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Properties of right fractional sum and right fractional difference operators and application

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Abstract

In this paper, the concepts of a right fractional sum and right fractional difference operators are introduced. Some basic properties of a right fractional sum and right fractional difference operators are proved. According to these properties of a right fractional sum and right fractional difference operators, we studied an initial problem and a boundary value problem with two-point boundary conditions. We hope that the present work will facilitate solving a fractional difference equation with right fractional difference operators.

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Keywords: right fractional sum; right fractional difference operator; initial problem; boundary value problem

1 Introduction

Recently, there appeared a number of papers on the discrete fractional calculus, which has helped to build up some of the basic theory of this area. For example, Atici and Eloe discussed the properties of the generalized falling function, a corresponding power rule for fractional delta-operators, and the commutativity of fractional sums in [1]. Goodrich studied a fractional boundary value problem in [2], which gave the existence results for a certain two-point boundary value problem of right-focal type for a fractional difference equation. The authors of [3] have developed a well-posed initial value problem and proposed multiple solution algorithms. An interesting recent paper by Atici and Sengül [4] addressed the use of fractional difference equations in tumor growth modeling. For recent studies in discrete fractional calculus involving initial boundary value problems, see [5–32].

From the above works, we can see in fact, although the discrete fractional calculus have been studied by many authors, to the best of our knowledge, that the properties of a right fractional difference operator have not been discussed. Our objective is twofold. On one hand we proceed to develop the theory of fractional difference calculus, namely we introduce the concepts of a right fractional sum and right fractional difference operators and prove some basic properties of a right fractional sum and right fractional difference operators. On the other hand, according to these properties of a right fractional sum and right fractional difference operators, we studied an initial problem and a boundary value problem with two-point boundary conditions. The proofs are similar to those of earlier work by both Goodrich [2] and Holm [10]; there nonetheless is something new and interesting here. The contributions of this article aim to initiate the study of right fractional difference operator. This interest is in part due to the useful applications of the fractional calculus together with its interesting and often nontrivial mathematical theory. On the other hand, it might be of interest to see what happens in the case of more complicated boundary conditions or a higher-order problem. Due to the lack of commutativity of the right fractional difference, the sequential boundary value problem is of interest. Furthermore, combining the results of Goodrich [2] and Holm [10], there seem to be considerable possibilities for future work to address the sequential boundary value problems with left and right fractional difference operators, and such investigations might provide interesting future work. We believe that the present work facilitates solving a fractional difference equation with a right fractional difference operator.

2 Right fractional sum and right fractional difference operators

In this section, the concepts of a right fractional sum and right fractional difference operators are introduced and their some basic properties are proved.

Denote ${}_b\mathbb{N} := \{b\} - \mathbb{N}_0 = \{\dots, b-2, b-1, b\}$, $b \in \mathbb{R}$.

Definition 1 (see [2]) We define $t^{\underline{\nu}} := \frac{\Gamma(t+1)}{\Gamma(t+1-\nu)}$, for any t and ν for which the right-hand side is defined. We also appeal to the convention that if $t+1-\nu$ is a pole of the *Gamma function* and if $t+1$ is not a pole, then $t^{\underline{\nu}} = 0$.

Definition 2 The *νth order right fractional sum* of a function f defined on ${}_b\mathbb{N}$, for $\nu > 0$, is defined to be

$$ {}_b\nabla^{-\nu} f(t) = \frac{1}{\Gamma(\nu)} \sum_{s=t+\nu}^{b} (s-t-1)^{\underline{\nu-1}} f(s), \quad t \in {}_{b-\nu}\mathbb{N}. \tag{2.1}$$

We also define the *trivial right sum* by ${}_b\nabla^{-0} f(t) = f(t)$, for $t \in {}_b\mathbb{N}$.

Definition 3 Let $f :_b \mathbb{N} \to \mathbb{R}$ and $\nu > 0$ be given, and let $N \in \mathbb{N}$ be chosen such that $N-1 < \nu \le N$. The *νth order right fractional difference* of f is given by

$$ \left({}_b\nabla^{\nu} f\right)(t) = {}_b\nabla^{\nu} f(t) = (-1)^N \nabla^N {}_b\nabla^{-(N-\nu)} f(t), \quad t \in {}_{b-N+\nu}\mathbb{N}. \tag{2.2}$$

Using Definition 3 together with a function $f : {}_b\mathbb{N} \to \mathbb{R}$ and an order $\nu > 0$ with $N-1 < \nu \le N$, we may calculate the domain of the νth order right fractional difference as

$$ D\left\{{}_b\nabla^{\nu} f\right\} = D\left\{\nabla^N {}_b\nabla^{-(N-\nu)} f\right\} = D\left\{{}_b\nabla^{-(N-\nu)} f\right\} = {}_{b-N+\nu}\mathbb{N}.$$

Moreover, the domains of all four sum and difference compositions are given below.

Let $f : {}_b\mathbb{N} \to \mathbb{R}$ and $\nu, \mu > 0$ be given. Let $N, M \in \mathbb{N}_0$ be chosen so that $N-1 < \nu \le N$ and $M-1 < \mu \le M$. Then

$$ D\left\{{}_{b-\mu}\nabla^{-\nu} {}_b\nabla^{-\mu} f\right\} = {}_{b-\mu-\nu}\mathbb{N}, \qquad D\left\{{}_{b-\mu}\nabla^{\nu} {}_b\nabla^{-\mu} f\right\} = {}_{b-\mu-N+\nu}\mathbb{N},$$

$$ D\left\{{}_{b-M+\mu}\nabla^{-\nu} {}_b\nabla^{\mu} f\right\} = {}_{b-M+\mu-\nu}\mathbb{N}, \qquad D\left\{{}_{b-M+\mu}\nabla^{\nu} {}_b\nabla^{\mu} f\right\} = {}_{b-M+\mu-N+\nu}\mathbb{N}.$$

Theorem 1 *Let $f : {}_b\mathbb{N} \to \mathbb{R}$ and $\nu > 0$ be given, with $N-1 < \nu \leq N$. The following two definitions for the right fractional difference ${}_b\nabla^\nu f : {}_{b-N+\nu}\mathbb{N} \to \mathbb{R}$ are equivalent*:

$$ {}_b\nabla^\nu f(t) = (-1)^N \nabla_b^N \nabla^{-(N-\nu)} f(t), \tag{2.3}$$

$$ {}_b\nabla^\nu f(t) = \begin{cases} \frac{1}{\Gamma(-\nu)} \sum_{s=t-\nu}^{b} (s-t-1)^{\underline{-\nu-1}} f(s), & N-1 < \nu \leq N, \\ (-1)^N \nabla^N f(t), & \nu = N. \end{cases} \tag{2.4}$$

Proof Let f and ν be given as in the statement of the theorem. We are proposing two definitions (2.3) and (2.4) and demonstrating that they are identically equal.

If $\nu = N$, then (2.3) and (2.4) are clearly equivalent, since in this case,

$$ {}_b\nabla^\nu f(t) = (-1)^N \nabla^N {}_b\nabla^{-(N-\nu)} f(t) = (-1)^N \nabla^N {}_b\nabla^{-0} f(t) = (-1)^N \nabla^N f(t). $$

If $N-1 < \nu < N$, then a direct application of (2.3) yields

$$ \begin{aligned} {}_b\nabla^\nu f(t) &= (-1)^N \nabla^N {}_b\nabla^{-(N-\nu)} f(t) \\ &= (-1)^N \nabla^N \left(\frac{1}{\Gamma(N-\nu)} \sum_{s=t+N-\nu}^{b} (s-t-1)^{\underline{N-\nu-1}} f(s) \right) \\ &= (-1)^N \nabla^{N-1} \nabla \left(\frac{1}{\Gamma(N-\nu)} \sum_{s=t+N-\nu}^{b} (s-t-1)^{\underline{N-\nu-1}} f(s) \right) \\ &= (-1)^N \nabla^{N-1} \left(\frac{1}{\Gamma(N-\nu)} \sum_{s=t+N-\nu}^{b} (s-t-1)^{\underline{N-\nu-1}} f(s) \right. \\ &\quad \left. - \frac{1}{\Gamma(N-\nu)} \sum_{s=t+N-\nu-1}^{b} (s-t)^{\underline{N-\nu-1}} f(s) \right) \\ &= (-1)^{N-1} \nabla^{N-1} \left(\frac{1}{\Gamma(N-\nu-1)} \sum_{s=t+N-\nu-1}^{b} (s-t-1)^{\underline{N-\nu-2}} f(s) \right) \\ &\ \ \vdots \\ &= \frac{1}{\Gamma(-\nu)} \sum_{s=t-\nu}^{b} (s-t-1)^{\underline{-\nu-1}} f(s). \end{aligned} $$

□

Lemma 1 *Let $b \in \mathbb{R}$ and $\mu > 0$ be given. Then*

$$ \nabla(b-t)^{\underline{\mu}} = -\mu(b-t)^{\underline{\mu-1}} \tag{2.5}$$

for any t, for which both sides are well defined.

Furthermore, for $\nu > 0$ with $N-1 < \nu \leq N$,

$$ {}_{b-\mu}\nabla^{-\nu}(b-t)^{\underline{\mu}} = \mu^{\underline{-\nu}}(b-t)^{\underline{\mu+\nu}}, \quad t \in {}_{b-\mu-\nu}\mathbb{N} \tag{2.6}$$

and

$$ {}_{b-\mu}\nabla^{\nu}(b-t)^{\underline{\mu}} = \mu^{\underline{\nu}}(b-t)^{\underline{\mu-\nu}}, \quad t \in {}_{b-\mu-N+\nu}\mathbb{N}. \tag{2.7}$$

Proof It is easy to show (2.5) using the definition of the nabla difference and properties of the gamma function. For (2.6) and (2.7), we first note that $(b-t)^{\underline{\mu}}$, $(b-t)^{\underline{\mu+\nu}}$, and $(b-t)^{\underline{\mu-\nu}}$ are all well defined and positive on their respective domains ${}_{b-\mu}\mathbb{N}$, ${}_{b-\mu-\nu}\mathbb{N}$, ${}_{b-\mu-N+\nu}\mathbb{N}$.

We next show that (2.6) and (2.7) hold. The course of this proof as regards the techniques heavily relies on Holm [10]. For convenience of the reader, we display it as follows.

For $\nu = 1$, we see from direct calculation that

$$\begin{aligned}
{}_{b-\mu}\nabla^{-1}(b-t)^{\underline{\mu}} &= {}_{b-\mu}\nabla^{-1}\left(-\frac{\nabla(b-t)^{\underline{\mu+1}}}{\mu+1}\right) \\
&= \sum_{s=t+1}^{b-\mu}(s-t-1)^{\underline{0}}\left(-\frac{\nabla(b-s)^{\underline{\mu+1}}}{\mu+1}\right) \\
&= \sum_{s=t+1}^{b-\mu}\left(\frac{(b-s+1)^{\underline{\mu+1}}}{\mu+1}-\frac{(b-s)^{\underline{\mu+1}}}{\mu+1}\right) \\
&= \frac{(b-t)^{\underline{\mu+1}}}{\mu+1}-\frac{\mu^{\underline{\mu+1}}}{\mu+1} \\
&= \mu^{\underline{-1}}(b-t)^{\underline{\mu+1}}.
\end{aligned}$$

For $\nu \in (0,1) \cup (1,+\infty)$, define for $t \in {}_{b-\mu-\nu}\mathbb{N}$ the functions

$$g_1(t) = {}_{b-\mu}\nabla^{-\nu}(b-t)^{\underline{\mu}}$$

and

$$g_2(t) = \mu^{\underline{-\nu}}(b-t)^{\underline{\mu+\nu}}.$$

We will show that both g_1 and g_2 solve the well-posed, first-order initial value problem

$$\begin{cases} (b-t-\mu-\nu+1)\nabla g(t) + (\mu+\nu)g(t) = 0, & t \in {}_{b-\mu-\nu}\mathbb{N}, \\ g(b-\mu-\nu) = \Gamma(\mu+1). \end{cases} \tag{2.8}$$

Since

$$\begin{aligned}
g_1(b-\mu-\nu) &= \frac{1}{\Gamma(\nu)}\sum_{s=t+\nu}^{b-\mu}(s-t-1)^{\underline{\nu-1}}(b-s)^{\underline{\mu}}\Big|_{t=b-\mu-\nu} \\
&= \frac{1}{\Gamma(\nu)}\sum_{s=b-\mu}^{b-\mu}(s-b+\mu+\nu-1)^{\underline{\nu-1}}(b-s)^{\underline{\mu}} \\
&= \frac{1}{\Gamma(\nu)}(\nu-1)^{\underline{\nu-1}}\mu^{\underline{\mu}} \\
&= \Gamma(\mu+1)
\end{aligned}$$

and

$$g_2(b-\mu-\nu) = \mu^{\underline{-\nu}}(\mu+\nu)^{\underline{\mu+\nu}} = \Gamma(\mu+1),$$

both g_1 and g_2 satisfy the initial condition in (2.8).

An effort is required to show that g_1 satisfies the difference equation in (2.8). For $t \in {}_{b-\mu-\nu}\mathbb{N}$,

$$\begin{aligned}
\nabla g_1(t) &= \nabla\left(\frac{1}{\Gamma(\nu)}\sum_{s=t+\nu}^{b-\mu}(s-t-1)^{\underline{\nu-1}}(b-s)^{\underline{\mu}}\right) \\
&= \frac{1}{\Gamma(\nu)}\left(\sum_{s=t+\nu}^{b-\mu}(s-t-1)^{\underline{\nu-1}}(b-s)^{\underline{\mu}} - \sum_{s=t+\nu-1}^{b-\mu}(s-t)^{\underline{\nu-1}}(b-s)^{\underline{\mu}}\right) \\
&= \frac{-(\nu-1)}{\Gamma(\nu)}\sum_{s=t+\nu}^{b-\mu}(s-t-1)^{\underline{\nu-2}}(b-s)^{\underline{\mu}} - (b-t+1-\nu)^{\underline{\mu}}.
\end{aligned}$$

Also, we may manipulate g_1 directly to obtain

$$\begin{aligned}
g_1(t) &= \frac{1}{\Gamma(\nu)}\sum_{s=t+\nu}^{b-\mu}(s-t-1)^{\underline{\nu-1}}(b-s)^{\underline{\mu}} \\
&= \frac{1}{\Gamma(\nu)}\sum_{s=t+\nu}^{b-\mu}\bigl(s-t-1-(\nu-2)\bigr)(s-t-1)^{\underline{\nu-2}}(b-s)^{\underline{\mu}} \\
&= \frac{1}{\Gamma(\nu)}\sum_{s=t+\nu}^{b-\mu}\bigl[\bigl(b-t-(\mu+\nu)+1\bigr)-(b-s-\mu)\bigr](s-t-1)^{\underline{\nu-2}}(b-s)^{\underline{\mu}} \\
&= \frac{b-t-\mu-\nu+1}{\Gamma(\nu)}\sum_{s=t+\nu}^{b-\mu}(s-t-1)^{\underline{\nu-2}}(b-s)^{\underline{\mu}} \\
&\quad - \frac{1}{\Gamma(\nu)}\sum_{s=t+\nu}^{b-\mu}(b-s-\mu)(s-t-1)^{\underline{\nu-2}}(b-s)^{\underline{\mu}} \\
&= \frac{b-t-\mu-\nu+1}{\Gamma(\nu)}\sum_{s=t+\nu}^{b-\mu}(s-t-1)^{\underline{\nu-2}}(b-s)^{\underline{\mu}} \\
&\quad - \frac{1}{\Gamma(\nu)}\sum_{s=t+\nu}^{b-\mu}(s-t-1)^{\underline{\nu-2}}(b-s)^{\underline{\mu+1}} \\
&= h(t)-k(t),
\end{aligned}$$

where

$$\begin{cases} h(t) := \frac{b-t-\mu-\nu+1}{\Gamma(\nu)}\sum_{s=t+\nu}^{b-\mu}(s-t-1)^{\underline{\nu-2}}(b-s)^{\underline{\mu}}, \\ k(t) := \frac{1}{\Gamma(\nu)}\sum_{s=t+\nu}^{b-\mu}(s-t-1)^{\underline{\nu-2}}(b-s)^{\underline{\mu+1}}. \end{cases}$$

Summing $k(t)$ by parts, we obtain

$$\begin{aligned}
k(t) &= \frac{1}{\Gamma(\nu)}\sum_{s=t+\nu}^{b-\mu}(s-t-1)^{\underline{\nu-2}}(b-s)^{\underline{\mu+1}} \\
&= \frac{1}{\Gamma(\nu)}\sum_{s=t+\nu}^{b-\mu}(b-s)^{\underline{\mu+1}}\triangle_s\left(\frac{(s-t-1)^{\underline{\nu-1}}}{\nu-1}\right)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\Gamma(\nu)}(b-s)^{\underline{\mu+1}}\frac{(s-t)^{\underline{\nu-1}}}{\nu-1}\bigg|_{s=t+\nu}^{b-\mu+1} + \frac{1}{\Gamma(\nu)}\sum_{s=t+\nu}^{b-\mu}\frac{(s-t)^{\underline{\nu-1}}}{\nu-1}(\mu+1)(b-s-1)^{\underline{\mu}} \\
&= \frac{1}{\nu-1}\left[\frac{\mu+1}{\Gamma(\nu)}\sum_{s=t+\nu}^{b-\mu}(s-t)^{\underline{\nu-1}}(b-s-1)^{\underline{\mu}} - (b-t-\nu)^{\underline{\mu+1}}\right] \\
&= \frac{1}{\nu-1}\left[\frac{\mu+1}{\Gamma(\nu)}\sum_{u=t+\nu+1}^{b-\mu+1}(u-t-1)^{\underline{\nu-1}}(b-u)^{\underline{\mu}} - (b-t-\nu)^{\underline{\mu+1}}\right] \\
&= \frac{1}{\nu-1}\left[\frac{\mu+1}{\Gamma(\nu)}\sum_{s=t+\nu}^{b-\mu}(s-t-1)^{\underline{\nu-1}}(b-s)^{\underline{\mu}}\right. \\
&\quad \left. - \frac{\mu+1}{\Gamma(\nu)}(\nu-1)^{\underline{\nu-1}}(b-t-\nu)^{\underline{\mu}} - (b-t-\nu)^{\underline{\mu+1}}\right] \\
&= \frac{1}{\nu-1}\left[\frac{\mu+1}{\Gamma(\nu)}\sum_{s=t+\nu}^{b-\mu}(s-t-1)^{\underline{\nu-1}}(b-s)^{\underline{\mu}} - (b-t-\nu+1)^{\underline{\mu+1}}\right].
\end{aligned}
$$

It follows from the above work that

$$
\begin{aligned}
&-(b-t-\mu-\nu+1)\nabla g_1(t) = (\nu-1)h(t) + (b-t-\nu+1)^{\underline{\mu+1}}, \\
&(\mu+1)g_1(t) - (\nu-1)k(t) = (b-t-\nu+1)^{\underline{\mu+1}}.
\end{aligned}
$$

Hence,

$$
(b-t-\mu-\nu+1)\nabla g_1(t) + (\mu+\nu)g_1(t) = 0.
$$

Finally, g_2 also satisfies the difference equation (2.8):

$$
\begin{aligned}
(b-t-\mu-\nu+1)\nabla g_2(t) &= (b-t-\mu-\nu+1)\mu^{\underline{-\nu}}\big[(b-t)^{\underline{\mu+\nu}} - (b-t+1)^{\underline{\mu+\nu}}\big] \\
&= -(b-t-\mu-\nu+1)\mu^{\underline{-\nu}}(\mu+\nu)(b-t)^{\underline{\mu+\nu-1}} \\
&= -(\mu+\nu)g_2(t).
\end{aligned}
$$

By the uniqueness of the solutions to the well-posed initial value problem (2.8), we conclude that $g_1 \equiv g_2$ on ${}_{b-\mu-\nu}\mathbb{N}$.

We next employ (2.5) and (2.6) to show (2.7) follows. For $t \in {}_{b-\mu-N+\nu}\mathbb{N}$,

$$
\begin{aligned}
{}_{b-\mu}\nabla^{\nu}(b-t)^{\underline{\mu}} &= (-1)^N\nabla^N{}_{b-\mu}\nabla^{-(N-\nu)}(b-t)^{\underline{\mu}} \\
&= (-1)^N\nabla^N\big(\mu^{\underline{-(N-\nu)}}(b-t)^{\underline{\mu+N-\nu}}\big) \\
&= (-1)^N\nabla^N\left(\frac{\Gamma(\mu+1)}{\Gamma(\mu+1+N-\nu)}(b-t)^{\underline{\mu+N-\nu}}\right) \\
&= (-1)^N\nabla^{N-1}\left(\frac{\Gamma(\mu+1)}{\Gamma(\mu+1+N-\nu)}\big((b-t)^{\underline{\mu+N-\nu}} - (b-t+1)^{\underline{\mu+N-\nu}}\big)\right) \\
&= (-1)^{N-1}\nabla^{N-1}\left(\frac{\Gamma(\mu+1)}{\Gamma(\mu+N-\nu)}(b-t)^{\underline{\mu+N-\nu-1}}\right) \\
&\ \ \vdots
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\Gamma(\mu+1)}{\Gamma(\mu+1-\nu)}(b-t)^{\underline{\mu-\nu}} \\
&= \mu^{\underline{\nu}}(b-t)^{\underline{\mu-\nu}}. \qquad \square
\end{aligned}
$$

Theorem 2 *Let $f : {}_b\mathbb{N} \to \mathbb{R}$ be given and suppose $\mu, \nu > 0$. Then*

$$
{}_{b-\mu}\nabla^{-\nu}{}_b\nabla^{-\mu}f(t) = {}_b\nabla^{-\mu-\nu}f(t) = {}_{b-\nu}\nabla^{-\mu}{}_b\nabla^{-\nu}f(t), \quad t \in {}_{b-\mu-\nu}\mathbb{N}.
$$

Proof Suppose $f : {}_b\mathbb{N} \to \mathbb{R}$ and $\mu, \nu > 0$. Then for $t \in {}_{b-\mu-\nu}\mathbb{N}$,

$$
\begin{aligned}
{}_{b-\mu}\nabla^{-\nu}{}_b\nabla^{-\mu}f(t) &= \frac{1}{\Gamma(\nu)}\sum_{s=t+\nu}^{b-\mu}(s-t-1)^{\underline{\nu-1}}\frac{1}{\Gamma(\mu)}\sum_{u=s+\mu}^{b}(u-s-1)^{\underline{\mu-1}}f(u) \\
&= \frac{1}{\Gamma(\nu)\Gamma(\mu)}\sum_{s=t}^{b-(\mu+\nu)}\sum_{u=s+\mu+\nu}^{b}(s+\nu-t-1)^{\underline{\nu-1}}(u-s-\nu-1)^{\underline{\mu-1}}f(u) \\
&= \frac{1}{\Gamma(\nu)\Gamma(\mu)}\sum_{s=t}^{b-\mu-\nu}\sum_{r=s}^{b-\mu-\nu}(s+\nu-t-1)^{\underline{\nu-1}}(r+\mu-s-1)^{\underline{\mu-1}}f(r+\mu+\nu) \\
&= \frac{1}{\Gamma(\nu)\Gamma(\mu)}\sum_{r=t}^{b-\mu-\nu}\sum_{s=t}^{r}(s+\nu-t-1)^{\underline{\nu-1}}(r+\mu-s-1)^{\underline{\mu-1}}f(r+\mu+\nu) \\
&= \frac{1}{\Gamma(\nu)\Gamma(\mu)}\sum_{r=t}^{b-\mu-\nu}f(r+\mu+\nu) \\
&\quad \times \sum_{s=t-\nu+\nu}^{r}\bigl(s-(t-\nu)-1\bigr)^{\underline{\nu-1}}(r+\mu-s-1)^{\underline{\mu-1}} \\
&= \frac{1}{\Gamma(\mu)}\sum_{r=t}^{b-\mu-\nu}f(r+\mu+\nu)\,{}_{(r+\mu-1)-(\mu-1)}\nabla^{-\nu}(r+\mu-1-t)^{\underline{\mu-1}}\Big|_{t-\nu} \\
&= \frac{1}{\Gamma(\mu)}\sum_{r=t}^{b-\mu-\nu}f(r+\mu+\nu)(\mu-1)^{\underline{-\nu}}\bigl(r+\mu-1-(t-\nu)\bigr)^{\underline{\mu+\nu-1}} \\
&= \frac{1}{\Gamma(\nu+\mu)}\sum_{r=t+\mu+\nu}^{b}(r-t-1)^{\underline{\mu+\nu-1}}f(r) \\
&= {}_b\nabla^{-\mu-\nu}f(t).
\end{aligned}
$$

Since ν and μ are arbitrary, we conclude more generally that

$$
{}_{b-\mu}\nabla^{-\nu}{}_b\nabla^{-\mu}f(t) = {}_b\nabla^{-\mu-\nu}f(t) = {}_{b-\nu}\nabla^{-\mu}{}_b\nabla^{-\nu}f(t), \quad t \in {}_{b-\mu-\nu}\mathbb{N}. \qquad \square
$$

Lemma 2 *Let $f : {}_b\mathbb{N} \to \mathbb{R}$ be given. For any $k \in \mathbb{N}_0$ and $\mu > 0$ with $M-1 < \mu \le M$, we have*

$$
{}_{b-\mu}\nabla^{k}\bigl({}_b\nabla^{-\mu}f(t)\bigr) = {}_b\nabla^{k-\mu}f(t), \quad t \in {}_{b-\mu}\mathbb{N}, \tag{2.9}
$$

$$
{}_{b-M+\mu}\nabla^{k}\bigl({}_b\nabla^{\mu}f(t)\bigr) = {}_b\nabla^{k+\mu}f(t), \quad t \in {}_{b-M+\mu}\mathbb{N}. \tag{2.10}
$$

Proof Let f, μ, M, and k be as given in the statement of the lemma. We first prove (2.9). We consider two cases.

Case 1. $\mu = M$.

Observe that for $t \in {}_{b-1}\mathbb{N}$,

$$\nabla_b \nabla^{-1} f(t) = \nabla\left(\sum_{s=t+1}^{b} f(s)\right) = -f(t).$$

Furthermore, for any $k \in \mathbb{N}^+$ and $t \in {}_{b-k}\mathbb{N}$, by Theorem 2, we have

$$\begin{aligned} \nabla^k{}_b\nabla^{-k} f(t) &= \nabla^{k-1}\big(\nabla_{b-k+1}\nabla^{-1}\big({}_b\nabla^{-(k-1)} f(t)\big)\big) \\ &= -\nabla^{k-1}\big({}_b\nabla^{-(k-1)} f(t)\big) \\ &\ \ \vdots \\ &= (-1)^k f(t). \end{aligned}$$

Therefore, for any $t \in {}_{b-M}\mathbb{N}$, we obtain

$$\begin{aligned} {}_{b-M}\nabla^k{}_b\nabla^{-M} f(t) &= (-1)^k \nabla^k{}_b\nabla^{-M} f(t) \\ &= (-1)^k \nabla^{k-M}\nabla^M{}_b\nabla^{-M} f(t) \\ &= (-1)^k \nabla^{k-M} f(t) \\ &= {}_b\nabla^{k-M} f(t), \quad \text{if } k \geq M; \\ {}_{b-M}\nabla^k{}_b\nabla^{-M} f(t) &= (-1)^k \nabla^k{}_b\nabla^{-M} f(t) \\ &= (-1)^k \nabla^k_{b-M+k}\nabla^{-k}{}_b\nabla^{-(M-k)} f(t) \\ &= {}_b\nabla^{k-M} f(t), \quad \text{if } k < M. \end{aligned}$$

Case 2. $M - 1 < \mu < M$.

$$\begin{aligned} \nabla_b\nabla^{-\mu} f(t) &= \nabla \frac{1}{\Gamma(\mu)} \sum_{s=t+\mu}^{b} (s-t-1)^{\underline{\mu-1}} f(s) \\ &= \frac{1}{\Gamma(\mu)}\left[\sum_{s=t+\mu}^{b} (s-t-1)^{\underline{\mu-1}} f(s) - \sum_{s=t+\mu-1}^{b} (s-t)^{\underline{\mu-1}} f(s)\right] \\ &= -\frac{1}{\Gamma(\mu-1)} \sum_{s=t+\mu-1}^{b} (s-t-1)^{\underline{\mu-2}} f(s) \\ &= -{}_b\nabla^{1-\mu} f(t). \end{aligned}$$

Repeating the above process, we may see that (2.9) holds.

Next, by (2.9), we get

$$\begin{aligned} {}_{b-M+\mu}\nabla^k\big({}_b\nabla^\mu f(t)\big) &= (-1)^k\nabla^k(-1)^M\nabla^M\big({}_b\nabla^{-(M-\mu)} f(t)\big) \\ &= (-1)^{k+M}\nabla^{k+M}\big({}_b\nabla^{-(M-\mu)} f(t)\big) \\ &= {}_{b-M+\mu}\nabla^{k+M}\big({}_b\nabla^{-(M-\mu)} f(t)\big) = {}_b\nabla^{k+\mu} f(t), \end{aligned}$$

and hence (2.10) holds. □

Theorem 3 *Let* $f : {}_b\mathbb{N} \to \mathbb{R}$ *be given and suppose* $\mu, \nu > 0$ *with* $N-1 < \nu < N$. *Then*

$$ {}_{b-\mu}\nabla^{\nu}{}_b\nabla^{-\mu}f(t) = {}_b\nabla^{\nu-\mu}f(t), \quad t \in {}_{b-\mu-N+\nu}\mathbb{N}. \tag{2.11}$$

Proof Let f, ν, N, and μ be given as in the statement of the theorem and let $t \in {}_{b-\mu-N+\nu}\mathbb{N}$. Then by Theorem 2 and Lemma 2, we have

$$\begin{aligned} {}_{b-\mu}\nabla^{\nu}{}_b\nabla^{-\mu}f(t) &= (-1)^N\nabla^N{}_{b-\mu}\nabla^{-(N-\nu)}{}_b\nabla^{-\mu}f(t) \\ &= (-1)^N\nabla^N{}_b\nabla^{-(N-\nu)-\mu}f(t) \\ &= {}_b\nabla^{\nu-\mu}f(t). \end{aligned}$$

□

Theorem 4 *Let* $f : {}_b\mathbb{N} \to \mathbb{R}$ *be given and suppose* $k \in \mathbb{N}_0$ *be given. Then for* $t \in {}_{b-\nu}\mathbb{N}$,

$$ {}_b\nabla^{-\nu}{}_b\nabla^k f(t) = {}_b\nabla^{k-\nu}f(t) - \sum_{j=0}^{k-1}\frac{{}_b\nabla^j f(b)}{\Gamma(\nu-k+j+1)}(b-t)^{\underline{\nu-k+j}}. \tag{2.12}$$

Moreover, if $\mu > 0$ *with* $M-1 < \mu < M$, *then for* $t \in {}_{b-M+\mu-\nu}\mathbb{N}$,

$$ {}_{b-M+\mu}\nabla^{-\nu}{}_b\nabla^{\mu}f(t) = {}_b\nabla^{\mu-\nu}f(t) - \sum_{j=0}^{M-1}\frac{{}_b\nabla^{j-M+\mu}f(b-M+\mu)}{\Gamma(\nu-M+j+1)}(b-M+\mu-t)^{\underline{\nu-M+j}}. \tag{2.13}$$

Proof We first consider (2.12). Let $k \in \mathbb{N}_0$ be given. Then

$$\begin{aligned} {}_b\nabla^{-\nu}{}_b\nabla^k f(t) &= (-1)^k{}_b\nabla^{-\nu}\nabla^k f(t) \\ &= (-1)^k\frac{1}{\Gamma(\nu)}\sum_{s=t+\nu}^{b}(s-t-1)^{\underline{\nu-1}}\nabla^k f(s) \\ &= (-1)^k\frac{1}{\Gamma(\nu)}\left(\sum_{s=t+\nu}^{b}(s-t-1)^{\underline{\nu-1}}\triangle\big(\nabla^{k-1}f(s-1)\big)\right) \\ &= (-1)^k\frac{1}{\Gamma(\nu)}\left((s-t-1)^{\underline{\nu-1}}\nabla^{k-1}f(s-1)\Big|_{s=t+\nu}^{b+1}\right. \\ &\quad\left. - \sum_{s=t+\nu}^{b}(\nu-1)(s-t-1)^{\underline{\nu-2}}\nabla^{k-1}f(s)\right) \\ &= (-1)^k\left(-\frac{1}{\Gamma(\nu-1)}\sum_{s=t+\nu-1}^{b}(s-t-1)^{\underline{\nu-2}}\nabla^{k-1}f(s) + \frac{\nabla^{k-1}f(b)}{\Gamma(\nu)}(b-t)^{\underline{\nu-1}}\right) \\ &= (-1)^{k-1}\frac{1}{\Gamma(\nu-1)}\sum_{s=t+\nu-1}^{b}(s-t-1)^{\underline{\nu-2}}\nabla^{k-1}f(s) - \frac{{}_b\nabla^{k-1}f(b)}{\Gamma(\nu)}(b-t)^{\underline{\nu-1}} \\ &= (-1)^{k-2}\frac{1}{\Gamma(\nu-2)}\sum_{s=t+\nu-2}^{b}(s-t-1)^{\underline{\nu-3}}\nabla^{k-2}f(s) \\ &\quad - \frac{{}_b\nabla^{k-2}f(b)}{\Gamma(\nu-1)}(b-t)^{\underline{\nu-2}} - \frac{{}_b\nabla^{k-1}f(b)}{\Gamma(\nu)}(b-t)^{\underline{\nu-1}} \\ &\quad \vdots \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\Gamma(\nu-k)} \sum_{s=t+\nu-k}^{b} (s-t-1)^{\underline{\nu-k-1}} f(s) \\
&\quad - \sum_{j=0}^{k-1} \frac{{}_b\nabla^j f(b)}{\Gamma(\nu-k+j+1)} (b-t)^{\underline{\nu-k+j}}.
\end{aligned}$$

We next consider (2.13). Suppose that $\nu, \mu > 0$ with $M-1 < \mu \leq M$. Defining

$$g(t) := {}_b\nabla^{-(M-\mu)} f(t) \quad \text{and} \quad a := b - M + \mu,$$

where a is the domain of the first point g, we have, for $t \in {}_{b-M+\mu-\nu}\mathbb{N}$,

$$\begin{aligned}
{}_{b-M+\mu}\nabla^{-\nu}{}_b\nabla^{\mu} f(t) &= {}_{b-M+\mu}\nabla^{-\nu}(-1)^M \nabla^M {}_b\nabla^{-(M-\mu)} f(t) \\
&= (-1)^M_{b-M+\mu}\nabla^{-\nu}\nabla^M g(t) \\
&= {}_{b-M+\mu}\nabla^{-\nu}{}_{b-M+\mu}\nabla^M g(t) \\
&= {}_{b-M+\mu}\nabla^{M-\nu} g(t) - \sum_{j=0}^{M-1} \frac{{}_{b-M+\mu}\nabla^j g(a)}{\Gamma(\nu-M+j+1)} (a-t)^{\underline{\nu-M+j}} \\
&= {}_{b-M+\mu}\nabla^{M-\nu}{}_b\nabla^{-(M-\mu)} f(t) \\
&\quad - \sum_{j=0}^{M-1} \frac{{}_{b-M+\mu}\nabla^j {}_b\nabla^{-(M-\mu)} f(a)}{\Gamma(\nu-M+j+1)} (a-t)^{\underline{\nu-M+j}} \\
&= {}_b\nabla^{\mu-\nu} f(t) - \sum_{j=0}^{M-1} \frac{{}_b\nabla^{-(M-\mu)+j} f(b-M+\mu)}{\Gamma(\nu-M+j+1)} (b-M+\mu-t)^{\underline{\nu-M+j}}. \qquad \square
\end{aligned}$$

Theorem 5 *Let $f : {}_b\mathbb{N} \to \mathbb{R}$ be given and suppose $\mu, \nu > 0$ with $N-1 < \nu \leq N$ and $M-1 < \mu \leq M$. Then for $t \in {}_{b-M+\mu-N+\nu}\mathbb{N}$,*

$$ {}_{b-M+\mu}\nabla^{\nu}{}_b\nabla^{\mu} f(t) = {}_b\nabla^{\mu+\nu} f(t) - \sum_{j=0}^{M-1} \frac{{}_b\nabla^{j-M+\mu} f(b-M+\mu)}{\Gamma(-\nu-M+j+1)} (b-M+\mu-t)^{\underline{-\nu-M+j}}, \quad (2.14)$$

where in agreement with both rule (2.14) *and the standard convention on t, the terms in the summation vanish in the case $\nu \in \mathbb{N}_0$.*

Proof Let f, ν, and μ be given as in the statement of the theorem. Recall that Lemma 2 proves (2.14) in the case when $\nu = N$. On the other hand, if $N-1 < \nu < N$, then by Theorem 4, we have for $t \in {}_{b-M+\mu-N+\nu}\mathbb{N}$,

$$\begin{aligned}
{}_{b-M+\mu}\nabla^{\nu}{}_b\nabla^{\mu} f(t) &= (-1)^N \nabla^N {}_{b-M+\mu}\nabla^{-(N-\nu)}{}_b\nabla^{\mu} f(t) \\
&= (-1)^N \nabla^N \Bigg[{}_b\nabla^{-N+\nu+\mu} f(t) - \sum_{j=0}^{M-1} \frac{{}_b\nabla^{j-M+\mu} f(b-M+\mu)}{\Gamma(N-\nu-M+j+1)} \\
&\quad \times (b-M+\mu-t)^{\underline{N-\nu-M+j}} \Bigg]
\end{aligned}$$

$$= {}_b\nabla^{\mu+\nu} f(t) - \sum_{j=0}^{M-1} \frac{{}_b\nabla^{j-M+\mu} f(b-M+\mu)}{\Gamma(N-\nu-M+j+1)}$$
$$\times (-1)^N \nabla^N (b-M+\mu-t)^{\underline{N-\nu-M+j}}$$
$$= {}_b\nabla^{\mu+\nu} f(t) - \sum_{j=0}^{M-1} \frac{{}_b\nabla^{j-M+\mu} f(b-M+\mu)}{\Gamma(-\nu-M+j+1)} (b-M+\mu-t)^{\underline{-\nu-M+j}}. \qquad \square$$

3 Application

Theorem 6 *Let $f : {}_b\mathbb{N} \to \mathbb{R}$ and $\nu > 0$ be given with $N-1 < \nu < N$, consider the initial value problem of the νth right fractional difference equation*

$$\begin{cases} {}_b\nabla^{\nu} y(t) = f(t), & t \in {}_b\mathbb{N}, \\ \nabla^i y(b) = A_i, & i = 0,1,\ldots,N-1, A_i \in \mathbb{R}. \end{cases} \tag{3.1}$$

The general solution to (3.1) *is*

$$y(t) = {}_{b-N+\nu}\nabla^{-\nu} f(t) + \sum_{i=0}^{N-1} \alpha_i (b-N+\nu-t)^{\underline{i+\nu-N}}, \tag{3.2}$$

where $\{\alpha_j\}_{j=0}^{N-1}$ are N real constants. Moreover, the unique solution to (3.1) *is* (3.2) *with particular constants,*

$$\alpha_j = \frac{1}{\Gamma(\nu-N+j+1)} \sum_{k=0}^{j} \sum_{i=0}^{j-k} (-1)^{k+i} \binom{j-N+\nu}{k} \binom{j-k}{i} A_i,$$
$$\binom{\alpha}{\beta} = \frac{\Gamma(\alpha+1)}{\Gamma(\beta+1)\Gamma(\alpha-\beta+1)}.$$

Proof By Theorem 4, we have

$$y(t) = {}_{b-N+\nu}\nabla^{-\nu} f(t) + \sum_{j=0}^{N-1} \frac{{}_b\nabla^{j-N+\nu} y(b-N+\nu)}{\Gamma(\nu-N+j+1)} (b-N+\nu-t)^{\underline{\nu-N+j}}.$$

We have

$$\begin{aligned} {}_b\nabla^{j-N+\nu} y(b-N+\nu) &= \frac{1}{\Gamma(N-\nu-j)} \sum_{s=b-j}^{b} (s-b+N-\nu-1)^{\underline{-j+N-\nu-1}} y(s) \\ &= \frac{1}{\Gamma(N-\nu-j)} \sum_{k=0}^{j} (k-j+N-\nu-1)^{\underline{-j+N-\nu-1}} y(b-j+k) \\ &= \frac{1}{\Gamma(N-\nu-j)} \sum_{k=0}^{j} \frac{\Gamma(k-j+N-\nu)}{\Gamma(k+1)} y(b-j+k) \\ &= \sum_{k=0}^{j} \frac{(k-j+N-\nu-1)(k-j+N-\nu-2)\cdots(N-\nu-j)}{\Gamma(k+1)} \\ &\quad \times y(b-j+k) \\ &= \sum_{k=0}^{j} (-1)^k \binom{\nu+j-N}{k} y(b-j+k) \end{aligned}$$

$$= \sum_{k=0}^{j} (-1)^k \binom{\nu + j - N}{k} \sum_{i=0}^{j-k} (-1)^i \binom{j-k}{i} \nabla^i y(b)$$

$$= \sum_{k=0}^{j} \sum_{i=0}^{j-k} (-1)^{i+k} \binom{\nu + j - N}{k} \binom{j-k}{i} A_i.$$

Therefore, $\alpha_j = \frac{1}{\Gamma(\nu-N+j+1)} \sum_{k=0}^{j} \sum_{i=0}^{j-k} (-1)^{i+k} \binom{\nu+j-N}{k} \binom{j-k}{i} A_i$.

We consider two-point boundary value problem

$$\begin{cases} {}_b\nabla^\nu y(t) + f(t-\nu+1, y(t-\nu+1)) = 0, & t \in [\nu-1, b+\nu-1]_{\mathbb{N}_{\nu-1}}, \\ y(-1) = y(b) = 0, \end{cases} \tag{3.3}$$

where $f : [0,b]_{\mathbb{N}_0} \times \mathbb{R} \to \mathbb{R}$ is a continuous function and $\nu \in (1,2]$, $b \in \mathbb{N}$, $b > 2$.

By Theorem 4, we have

$$\begin{aligned} y(t) &= -{}_{b-2-\nu}\nabla^{-\nu} f\bigl(t-\nu+1, y(t-\nu+1)\bigr) \\ &\quad + C_1 (b-2+\nu-t)^{\underline{\nu-1}} + C_2 (b-2+\nu-t)^{\underline{\nu-2}}, \\ &= -\frac{1}{\Gamma(\nu)} \sum_{s=t+\nu}^{b-2+\nu} (s-t-1)^{\underline{\nu-1}} f\bigl(s-\nu+1, y(s-\nu+1)\bigr) \\ &\quad + C_1 (b-2+\nu-t)^{\underline{\nu-1}} + C_2 (b-2+\nu-t)^{\underline{\nu-2}}, \\ y(b) &= C_2 \Gamma(\nu-1) = 0, \quad C_2 = 0, \\ y(-1) &= -\frac{1}{\Gamma(\nu)} \sum_{s=\nu-1}^{b+\nu-2} s^{\underline{\nu-1}} f\bigl(s-\nu+1, y(s-\nu+1)\bigr) + C_1 (b-1+\nu)^{\underline{\nu-1}} = 0, \\ C_1 &= \frac{1}{(b+\nu-1)^{\underline{\nu-1}} \Gamma(\nu)} \sum_{s=\nu-1}^{b+\nu-2} s^{\underline{\nu-1}} f\bigl(s-\nu+1, y(s-\nu+1)\bigr), \end{aligned}$$

$$y(t) = \sum_{s=\nu-1}^{b+\nu-2} G(t,s) f\bigl(s-\nu+1, y(s-\nu+1)\bigr), \tag{3.4}$$

where

$$G(t,s) = \frac{1}{\Gamma(\nu)} \begin{cases} \frac{s^{\underline{\nu-1}} (b-2+\nu-t)^{\underline{\nu-1}}}{(b+\nu-1)^{\underline{\nu-1}}} - (s-t-1)^{\underline{\nu-1}}, & \nu-1 \le t+\nu-1 < s \le b+\nu-2, \\ \frac{s^{\underline{\nu-1}} (b-2+\nu-t)^{\underline{\nu-1}}}{(b+\nu-1)^{\underline{\nu-1}}}, & \nu-1 \le s \le t+\nu-1 \le b+\nu-2. \end{cases} \qquad \square$$

Theorem 7 *The Green's function $G(t,s)$ satisfies the following conditions*:

(i) $G(t,s) > 0$ *for* $t \in [0, b-1]_{\mathbb{N}_0}$ *and* $s \in [\nu, b+\nu-2]_{\mathbb{N}_\nu}$;

(ii) $\max_{t \in [0,b]_{\mathbb{N}_0}} G(t,s) = G(s-\nu+1, s)$, *for* $s \in [\nu, b+\nu-2]_{\mathbb{N}_\nu}$;

(iii) *there exists a positive number* $\gamma \in (0,1)$ *such that* $\min_{t \in [\frac{b}{4}, \frac{3b}{4}]} G(t,s) \ge \gamma \max_{t \in [-1,b]_{\mathbb{N}_{-1}}} G(t,s) = \gamma G(s-\nu+1, s)$, *for* $s \in [\nu-1, b+\nu-2]_{\mathbb{N}_{\nu-1}}$.

Proof (i) One can see that $\triangle_t G(t,s) < 0$ for $\nu-1 \le s \le t+\nu-1$, and $\triangle_t G(t,s) > 0$ for $\nu-1 \le t+\nu-1 < s$. Indeed, for $\nu-1 \le t+\nu-1 < s$, we have

$$\triangle_t G(t,s) = \frac{\nu-1}{(b-1+\nu)^{\underline{\nu-1}}} \bigl[(s-t-2)^{\underline{\nu-2}} (b-1+\nu)^{\underline{\nu-1}} - (b+\nu-t-3)^{\underline{\nu-2}} s^{\underline{\nu-1}} \bigr].$$

Thus, $\triangle_t G(t,s) > 0$ if and only if

$$\frac{(s-t-2)^{\underline{\nu-2}}(b-1+\nu)^{\underline{\nu-1}}}{(b+\nu-t-3)^{\underline{\nu-2}}s^{\underline{\nu-1}}} > 1.$$

The inequality follows from the fact that $t^{\underline{\alpha}}$ is increasing and $t^{\underline{-\alpha}}$ is decreasing if $0 < \alpha \leq 1$. Since

$$G(-1,s) = G(b,s) = 0$$

and

$$\begin{aligned} G(s-\nu+1,s) &= \frac{s^{\underline{\nu-1}}(b+\nu-2-s+\nu-1)^{\underline{\nu-1}}}{(b+\nu-1)^{\underline{\nu-1}}} \\ &= \frac{s^{\underline{\nu-1}}(b+2\nu-s-3)^{\underline{\nu-1}}}{(b+\nu-1)^{\underline{\nu-1}}} > 0, \quad s \in [\nu-1, b+\nu-2]_{\mathbb{N}_{\nu-1}}, \end{aligned}$$

(i), (ii) are proved.

Next, we prove the (iii). Clearly,

$$\frac{G(t,s)}{G(s-\nu+1,s)} = \begin{cases} \frac{(b+\nu-2-t)^{\underline{\nu-1}}}{(b+2\nu-s-3)^{\underline{\nu-1}}} - \frac{(s-t-1)^{\underline{\nu-1}}(b+\nu-1)^{\underline{\nu-1}}}{s^{\underline{\nu-1}}(b+2\nu-s-3)^{\underline{\nu-1}}}, & \nu-1 \leq t+\nu-1 < s \leq b+\nu-2, \\ \frac{(b+\nu-2-t)^{\underline{\nu-1}}}{(b+2\nu-s-3)^{\underline{\nu-1}}}, & \nu-1 \leq s \leq t+\nu-1 \leq b+\nu-2. \end{cases}$$

For $\nu-1 \leq s \leq t+\nu-1 \leq b+\nu-2$ and $\frac{b}{4} \leq t \leq \frac{3b}{4}$,

$$\frac{G(t,s)}{G(s-\nu+1,s)} = \frac{(b+\nu-2-t)^{\underline{\nu-1}}}{(b+2\nu-s-3)^{\underline{\nu-1}}} \geq \frac{(b+\nu-2-\frac{3b}{4})^{\underline{\nu-1}}}{(b+\nu-2)^{\underline{\nu-1}}}.$$

For $\nu-1 \leq t+\nu-1 < s \leq b+\nu-2$ and $\frac{b}{4} \leq t \leq \frac{3b}{4}$, we know that $G(t,s)$ is increasing with respect to t, hence we have

$$\begin{aligned} \frac{G(t,s)}{G(s-\nu+1,s)} &\geq \frac{(b+\nu-2-\frac{b}{4})^{\underline{\nu-1}}}{(b+2\nu-s-3)^{\underline{\nu-1}}} - \frac{(s-\frac{b}{4}-1)^{\nu-1}(b+\nu-1)^{\underline{\nu-1}}}{s^{\underline{\nu-1}}(b+2\nu-s-3)^{\underline{\nu-1}}} \\ &= \frac{1}{(b+2\nu-s-3)^{\underline{\nu-1}}} \\ &\quad \times \left[\left(\frac{3b}{4}+\nu-2\right)^{\underline{\nu-1}} - \frac{1}{s^{\underline{\nu-1}}}\left(s-\frac{b}{4}-1\right)^{\underline{\nu-1}}(b+\nu-1)^{\underline{\nu-1}}\right] \\ &\geq \frac{1}{(b+\nu-2)^{\underline{\nu-1}}}\left[\left(\frac{3b}{4}+\nu-2\right)^{\underline{\nu-1}} - \frac{(\frac{3b}{4}+\nu-3)^{\underline{\nu-1}}}{(b+\nu-2)^{\underline{\nu-1}}}(b+\nu-1)^{\underline{\nu-1}}\right] \\ &> \frac{1}{(b+\nu-2)^{\underline{\nu-1}}}\left[\left(\frac{3b}{4}+\nu-2\right)^{\underline{\nu-1}} - \frac{(\frac{3b}{4}+\nu-2)^{\underline{\nu-1}}}{(b+\nu-1)^{\underline{\nu-1}}}(b+\nu-1)^{\underline{\nu-1}}\right] = 0, \end{aligned}$$

since $\frac{(s-\frac{b}{4}-1)^{\underline{\nu-1}}}{s^{\underline{\nu-1}}}$ is increasing for s.

Thus

$$\min_{t\in[\frac{b}{4},\frac{3b}{4}]} G(t,s) \geq \gamma \max_{t\in[-1,b]_{\mathbb{N}_{-1}}} G(t,s) = \gamma G(s-\nu+1,s),$$

where

$$\gamma = \min\left\{ \frac{(\frac{b}{4}+\nu-2)^{\underline{\nu-1}}}{(b+\nu-2)^{\underline{\nu-1}}}, \frac{1}{(b+\nu-2)^{\underline{\nu-1}}}\left[\left(\frac{3b}{4}+\nu-2\right)^{\underline{\nu-1}} - \frac{(\frac{3b}{4}+\nu-3)^{\underline{\nu-1}}}{(b+\nu-2)^{\underline{\nu-1}}}(b+\nu-1)^{\underline{\nu-1}}\right]\right\}. \tag{3.5}$$

□

Below we shall employ the following fixed point result.

Lemma 3 (see [2]) *Let $\mathbb{B}$ be a Banach space, and let $\mathcal{P} \subset \mathbb{B}$ be a cone. Assume Ω_1 and Ω_2 are open discs contained in $\mathbb{B}$ with $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$ and let $A : \mathcal{P} \cap (\overline{\Omega}_2 \backslash \Omega_1) \to \mathcal{P}$ be a completely continuous operator such that, either*

(i) $\|Ay\| \le \|y\|$, $y \in \mathcal{P} \cap \partial\Omega_1$ *and* $\|Ay\| \ge \|y\|$, $y \in \mathcal{P} \cap \partial\Omega_2$ *or*

(ii) $\|Ay\| \ge \|y\|$, $y \in \mathcal{P} \cap \partial\Omega_1$ *and* $\|Ay\| \le \|y\|$, $y \in \mathcal{P} \cap \partial\Omega_2$.

Then A has least one fixed point in $\mathcal{P} \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

Clearly, finding a solution $y(t)$ of the FBVP (3.3) is equivalent to finding a solution of the summation equation (3.4).

For our purpose, define the Banach space $\mathcal{B}$ by

$$\mathcal{B} = \{y : [0,b]_{\mathbb{N}_0} \to \mathbb{R} : y(-1) = y(b) = 0\},$$

with norm $\|y\| = \max_{t\in[-1,b]_{\mathbb{N}_{-1}}} |y(t)|$.

Let γ be defined by (3.5) and define cones $\mathcal{P}$ and $\mathcal{P}_0$ in $\mathbb{B}$ by

$$\mathcal{P} = \{y \in \mathbb{B} : y(t) \ge 0 \text{ for } t \in [-1,b]_{\mathbb{N}_{-1}}\},$$

$$\mathcal{P}_0 = \Big\{y \in \mathcal{P} : \min_{t\in[\frac{b}{4},\frac{3b}{4}]} y(t) \ge \gamma\|y\|\Big\}.$$

Thus, y is a solution of the boundary value problem (3.3) if and only if y is a fixed point of the operator $T : \mathbb{B} \to \mathbb{B}$ defined by

$$Ty(t) = \sum_{s=\nu-1}^{b+\nu-2} G(t,s) f\big(s-\nu+1, y(s-\nu+1)\big), \quad t \in [-1,b]_{\mathbb{N}_{-1}}.$$

We state three hypotheses that will be used below.

(H_1) $f(t-\nu+1,x) \ge 0$, $(t,x) \in [\nu-1, b+\nu-1]_{\mathbb{N}_{\nu-1}} \times [0,+\infty)$;

(H_2) $f(t-\nu+1,x) = h(t-\nu+1)g(x)$, where h is a positive function, g is a nonnegative function, and $\lim_{x\to 0^+} \frac{g(x)}{x} = 0$, $\lim_{x\to\infty} \frac{g(x)}{x} = \infty$;

(H_3) $f(t-\nu+1,x) = h(t-\nu+1)g(x)$, where h is a positive function, g is a nonnegative function, and $\lim_{x\to 0^+} \frac{g(x)}{x} = \infty$, $\lim_{x\to\infty} \frac{g(x)}{x} = 0$.

Lemma 4 *Assume condition* (H_1) *holds. Then $Ty \in \mathcal{P}_0$ for all $y \in \mathcal{P}$. In particular, the operator T leaves the cone $\mathcal{P}_0$ invariant.*

Proof For all $y \in \mathcal{P}$, by Theorem 7 and (H_1), we have $Ty(t) \geq 0$ for all $t \in [-1, b]_{\mathbb{N}_{-1}}$. Further, it follows immediately from Theorem 7(iii) that

$$\min_{t\in[\frac{b}{4},\frac{3b}{4}]} (Ty)(t) \geq \gamma \sum_{s=\nu-1}^{b+\nu-2} \max_{t\in[-1,b]_{\mathbb{N}_{-1}}} G(t,s) f\big(s-\nu+1, y(s-\nu+1)\big) \geq \gamma \|Ty\|.$$

Therefore, $Ty \in \mathcal{P}_0$. □

Theorem 8 *Assume that conditions* (H_1) *and* (H_2) *are satisfied. Then the FBVP* (3.3) *has at least one solution* $y \neq 0 \in \mathcal{P}_0$.

Theorem 9 *Assume that conditions* (H_1) *and* (H_3) *are satisfied. Then the FBVP* (3.3) *has at least one solution* $y \neq 0 \in \mathcal{P}_0$.

The proofs of Theorem 8 and 9 are similar to Theorems 4.1 and 4.2 in [24] and are skipped.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
CH and ZX worked together in the derivation of the mathematical results. Both authors read and approved the final manuscript.

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