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Existence of a class of fractional difference equations with two point boundary value problem

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Abstract

This paper studies a fractional difference equation of two point boundary value problem (BVP) type, which is recognized as the 'discrete' BVP. Certain cases are expressed under which the discrete boundary value problems (DBVP) will have a single solution. The novelty hither comprises a method selection of metric and employment of Hölder's inequality. This attitude allows the related functions to be contractive, which were earlier non-contractive in classical regularities. This consequently qualifies an enhanced application of Banach's fixed point theorem for classifying a more extensive framework of issues than those which appeared in the current designs.

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1 Introduction

In [1], Diaz and Osler concluded that the fractional difference by choice is a normal method of letting the index of differences, in the criterion appearance for the n th difference, to be any number (real or complex). Farther along, Hirota [2] employed the fractional difference operator (FDO) for any real number utilizing a Taylor's series. In [3], Nagai assumed different description of FDO by adapting Hirota's [2] concept. Newly, in [4], Deekshitulu and Mohan improved Nagai's definition [3]. In [5], Jumarie suggested another formula of fractional difference operator, of which the leading features are a new fractional Taylor series and its companion Rolle's formula which are employed to non-differentiable functions. In [6, 7], the authors generalized Gâteaux derivative by employing a fractional discrete operator for a Jumarie fractional operator. The method of finding solutions was based on critical point theorems of finite dimensional Banach spaces. Other techniques can be found in [8] and [9].

Investigators in the areas of control systems, neural networks, computer science, food processing and economics rely on mathematical modeling because it surely affects non-linear difference equations. Consequently, many novelists have widely established various procedures and patterns, such as fixed point theorems, upper and lower solutions and Brouwer degree, to consider discrete problems [10, 11]. The investigation of fractional differential equations (FDE) was started to establish the existence and uniqueness of findings

for various classes of FDE. Moreover, the theory of integro-differential equations has been almost settled parallel to the theory of FDE. However, the theory of fractional difference equations led to a very minor development of itself.

This paper aims to study a fractional difference equation of two point BVP type, which was realized as the discrete BVP. Certain classes are formulated in which the discrete boundary value problems will have a single solution. The novelty hither comprises a method selection of metric and employment of Hölder’s inequality. This investigation allows the related functions to be contractive, which were earlier non-contractive in classical regularities. This work, unlike those which appeared in the current designs, consequently grants the enhanced applications of Banach’s fixed point theorem for classifying an extensive framework of issues.

2 Main methods

This section concerns some concepts as well as preliminaries.

Definition 2.1 Let $\wp \in \mathbb{R}$ and $k \in \mathbb{Z}$ such that $k - 1 < \wp \leq k$. The FDO ∇ of order \wp , with step length κ , is introduced as

$$\nabla^\wp \mu(n) := \begin{cases} \kappa^{k-\wp} \sum_{j=0}^{n-1} \binom{\wp-k}{j} (-1)^j \nabla^k \mu(n-j), & \wp > 0, \\ \mu(n), & \wp = 0, \\ \kappa^{-\wp} \sum_{j=0}^{n-1} \binom{\wp}{j} (-1)^j \mu(n-j), & \wp < 0, \end{cases}$$

where the extended binomial coefficient, for $\varepsilon \in \mathbb{R}$, $n \in \mathbb{Z}$, is defined by

$$\binom{\varepsilon}{n} = \begin{cases} \frac{\Gamma(\varepsilon+1)}{\Gamma(\varepsilon+1-n)\Gamma(n+1)}, & n > 0, \\ 1, & n = 0, \\ 0, & n < 0. \end{cases}$$

The above formula is hard to study the properties of findings (solutions). Therefore, Mohan and Deekshitulu [12] introduced the following formula for $\kappa = k = 1$.

Definition 2.2 The FDO of order $0 < \wp < 1$ is given by

$$\nabla^{-\wp} \mu(n) = \sum_{j=0}^{n-1} \binom{j + \wp - 1}{j} \mu(n-j).$$

And the FDO of order \wp is given by

$$\begin{aligned} \nabla^\wp \mu(n) &= \sum_{j=0}^{n-1} \binom{j - \wp}{j} \nabla \mu(n-j) \\ &= \sum_{j=1}^n \binom{n-j-\wp-1}{n-j} \mu(j) - \binom{n-\wp-1}{n-1} \mu(0). \end{aligned}$$

Remark 2.1 The solution for the nonlinear fractional difference equation

$$\nabla^\wp \mu(n + 1) = \phi(n, \mu(n)), \quad \mu(0) = \mu_0,$$

is given by the form

$$\mu(n) = \mu_0 + \sum_{j=0}^{n-1} \beta(n-1, \wp; j) \phi(j, \mu(j)),$$

where

$$\beta(m, \wp; j) := \binom{m-j+\wp-1}{m-j}, \quad 0 \leq j \leq m,$$

stands as the extended binomial coefficient and $\phi : \mathbb{N}_0 \rightarrow \mathbb{R}$ denotes a function of n . It is clear that $0 \leq \beta(m, \wp; j) \leq 1$ for $\wp \in [0, 1]$. Moreover, the coefficient $\beta(m, \wp; j)$ satisfies

$$\begin{aligned} \sum_{j=0}^m \beta(m, \wp; j) &= \sum_{j=0}^m \frac{\Gamma(m-j+\wp)}{\Gamma(\wp)\Gamma(m-j+1)} \\ &= \frac{1}{\Gamma(m+1)} \sum_{j=0}^m \binom{m}{j} (1)_j (\wp)_{m-j} \\ &= \frac{1}{\Gamma(m+1)} (1+\wp)_m \\ &= \binom{m+\wp}{m}, \end{aligned}$$

where $(x)_m := \frac{\Gamma(x+m)}{\Gamma(x)}$ is the Pochhammer symbol, which satisfies

$$(x+y)_m = \sum_{j=0}^m \binom{m}{j} (x)_{m-j} (y)_j.$$

Example 2.1 Consider the problem

$$\nabla^\wp \mu(n+1) = \alpha(n)\mu(n) + \gamma(n), \quad \mu(0) = \mu_0,$$

where α and γ are nonnegative functions on n ; thus the solution can be expressed by

$$\begin{aligned} \mu(n) &= \mu_0 \prod_{j=0}^{n-1} [1 + \beta(n-1, \wp; j)\alpha(j)] \\ &\quad + \sum_{j=0}^{n-1} \beta(n-1, \wp; j)\gamma(j) \prod_{k=j+1}^{n-1} [1 + \beta(n-1, \wp; k)\alpha(k)]. \end{aligned}$$

This paper studies a boundary value problem that includes a nonlinear difference equation. Let $\phi : [0, N] \times \mathbb{R} \rightarrow \mathbb{R}$ be continuous and consider the discrete BVP

$$\begin{aligned} \frac{\nabla^\wp \mu_{i+1}}{\ell^\wp} &= \phi(t_i, \mu_i), \quad i = 0, 1, \dots, n \\ (a\mu_0 + b\mu_n &= c, a + b \neq 0), \end{aligned} \tag{1}$$

where a, b, c are constants and $0 < \wp < 1, 0 < \ell = N/n < N, t_i = i\ell, i = 0, 1, \dots, n$.

3 Main findings

The main result is as follows.

Theorem 3.1 *Let $\phi : [0, N] \times \mathbb{R} \rightarrow \mathbb{R}$ be continuous and $a + b \neq 0$. Assume that there exist constants $L > 0$, $\lambda \in (0, N)$ and $p > 1$. If*

$$|\phi(t, \mu) - \phi(t, \nu)| \leq L|\mu - \nu|, \quad \forall t \in [0, N], (\mu, \nu) \in \mathbb{R}^2$$

and

$$\lambda L \bar{\Phi} < 1,$$

where

$$\bar{\Phi} := \max_i \left\{ \sum_{j=0}^{n-1} |\Phi(i, j)| \right\}$$

with

$$\Phi(i, j) := \begin{cases} \beta(i - 1, \wp; j) - b\beta(n - 1, \wp; j), & j \leq i - 1, \\ b\beta(n - 1, \wp; j), & j \geq i, \end{cases}$$

then problem (1) has a unique solution for each $|\frac{\ell^{\wp}}{a+b}| \leq \lambda$.

Proof Consider the metric space $(\Xi := \mathbb{R}^{n+1}, d_p)$ and for all $\mu, \nu \in \Xi$,

$$d_p(\mu, \nu) = \sum_{i=1}^n (|\mu_i - \nu_i|^p)^{1/p}.$$

It is easy to show that (Ξ, d_p) is a complete metric space for all $p \geq 1$. In view of Remark 2.1, it can be shown that the discrete BVP (1) is equivalent to the summation equation

$$\mu_i = \frac{c}{a + b} + \frac{\ell^{\wp}}{a + b} \sum_{j=0}^{n-1} \Phi(i, j)\phi(t_j, \mu_j), \quad i = 0, 1, \dots, n, \tag{2}$$

where

$$\Phi(i, j) = \begin{cases} \beta(i - 1, \wp; j) - b\beta(n - 1, \wp; j), & j \leq i - 1, \\ b\beta(n - 1, \wp; j), & j \geq i. \end{cases}$$

Our aim is to prove that (2) has a unique solution. We define the operator

$$(\Lambda\mu)(i) = \frac{c}{a + b} + \frac{\ell^{\wp}}{a + b} \sum_{j=0}^{n-1} \Phi(i, j)\phi(t_j, \mu_j), \quad i = 0, 1, \dots, n. \tag{3}$$

So that

$$(\Lambda\mu) = ((\Lambda\mu)(0), \dots, (\Lambda\mu)(n)).$$

Obviously, $\Lambda(\mu)$ is bounded

$$\begin{aligned} |(\Lambda\mu)| &= \left| \frac{c}{a+b} + \frac{\ell^\wp}{a+b} \sum_{j=0}^{n-1} \Phi(i,j)\phi(t_j, \mu_j) \right| \\ &\leq \left| \frac{c}{a+b} \right| + \left| \frac{\ell^\wp}{a+b} \right| \sum_{j=0}^{n-1} |\Phi(i,j)| |\phi(t_j, \mu_j) - \phi(t_j, o) + \phi(t_j, o)| \\ &\leq \left| \frac{c}{a+b} \right| + \left| \frac{\ell^\wp}{a+b} \right| \sum_{j=0}^{n-1} |\Phi(i,j)| (L|\mu| + |\phi(t, o)|) \\ &\leq \left| \frac{c}{a+b} \right| + \lambda \bar{\Phi} (L|\mu| + |\phi(t, o)|), \end{aligned}$$

where o is the zero element in \mathbb{R}^{n+1} .

We aim to show that $(\Lambda\mu) = \mu$ for a unique $\mu \in \Xi$. Since ϕ is well defined on $[0, N] \times \mathbb{R}$, it follows that $\Lambda : \Xi \rightarrow \Xi$. We proceed to show that Λ is a contraction mapping on Ξ with respect d_p , i.e., there is a constant $0 < \sigma < 1$ such that

$$d_p(\Lambda\mu, \Lambda\nu) \leq \sigma d_p(\mu, \nu), \quad \forall \mu, \nu \in \mathbb{R}^{n+1}.$$

A computation yields, for $0 < \wp < 1$,

$$\begin{aligned} |(\Lambda\mu) - (\Lambda\nu)| &= \left| \frac{\ell^\wp}{a+b} \sum_{j=0}^{n-1} \Phi(i,j)\phi(t_j, \mu_j) - \frac{\ell^\wp}{a+b} \sum_{j=0}^{n-1} \Phi(i,j)\phi(t_j, \nu_j) \right| \\ &\leq \left| \frac{\ell^\wp}{a+b} \right| \sum_{j=0}^{n-1} |\Phi(i,j)| |\phi(t_j, \mu_j) - \phi(t_j, \nu_j)| \\ &\leq L \left| \frac{\ell^\wp}{a+b} \right| \sum_{j=0}^{n-1} |\Phi(i,j)| |\mu_j - \nu_j|. \end{aligned}$$

This yields

$$|(\Lambda\mu) - (\Lambda\nu)| \leq L\bar{\Phi} \left| \frac{\ell^\wp}{a+b} \right| \left(\sum_{j=0}^{n-1} |\mu_j - \nu_j|^p \right)^{1/p}. \tag{4}$$

Hence, we obtain

$$|(\Lambda\mu) - (\Lambda\nu)| \leq L\bar{\Phi} \left| \frac{\ell^\wp}{a+b} \right| d_p(\mu_i, \nu_i).$$

Consequently, we have

$$\begin{aligned} |(\Lambda\mu) - (\Lambda\nu)|^p &\leq \left(L\bar{\Phi} \left| \frac{\ell^\wp}{a+b} \right| \right)^p d_p^p(\mu_i, \nu_i) \\ &\leq (\lambda L\bar{\Phi})^p d_p^p(\mu_i, \nu_i). \end{aligned}$$

Thus, for all $\mu, \nu \in \mathbb{R}^{n+1}$, we attain

$$d_p((\Lambda\mu), (\Lambda\nu)) \leq \sigma d_p(\mu, \nu),$$

where $0 < \sigma := \lambda L \bar{\Phi} < 1$. Thus, Λ is a contraction mapping on \mathbb{R}^{n+1} with respect to the d_p metric and therefore, all of the assumptions of Banach’s fixed point theorem are accorded. So, there is a unique $\mu \in \mathbb{R}^{n+1}$ such that $\Lambda\mu = \mu$, giving the existence of a unique solution to problem (1). \square

Theorem 3.2 *Let $\phi : [0, N] \times \mathbb{R} \rightarrow \mathbb{R}$ be continuous and $a + b \neq 0$. Assume that there exist constants $L > 0, \lambda \in (0, N)$; $p > 1, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. If*

$$|\phi(t, \mu) - \phi(t, \nu)| \leq L|\mu - \nu|, \quad \forall t \in [0, N], (\mu, \nu) \in \mathbb{R}^2$$

and

$$\frac{\lambda L}{|a + b|} \left(\sum_{i=1}^n (i|a|^q + n|b|^q)^{p/q} \right)^{1/p} < 1, \quad |a| \geq 1,$$

then problem (1) has a unique solution for each $\ell^\varphi \leq \lambda$.

Proof Operator (3) implies

$$\begin{aligned} |(\Lambda\mu) - (\Lambda\nu)| &\leq \left| \frac{\ell^\varphi}{a + b} \right| \sum_{j=0}^{n-1} |\Phi(i, j)| |\phi(t_j, \mu_j) - \phi(t_j, \nu_j)| \\ &\leq \frac{\lambda}{|a + b|} L \sum_{j=0}^{n-1} |\Phi(i, j)| |\mu_j - \nu_j|. \end{aligned}$$

An application of Hölder’s inequality yields

$$|(\Lambda\mu) - (\Lambda\nu)| \leq \frac{\lambda}{|a + b|} L \left(\sum_{j=0}^{n-1} |\mu_j - \nu_j|^p \right)^{1/p} \left(\sum_{j=0}^{n-1} |\Phi(i, j)|^q \right)^{1/q}. \tag{5}$$

By applying the boundary condition and taking account of $0 \leq \beta(m, \varphi; j) \leq 1$, we obtain

$$\left(\sum_{j=0}^{n-1} |\Phi(i, j)|^q \right) \leq i|a|^q + n|b|^q, \quad i = 0, 1, \dots, n.$$

Thus, we attain

$$|(\Lambda\mu) - (\Lambda\nu)| \leq d_p(\mu_i, \nu_i) \frac{\lambda L}{|a + b|} (i|a|^q + n|b|^q)^{1/q}.$$

Consequently, this leads to

$$|(\Lambda\mu) - (\Lambda\nu)|^p \leq d_p^p(\mu_i, \nu_i) \left(\frac{\lambda L}{|a + b|} \right)^p (i|a|^q + n|b|^q)^{p/q}.$$

For all $u, v \in \mathbb{R}^{n+1}$, we have

$$\begin{aligned} d_p((\Lambda\mu), (\Lambda\nu)) &= \left(\sum_{i=1}^n |(\Lambda\mu)_i - (\Lambda\nu)_i|^p \right)^{1/p} \\ &\leq d_p(\mu, \nu) \left(\frac{\lambda L}{|a+b|} \right) \left(\sum_{i=1}^n (i|a|^q + n|b|^q)^{p/q} \right)^{1/p} \\ &\leq d_p(\mu, \nu) \left(\frac{\lambda L}{|a+b|} \right) \left(\sum_{i=1}^n (i|a|^q + n|b|^q)^{p/q} \right)^{1/p}. \end{aligned}$$

Thus, we conclude that

$$d_p((\Lambda\mu), (\Lambda\nu)) \leq \sigma d_p(\mu, \nu),$$

with

$$\sigma := \left(\frac{\lambda L}{|a+b|} \right) \left(\sum_{i=1}^n (i|a|^q + n|b|^q)^{p/q} \right)^{1/p} < 1.$$

Hence, Λ is a contraction mapping on \mathbb{R}^{n+1} with respect to the d_p metric and so, all of the conditions of Banach's fixed point theorem are achieved. Therefore, there is a unique $\mu \in \mathbb{R}^{n+1}$ such that $\Lambda\mu = \mu$, yielding the existence of a unique solution to problem (1). \square

As a special case, for $p = q = 2$, we have the following result.

Corollary 3.1 *Let $\phi : [0, N] \times \mathbb{R} \rightarrow \mathbb{R}$ be continuous and $a + b \neq 0$. If there exist $L > 0$ and $\lambda \in (0, N)$ such that*

$$\begin{aligned} |\phi(t, \mu) - \phi(t, \nu)| &\leq L|\mu - \nu|, \quad \forall t \in [0, N], \forall (\mu, \nu) \in \mathbb{R}^2; \\ L\sqrt{N^2 + \lambda N} &< \frac{\sqrt{2}|a+b|}{\sqrt{a^2 + b^2}}, \quad \wp \in (0, 1), \end{aligned}$$

then problem (1) has a unique solution.

Example 3.1 Consider the problem

$$\begin{aligned} \frac{\nabla^{0.5}\mu}{\ell^{0.5}} &= \frac{1}{2}(t + \mu), \quad N = 1 \\ (a\mu_0 + b\mu_n &= 0, a + b \neq 0), \end{aligned} \tag{6}$$

where $0 < \ell = 1/4 < 1$. For $\lambda = |\frac{\ell^{\wp}}{2}| \approx 0.25$, $a = b = 1$, then the condition of Corollary 3.1 is satisfied for

$$L\sqrt{N^2 + \lambda N} = 0.56 < \frac{\sqrt{2}|a+b|}{\sqrt{a^2 + b^2}} = 2, \quad L = 0.5.$$

Hence problem (6) has a unique solution.

Theorem 3.3 Let $\phi : [0, N] \times \mathbb{R} \rightarrow \mathbb{R}$ be continuous and $a + b \neq 0$. Assume that there exist constants $L > 0, \lambda \in (0, N)$. If

$$|\phi(t, \mu) - \phi(t, \nu)| \leq L|\mu - \nu|, \quad \forall t \in [0, N], (\mu, \nu) \in \mathbb{R}^2$$

and

$$\lambda LN\bar{\Phi} < 1,$$

then problem (1) has a unique solution for each $|\frac{\ell^{\wp}}{a+b}| \leq \lambda$.

Proof Consider the metric space $(\Xi := \mathbb{R}^{n+1}, \partial)$ and for all $\mu, \nu \in \Xi$,

$$\partial(\mu, \nu) = \max_i |\mu_i - \nu_i|.$$

It is easy to show that (Ξ, ∂) is a complete metric space. Operator (3) yields

$$\begin{aligned} |(\Lambda\mu) - (\Lambda\nu)| &\leq \left| \frac{\ell^{\wp}}{a+b} \right| \sum_{j=0}^{n-1} |\Phi(i, j)| |\phi(t_j, \mu_j) - \phi(t_j, \nu_j)| \\ &\leq \left| \frac{\ell^{\wp}}{a+b} \right| L \sum_{j=0}^{n-1} |\Phi(i, j)| |\mu_j - \nu_j| \\ &\leq \left| \frac{\ell^{\wp}}{a+b} \right| L \sum_{j=0}^{n-1} |\Phi(i, j)| |\mu_j - \nu_j| \\ &\leq \left| \frac{\ell^{\wp}}{a+b} \right| L\bar{\Phi} \sum_{j=0}^{n-1} |\mu_j - \nu_j|. \end{aligned}$$

Taking the max, we obtain

$$\partial(\Lambda\mu, \Lambda\nu) \leq \lambda LN\bar{\Phi}\partial(\mu, \nu).$$

Thus, we conclude that

$$\partial(\Lambda\mu, \Lambda\nu) \leq \sigma\partial(\mu, \nu)$$

with

$$\sigma := \lambda LN\bar{\Phi} < 1.$$

Hence, Λ is a contraction mapping on \mathbb{R}^{n+1} with respect to the ∂ metric and so, all of the conditions of Banach’s fixed point theorem are satisfied. Hence, there is a unique $\mu \in \mathbb{R}^{n+1}$ such that $\Lambda\mu = \mu$, implying the existence of a unique solution to problem (1). \square

Example 3.2 Consider problem (6) with

$$(a\mu_0 + b\mu_n = 0, a + b \neq 0)$$

and $0 < \ell = 1/4 < 1$. For $\lambda = 0.25, N = a = b = 1$, then the condition of Theorem 3.3 is achieved for

$$LN\lambda\bar{\Phi} = 0.125 < 1, \quad \bar{\Phi} = 1, L = 0.5.$$

Hence problem (6) has a unique solution.

Example 3.3 Assume problem (6) with

$$(a\mu_0 + b\mu_n = 0, a + b \neq 0)$$

and $0 < \ell = 1/4 < 1$. If $\lambda = 0.9, N = 1, a = -1, b = 2$, then the condition of Theorem 3.3 is satisfied for

$$LN\lambda\bar{\Phi} = 0.9 < 1, \quad \bar{\Phi} = 2, L = 0.5.$$

Hence, problem (6) has a unique solution. It is interesting to note that the condition of Corollary 3.1 does not hold for

$$L\sqrt{N^2 + \lambda N} = 0.6892 > \frac{\sqrt{2}|a + b|}{\sqrt{a^2 + b^2}} = 0.632.$$

Next, we study a boundary value problem that includes a nonlinear difference equation of Volterra type of order \wp . Let $\phi : [0, N] \times \mathbb{R} \rightarrow \mathbb{R}$ be continuous and consider the discrete BVP

$$\frac{\nabla^\wp \mu_{i+1}}{\ell^\wp} = \phi(t_i, \mu_i) + \sum_{k=0}^{i-1} f(t_i, t_k; \mu_k) \tag{7}$$

$$(a\mu_0 + b\mu_n = c, a + b \neq 0),$$

where a, b, c are constants, $0 < \ell = N/n < N, t_i = i\ell$, and ϕ, f are any two functions defined for $i, k \in \mathbb{N}_0, k \leq i$. The solution of problem (7) can be expressed as follows:

$$\mu_i = \frac{c}{a + b} + \frac{\ell^\wp}{a + b} \sum_{j=0}^{n-1} \Phi(i, j) \left[\phi(t_j, \mu_j) + \sum_{k=0}^{j-1} f(t_j, t_k; \mu_k) \right], \quad i = 0, 1, \dots, n, \tag{8}$$

where

$$\Phi(i, j) := \begin{cases} \beta(i - 1, \wp; j) - b\beta(n - 1, \wp; j), & j \leq i - 1, \\ b\beta(n - 1, \wp; j), & j \geq i. \end{cases}$$

We have the following existence result.

Theorem 3.4 Let $\phi : [0, N] \times \mathbb{R} \rightarrow \mathbb{R}$ and $f : [0, N] \times [0, N] \times \mathbb{R} \rightarrow \mathbb{R}$ be continuous and $a + b \neq 0$. Assume that there exist constants $L_1 > 0, L_2 > 0, \lambda \in (0, N); p > 1, q > 1$ such that

$\frac{1}{p} + \frac{1}{q} = 1$. If

$$|\phi(t, \mu) - \phi(t, \nu)| \leq L_1 |\mu - \nu|, \quad \forall t \in [0, N], (\mu, \nu) \in \mathbb{R}^2,$$

$$\sum_{k=0}^{j-1} |f(t_j, t_k; \mu_k) - f(t_j, t_k; \nu_k)| \leq L_2 |\mu - \nu|, \quad \forall t \in [0, N], (\mu, \nu) \in \mathbb{R}^2$$

and

$$\left(\frac{\lambda(L_1 + L_2)}{|a + b|}\right) \left(\sum_{i=1}^n (i|a|^q + n|b|^q)^{p/q}\right)^{1/p} < 1, \quad |a| \geq 1,$$

then problem (7) has a unique solution for each $0 < \ell^\wp \leq \lambda$.

Proof Consider the complete metric space $(\Xi := \mathbb{R}^{n+1}, d_p)$ and for all $\mu, \nu \in \Xi$,

$$d_p(\mu, \nu) = \sum_{i=1}^n (|\mu_i - \nu_i|^p)^{1/p}.$$

Our aim is to prove that (7) has a unique solution. We define the operator

$$(\Upsilon\mu)(i) = \frac{c}{a + b} + \frac{\ell^\wp}{a + b} \sum_{j=0}^{n-1} \Phi(i, j) \left[\phi(t_j, \mu_j) + \sum_{k=0}^{j-1} f(t_j, t_k; \mu_k) \right]$$

$$(i = 0, 1, \dots, n, 0 < \wp < 1) \tag{9}$$

such that

$$(\Upsilon\mu) = ((\Upsilon\mu)(0), \dots, (\Upsilon\mu)(n)).$$

Thus the operator (9) implies

$$|(\Upsilon\mu) - (\Upsilon\nu)| \leq \left| \frac{\ell^\wp}{a + b} \right| \sum_{j=0}^{n-1} |\Phi(i, j)|$$

$$\times \left[|\phi(t_j, \mu_j) - \phi(t_j, \nu_j)| + \sum_{k=0}^{j-1} |f(t_j, t_k; \mu_k) - f(t_j, t_k; \nu_k)| \right]$$

$$\leq \left| \frac{\ell^\wp}{a + b} \right| (L_1 + L_2) \sum_{j=0}^{n-1} |\Phi(i, j)| |\mu_j - \nu_j|, \quad \wp \in (0, 1).$$

By employing Hölder’s inequality, we receive

$$|(\Upsilon\mu) - (\Upsilon\nu)| \leq \left| \frac{\ell^\wp}{a + b} \right| (L_1 + L_2) \left(\sum_{j=0}^{n-1} |\mu_j - \nu_j|^p\right)^{1/p} \left(\sum_{j=0}^{n-1} |\Phi(i, j)|^q\right)^{1/q}$$

$$\leq \left| \frac{\ell^\wp}{a + b} \right| (L_1 + L_2) \left(\sum_{j=0}^{n-1} |\mu_j - \nu_j|^p\right)^{1/p} \left(\sum_{j=0}^{n-1} |\Phi(i, j)|^q\right)^{1/q}. \tag{10}$$

By applying the boundary condition and the fact that $\beta(m, \wp; j) < 1$,

$$\left(\sum_{j=0}^{n-1} |\Phi(i, j)|^q \right) \leq i|a|^q + n|b|^q, \quad i = 0, 1, \dots, n, \tag{11}$$

and utilizing (10), we obtain

$$|(\Upsilon\mu) - (\Upsilon\nu)| \leq d_p(\mu_i, \nu_i) \frac{\ell^{\wp}(L_1 + L_2)}{|a + b|} (i|a|^q + (n - i)|b|^q)^{1/q}.$$

Consequently, this leads to

$$|(\Upsilon\mu) - (\Upsilon\nu)|^p \leq d_p^p(\mu_i, \nu_i) \left(\frac{\ell^{\wp}(L_1 + L_2)}{|a + b|} \right)^p (i|a|^q + n|b|^q)^{p/q}.$$

Now, for all $u, v \in \mathbb{R}^{n+1}$, we have

$$\begin{aligned} d_p((\Upsilon\mu), (\Upsilon\nu)) &= \left(\sum_{i=1}^n |(\Upsilon\mu)(i) - (\Upsilon\nu)(i)|^p \right)^{1/p} \\ &\leq d_p(\mu, \nu) \left(\frac{\ell^{\wp}(L_1 + L_2)}{|a + b|} \right) \left(\sum_{i=1}^n (i|a|^q + n|b|^q)^{p/q} \right)^{1/p} \\ &\leq d_p(\mu, \nu) \left(\frac{\lambda(L_1 + L_2)}{|a + b|} \right) \left(\sum_{i=1}^n (i|a|^q + n|b|^q)^{p/q} \right)^{1/p}. \end{aligned}$$

Thus, we conclude that

$$d_p((\Upsilon\mu), (\Upsilon\nu)) \leq \sigma d_p(\mu, \nu),$$

with

$$\sigma := \left(\frac{\lambda(L_1 + L_2)}{|a + b|} \right) \left(\sum_{i=1}^n (i|a|^q + n|b|^q)^{p/q} \right)^{1/p} < 1.$$

Hence, Υ is a contraction mapping on \mathbb{R}^{n+1} with respect to the d_p metric and so, all of the conditions of Banach’s fixed point theorem are satisfied. Therefore, there is a unique $\mu \in \mathbb{R}^{n+1}$ such that $\Upsilon\mu = \mu$, yielding the existence of a unique solution to problem (7). \square

Competing interests

The authors declare that there is no conflict of interests regarding the publication of this article.

Authors’ contributions

Both authors jointly worked on deriving the results and approved the final manuscript.

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