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Oscillation and asymptotic behavior of third-order neutral differential equations with distributed deviating arguments

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Abstract

By employing a generalized Riccati transformation and integral averaging technique, two Philos-type criteria are obtained which ensure that every solution of a class of third-order neutral differential equations with distributed deviating arguments is either oscillatory or converges to zero. These results extend and improve related criteria reported in the literature. Two illustrative examples are provided.

MSC: 34K11

Keywords: oscillation; asymptotic behavior; third-order neutral differential equation; distributed deviating argument; generalized Riccati transformation

1 Introduction

Differential equations with distributed deviating arguments are often used for modeling various problems arising in the engineering and natural sciences. Therefore, analysis of qualitative properties of solutions to such equations is crucial for applications; see Wang [1]. On the basis of these background details, we investigate the oscillation and asymptotic behavior of a third-order neutral differential equation with distributed deviating arguments

$$\left[r(t)\left(\left[x(t)+\int_a^b p(t,\xi)x\bigl(\tau(t,\xi)\bigr)\,d\xi\right]''\right)^{\alpha}\right]'+\int_c^d q(t,\xi)f\bigl(x\bigl(\sigma(t,\xi)\bigr)\bigr)\,d\xi=0,\quad t\ge t_0,\tag{1.1}$$

where $\alpha \ge 1$ is the ratio of odd positive integers. Throughout, we suppose that the following assumptions hold.

(A_1) $r(t)\in C^1([t_0,\infty),(0,\infty))$, $r'(t)\ge 0$, $\int_{t_0}^{\infty} r^{-1/\alpha}(s)\,ds=\infty$;
(A_2) $p(t,\xi)\in C([t_0,\infty)\times[a,b],[0,\infty))$, $0\le\int_a^b p(t,\xi)\,d\xi\le P<1$;
(A_3) $\tau(t,\xi)\in C([t_0,\infty)\times[a,b],R)$ is a nondecreasing function for ξ satisfying $\tau(t,\xi)\le t$ and $\liminf_{t\to\infty}\tau(t,\xi)=\infty$ for $\xi\in[a,b]$;
(A_4) $q(t,\xi)\in C([t_0,\infty)\times[c,d],[0,\infty))$;
(A_5) $\sigma(t,\xi)\in C([t_0,\infty)\times[c,d],R)$ is a nondecreasing function for ξ satisfying $\sigma(t,\xi)\le t$ and $\liminf_{t\to\infty}\sigma(t,\xi)=\infty$ for $\xi\in[c,d]$;

(A_6) $f(x) \in C(R,R)$ and there exists a positive constant K such that $f(x)/x^\alpha \geq K$ for all $x \neq 0$.

Define a new function $z(t)$ by

$$z(t) = x(t) + \int_a^b p(t,\xi)x\bigl(\tau(t,\xi)\bigr)\,d\xi.$$

By a solution of (1.1) we mean a nontrivial function $x(t) \in C([T_x,\infty),R)$, $T_x \geq t_0$, which has the properties $z(t) \in C^2([T_x,\infty),R)$ and $r(t)(z''(t))^\alpha \in C^1([T_x,\infty),R)$ for $T_x \geq t_0$. Our attention is restricted to those solutions of (1.1) which satisfy $\sup\{|x(t)| : t \geq T\} > 0$ for any $T \geq T_x$. A solution $x(t)$ of (1.1) is said to be oscillatory on $[T_x,\infty)$ if it is neither eventually positive nor eventually negative. Otherwise it is called nonoscillatory. Equation (1.1) is called oscillatory if all its solutions are oscillatory.

Recently, there has been much research activity concerning the oscillation and asymptotic properties of various classes of differential equations; see, *e.g.*, [1–21] and the references cited therein. So far, there are few results dealing with the asymptotic behavior of third-order neutral differential equations with distributed deviating arguments, we refer the reader to [17, 19]. The third-order neutral differential equation

$$\bigl[r(t)\bigl(\bigl[x(t) + p(t)x\bigl(\tau(t)\bigr)\bigr]''\bigr)^\alpha\bigr]' + q(t)f\bigl(x\bigl(\sigma(t)\bigr)\bigr) = 0$$

and its special cases have been studied by Baculíková and Džurina [5, 6], Candan [7], Grace *et al.* [9], Jiang and Li [10], and Li *et al.* [14]. Using Riccati transformation, Zhang *et al.* [19] considered a class of third-order neutral differential equations

$$\left[r(t)\left(x(t) + \int_a^b p(t,\xi)x\bigl(\tau(t,\xi)\bigr)\,d\xi\right)''\right]' + \int_c^d q(t,\xi)f\bigl(x\bigl(\sigma(t,\xi)\bigr)\bigr)\,d\xi = 0, \tag{1.2}$$

and they obtained several Philos-type (see [15]) criteria for (1.2), whereas Şenel and Utku [17] studied (1.1).

In the study of oscillation of differential equations, there are two techniques which are used to reduce the higher-order equations to the first-order Riccati equations (or inequalities). One of them is the Riccati transformation technique which has been recently extended to dynamic equations on time scales; see, *e.g.*, Şenel and Utku [17]. The other one is termed the generalized Riccati transformation technique; we refer the reader to Li [11], Li *et al.* [12], Li and Saker [13], and the related references cited therein. In particular, Li [11] used the generalized Riccati substitution and established several oscillation criteria for a second-order ordinary differential equation

$$\bigl(r(t)x'(t)\bigr)' + q(t)x(t) = 0.$$

Furthermore, he proved that the equation

$$\left(\frac{1}{t}x'(t)\right)' + \frac{1}{t^3}x(t) = 0$$

is oscillatory and showed that the results established by the Riccati transformation technique cannot be applied.

In the special case when $\alpha = 1$, (1.1) reduces to (1.2). Now the following question arises. Could we obtain new Philos-type oscillation criteria for (1.1) by using a generalized Riccati transformation which differs from that of [19]? Motivated by Li [11], Li *et al.* [12], and Li and Saker [13], our purpose in this paper is to give a positive answer to this question. In Section 2, four lemmas are given to prove the main results. In Section 3, we establish two Philos-type theorems for (1.1). In Section 4, two examples and some conclusions are presented to illustrate the main results. As customary, all functional inequalities considered in this paper are supposed to hold for all t large enough.

2 Some lemmas

Lemma 2.1 *Suppose that conditions* (A_1)-(A_6) *are satisfied and let* $x(t)$ *be a positive solution of* (1.1). *Then* $z(t)$ *has only one of the following two properties*:

(I) $z(t) > 0, z'(t) > 0, z''(t) > 0, z'''(t) \leq 0$;

(II) $z(t) > 0, z'(t) < 0, z''(t) > 0, z'''(t) \leq 0$,

for $t \geq t_1$, *where* $t_1 \geq t_0$ *is sufficiently large.*

Proof Assume that $x(t)$ is a positive solution of (1.1). Then there exists a $t_1 \geq t_0$ such that, for $t \geq t_1$,

$$x(t) > 0, \qquad x\bigl(\tau(t,\xi)\bigr) > 0, \quad \xi \in [a,b], \quad \text{and} \quad x\bigl(\sigma(t,\xi)\bigr) > 0, \quad \xi \in [c,d].$$

From (1.1) and the definition of $z(t)$, we have $z(t) > 0$ and

$$\bigl[r(t)\bigl(z''(t)\bigr)^{\alpha}\bigr]' = -\int_c^d q(t,\xi)f\bigl(x\bigl(\sigma(t,\xi)\bigr)\bigr)\,d\xi \leq 0.$$

Thus $r(t)(z''(t))^{\alpha}$ is nonincreasing and of one sign. Therefore, $z''(t)$ is also of one sign and so we have two possibilities: $z''(t) < 0$ or $z''(t) > 0$ for $t \geq t_2 \geq t_1$. We assert that $z''(t) > 0$ for $t \geq t_2$. Otherwise, there exists a constant $M > 0$ such that, for $t \geq t_2$,

$$z''(t) \leq -M^{\frac{1}{\alpha}} \frac{1}{r^{\frac{1}{\alpha}}(t)} < 0.$$

Integrating this inequality from t_2 to t, we obtain

$$z'(t) \leq z'(t_2) - M^{\frac{1}{\alpha}} \int_{t_2}^{t} \frac{1}{r^{\frac{1}{\alpha}}(s)}\,ds.$$

Letting $t \to \infty$ and using (A_1), we get $\lim_{t\to\infty} z'(t) = -\infty$. Thus $z'(t) < 0$ eventually. But conditions $z''(t) < 0$ and $z'(t) < 0$ imply that $z(t) < 0$, which contradicts our assumption $z(t) > 0$. Hence, $z''(t) > 0$ for $t \geq t_2$. Furthermore, we have, for $t \geq t_2$,

$$\bigl[r(t)\bigl(z''(t)\bigr)^{\alpha}\bigr]' = r'(t)\bigl(z''(t)\bigr)^{\alpha} + \alpha r(t)\bigl(z''(t)\bigr)^{\alpha-1}z'''(t) \leq 0.$$

This yields $z'''(t) \leq 0$ for $t \geq t_2$ due to condition (A_1). Therefore, $z(t)$ has only one of the two properties (I) and (II). This completes the proof. □

Lemma 2.2 *Let $x(t)$ be a positive solution of* (1.1) *and assume that corresponding $z(t)$ has the property* (II). *If*

$$\int_{t_0}^{\infty}\int_{v}^{\infty}\left[\frac{1}{r(u)}\int_{u}^{\infty}\int_{c}^{d}q(s,\xi)\,d\xi\,ds\right]^{\frac{1}{\alpha}}du\,dv=\infty, \tag{2.1}$$

then $\lim_{t\to\infty}x(t)=0$.

Proof Let $x(t)$ be a positive solution of (1.1). Since $z(t)$ has the property (II), there exists a finite constant $l\geq 0$ such that $\lim_{t\to\infty}z(t)=l\geq 0$. We prove that $l=0$. Assume now that $l>0$. Then we have $l+\varepsilon>z(t)>l$ for all $\varepsilon>0$. Choose $0<\varepsilon<l(1-P)/P$. It is easy to verify that

$$\begin{aligned}x(t)&=z(t)-\int_a^b p(t,\xi)x\bigl(\tau(t,\xi)\bigr)\,d\xi\\&\geq l-\int_a^b p(t,\xi)z\bigl(\tau(t,\xi)\bigr)\,d\xi\geq l-z\bigl(\tau(t,a)\bigr)\int_a^b p(t,\xi)\,d\xi\\&\geq l-P(l+\varepsilon)=N(l+\varepsilon)>Nz(t),\end{aligned}\tag{2.2}$$

where $N=(l-P(l+\varepsilon))/(l+\varepsilon)>0$. Using ($A_6$) and (2.2), we conclude that

$$\bigl[r(t)\bigl(z''(t)\bigr)^{\alpha}\bigr]'\leq -KN^{\alpha}\int_c^d q(t,\xi)z^{\alpha}\bigl(\sigma(t,\xi)\bigr)\,d\xi.$$

Noting that $z(t)$ has the property (II) and using (A_5), we have

$$\bigl[r(t)\bigl(z''(t)\bigr)^{\alpha}\bigr]'\leq -KN^{\alpha}z^{\alpha}\bigl(\sigma(t,d)\bigr)\int_c^d q(t,\xi)\,d\xi=-q_1(t)z^{\alpha}\bigl(\sigma_1(t)\bigr), \tag{2.3}$$

where $q_1(t)=KN^{\alpha}\int_c^d q(t,\xi)\,d\xi$ and $\sigma_1(t)=\sigma(t,d)$. Integrating inequality (2.3) from t to ∞, we obtain

$$r(t)\bigl(z''(t)\bigr)^{\alpha}\geq\int_t^{\infty}q_1(s)z^{\alpha}\bigl(\sigma_1(s)\bigr)\,ds.$$

By virtue of $z(\sigma_1(t))\geq l$,

$$z''(t)\geq l\left[\frac{1}{r(t)}\int_t^{\infty}q_1(s)\,ds\right]^{\frac{1}{\alpha}}. \tag{2.4}$$

Integrating inequality (2.4) from t to ∞, we have

$$-z'(t)\geq l\int_t^{\infty}\left[\frac{1}{r(u)}\int_u^{\infty}q_1(s)\,ds\right]^{\frac{1}{\alpha}}du.$$

Integrating the latter inequality from t_1 to ∞, we obtain

$$z(t_1)\geq l\int_{t_1}^{\infty}\int_v^{\infty}\left[\frac{1}{r(u)}\int_u^{\infty}q_1(s)\,ds\right]^{\frac{1}{\alpha}}du\,dv,$$

which contradicts (2.1). Hence $l = 0$ and $\lim_{t\to\infty} z(t) = 0$. Then it follows from $0 \le x(t) \le z(t)$ that $\lim_{t\to\infty} x(t) = 0$. The proof is complete. □

Lemma 2.3 ([5], Lemma 3) *Assume that $u(t) > 0$, $u'(t) > 0$, and $u''(t) < 0$ for $t \ge t_0$. If $\sigma(t) \in C([t_0,\infty),[0,\infty))$, $\sigma(t) \le t$, and $\lim_{t\to\infty}\sigma(t) = \infty$, then, for every $\beta \in (0,1)$, there exists a $T_\beta \ge t_0$ such that, for $t \ge T_\beta$,*

$$u\big(\sigma(t)\big) \ge \beta \frac{\sigma(t)}{t} u(t).$$

Lemma 2.4 *Assume that $u(t) > 0$, $u'(t) > 0$, $u''(t) > 0$, and $u'''(t) \le 0$ for $t \ge t_0$. Then, for every $\gamma \in (0,1)$, there exists a $T_\gamma \ge t_0$ such that, for $t \ge T_\gamma$,*

$$u(t) \ge \frac{1}{2}\gamma t u'(t).$$

Proof The proof is similar to that of Baculíková and Džurina ([5], Lemma 4), and hence it is omitted. □

3 Main results

Let

$$D = \big\{(t,s) \in R^2 : t \ge s \ge t_0\big\} \quad \text{and} \quad D_0 = \big\{(t,s) \in R^2 : t > s \ge t_0\big\}.$$

The function $H(t,s) \in C(D,R)$ is said to belong to the class X (denoted by $H \in X$) if it satisfies

(i) $H(t,t) = 0$, $t \ge t_0$, $H(t,s) > 0$, $(t,s) \in D_0$;

(ii) $\partial H(t,s)/\partial s \le 0$, there exist $\rho(t) \in C^1([t_0,\infty),(0,\infty))$, $b(t) \in C^1([t_0,\infty),[0,\infty))$, and $h(t,s) \in C(D_0,R)$ satisfying

$$-\frac{\partial H(t,s)}{\partial s} = H(t,s)\left[\frac{\rho'(s)}{\rho(s)} + (\alpha+1)b^{\frac{1}{\alpha}}(s)\right] + h(t,s).$$

Theorem 3.1 *Assume that conditions* (A_1)-(A_6) *and* (2.1) *are satisfied. If there exists a function $H \in X$ such that, for some $\beta \in (0,1)$ and $\gamma \in (0,1)$,*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left[H(t,s)\psi(s) - \frac{1}{(\alpha+1)^{\alpha+1}} \frac{\rho(s)r(s)|h(t,s)|^{\alpha+1}}{H^\alpha(t,s)}\right] ds = \infty, \tag{3.1}$$

where $\sigma_2(t) = \sigma(t,c)$ and

$$\begin{aligned}\psi(t) &= K(1-P)^\alpha \rho(t)\left(\frac{1}{2}\beta\gamma \frac{\sigma_2^2(t)}{t}\right)^\alpha \int_c^d q(t,\xi)\,d\xi \\ &\quad + \rho(t)r(t)b^{1+\frac{1}{\alpha}}(t) - \rho(t)\big(r(t)b(t)\big)',\end{aligned} \tag{3.2}$$

then every solution $x(t)$ of (1.1) *is either oscillatory or satisfies $\lim_{t\to\infty} x(t) = 0$.*

Proof Assume that (1.1) has a nonoscillatory solution $x(t)$. Without loss of generality, we may assume that $x(t)$ is an eventually positive solution of (1.1). By Lemma 2.1, we observe that $z(t)$ satisfies either (I) or (II) for $t \ge t_1$. We consider each of two cases separately.

Suppose first that $z(t)$ has the property (I). Then we obtain

$$\begin{aligned} x(t) &= z(t) - \int_a^b p(t,\xi)x\bigl(\tau(t,\xi)\bigr)\,d\xi \\ &\geq z(t) - \int_a^b p(t,\xi)z\bigl(\tau(t,\xi)\bigr)\,d\xi \geq z(t) - z\bigl(\tau(t,b)\bigr)\int_a^b p(t,\xi)\,d\xi \\ &\geq \left(1 - \int_a^b p(t,\xi)\,d\xi\right)z(t) \geq (1-P)z(t). \end{aligned} \tag{3.3}$$

Using (A_5), (A_6), and (3.3), we have

$$\begin{aligned} \bigl[r(t)\bigl(z''(t)\bigr)^\alpha\bigr]' &\leq -K\int_c^d q(t,\xi)x^\alpha\bigl(\sigma(t,\xi)\bigr)\,d\xi \\ &\leq -K(1-P)^\alpha \int_c^d q(t,\xi)z^\alpha\bigl(\sigma(t,\xi)\bigr)\,d\xi \\ &\leq -K(1-P)^\alpha z^\alpha\bigl(\sigma(t,c)\bigr)\int_c^d q(t,\xi)\,d\xi = -q_2(t)z^\alpha\bigl(\sigma_2(t)\bigr), \end{aligned} \tag{3.4}$$

where $q_2(t) = K(1-P)^\alpha \int_c^d q(t,\xi)\,d\xi$ and $\sigma_2(t) = \sigma(t,c)$. Define a generalized Riccati transformation $\omega(t)$ by

$$\omega(t) = \rho(t)\left[\frac{r(t)(z''(t))^\alpha}{(z'(t))^\alpha} + r(t)b(t)\right], \quad t \geq t_1. \tag{3.5}$$

Then we have $\omega(t) > 0$ and

$$\begin{aligned} \omega'(t) &= \rho'(t)\left[\frac{r(t)(z''(t))^\alpha}{(z'(t))^\alpha} + r(t)b(t)\right] + \rho(t)\left[\frac{r(t)(z''(t))^\alpha}{(z'(t))^\alpha} + r(t)b(t)\right]' \\ &= \frac{\rho'(t)}{\rho(t)}\omega(t) + \rho(t)\bigl(r(t)b(t)\bigr)' + \rho(t)\left[\frac{r(t)(z''(t))^\alpha}{(z'(t))^\alpha}\right]' \\ &= \frac{\rho'(t)}{\rho(t)}\omega(t) + \rho(t)\bigl(r(t)b(t)\bigr)' + \rho(t)\frac{[r(t)(z''(t))^\alpha]'}{(z'(t))^\alpha} - \alpha\rho(t)r(t)\left(\frac{z''(t)}{z'(t)}\right)^{\alpha+1}. \end{aligned} \tag{3.6}$$

By virtue of (3.5), we conclude that

$$\frac{z''(t)}{z'(t)} = \frac{1}{r^{\frac{1}{\alpha}}(t)}\left(\frac{\omega(t)}{\rho(t)} - r(t)b(t)\right)^{\frac{1}{\alpha}}. \tag{3.7}$$

Combining (3.4), (3.6), and (3.7), we have

$$\begin{aligned} \omega'(t) &\leq \frac{\rho'(t)}{\rho(t)}\omega(t) + \rho(t)\bigl(r(t)b(t)\bigr)' - \rho(t)q_2(t)\frac{z^\alpha(\sigma_2(t))}{(z'(t))^\alpha} \\ &\quad - \frac{\alpha\rho(t)}{r^{\frac{1}{\alpha}}(t)}\left(\frac{\omega(t)}{\rho(t)} - r(t)b(t)\right)^{1+\frac{1}{\alpha}}. \end{aligned} \tag{3.8}$$

Using Lemma 2.4, for every $\gamma \in (0,1)$, there exists a $T_\gamma \geq t_1$ such that, for $t \geq T_\gamma$,

$$z\bigl(\sigma_2(t)\bigr) \geq \frac{1}{2}\gamma\sigma_2(t)z'\bigl(\sigma_2(t)\bigr). \tag{3.9}$$

From Lemma 2.3, for every $\beta \in (0,1)$, there exists a $T_\beta \geq T_\gamma$ such that, for $t \geq T_\beta$,

$$\frac{1}{z'(t)} \geq \frac{\beta\sigma_2(t)}{tz'(\sigma_2(t))}. \tag{3.10}$$

Define

$$A^* = \frac{\omega(t)}{\rho(t)} \quad \text{and} \quad B^* = r(t)b(t).$$

Using the inequality (see [13])

$$(A^*)^{1+\frac{1}{\alpha}} - (A^* - B^*)^{1+\frac{1}{\alpha}} \leq (B^*)^{\frac{1}{\alpha}}\left[\left(1+\frac{1}{\alpha}\right)A^* - \frac{1}{\alpha}B^*\right], \quad A^*B^* \geq 0, \alpha = \frac{odd}{odd} \geq 1,$$

we have

$$\left(\frac{\omega(t)}{\rho(t)} - r(t)b(t)\right)^{1+\frac{1}{\alpha}} \geq \frac{\omega^{1+\frac{1}{\alpha}}(t)}{\rho^{1+\frac{1}{\alpha}}(t)} + \frac{1}{\alpha}\big(r(t)b(t)\big)^{1+\frac{1}{\alpha}} - \left(1+\frac{1}{\alpha}\right)\frac{(r(t)b(t))^{\frac{1}{\alpha}}}{\rho(t)}\omega(t). \tag{3.11}$$

Using inequalities (3.8)-(3.11), for $t \geq T \geq T_\beta$, we have

$$\begin{aligned}
\omega'(t) &\leq \rho(t)\big(r(t)b(t)\big)' - \rho(t)q_2(t)\left(\frac{1}{2}\beta\gamma\frac{\sigma_2^2(t)}{t}\right)^\alpha - \rho(t)r(t)b^{1+\frac{1}{\alpha}}(t) \\
&\quad + \left[\frac{\rho'(t)}{\rho(t)} + (\alpha+1)b^{\frac{1}{\alpha}}(t)\right]\omega(t) - \frac{\alpha}{(\rho(t)r(t))^{\frac{1}{\alpha}}}\omega^{1+\frac{1}{\alpha}}(t) \\
&= -\psi(t) + A(t)\omega(t) - B(t)\omega^{1+\frac{1}{\alpha}}(t),
\end{aligned} \tag{3.12}$$

where $\psi(t)$ is defined as in (3.2), $A(t) = (\rho'(t)/\rho(t)) + (\alpha+1)b^{1/\alpha}(t)$, and $B(t) = \alpha/(\rho(t)r(t))^{1/\alpha}$. Multiplying inequality (3.12) by $H(t,s)$ and integrating the resulting inequality from T to t, we have

$$\begin{aligned}
\int_T^t H(t,s)\psi(s)\,ds &\leq \int_T^t H(t,s)\big(-\omega'(s) + A(s)\omega(s) - B(s)\omega^{1+\frac{1}{\alpha}}(s)\big)\,ds \\
&= H(t,T)\omega(T) + \int_T^t \left(\frac{\partial H(t,s)}{\partial s} + H(t,s)A(s)\right)\omega(s)\,ds \\
&\quad - \int_T^t H(t,s)B(s)\omega^{1+\frac{1}{\alpha}}(s)\,ds \\
&= H(t,T)\omega(T) - \int_T^t h(t,s)\omega(s)\,ds - \int_T^t H(t,s)B(s)\omega^{1+\frac{1}{\alpha}}(s)\,ds \\
&\leq H(t,T)\omega(T) + \int_T^t \big[\big|h(t,s)\big|\omega(s) - H(t,s)B(s)\omega^{1+\frac{1}{\alpha}}(s)\big]\,ds.
\end{aligned} \tag{3.13}$$

Letting $C = |h(t,s)|$, $D = H(t,s)B(s)$, and using the inequality (see [13])

$$C\omega - D\omega^{1+\frac{1}{\alpha}} \leq \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}}\frac{C^{\alpha+1}}{D^\alpha}, \quad D > 0,$$

we obtain

$$\int_T^t H(t,s)\psi(s)\,ds \le H(t,T)\omega(T) + \int_T^t \frac{1}{(\alpha+1)^{\alpha+1}} \frac{\rho(s)r(s)|h(t,s)|^{\alpha+1}}{H^\alpha(t,s)}\,ds.$$

Hence

$$\frac{1}{H(t,T)} \int_T^t \left[H(t,s)\psi(s) - \frac{1}{(\alpha+1)^{\alpha+1}} \frac{\rho(s)r(s)|h(t,s)|^{\alpha+1}}{H^\alpha(t,s)} \right] ds \le \omega(T) \tag{3.14}$$

for all sufficiently large t, which contradicts (3.1).

Assume now that $z(t)$ has the property (II). By Lemma 2.2, we have $\lim_{t\to\infty} x(t) = 0$. The proof is complete. □

It may happen that assumption (3.1) in Theorem 3.1 fails to hold. Consequently, Theorem 3.1 cannot be applied. The following theorem provides a new oscillation criterion for (1.1).

Theorem 3.2 *Let conditions* (A_1)-(A_6) *and* (2.1) *be satisfied. Assume that there exists a function* $H \in X$ *such that*

$$0 < \inf_{s\ge t_0} \left\{ \liminf_{t\to\infty} \frac{H(t,s)}{H(t,t_0)} \right\} \le \infty \tag{3.15}$$

and

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^t \frac{\rho(s)r(s)|h(t,s)|^{\alpha+1}}{H^\alpha(t,s)}\,ds < \infty \tag{3.16}$$

hold. If there exists a function $\varphi(t) \in C([t_0,\infty),R)$ *such that, for all* $T \ge t_0$,

$$\limsup_{t\to\infty} \int_{t_0}^t \rho^{-\frac{1}{\alpha}}(s) r^{-\frac{1}{\alpha}}(s) \left[\varphi_+(s)\right]^{\frac{\alpha+1}{\alpha}}\,ds = \infty \tag{3.17}$$

and

$$\limsup_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \left[H(t,s)\psi(s) - \frac{1}{(\alpha+1)^{\alpha+1}} \frac{\rho(s)r(s)|h(t,s)|^{\alpha+1}}{H^\alpha(t,s)} \right] ds \ge \varphi(T), \tag{3.18}$$

where $\psi(t)$ *is defined by* (3.2) *and* $\varphi_+(t) = \max\{\varphi(t), 0\}$, *then the conclusion of Theorem* 3.1 *remains intact.*

Proof Assuming that $z(t)$ has the property (I) and proceeding as in the proof of Theorem 3.1, we have (3.13) and (3.14) for all $t > T$. Hence, by virtue of (3.14),

$$\limsup_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \left[H(t,s)\psi(s) - \frac{1}{(\alpha+1)^{\alpha+1}} \frac{\rho(s)r(s)|h(t,s)|^{\alpha+1}}{H^\alpha(t,s)} \right] ds \le \omega(T) \tag{3.19}$$

for all $t > T$. Thus, by (3.18) and (3.19), we have

$$\varphi(T) \le \omega(T) \tag{3.20}$$

and

$$\limsup_{t\to\infty} \frac{1}{H(t,T)} \int_T^t H(t,s)\psi(s)\,ds \ge \varphi(T). \tag{3.21}$$

From (3.20), we obtain

$$\int_T^\infty \rho^{-\frac{1}{\alpha}}(s) r^{-\frac{1}{\alpha}}(s) \omega^{\frac{\alpha+1}{\alpha}}(s)\,ds \ge \int_T^\infty \rho^{-\frac{1}{\alpha}}(s) r^{-\frac{1}{\alpha}}(s) \big[\varphi_+(s)\big]^{\frac{\alpha+1}{\alpha}}\,ds \tag{3.22}$$

and hence, by (3.17),

$$\int_T^\infty \rho^{-\frac{1}{\alpha}}(s) r^{-\frac{1}{\alpha}}(s) \omega^{\frac{\alpha+1}{\alpha}}(s)\,ds = \infty. \tag{3.23}$$

To complete the proof, we show that (3.23) does not hold. Let

$$u(t) = \frac{\alpha}{H(t,T)} \int_T^t H(t,s) \rho^{-\frac{1}{\alpha}}(s) r^{-\frac{1}{\alpha}}(s) \omega^{\frac{\alpha+1}{\alpha}}(s)\,ds$$

and

$$v(t) = \frac{1}{H(t,T)} \int_T^t \big|h(t,s)\big| \omega(s)\,ds$$

for all $t > T$. It follows from (3.13) and (3.21) that

$$\begin{aligned} \liminf_{t\to\infty} \big[u(t) - v(t)\big] &\le \omega(T) - \limsup_{t\to\infty} \frac{1}{H(t,T)} \int_T^t H(t,s)\psi(s)\,ds \\ &\le \omega(T) - \varphi(T) < \infty. \end{aligned} \tag{3.24}$$

Now by (3.15), there exists a positive constant ξ_1 satisfying

$$\inf_{s\ge t_0} \left\{ \liminf_{t\to\infty} \frac{H(t,s)}{H(t,t_0)} \right\} > \xi_1 > 0. \tag{3.25}$$

Let ξ_2 be an any positive constant. It follows from (3.23) that, for all $t \ge T_1$,

$$\int_T^t \rho^{-\frac{1}{\alpha}}(s) r^{-\frac{1}{\alpha}}(s) \omega^{\frac{\alpha+1}{\alpha}}(s)\,ds \ge \frac{\xi_2}{\alpha \xi_1},$$

where $T_1 > T$ is sufficiently large. Therefore, for all $t \ge T_1$,

$$\begin{aligned} u(t) &= \frac{\alpha}{H(t,T)} \int_T^t H(t,s)\,d\left(\int_T^s \rho^{-\frac{1}{\alpha}}(\zeta) r^{-\frac{1}{\alpha}}(\zeta) \omega^{\frac{\alpha+1}{\alpha}}(\zeta)\,d\zeta \right) \\ &\ge \frac{\alpha}{H(t,T)} \int_{T_1}^t \left(-\frac{\partial H(t,s)}{\partial s} \right) \left(\int_T^s \rho^{-\frac{1}{\alpha}}(\zeta) r^{-\frac{1}{\alpha}}(\zeta) \omega^{\frac{\alpha+1}{\alpha}}(\zeta)\,d\zeta \right) ds \\ &\ge \frac{\xi_2}{\xi_1 H(t,T)} \int_{T_1}^t \left(-\frac{\partial H(t,s)}{\partial s} \right) ds = \frac{\xi_2 H(t,T_1)}{\xi_1 H(t,T)} \ge \frac{\xi_2 H(t,T_1)}{\xi_1 H(t,t_0)}. \end{aligned}$$

By (3.25), there exists a $T_2 \geq T_1$ such that $H(t, T_1)/H(t, t_0) \geq \xi_1$ for all $t \geq T_2$. Thus $u(t) \geq \xi_2$ for all $t \geq T_2$. Since ξ_2 is an arbitrary constant,

$$\lim_{t\to\infty} u(t) = \infty. \tag{3.26}$$

Consider now a sequence $\{t_i\}_{i=1}^{\infty}$ in (T, ∞) with $\lim_{i\to\infty} t_i = \infty$ such that

$$\lim_{i\to\infty} \bigl[u(t_i) - v(t_i)\bigr] = \liminf_{t\to\infty} \bigl[u(t) - v(t)\bigr] < \infty.$$

By virtue of (3.24), there exist a natural number N_0 and a constant $L > 0$ such that, for all $i > N_0$,

$$u(t_i) - v(t_i) \leq L. \tag{3.27}$$

It follows from (3.26) that

$$\lim_{i\to\infty} u(t_i) = \infty. \tag{3.28}$$

Combining (3.27) and (3.28), we conclude that

$$\lim_{i\to\infty} v(t_i) = \infty \tag{3.29}$$

and, for i large enough,

$$\frac{v(t_i)}{u(t_i)} > \frac{1}{2}. \tag{3.30}$$

From (3.29) and (3.30), we obtain

$$\lim_{i\to\infty} \frac{v^{\alpha+1}(t_i)}{u^{\alpha}(t_i)} = \infty. \tag{3.31}$$

On the other hand, by Hölder's inequality, we have

$$\begin{aligned} v(t_i) \leq{}& \left\{\frac{\alpha}{H(t_i, T)} \int_T^{t_i} H(t_i, s)\rho^{-\frac{1}{\alpha}}(s) r^{-\frac{1}{\alpha}}(s)\omega^{\frac{\alpha+1}{\alpha}}(s)\,ds\right\}^{\frac{\alpha}{\alpha+1}} \\ &\times \left\{\frac{1}{\alpha^{\alpha} H(t_i, T)} \int_T^{t_i} \frac{\rho(s)r(s)|h(t_i, s)|^{\alpha+1}}{H^{\alpha}(t_i, s)}\,ds\right\}^{\frac{1}{\alpha+1}}. \end{aligned}$$

Therefore, for all i large enough,

$$\frac{v^{\alpha+1}(t_i)}{u^{\alpha}(t_i)} \leq \frac{1}{\alpha^{\alpha}\xi_1 H(t_i, t_0)} \int_{t_0}^{t_i} \frac{\rho(s)r(s)|h(t_i, s)|^{\alpha+1}}{H^{\alpha}(t_i, s)}\,ds. \tag{3.32}$$

From (3.31) and (3.32), we deduce that

$$\lim_{i\to\infty} \frac{1}{H(t_i, t_0)} \int_{t_0}^{t_i} \frac{\rho(s)r(s)|h(t_i, s)|^{\alpha+1}}{H^{\alpha}(t_i, s)}\,ds = \infty,$$

so

$$\lim_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\frac{\rho(s)r(s)|h(t,s)|^{\alpha+1}}{H^{\alpha}(t,s)}\,ds=\infty,$$

which contradicts (3.16). Therefore, (3.23) cannot hold. By virtue of (3.22), we get

$$\int_T^{\infty}\rho^{-\frac{1}{\alpha}}(s)r^{-\frac{1}{\alpha}}(s)\big[\varphi_+(s)\big]^{\frac{\alpha+1}{\alpha}}\,ds<\infty,$$

which contradicts (3.17).

Suppose that $z(t)$ has the property (II). By Lemma 2.2, we obtain $\lim_{t\to\infty}x(t)=0$. This completes the proof. □

4 Examples and conclusions

Example 4.1 For $t\geq 1$, consider a third-order differential equation

$$\left[t^2\left(\left[x(t)+\int_{\frac{1}{2}}^{1}\frac{4\xi}{3t^2}x\left(\frac{t+\xi}{3}\right)d\xi\right]''\right)^3\right]'+\int_0^1\frac{32q_0\xi}{t^5}x^3\left(\frac{t+\xi}{2}\right)d\xi=0, \tag{4.1}$$

where $q_0>0$ is a constant. Let $\alpha=3$, $a=1/2$, $b=1$, $c=0$, $d=1$, $r(t)=t^2$, $p(t,\xi)=4\xi/(3t^2)$, $\tau(t,\xi)=(t+\xi)/3$, $q(t,\xi)=32q_0\xi/t^5$, and $\sigma(t,\xi)=(t+\xi)/2$. Then

$$\int_a^b p(t,\xi)\,d\xi=\int_{\frac{1}{2}}^{1}\frac{4\xi}{3t^2}\,d\xi=\frac{1}{2t^2}\leq\frac{1}{2}\quad\text{and}\quad\sigma_2(t)=\sigma(t,0)=\frac{t}{2}.$$

It is not difficult to verify that

$$\int_1^{\infty}\frac{1}{s^{\frac{2}{3}}}\,ds=\infty\quad\text{and}\quad\int_1^{\infty}\int_v^{\infty}\left[\frac{1}{u^2}\int_u^{\infty}\int_0^1\frac{32q_0\xi}{s^5}\,d\xi\,ds\right]^{\frac{1}{3}}du\,dv=\infty.$$

Therefore, the conditions (A_1)-(A_6) and (2.1) are satisfied. Furthermore, we choose $K=1$, $P=1/2$, $\rho(t)=t$, $b(t)=0$, and $H(t,s)=(t-s)^4$. Then $h(t,s)=(t-s)^3(5-ts^{-1})$,

$$\psi(t)=\left(1-\frac{1}{2}\right)^3 t\left(\frac{\beta\gamma(\frac{t}{2})^2}{2t}\right)^3\int_0^1\frac{32q_0\xi}{t^5}\,d\xi=\frac{q_0(\beta\gamma)^3}{256t},$$

and

$$\begin{aligned}
&\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\left[H(t,s)\psi(s)-\frac{1}{(\alpha+1)^{\alpha+1}}\frac{\rho(s)r(s)|h(t,s)|^{\alpha+1}}{H^{\alpha}(t,s)}\right]ds\\
&\quad=\limsup_{t\to\infty}\frac{1}{(t-1)^4}\int_1^t\left[\frac{q_0(\beta\gamma)^3}{256}(t-s)^4\frac{1}{s}-\frac{1}{256}s^3\left(5-ts^{-1}\right)^4\right]ds\\
&\quad=\limsup_{t\to\infty}\frac{1}{(t-1)^4}\int_1^t\left[\frac{q_0(\beta\gamma)^3}{256}\left(t^4s^{-1}-4t^3+6t^2s-4ts^2+s^3\right)\right.\\
&\qquad\left.-\left(\frac{1}{256}t^4s^{-1}-\frac{5}{64}t^3+\frac{75}{128}t^2s-\frac{125}{64}ts^2+\frac{625}{256}s^3\right)\right]ds=\infty,
\end{aligned}$$

if $q_0 > 1/(\beta\gamma)^3$ for some $\beta \in (0,1)$ and $\gamma \in (0,1)$. Hence, by Theorem 3.1, every solution $x(t)$ of (4.1) is either oscillatory or converges to zero as $t \to \infty$ in the case where $q_0 > 1{,}000{,}000/(531{,}441) \approx 1.9$ (by letting $\beta = \gamma = 9/10$). Observe that the results reported in [19] cannot be applied to (4.1) since $\alpha = 3$.

Example 4.2 For $t \geq 1$, consider a third-order differential equation

$$\left[x(t) + \int_{\frac{1}{2}}^{1} \frac{4\xi}{3t^2} x\left(\frac{t+\xi}{3}\right) d\xi\right]''' + \int_0^1 \frac{16q_0\xi}{t^3} x\left(\frac{t+\xi}{2}\right) d\xi = 0, \tag{4.2}$$

where $q_0 > 0$ is a constant. Let $\alpha = 1$, $a = 1/2$, $b = 1$, $c = 0$, $d = 1$, $r(t) = 1$, $p(t,\xi) = 4\xi/(3t^2)$, $\tau(t,\xi) = (t+\xi)/3$, $q(t,\xi) = 16q_0\xi/t^3$, and $\sigma(t,\xi) = (t+\xi)/2$. Then

$$\int_a^b p(t,\xi)\,d\xi = \int_{\frac{1}{2}}^{1} \frac{4\xi}{3t^2}\,d\xi = \frac{1}{2t^2} \leq \frac{1}{2} \quad \text{and} \quad \sigma_2(t) = \sigma(t,0) = \frac{t}{2}.$$

It is easy to verify that

$$\int_1^\infty \frac{1}{r(s)}\,ds = \infty \quad \text{and} \quad \int_1^\infty \int_v^\infty \left[\int_u^\infty \int_0^1 q(s,\xi)\,ds\right] du\,dv = \infty.$$

Therefore, the conditions (A_1)-(A_6) and (2.1) are satisfied. Furthermore, we choose $K = 1$, $P = 1/2$, $\rho(t) = t$, $b(t) = 1/t$, and $H(t,s) = (t-s)^2$. Then $h(t,s) = (t-s)(5 - 3ts^{-1})$,

$$\psi(t) = \left(1 - \frac{1}{2}\right) t \left(\frac{\beta\gamma(\frac{t}{2})^2}{2t}\right) \int_0^1 \frac{16q_0\xi}{t^3}\,d\xi + t\left(\frac{1}{t}\right)^2 - t\left(\frac{1}{t}\right)' = \left(\frac{1}{2}q_0\beta\gamma + 2\right)\frac{1}{t},$$

and

$$\begin{aligned}
&\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^t \left[H(t,s)\psi(s) - \frac{1}{(\alpha+1)^{\alpha+1}} \frac{\rho(s)r(s)|h(t,s)|^{\alpha+1}}{H^\alpha(t,s)}\right] ds \\
&\quad = \limsup_{t\to\infty} \frac{1}{(t-1)^2} \int_1^t \left[\left(\frac{1}{2}q_0\beta\gamma + 2\right)(t-s)^2\frac{1}{s} - \frac{1}{4}s\left(5 - 3ts^{-1}\right)^2\right] ds \\
&\quad = \limsup_{t\to\infty} \frac{1}{(t-1)^2} \int_1^t \left[\left(\frac{1}{2}q_0\beta\gamma + 2\right)\left(t^2s^{-1} - 2t + s\right) - \frac{1}{4}\left(25s - 30t + 9t^2s^{-1}\right)\right] ds \\
&\quad = \infty,
\end{aligned}$$

if $q_0 > 1/(2\beta\gamma)$ for some $\beta \in (0,1)$ and $\gamma \in (0,1)$. Therefore, by Theorem 3.1, every solution $x(t)$ of (4.2) is either oscillatory or converges to zero as $t \to \infty$ in the case where $q_0 > 50/81 \approx 0.62$ (by letting $\beta = \gamma = 9/10$).

Remark 4.1 With an appropriate choice of the function H, one can derive from Theorems 3.1 and 3.2 a number of oscillation criteria for (1.1). For example, consider a Kamenev-type function $H(t,s)$ by $H(t,s) = (t-s)^{n-1}$, $(t,s) \in D$, where $n > 2$ is an integer. The remainder of the details are left to the reader.

Remark 4.2 Theorems 3.1 and 3.2 reported in this paper reduce to ([19], Theorems 3.1 and 3.2), respectively, when letting $\alpha = 1$ and $b(t) = 0$.

Remark 4.3 Note that Theorems 3.1 and 3.2 ensure that every solution $x(t)$ to (1.1) is either oscillatory or satisfies $\lim_{t\to\infty} x(t) = 0$ and, unfortunately, these results cannot distinguish solutions with different behaviors. Since the sign of the derivative $z'(t)$ is not fixed, it is not easy to establish sufficient conditions which guarantee that all solutions to (1.1) are just oscillatory and do not satisfy $\lim_{t\to\infty} x(t) = 0$. Neither is it possible to use the technique exploited in this paper for proving that all solutions of (1.1) satisfy $\lim_{t\to\infty} x(t) = 0$. Hence, these two interesting problems are left for future research.

Remark 4.4 It would be interesting to find a different method to investigate (1.1) when $0 < \alpha < 1$. It would also be of interest to find another method to study (1.1) in the case where $\int_{t_0}^{\infty} r^{-1/\alpha}(s)\,ds < \infty$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All four authors contributed equally to this work. They all read and approved the final version of the manuscript.

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