1)(dx_2),\quad \gamma\in[0,1]$$

and

$$\begin{aligned}
g(\gamma)\le\min\biggl\{&\bigl[\gamma^s+(1-\gamma)^s\bigr]\frac{1}{(a_2-a_1)}\int_{a_1}^{a_2} f(x_1)(dx_1),\\
&\frac{1}{(1+s)^2}\bigl[f(a_1)+f\bigl(\gamma a_1+(1-\gamma)a_2\bigr)+f\bigl((1-\gamma)a_1+\gamma a_2\bigr)+f(a_2)\bigr]\biggr\}.
\end{aligned}$$

Theorem 3.4 *Let us consider that a sum of A belongs to* GK_s^2,

$$A=\sum_{i=1}^n a_i(\gamma),$$

where

$$a_i(\gamma)=\frac{1}{(\Gamma(1+\alpha))^2}\int_{a_1}^{a_2}\int_{a_1}^{a_2} f_i\bigl(\gamma x_1+(1-\gamma)x_2\bigr)(dx_1)^\alpha(dx_2)^\alpha,$$

then

(i) $\sup(A)=2^\alpha\sum_{i=1}^n a_i(0)=2^\alpha\sum_{i=1}^n a_i(1)$,
(ii) *A is symmetric about* $\gamma=\frac{1}{2}$,
(iii) $A\in GK_s^2$.

Proof (i)

$$\begin{aligned} a_i(\gamma) &\le \gamma^{\alpha s}\frac{1}{(\Gamma(1+\alpha))^2}\int_{a_1}^{a_2}\int_{a_1}^{a_2} f_i(x_1)(dx_1)^\alpha(dx_2)^\alpha \\ &\quad + (1-\gamma)^{\alpha s}\frac{1}{(\Gamma(1+\alpha))^2}\int_{a_1}^{a_2}\int_{a_1}^{a_2} f_i(x_2)(dx_1)^\alpha(dx_2)^\alpha, \quad \forall i. \end{aligned}$$

Since f_i are generalized s-convex functions, we get

$$\begin{aligned} a_i(\gamma) &\le \bigl(\gamma^{\alpha s} + (1-\gamma)^{\alpha s}\bigr)\frac{1}{(\Gamma(1+\alpha))^2}\int_{a_1}^{a_2}\int_{a_1}^{a_2} f_i(x_1)(dx_1)^\alpha(dx_2)^\alpha \\ &\le \frac{2^\alpha}{(\Gamma(1+\alpha))^2}\int_{a_1}^{a_2}\int_{a_1}^{a_2} f_i(x_1)(dx_1)^\alpha(dx_2)^\alpha \\ &= 2^\alpha a_i(0) = 2^\alpha a_i(1). \end{aligned}$$

(ii) $a_i(\gamma)$ is symmetric about $\gamma = \frac{1}{2}$ since $a_i(\gamma) = a_i(1-\gamma)$, $\forall i$.
Then $A(1-\gamma) = A(\gamma)$ and A also is.
(iii) Since $a_i(\gamma x_1 + (1-\gamma)x_2) \le \gamma^{\alpha s} a_i(x_1) + (1-\gamma)^{\alpha s} a_i(x_2)$, then

$$\begin{aligned} A\bigl(\gamma x_1 + (1-\gamma)x_2\bigr) &= \sum_{i=1}^{n} a_i\bigl(\gamma x_1 + (1-\gamma)x_2\bigr) \\ &\le \gamma^{\alpha s}\sum_{i=1}^{n} a_i(x_1) + (1-\gamma)^{\alpha s}\sum_{i=1}^{n} a_i(x_2) \\ &= \gamma^{\alpha s}A(x_1) + (1-\gamma)^{\alpha s}A(x_2), \end{aligned}$$

that is, $A \in GK_s^2$. □

4 Applications to special means

We now consider the applications of our theorems to the following generalized means:

$$\begin{aligned} A(a_1,a_2) &= \frac{a_1^\alpha + a_2^\alpha}{2^\alpha}, \quad a_1,a_2 \ge 0, \\ K(a_1,a_2) &= \left(\frac{a_1^{2\alpha} + a_2^{2\alpha}}{2^\alpha}\right)^{\frac{1}{2}}, \quad a_1,a_2 \ge 0 \end{aligned}$$

and

$$G(a_1,a_2) = \bigl(a_1^\alpha a_2^\alpha\bigr)^{\frac{1}{2}}, \quad a_1,a_2 \ge 0.$$

In [23], the following example is given.
Let $0 < s < 1$ and $a_1^\alpha, a_2^\alpha, a_3^\alpha \in \mathbb{R}^\alpha$. Define, for $x \in \mathbb{R}_+$,

$$f(n) = \begin{cases} a_1^\alpha, & n = 0, \\ a_2^\alpha n^{s\alpha} + a_3^\alpha, & n > 0. \end{cases}$$

If $a_2^\alpha \ge 0^\alpha$ and $0^\alpha \le a_3^\alpha \le a_1^\alpha$, then $f \in GK_s^2$.

Proposition 4.1 *Let $a_1, a_2 \in \mathbb{R}_+$, $a_1 < a_2$ and $a_2 - a_1 \leq 1$, then the following inequalities hold*:

$$\frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)\Gamma(1+3\alpha)}\left[K^2(a_1,a_2)+\frac{1}{2^\alpha}G^2(a_1,a_2)\right] \geq 2^\alpha\left[\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\right]^2 A^2(a_1,a_2), \tag{10}$$

$$\begin{aligned}&\left[\left(1-\gamma+\gamma^2\right)^\alpha K^2(a_1,a_2)+\gamma^\alpha(1-\gamma)^\alpha G^2(a_1,a_2)\right]\\ &\quad-\frac{\gamma^{2\alpha}+(1-\gamma)^\alpha}{2^\alpha\Gamma(1+3\alpha)}\left[\frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)}\right]^2\left[K^2(a_1,a_2)+\frac{1}{2^\alpha}G^2(a_1,a_2)\right]\\ &\geq 2^\alpha\gamma^\alpha(1-\gamma)^\alpha A^2(a_1,a_2).\end{aligned} \tag{11}$$

Proof If $f \in GK_s^2$ on $[a_1, a_2]$ for some $\gamma \in [0,1]$ and $s \in (0,1]$, then, in Theorem 3.3, if $f: [0,1] \to [0^\alpha, 1^\alpha]$, $f(x) = x^{2\alpha}$, where $x \in [a_1, a_2]$ and $s = 1$, so

$$\begin{aligned}&\frac{1}{(\Gamma(1+\alpha))^2}\frac{1}{(a_2-a_1)^{2\alpha}}\int_{a_1}^{a_2}\int_{a_1}^{a_2}\left(\gamma x_1+(1-\gamma)x_2\right)^{2\alpha}(dx_1)^\alpha(dx_2)^\alpha\\ &\quad=\left(\gamma^{2\alpha}+(1-\gamma)^{2\alpha}\right)\frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)\Gamma(1+3\alpha)}\left(a_2^{2\alpha}+a_1^\alpha a_2^\alpha+a_1^{2\alpha}\right)\\ &\qquad+2^\alpha\gamma^\alpha(1-\gamma)^\alpha\left[\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\right]^\alpha\left(a_2^\alpha+a_1^\alpha\right)^2.\end{aligned}$$

Then, by Theorem 3.3, we get

$$\frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)\Gamma(1+3\alpha)}\left(a_2^{2\alpha}+a_1^{2\alpha}+a_1^\alpha a_2^\alpha\right) \geq \left[\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\right]^2\left(a_2^\alpha+a_1^\alpha\right)^2.$$

Then we obtain inequality (10).

By applying Theorem 3.3, we obtain inequality (11) as follows:

$$\begin{aligned}&\left[\left(1-\gamma+\gamma^2\right)^\alpha\left(\frac{a_2^{2\alpha}+a_1^{2\alpha}}{2^\alpha}\right)+\gamma^\alpha(1-\gamma)^\alpha a_1^\alpha a_2^\alpha\right]\\ &\quad-\frac{\gamma^{2\alpha}+(1-\gamma)^{2\alpha}}{2^\alpha\Gamma(1+3\alpha)}\left[\frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)}\right]^2\left[\left(\frac{a_2^{2\alpha}+a_1^{2\alpha}}{2^\alpha}\right)+\frac{1}{2^\alpha}a_1^\alpha a_2^\alpha\right]\\ &\geq 2^\alpha\gamma^\alpha(1-\gamma)^\alpha\left(\frac{a_2^\alpha+a_1^\alpha}{2^\alpha}\right)^2.\end{aligned}$$ □

5 Conclusion

In this article, we have established some new integral inequalities of generalized Hermite-Hadamard type for generalized s-convex functions in the second sense on fractal sets $\mathbb{R}^\alpha$, $0 < \alpha < 1$. In particular, our results extend some important inequalities in a classical situation; when $\alpha = 1$, some relationships between these inequalities and the classical inequalities have been established. Finally, we have also given some applications for these inequalities on fractal sets.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors jointly worked on deriving the results and approved the final manuscript.

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