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# Degenerate, partially degenerate and totally degenerate Daehee numbers and polynomials

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## Abstract

Recently, Kim and Seo considered the partially degenerate Bernoulli numbers and polynomials of the first kind and of the second kind and investigated some properties of these numbers and polynomials. In this paper, we consider degenerate Daehee numbers and polynomials and extend them to partially and totally degenerate Daehee numbers and polynomials. We give some properties and identities and relate those numbers and polynomials with Daehee polynomials and Bernoulli polynomials of the second kind. Some  $q$ -analogues of the degenerate Bernoulli polynomials are studied by Kim *et al.* This research influenced us to study  $q$ -analogues of the degenerate Daehee numbers and polynomials. We also study  $q$ -analogues of partially and totally degenerate Daehee numbers and polynomials. And we give some formulas related to those numbers and polynomials.

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## 1 Introduction

Let  $p$  be a fixed prime number. Throughout this paper,  $\mathbb{Z}_p$ ,  $\mathbb{Q}_p$  and  $\mathbb{C}_p$  will denote the ring of  $p$ -adic integers, the field of  $p$ -adic rational numbers and the completion of the algebraic closure of  $\mathbb{Q}_p$ , respectively. The  $p$ -adic norm is normalized as  $|p|_p = \frac{1}{p}$ . Let  $\text{UD}(\mathbb{Z}_p)$  be the space of all  $\mathbb{C}_p$ -valued uniformly differentiable functions of  $\mathbb{Z}_p$ .

The ordinary  $p$ -adic invariant integral on  $\mathbb{Z}_p$  is given by

$$I_0(f) = \int_{\mathbb{Z}_p} f(x) d\mu_0(x) = \lim_{N \rightarrow \infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} f(x). \quad (1)$$

The Daehee polynomials are given by the generating function to be

$$\sum_{n=0}^{\infty} D_n(x) \frac{t^n}{n!} = \frac{\log(1+t)}{t} (1+t)^x \quad (\text{see [1, 2]}). \quad (2)$$

There are many works related with Daehee numbers and polynomials (see [1–8]).

For  $\lambda, t \in \mathbb{C}_p$  with  $|\lambda t| < p^{-\frac{1}{p-1}}$ , the Daehee polynomials with parameter  $\lambda$  are defined by the generating function to be

$$\sum_{n=0}^{\infty} D_n(x|\lambda) \frac{t^n}{n!} = \frac{\log(1 + \lambda t)^{\frac{1}{\lambda}}}{(1 + \lambda t)^{\frac{1}{\lambda}} - 1} (1 + \lambda t)^{\frac{x}{\lambda}} \quad (\text{see [9–12]}). \tag{3}$$

When  $\lambda = 1$ ,  $D_n(x) = D_n(x|1)$  are the Daehee polynomials in (2) ( $n \geq 0$ ).

As it is well known, Bernoulli polynomials of the first kind  $B_n(x)$  and of the second kind  $b_n(x)$  are defined by the generating function to be

$$\sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} = \frac{t}{e^t - 1} e^{tx}, \quad \sum_{n=0}^{\infty} b_n(x) \frac{t^n}{n!} = \frac{t}{\log(1 + t)} (1 + t)^x, \tag{4}$$

respectively (see [13–16]). When  $x = 0$ ,  $B_n = B_n(0)$  and  $b_n = b_n(0)$  are the Bernoulli numbers of the first kind and of the second kind, respectively.

In [17], the degenerate Bernoulli polynomials are defined by Carlitz to be

$$\sum_{n=0}^{\infty} \beta_n(x|\lambda) \frac{t^n}{n!} = \frac{t}{(1 + \lambda t)^{\frac{1}{\lambda}} - 1} (1 + \lambda t)^{\frac{x}{\lambda}}. \tag{5}$$

Let  $q$  be an indeterminate such that  $|1 - q|_p < p^{-\frac{1}{p-1}}$ . The  $q$ -extension of the number  $x$  is defined as  $[x]_q = \frac{1 - q^x}{1 - q}$ . Note that  $\lim_{q \rightarrow 1} [x]_q = x$ . For  $f \in \text{UD}(\mathbb{Z}_p)$ , the  $p$ -adic  $q$ -integral on  $\mathbb{Z}_p$  is defined [18] by Kim to be

$$I_q(f) = \int_{\mathbb{Z}_p} f(x) d\mu_q(x) = \lim_{N \rightarrow \infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x) q^x. \tag{6}$$

From (6), we can derive the following integral equation:

$$qI_q(f_1) - I_q(f) = (q - 1)f(0) + \frac{q - 1}{\log q} f'(0) \quad (\text{see [18]}), \tag{7}$$

where  $f_1(x) = f(x + 1)$ .

When we study  $q$ -analogues of Bernoulli polynomials or Daehee polynomials, the above (6) and (7) are very useful (see [10, 13, 19–22]).

Now we recall from [19] that the  $q$ -analogues of Bernoulli polynomials of the second kind  $b_{n,q}(x)$ , degenerate Bernoulli polynomials of the first kind  $\beta_{n,q}(x|\lambda)$ , of the second kind  $b_{n,q}(x|\lambda)$  and  $\lambda$ -Daehee polynomials  $D_{n,q}(x|\lambda)$  are generated by

$$\begin{aligned} \sum_{n=0}^{\infty} b_{n,q}(x) \frac{t^n}{n!} &= \frac{q(1 + t) - 1}{\log q + \log(1 + t)} (1 + t)^x, \\ \sum_{n=0}^{\infty} \beta_{n,q}(x|\lambda) \frac{t^n}{n!} &= \frac{t}{q(1 + \lambda t)^{\frac{1}{\lambda}} - 1} (1 + \lambda t)^{\frac{x}{\lambda}}, \\ \sum_{n=0}^{\infty} b_{n,q}(x|\lambda) \frac{t^n}{n!} &= \frac{t}{(q - 1) + \frac{q-1}{\log q} \log(1 + \lambda t)^{\frac{1}{\lambda}}} (1 + \lambda t)^{\frac{x}{\lambda}}, \\ \sum_{n=0}^{\infty} D_{n,q}(x|\lambda) \frac{t^n}{n!} &= \frac{(q - 1) + \frac{q-1}{\log q} \log(1 + \lambda t)^{\frac{1}{\lambda}}}{q(1 + \lambda t)^{\frac{1}{\lambda}} - 1} (1 + \lambda t)^{\frac{x}{\lambda}}, \end{aligned} \tag{8}$$

respectively. When  $x = 0$ ,  $b_{n,q} = b_{n,q}(0)$ ,  $\beta_{n,q}(\lambda) = \beta_{n,q}(0|\lambda)$ ,  $b_{n,q}(\lambda) = b_{n,q}(0|\lambda)$  and  $D_{n,q}(\lambda) = D_{n,q}(0|\lambda)$  are called the  $q$ -analogues of Bernoulli numbers of the second kind, degenerate Bernoulli numbers of the first kind, of the second kind and  $\lambda$ -Daehee numbers, respectively. Note that  $\lim_{q \rightarrow 1} b_{n,q}(x) = b_n(x)$ .

The purpose of this paper is to study the degenerate, partially degenerate and totally degenerate Daehee numbers and polynomials. In addition, we study  $q$ -analogues of those numbers and polynomials.

### 2 On degenerate Daehee polynomials

For  $\lambda, t \in \mathbb{C}_p$  with  $|\lambda t| < p^{-\frac{1}{p-1}}$ , we consider the degenerate Daehee polynomials which are given by the generating function to be

$$\frac{\log(1+t)}{\log(1+\lambda t)^{\frac{1}{\lambda}}}(1+t)^x = \sum_{n=0}^{\infty} d_n(x|\lambda) \frac{t^n}{n!}. \tag{9}$$

When  $x = 0$ ,  $d_n(\lambda) = d_n(0|\lambda)$  are called the degenerate Daehee numbers,

$$\lim_{\lambda \rightarrow 0} \frac{\log(1+t)}{\log(1+\lambda t)^{\frac{1}{\lambda}}}(1+t)^x = \frac{\log(1+t)}{t}(1+t)^x = \sum_{n=0}^{\infty} D_n(x) \frac{t^n}{n!}.$$

Thus,  $\lim_{\lambda \rightarrow 0} d_n(x|\lambda) = D_n(x)$ .

The degenerate Daehee numbers and polynomials are constructed by degenerating only denominator of Daehee numbers and polynomials. This idea comes from the observation of degenerate Bernoulli numbers and polynomials which are defined and studied by Carlitz.

From the degenerate Daehee polynomials in (9),

$$\begin{aligned} \sum_{n=0}^{\infty} d_n(x|\lambda) \frac{t^n}{n!} &= \left( \sum_{m=0}^{\infty} d_m(\lambda) \frac{t^m}{m!} \right) \left( \sum_{l=0}^{\infty} (x)_l \frac{t^l}{l!} \right) \\ &= \sum_{n=0}^{\infty} \left( \sum_{m=0}^n \binom{n}{m} d_m(\lambda)(x)_{n-m} \right) \frac{t^n}{n!} \end{aligned} \tag{10}$$

and

$$\begin{aligned} \sum_{n=0}^{\infty} d_n(x+y|\lambda) \frac{t^n}{n!} &= \frac{\log(1+t)}{\log(1+\lambda t)^{\frac{1}{\lambda}}}(1+t)^{x+y} \\ &= \left( \sum_{l=0}^{\infty} d_l(x|\lambda) \frac{t^l}{l!} \right) \left( \sum_{m=0}^{\infty} (y)_m \frac{t^m}{m!} \right) \\ &= \sum_{n=0}^{\infty} \left( \sum_{l=0}^n \binom{n}{l} d_l(x|\lambda)(y)_{n-l} \right) \frac{t^n}{n!}. \end{aligned} \tag{11}$$

We note that (11) implies that the degenerate Daehee polynomials are a Sheffer sequence. Thus, from (10) and (11), we have the following theorem.

**Theorem 1** For  $n \geq 0$ , we have

$$d_n(x|\lambda) = \sum_{m=0}^n \binom{n}{m} d_m(\lambda)(x)_{n-m}$$

and

$$d_n(x + y|\lambda) = \sum_{m=0}^n \binom{n}{m} d_m(x|\lambda)(y)_{n-m}.$$

From (9), we consider the following:

$$\begin{aligned} \frac{\log(1+t)}{\log(1+\lambda t)^{\frac{1}{\lambda}}}(1+t)^x &= \frac{\log(1+t)}{t} \frac{\lambda t}{\log(1+\lambda t)}(1+t)^x \\ &= \left( \sum_{l=0}^{\infty} D_l(x) \frac{t^l}{l!} \right) \left( \sum_{m=0}^{\infty} B_m^{(m)}(1) \frac{\lambda^m t^m}{m!} \right) \\ &= \sum_{n=0}^{\infty} \left( \sum_{l=0}^n \binom{n}{l} D_l(x) B_{n-l}^{(n-l)}(1) \lambda^{n-l} \right) \frac{t^n}{n!}, \end{aligned} \tag{12}$$

where  $B_n^{(r)}(x)$  are the Bernoulli polynomials of order  $r$  given by the generating function to be

$$\left( \frac{t}{e^t - 1} \right)^r e^{xt} = \sum_{n=0}^{\infty} B_n^{(r)}(x) \frac{t^n}{n!}.$$

By comparing the coefficients of both sides of (12), we have the following result.

**Theorem 2** For  $n \geq 0$ , we have

$$d_n(x|\lambda) = \sum_{l=0}^n \binom{n}{l} D_l(x) B_{n-l}^{(n-l)}(1) \lambda^{n-l}.$$

From (9), for  $\lambda, t \in \mathbb{C}_p$  with  $|\lambda t| < p^{-\frac{1}{p-1}}$ , we observe the following identity:

$$\begin{aligned} \frac{\log(1+t)}{\log(1+\lambda t)^{\frac{1}{\lambda}}}(1+t)^x &\int_{\mathbb{Z}_p} (1+\lambda t)^{\frac{x}{\lambda}} d\mu_0(x) \\ &= \left( \sum_{m=0}^{\infty} d_m(x|\lambda) \frac{t^m}{m!} \right) \left( \sum_{l=0}^{\infty} D_l(\lambda) \frac{t^l}{l!} \right) \\ &= \sum_{n=0}^{\infty} \left( \sum_{m=0}^n \binom{n}{m} d_m(x|\lambda) D_{n-m}(\lambda) \right) \frac{t^n}{n!}. \end{aligned} \tag{13}$$

On the other hand, by (2) and (5), we have the following:

$$\begin{aligned} \frac{\log(1+t)}{\log(1+\lambda t)^{\frac{1}{\lambda}}}(1+t)^x &\int_{\mathbb{Z}_p} (1+\lambda t)^{\frac{x}{\lambda}} d\mu_0(x) \\ &= \frac{t}{(1+\lambda t)^{\frac{1}{\lambda}} - 1} \frac{\log(1+t)}{t} (1+t)^x \end{aligned}$$

$$\begin{aligned}
 &= \left( \sum_{m=0}^{\infty} \beta_m(\lambda) \frac{t^m}{m!} \right) \left( \sum_{l=0}^{\infty} D_l(x) \frac{t^l}{l!} \right) \\
 &= \sum_{n=0}^{\infty} \left( \sum_{m=0}^n \binom{n}{m} \beta_m(\lambda) D_{n-m}(x) \right) \frac{t^n}{n!}.
 \end{aligned} \tag{14}$$

Therefore, by (13) and (14), we obtain the following theorem.

**Theorem 3** For  $n \geq 0$ , we have

$$\sum_{m=0}^n \binom{n}{m} d_m(x|\lambda) D_{n-m}(\lambda) = \sum_{m=0}^n \binom{n}{m} \beta_m(\lambda) D_{n-m}(x).$$

Let us take  $\lambda = 1$  and replace  $t$  by  $e^{(e^t-1)} - 1$  in (9). Then we have

$$\begin{aligned}
 e^{(e^t-1)x} &= \sum_{n=0}^{\infty} d_n(x|1) \frac{1}{n!} (e^{(e^t-1)} - 1)^n \\
 &= \sum_{n=0}^{\infty} d_n(x|1) \frac{1}{n!} \sum_{m=n}^{\infty} S_2(m, n) \frac{1}{m!} (e^t - 1)^m \\
 &= \sum_{m=0}^{\infty} \left( \sum_{n=0}^m d_n(x|1) S_2(m, n) \right) \frac{1}{m!} (e^t - 1)^m \\
 &= \sum_{m=0}^{\infty} \left( \sum_{n=0}^m d_n(x|1) S_2(m, n) \right) \frac{1}{m!} m! \sum_{k=m}^{\infty} S_2(k, m) \frac{t^k}{k!} \\
 &= \sum_{k=0}^{\infty} \left( \sum_{m=0}^k \sum_{n=0}^m d_n(x|1) S_2(m, n) S_2(k, m) \right) \frac{t^k}{k!}.
 \end{aligned} \tag{15}$$

On the other hand,

$$e^{(e^t-1)x} = \sum_{k=0}^{\infty} \text{Bel}_k(x) \frac{t^k}{k!}, \tag{16}$$

where  $\text{Bel}_k(x)$  are called the  $k$ th Bell polynomial (see [23]).

Therefore, by (15) and (16), we obtain the following theorem.

**Theorem 4** For  $n \geq 0$ , we have

$$\text{Bel}_k(x) = \sum_{m=0}^k \sum_{n=0}^m d_n(x|1) S_2(m, n) S_2(k, m).$$

For  $r \in \mathbb{N}$ , we define the higher-order degenerate Daehee polynomials which are given by the generating function to be

$$\left( \frac{\log(1+t)}{\log(1+\lambda t)^{1/\lambda}} \right)^r (1+t)^x = \sum_{n=0}^{\infty} d_n^{(r)}(x|\lambda) \frac{t^n}{n!}. \tag{17}$$

When  $x = 0$ ,  $d_n^{(r)}(\lambda) = d_n^{(r)}(0|\lambda)$  are called the degenerate Daehee numbers of order  $r$ .

From (17), we note that

$$\begin{aligned} \sum_{n=0}^{\infty} d_n^{(r)}(x|\lambda) \frac{t^n}{n!} &= \left(\frac{\lambda t}{\log(1 + \lambda t)}\right)^r \left(\frac{\log(1 + t)}{t}\right)^r (1 + t)^x \\ &= \left(\sum_{m=0}^{\infty} B_m^{(m-r+1)}(1) \frac{\lambda^m t^m}{m!}\right) \left(\sum_{l=0}^{\infty} D_l^{(r)}(x) \frac{t^l}{l!}\right) \\ &= \sum_{n=0}^{\infty} \left(\sum_{m=0}^n \binom{n}{m} B_m^{(m-r+1)}(1) \lambda^m D_{n-m}^{(r)}(x)\right) \frac{t^n}{n!}. \end{aligned} \tag{18}$$

Thus, we have the following theorem.

**Theorem 5** For  $n \geq 0$ , we have

$$d_n^{(r)}(x|\lambda) = \sum_{m=0}^n \binom{n}{m} B_m^{(m-r+1)}(1) \lambda^m D_{n-m}^{(r)}(x).$$

Specially, for the higher-order degenerate Daehee numbers case, we have the following observation:

$$\begin{aligned} \sum_{n=0}^{\infty} d_n^{(r)}(\lambda) \frac{t^n}{n!} &= \left(\frac{\lambda t}{\log(1 + \lambda t)}\right)^r \left(\frac{\log(1 + t)}{t}\right)^r \\ &= \left(\sum_{m=0}^{\infty} B_m^{(m-r+1)}(1) \frac{\lambda^m t^m}{m!}\right) \left(r! \sum_{l=0}^{\infty} S_1(l + r, l) \frac{t^l}{(l + r)!}\right) \\ &= \left(\sum_{m=0}^{\infty} B_m^{(m-r+1)}(1) \frac{\lambda^m t^m}{m!}\right) \left(\sum_{l=0}^{\infty} \frac{S_1(l + r, r)}{\binom{l+r}{r}} \frac{t^l}{l!}\right) \\ &= \sum_{n=0}^{\infty} \left(\sum_{l=0}^n \binom{n}{l} \frac{S_1(l + r, r)}{\binom{l+r}{r}} \lambda^{n-l} B_{n-l}^{(n-l-r+1)}(1)\right) \frac{t^n}{n!}. \end{aligned} \tag{19}$$

Thus, by (19), we have the following theorem.

**Theorem 6** For  $n \geq 0$ , we have

$$d_n^{(r)}(\lambda) = \sum_{l=0}^n \binom{n}{l} \frac{S_1(l + r, r)}{\binom{l+r}{r}} \lambda^{n-l} B_{n-l}^{(n-l-r+1)}(1)$$

and

$$d_n^{(r)}(x|\lambda) = \sum_{l=0}^n \binom{n}{l} \frac{S_1(l + r, r)}{\binom{l+r}{r}} \lambda^{n-l} B_{n-l}^{(n-l-r+1)}(x + 1).$$

### 3 On partially degenerate Daehee polynomials

For  $\lambda, t \in \mathbb{C}_p$  with  $|\lambda t| < p^{-\frac{1}{p-1}}$ , we consider the partially degenerate Daehee polynomials denoted by  $\tilde{d}_n(x|\lambda)$  and given by the generating function to be

$$\frac{(1 + t)^\lambda - 1}{\lambda t} (1 + t)^x = \sum_{n=0}^{\infty} \tilde{d}_n(x|\lambda) \frac{t^n}{n!}. \tag{20}$$

When  $x = 0$ ,  $\tilde{d}_n(\lambda) = \tilde{d}_n(0|\lambda)$  are called the partially degenerate Daehee numbers,

$$\lim_{\lambda \rightarrow 0} \frac{(1+t)^\lambda - 1}{\lambda t} (1+t)^x = \frac{\log(1+t)}{t} (1+t)^x = \sum_{n=0}^{\infty} D_n(x) \frac{t^n}{n!}.$$

Thus,  $\lim_{\lambda \rightarrow 0} \tilde{d}_n(x|\lambda) = D_n(x)$ .

In this section, we concentrate on the polynomials obtained by degenerating only the numerator part of Daehee numbers and polynomials. We call these numbers and polynomials partially degenerate Daehee numbers and polynomials. The motivation for this partially degenerate concept comes from the Daehee polynomials with  $\lambda$ -parameter  $D_n(x|\lambda)$ . As we know,

$$\lim_{\lambda \rightarrow 0} D_n(x|\lambda) = B_n(x).$$

So, the Daehee polynomials with  $\lambda$ -parameter  $D_n(x|\lambda)$  are a wonderful degenerating function for Bernoulli polynomials.

From the generating function for the partially degenerate Daehee polynomials (20), we can prove the following theorem similarly to (10) and (11).

**Theorem 7** *For  $n \geq 0$ , we have*

$$\tilde{d}_n(x|\lambda) = \sum_{m=0}^n \binom{n}{m} \tilde{d}_m(\lambda)(x)_{n-m}$$

and

$$\tilde{d}_n(x+y|\lambda) = \sum_{m=0}^n \binom{n}{m} \tilde{d}_m(x|\lambda)(y)_{n-m}.$$

From (20), we observe the partially degenerate Daehee numbers. Then we have the following identity:

$$\begin{aligned} & \frac{(1+\lambda t)^{\frac{1}{\lambda}} - 1}{\lambda t} \int_{\mathbb{Z}_p} (1+\lambda t)^{\frac{x}{\lambda}} d\mu_0(x) \\ &= \left( \sum_{m=0}^{\infty} \tilde{d}_m(\lambda) \frac{t^m}{m!} \right) \left( \sum_{l=0}^{\infty} D_l(\lambda) \frac{t^l}{l!} \right) \\ &= \sum_{n=0}^{\infty} \left( \sum_{m=0}^n \binom{n}{m} \tilde{d}_m(\lambda) D_{n-m}(\lambda) \right) \frac{t^n}{n!}. \end{aligned} \tag{21}$$

On the other hand,

$$\begin{aligned} & \frac{(1+\lambda t)^{\frac{1}{\lambda}} - 1}{\lambda t} \int_{\mathbb{Z}_p} (1+\lambda t)^{\frac{x}{\lambda}} d\mu_0(x) \\ &= \frac{(1+\lambda t)^{\frac{1}{\lambda}} - 1}{\lambda t} \frac{\log(1+\lambda t)^{\frac{1}{\lambda}}}{(1+\lambda t)^{\frac{1}{\lambda}} - 1} \\ &= \frac{\log(1+\lambda t)^{\frac{1}{\lambda}}}{\lambda t} = \sum_{l=0}^{\infty} \frac{(-1)^l}{l+1} \lambda^{l-1} \frac{t^l}{l!}. \end{aligned} \tag{22}$$

Therefore, by (21) and (22), we obtain the following theorem.

**Theorem 8** For  $n \geq 0$ , we have

$$\sum_{m=0}^n \binom{n}{m} \tilde{d}_m(x|\lambda) D_{n-m}(\lambda) = \frac{(-1)^n}{n+1} \lambda^{n-1}.$$

Let us take  $\lambda = 1$  and replace  $t$  by  $e^{(t-1)} - 1$  in (20). Then similarly to (15) we have the following theorem.

**Theorem 9** For  $n \geq 0$ , we have

$$\text{Bel}_k(x) = \sum_{m=0}^k \sum_{n=0}^m \tilde{d}_n(x|1) S_2(m, n) S_2(k, m).$$

**4 On totally degenerate Daehee polynomials**

For  $\lambda, t \in \mathbb{C}_p$  with  $|\lambda t| < p^{-\frac{1}{p-1}}$ , we consider the totally degenerate Daehee polynomials denoted by  $d_n^*(x|\lambda)$  and given by the generating function to be

$$\frac{(1+t)^\lambda - 1}{\log(1+\lambda t)} (1+t)^x = \sum_{n=0}^\infty d_n^*(x|\lambda) \frac{t^n}{n!}. \tag{23}$$

When  $x = 0$ ,  $d_n^*(\lambda) = d_n^*(0|\lambda)$  are called the totally degenerate Daehee numbers. Similar to the previous case, we can check that  $\lim_{\lambda \rightarrow 0} d_n^*(x|\lambda) = D_n(x)$ .

The totally degenerate Daehee numbers and polynomials are constructed by degenerating both numerator and denominator of Daehee numbers and polynomials. From the generating function for the totally degenerate Daehee polynomials (23), we can see that the totally degenerate Daehee polynomials are a Sheffer sequence. And we have the following theorem.

**Theorem 10** For  $n \geq 0$ , we have

$$d_n^*(x|\lambda) = \sum_{m=0}^n \binom{n}{m} d_m^*(\lambda)(x)_{n-m}$$

and

$$d_n^*(x+y|\lambda) = \sum_{m=0}^n \binom{n}{m} d_m^*(x|\lambda)(y)_{n-m}.$$

From (23), we consider the following:

$$\begin{aligned} \frac{(1+t)^\lambda - 1}{\log(1+\lambda t)} &= \frac{(1+t)^\lambda - 1}{\lambda t} \frac{\lambda t}{\log(1+\lambda t)} \\ &= \left( \sum_{l=0}^\infty \tilde{d}_l(\lambda) \frac{t^l}{l!} \right) \left( \sum_{m=0}^\infty B_m^{(m)}(1) \frac{\lambda^m t^m}{m!} \right) \\ &= \sum_{n=0}^\infty \left( \sum_{l=0}^n \binom{n}{l} \tilde{d}_l(\lambda) B_{n-l}^{(n-l)}(1) \lambda^{n-l} \right) \frac{t^n}{n!}. \end{aligned} \tag{24}$$



By comparing the coefficients of both sides of (24), we have the following result.

**Theorem 11** *For  $n \geq 0$ , we have*

$$d_n^*(x) = \sum_{l=0}^n \binom{n}{l} \tilde{d}_l(\lambda) B_{n-l}^{(n-l)}(1) \lambda^{n-l}.$$

From (23), we observe the following identity:

$$\begin{aligned} & \frac{(1+t)^\lambda - 1}{\log(1+\lambda t)} (1+t)^x \int_{\mathbb{Z}_p} (1+\lambda t)^{\frac{x}{\lambda}} d\mu_0(x) \\ &= \left( \sum_{m=0}^{\infty} d_m^*(x|\lambda) \frac{t^m}{m!} \right) \left( \sum_{l=0}^{\infty} D_l(\lambda) \frac{t^l}{l!} \right) \\ &= \sum_{n=0}^{\infty} \left( \sum_{m=0}^n \binom{n}{m} d_m^*(x|\lambda) D_{n-m}(\lambda) \right) \frac{t^n}{n!}. \end{aligned} \tag{25}$$

On the other hand, by (5) and (20), we have the following:

$$\begin{aligned} & \frac{(1+t)^\lambda - 1}{\log(1+\lambda t)} (1+t)^x \int_{\mathbb{Z}_p} (1+\lambda t)^{\frac{x}{\lambda}} d\mu_0(x) \\ &= \frac{t}{(1+\lambda t)^{\frac{1}{\lambda}} - 1} \frac{(1+t)^\lambda - 1}{\lambda t} (1+t)^x \\ &= \left( \sum_{m=0}^{\infty} \beta_m(\lambda) \frac{t^m}{m!} \right) \left( \sum_{l=0}^{\infty} \tilde{d}_l(x|\lambda) \frac{t^l}{l!} \right) \\ &= \sum_{n=0}^{\infty} \left( \sum_{m=0}^n \binom{n}{m} \beta_m(\lambda) \tilde{d}_{n-m}(x|\lambda) \right) \frac{t^n}{n!}. \end{aligned} \tag{26}$$

Therefore, by (25) and (26), we obtain the following theorem.

**Theorem 12** *For  $n \geq 0$ , we have*

$$\begin{aligned} & \sum_{m=0}^n \binom{n}{m} d_m^*(x|\lambda) D_{n-m}(\lambda) \\ &= \sum_{m=0}^n \binom{n}{m} \beta_m(\lambda) \tilde{d}_{n-m}(x|\lambda). \end{aligned}$$

For  $r \in \mathbb{N}$ , we define the higher-order totally degenerate Daehee polynomials which are given by the generating function to be

$$\left( \frac{(1+t)^\lambda - 1}{\log(1+\lambda t)} \right)^r (1+t)^x = \sum_{n=0}^{\infty} d_n^{*(r)}(x|\lambda) \frac{t^n}{n!}. \tag{27}$$

When  $x = 0$ ,  $d_n^{*(r)}(\lambda) = d_n^{*(r)}(0|\lambda)$  are called the totally degenerate Daehee numbers of order  $r$ .

From the generating function, we note that

$$\begin{aligned} \sum_{n=0}^{\infty} d_n^{*(r)}(x|\lambda) \frac{t^n}{n!} &= \left( \frac{\lambda t}{\log(1 + \lambda t)} \right)^r \left( \frac{(1+t)^\lambda - 1}{\lambda t} \right)^r (1+t)^x \\ &= \left( \sum_{m=0}^{\infty} B_m^{(m-r+1)}(1) \frac{\lambda^m t^m}{m!} \right) \left( \sum_{l=0}^{\infty} \tilde{d}_l^{(r)}(x) \frac{t^l}{l!} \right) \\ &= \sum_{n=0}^{\infty} \left( \sum_{m=0}^n \binom{n}{m} B_m^{(m-r+1)}(1) \lambda^m \tilde{d}_{n-m}^{(r)}(x) \right) \frac{t^n}{n!}. \end{aligned} \tag{28}$$

Thus, we have the following theorem.

**Theorem 13** For  $n \geq 0$ , we have

$$d_n^{*(r)}(x|\lambda) = \sum_{m=0}^n \binom{n}{m} B_m^{(m-r+1)}(1) \lambda^m \tilde{d}_{n-m}^{(r)}(x).$$

Now, we give a relation for the given numbers and polynomials, which are Daehee, degenerate Daehee, partially degenerate Daehee and totally degenerate Daehee numbers and polynomials as follows:

$$\sum_{k=0}^n \binom{n}{k} D_k d_{n-k}^* = \sum_{k=0}^n \binom{n}{k} d_k \tilde{d}_{n-k}$$

and

$$\sum_{k=0}^n \binom{n}{k} D_k(x) d_{n-k}^* = \sum_{k=0}^n \binom{n}{k} d_k(x) \tilde{d}_{n-k}.$$

### 5 $q$ -Analogues of degenerate, partially degenerate and totally degenerate Daehee polynomials

Some  $q$ -analogues of the degenerate Bernoulli polynomials are studied by Kim *et al.* in [19]. This research influenced us to study  $q$ -analogues of the degenerate Daehee numbers and polynomials. In this section, we study  $q$ -analogues of degenerate, partially degenerate and totally degenerate Daehee polynomials. And we give some formulas related to those polynomials.

For  $\lambda, t \in \mathbb{C}_p$  with  $|\lambda t| < p^{-\frac{1}{p-1}}$ , the  $q$ -analogues of degenerate Daehee polynomials denoted by  $d_{n,q}(x|\lambda)$  are given by the generating function to be

$$\frac{\log(1+t)}{(q-1) + \frac{q-1}{\log q} \log(1+\lambda t)^{\frac{1}{\lambda}}} (1+t)^x = \sum_{n=0}^{\infty} d_{n,q}(x|\lambda) \frac{t^n}{n!}. \tag{29}$$

When  $x = 0$ ,  $d_{n,q}(\lambda) = d_{n,q}(0|\lambda)$  are called the  $q$ -analogues of degenerate Daehee numbers.

We note that

$$\lim_{q \rightarrow 1} d_{n,q}(x|\lambda) = d_n(x|\lambda)$$

and

$$\lim_{\lambda \rightarrow 0, q \rightarrow 1} d_{n,q}(x|\lambda) = D_n(x).$$

From (9), we have the following theorem.

**Theorem 14** *For  $n \geq 0$ , we have*

$$d_{n,q}(x|\lambda) = \sum_{m=0}^n \binom{n}{m} d_{m,q}(\lambda)(x)_{n-m}$$

and

$$d_{n,q}(x + y|\lambda) = \sum_{m=0}^n \binom{n}{m} d_{m,q}(x|\lambda)(y)_{n-m}.$$

Also, by (8) and (29), we consider the following:

$$\begin{aligned} & \frac{\log(1+t)}{(q-1) + \frac{q-1}{\log q} \log(1+\lambda t)^{\frac{1}{\lambda}}} (1+t)^x \\ &= \frac{t}{(q-1) + \frac{q-1}{\log q} \log(1+\lambda t)^{\frac{1}{\lambda}}} \frac{\log(1+t)}{t} (1+t)^x \\ &= \left( \sum_{m=0}^{\infty} b_{m,q}(x|\lambda) \frac{t^m}{m!} \right) \left( \sum_{l=0}^{\infty} D_l \frac{t^l}{l!} \right) \\ &= \sum_{n=0}^{\infty} \left( \sum_{m=0}^n \binom{n}{m} b_{m,q}(x|\lambda) D_{n-m} \right) \frac{t^n}{n!}. \end{aligned} \tag{30}$$

Thus, we have the following theorem.

**Theorem 15** *For  $n \geq 0$ , we have*

$$d_{n,q}(x|\lambda) = \sum_{m=0}^n \binom{n}{m} b_{m,q}(x|\lambda) D_{n-m}.$$

From (8) and (29), we note that

$$\begin{aligned} & \frac{\log(1+t)}{(q-1) + \frac{q-1}{\log q} \log(1+\lambda t)^{\frac{1}{\lambda}}} \int_{\mathbb{Z}_p} (1+\lambda t)^{\frac{x+y}{\lambda}} d\mu_q(y) \\ &= \left( \sum_{m=0}^{\infty} \beta_{m,q}(x|\lambda) \frac{t^m}{m!} \right) \left( \sum_{l=0}^{\infty} D_l \frac{t^l}{l!} \right) \\ &= \sum_{n=0}^{\infty} \left( \sum_{m=0}^n \binom{n}{m} \beta_{m,q}(x|\lambda) D_{n-m}(\lambda) \right) \frac{t^n}{n!}. \end{aligned} \tag{31}$$

On the other hand,

$$\begin{aligned} & \frac{\log(1+t)}{(q-1) + \frac{q-1}{\log q} \log(1+\lambda t)^{\frac{1}{\lambda}}} \int_{\mathbb{Z}_p} (1+\lambda t)^{\frac{x+y}{\lambda}} d\mu_q(y) \\ &= \left( \sum_{m=0}^{\infty} d_{m,q}(\lambda) \frac{t^m}{m!} \right) \left( \sum_{l=0}^{\infty} D_{l,q}(x|\lambda) \frac{t^l}{l!} \right) \\ &= \sum_{n=0}^{\infty} \left( \sum_{m=0}^n \binom{n}{m} d_{m,q}(\lambda) D_{n-m,q}(x|\lambda) \right) \frac{t^n}{n!}. \end{aligned} \tag{32}$$

Therefore, by (31) and (32), we obtain the following result.

**Theorem 16** For  $n \geq 0$ , we have

$$\sum_{m=0}^n \binom{n}{m} \beta_{m,q}(x|\lambda) D_{n-m} = \sum_{m=0}^n \binom{n}{m} d_{m,q}(\lambda) D_{n-m,q}(x|\lambda).$$

We consider the  $q$ -analogues of partially degenerate Daehee polynomials by the generating function to be

$$\frac{(q-1) + \frac{q-1}{\log q} \frac{(1+t)^\lambda - 1}{\lambda}}{t} (1+t)^x = \sum_{n=0}^{\infty} \tilde{d}_{n,q}(x|\lambda) \frac{t^n}{n!}. \tag{33}$$

When  $x = 0$ ,  $\tilde{d}_{n,q}(\lambda) = \tilde{d}_{n,q}(0|\lambda)$  are called the  $q$ -analogues of partially degenerate Daehee numbers.

Also, we can see the following:

$$\lim_{q \rightarrow 1} \tilde{d}_{n,q}(x|\lambda) = \tilde{d}_n(x|\lambda)$$

and

$$\lim_{\lambda \rightarrow 0, q \rightarrow 1} \tilde{d}_{n,q}(x|\lambda) = D_n(x).$$

From the definition of  $q$ -analogues partially degenerate Daehee numbers and polynomials in (33), we have the following identities.

**Theorem 17** For  $n \geq 0$ , we have

$$\tilde{d}_{n,q}(x|\lambda) = \sum_{m=0}^n \binom{n}{m} \tilde{d}_{m,q}(\lambda)(x)_{n-m}$$

and

$$\tilde{d}_{n,q}(x+y|\lambda) = \sum_{m=0}^n \binom{n}{m} \tilde{d}_{m,q}(x|\lambda)(y)_{n-m}.$$

We can also consider the  $q$ -analogues of totally degenerate Daehee polynomials by the generating function to be

$$\frac{(q-1) + \frac{q-1}{\log q} \frac{(1+t)^\lambda - 1}{\lambda}}{(q-1) + \frac{q-1}{\log q} \log(1 + \lambda t)^{\frac{1}{\lambda}}} (1+t)^x = \sum_{n=0}^{\infty} d_{n,q}^*(x|\lambda) \frac{t^n}{n!}. \tag{34}$$

When  $x = 0$ ,  $d_{n,q}^*(\lambda) = d_{n,q}^*(0|\lambda)$  are called the  $q$ -analogues of totally degenerate Daehee numbers.

We can also see the following:

$$\lim_{q \rightarrow 1} d_{n,q}^*(x|\lambda) = d_n^*(x|\lambda)$$

and

$$\lim_{\lambda \rightarrow 0, q \rightarrow 1} d_{n,q}^*(x|\lambda) = D_n(x).$$

From (34), we have the following identities.

**Theorem 18** For  $n \geq 0$ , we have

$$d_{n,q}^*(x|\lambda) = \sum_{m=0}^n \binom{n}{m} d_{m,q}^*(\lambda)(x)_{n-m}$$

and

$$d_{n,q}^*(x + y|\lambda) = \sum_{m=0}^n \binom{n}{m} d_{m,q}^*(x|\lambda)(y)_{n-m}.$$

We give the relation among  $q$ -analogues of degenerate, partially degenerate, totally degenerate Daehee numbers and polynomials and Daehee numbers and polynomials.

Consider the following:

$$\begin{aligned} & \frac{(q-1) + \frac{q-1}{\log q} \frac{(1+t)^\lambda - 1}{\lambda}}{(q-1) + \frac{q-1}{\log q} \log(1 + \lambda t)^{\frac{1}{\lambda}}} \frac{\log(1+t)}{t} (1+t)^x \\ &= \frac{\log(1+t)}{(q-1) + \frac{q-1}{\log q} \log(1 + \lambda t)^{\frac{1}{\lambda}}} \frac{(q-1) + \frac{q-1}{\log q} \frac{(1+t)^\lambda - 1}{\lambda}}{t} (1+t)^x \\ &= \left( \sum_{m=0}^{\infty} d_{m,q}(x|\lambda) \frac{t^m}{m!} \right) \left( \sum_{l=0}^{\infty} \tilde{d}_{l,q}(\lambda) \frac{t^l}{l!} \right) \\ &= \sum_{n=0}^{\infty} \left( \sum_{m=0}^n \binom{n}{m} d_{m,q}(x|\lambda) \tilde{d}_{n-m,q}(\lambda) \right) \frac{t^n}{n!}. \end{aligned} \tag{35}$$

Thus we have the following theorem.

**Theorem 19** For  $n \geq 0$ , we have

$$\begin{aligned} \sum_{m=0}^n \binom{n}{m} d_{m,q}^*(x|\lambda) D_{n-m} &= \sum_{m=0}^n \binom{n}{m} d_{m,q}(x|\lambda) \tilde{d}_{n-m,q}(\lambda) \\ &= \sum_{m=0}^n \binom{n}{m} d_{m,q}^*(\lambda) D_{n-m}(x) \\ &= \sum_{m=0}^n \binom{n}{m} d_{m,q}(\lambda) \tilde{d}_{n-m}(x|\lambda). \end{aligned}$$

**Competing interests**

The author declares that they have no competing interests.

**Author's contributions**

The author carried out all work of this article and the main theorem. The author read and approved the final manuscript.

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