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A note on degenerate poly-Bernoulli numbers and polynomials

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Abstract
In this paper, we consider the degenerate poly-Bernoulli polynomials and present new and explicit formulas for computing them in terms of the degenerate Bernoulli polynomials and Stirling numbers of the second kind.

MSC: 11B68; 11B73; 11B83

Keywords: degenerate poly-Bernoulli polynomial; degenerate Bernoulli polynomial; Stirling number of the second kind

1 Introduction

For $\lambda \in \mathbb{C}$, Carlitz considered the degenerate Bernoulli polynomials given by the generating function

$$\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty}\beta_n(x \mid \lambda)\frac{t^n}{n!} \quad (\text{see [1–3]}). \tag{1.1}$$

When $x = 0$, $\beta_n(\lambda) = \beta_n(0 \mid \lambda)$ are called the degenerate Bernoulli numbers.

Thus, by (1.1), we get

$$\beta_n(x \mid \lambda) = \sum_{l=0}^{n}\binom{n}{l}\beta_l(\lambda)(x \mid \lambda)_{n-l}, \tag{1.2}$$

where $(x \mid \lambda)_n = x(x-\lambda)(x-2\lambda)\cdots(x-\lambda(n-1))$.

The classical polylogarithm function Li_k is

$$\mathrm{Li}_k(x) = \sum_{n=1}^{\infty}\frac{x^n}{n^k} \quad (k \in \mathbb{Z}; \text{see [2, 4–11]}). \tag{1.3}$$

From (1.1), we note that

$$\begin{aligned}\sum_{n=0}^{\infty}\lim_{\lambda\to 0}\beta_n(x \mid \lambda)\frac{t^n}{n!} \\ = \lim_{\lambda\to 0}\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{x}{\lambda}}\end{aligned}$$

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$$= \frac{t}{e^t - 1} e^{xt}$$
$$= \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, \tag{1.4}$$

where $B_n(x)$ are called the Bernoulli polynomials (see [1–27]).

Thus, by (1.4), we get

$$\lim_{\lambda \to 0} \beta_n(x \mid \lambda) = B_n(x) \quad (n \geq 0). \tag{1.5}$$

In [4, 14], the poly-Bernoulli polynomials are given by

$$\frac{\mathrm{Li}_k(1 - e^{-t})}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n^{(k)}(x) \frac{t^n}{n!}. \tag{1.6}$$

For $k = 1$, we have

$$\frac{\mathrm{Li}_1(1 - e^{-t})}{e^t - 1} e^{xt} = \frac{t}{e^t - 1} e^{xt}$$
$$= \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}. \tag{1.7}$$

By (1.4) and (1.7), we get $B_n^{(1)}(x) = B_n(x)$.

The Stirling numbers of the second kind are given by

$$x^n = \sum_{l=0}^{n} S_2(n, l)(x)_l \quad (\text{see [1–27]}), \tag{1.8}$$

and the Stirling numbers of the first kind are defined by

$$(x)_n = x(x - 1) \cdots (x - n + 1) = \sum_{l=0}^{n} S_1(n, l) x^l \quad (n \geq 0). \tag{1.9}$$

The purpose of this paper is to construct the degenerate poly-Bernoulli polynomials and present new and explicit formulas for computing them in terms of the degenerate Bernoulli polynomials and Stirling numbers of the second kind.

2 Degenerate poly-Bernoulli numbers and polynomials

For $\lambda \in \mathbb{C}$, $k \in \mathbb{Z}$, we consider the degenerate poly-Bernoulli polynomials given by the generating function

$$\frac{\mathrm{Li}_k(1 - e^{-t})}{(1 + \lambda t)^{\frac{1}{\lambda}} - 1} (1 + \lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} \beta_n^{(k)}(x \mid \lambda) \frac{t^n}{n!}. \tag{2.1}$$

When $x = 0$, $\beta_n^{(k)}(\lambda) = \beta_n^{(k)}(0 \mid \lambda)$ are called the degenerate poly-Bernoulli numbers. Note that $\beta_n^{(1)}(x \mid \lambda) = \beta_n(x \mid \lambda)$ and $\lim_{\lambda \to 0} \beta_n^{(k)}(x \mid \lambda) = B_n^{(k)}(x)$.

From (2.1), we can derive the following equation:

$$\begin{aligned}\sum_{n=0}^{\infty}\beta_n^{(k)}(x\mid\lambda)\frac{t^n}{n!}&=\left(\frac{\mathrm{Li}_k(1-e^{-t})}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\right)(1+\lambda t)^{\frac{x}{\lambda}}\\&=\left(\sum_{l=0}^{\infty}\beta_l^{(k)}(\lambda)\frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty}(x\mid\lambda)_m\frac{t^m}{m!}\right)\\&=\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}\beta_l^{(k)}(\lambda)(x\mid\lambda)_{n-l}\right)\frac{t^n}{n!}.\end{aligned}\tag{2.2}$$

Thus, by (2.2), we get

$$\beta_n^{(k)}(x\mid\lambda)=\sum_{l=0}^{n}\binom{n}{l}\beta_l^{(k)}(\lambda)(x\mid\lambda)_{n-l}.\tag{2.3}$$

Now, we observe that

$$\begin{aligned}&\frac{\mathrm{Li}_k(1-e^{-t})}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+t)^{\frac{x}{\lambda}}\\&\quad=\sum_{n=0}^{\infty}\beta_n^{(k)}(x\mid\lambda)\frac{t^n}{n!}\\&\quad=\frac{(1+t)^{\frac{x}{\lambda}}}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\int_0^t\underbrace{\frac{1}{e^y-1}\int_0^y\frac{1}{e^y-1}\int_0^y\cdots\frac{1}{e^y-1}\int_0^y}_{(k-2)\text{ times}}\frac{y}{e^y-1}\,dy\cdots dy.\end{aligned}\tag{2.4}$$

From (2.4), we have

$$\begin{aligned}&\sum_{n=0}^{\infty}\beta_n^{(2)}(x\mid\lambda)\frac{t^n}{n!}\\&\quad=\frac{(1+t)^{\frac{x}{\lambda}}}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\int_0^t\frac{y}{e^y-1}\,dy\\&\quad=\frac{(1+t)^{\frac{x}{\lambda}}}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\sum_{l=0}^{\infty}\frac{B_l}{l!}\int_0^t y^l\,dy\\&\quad=\left(\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{x}{\lambda}}\right)\left(\sum_{l=0}^{\infty}\frac{B_l}{l+1}\frac{t^l}{l!}\right)\\&\quad=\sum_{n=0}^{\infty}\left\{\sum_{l=0}^{n}\binom{n}{l}\frac{B_l}{l+1}\beta_{n-l}(x\mid\lambda)\right\}\frac{t^n}{n!},\end{aligned}\tag{2.5}$$

where $B_n=B_n(0)$ are the Bernoulli numbers.

By comparing the coefficients on both sides of (2.5), we obtain the following theorem.

Theorem 2.1 *For $n \geq 0$, we have*

$$\begin{aligned}\beta_n^{(2)}(x \mid \lambda) &= \sum_{l=0}^{n} \binom{n}{l} \frac{B_l}{l+1} \beta_{n-l}(x \mid \lambda) \\ &= \beta_n(x \mid \lambda) - \frac{n}{4}\beta_{n-1}(x \mid \lambda) + \sum_{l=2}^{n} \binom{n}{l} \frac{B_l}{l+1} \beta_{n-l}(x \mid \lambda).\end{aligned}$$

Moreover,

$$\beta_n^{(k)}(x \mid \lambda) = \sum_{l=0}^{n} \binom{n}{l} \beta_l^{(k)}(\lambda)(x \mid \lambda)_{n-l}.$$

By (2.4), we easily get

$$\begin{aligned}&\sum_{n=0}^{\infty} \beta_n^{(k)}(x \mid \lambda) \frac{t^n}{n!} \\ &\quad = \frac{\mathrm{Li}_k(1-e^{-t})}{(1+\lambda t)^{\frac{1}{\lambda}} - 1}(1+t)^{\frac{x}{\lambda}} \\ &\quad = \frac{t}{(1+\lambda t)^{\frac{1}{\lambda}} - 1}(1+t)^{\frac{x}{\lambda}} \frac{\mathrm{Li}_k(1-e^{-t})}{t}.\end{aligned} \tag{2.6}$$

We observe that

$$\begin{aligned}\frac{1}{t}\mathrm{Li}_k\left(1-e^{-t}\right) &= \frac{1}{t}\sum_{n=1}^{\infty} \frac{1}{n^k}\left(1-e^{-t}\right)^n \\ &= \frac{1}{t}\sum_{n=1}^{\infty} \frac{(-1)^n}{n^k} n! \sum_{l=n}^{\infty} S_2(l,n)\frac{(-t)^l}{l!} \\ &= \frac{1}{t}\sum_{l=1}^{\infty}\sum_{n=1}^{l} \frac{(-1)^{n+l}}{n^k} n! S_2(l,n)\frac{t^l}{l!} \\ &= \sum_{l=0}^{\infty}\sum_{n=1}^{l+1} \frac{(-1)^{n+l+1}}{n^k} n! \frac{S_2(l+1,n)}{l+1}\frac{t^l}{l!}.\end{aligned} \tag{2.7}$$

From (2.6) and (2.7), we have

$$\begin{aligned}&\sum_{n=0}^{\infty} \beta_n^{(k)}(x \mid \lambda) \frac{t^n}{n!} \\ &\quad = \left(\sum_{m=0}^{\infty} \beta_m(x \mid \lambda)\frac{t^m}{m!}\right)\left(\sum_{l=0}^{\infty}\left(\sum_{p=1}^{l+1} \frac{(-1)^{p+l+1}}{p^k} p! \frac{S_2(l+1,p)}{l+1}\right)\frac{t^l}{l!}\right) \\ &\quad = \sum_{n=0}^{\infty}\left\{\sum_{l=0}^{n}\binom{n}{l}\left(\sum_{p=1}^{l+1}\frac{(-1)^{p+l+1}p!}{p^k}\frac{S_2(l+1,p)}{l+1}\right)\beta_{n-l}(x \mid \lambda)\right\}\frac{t^n}{n!}.\end{aligned} \tag{2.8}$$

By comparing the coefficients on both sides of (2.8), we obtain the following theorem.

Theorem 2.2 *For $n \geq 0$, we have*

$$\beta_n^{(k)}(x \mid \lambda) = \sum_{l=0}^{n} \binom{n}{l} \left(\sum_{p=1}^{l+1} \frac{(-1)^{p+l+1} p!}{p^k} \frac{S_2(l+1,p)}{l+1} \right) \beta_{n-l}(x \mid \lambda).$$

It is easy to show that

$$\begin{aligned}
&\frac{\mathrm{Li}_k(1-e^{-t})}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{x+1}{\lambda}} - \frac{\mathrm{Li}_k(1-e^{-t})}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{x}{\lambda}} \\
&\quad = (1+\lambda t)^{\frac{x}{\lambda}} \mathrm{Li}_k\bigl(1-e^{-t}\bigr) \\
&\quad = \left(\sum_{l=0}^{\infty} (x \mid \lambda)_l \frac{t^l}{l!} \right) \left(\sum_{m=1}^{\infty} \frac{(1-e^{-t})^m}{m^k} \right) \\
&\quad = \left(\sum_{l=0}^{\infty} (x \mid \lambda)_l \frac{t^l}{l!} \right) \left(\sum_{m=0}^{\infty} \frac{(1-e^{-t})^{m+1}}{(m+1)^k} \right) \\
&\quad = \left(\sum_{l=0}^{\infty} (x \mid \lambda)_l \frac{t^l}{l!} \right) \left(\sum_{p=1}^{\infty} \left(\sum_{m=0}^{p-1} \frac{(-1)^{m+p+1}}{(m+1)^k} (m+1)! S_2(p,m+1) \right) \frac{t^p}{p!} \right) \\
&\quad = \sum_{n=1}^{\infty} \left\{ \sum_{p=1}^{n} \sum_{m=0}^{p-1} \frac{(-1)^{m+p+1}}{(m+1)^k} (m+1)! S_2(p,m+1) \binom{n}{p} (x \mid \lambda)_{n-p} \right\} \frac{t^n}{n!}. \qquad (2.9)
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\frac{\mathrm{Li}_k(1-e^{-t})}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{x+1}{\lambda}} - \frac{\mathrm{Li}_k(1-e^{-t})}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{x}{\lambda}} \\
&\quad = \sum_{n=0}^{\infty} \bigl\{ \beta_n^{(k)}(x+1 \mid \lambda) - \beta_n^{(k)}(x \mid \lambda) \bigr\} \frac{t^n}{n!}. \qquad (2.10)
\end{aligned}$$

Therefore, by (2.9) and (2.10), we obtain the following theorem.

Theorem 2.3 *For $n \geq 1$, we have*

$$\begin{aligned}
&\beta_n^{(k)}(x+1 \mid \lambda) - \beta_n^{(k)}(x \mid \lambda) \\
&\quad = \sum_{p=1}^{n} \left(\sum_{m=0}^{p-1} \frac{(-1)^{m+k+1}}{(m+1)^k} (m+1)! S_2(k+m+1) \right) \binom{n}{p} (x \mid \lambda)_{n-p}.
\end{aligned}$$

Now, we note that

$$\begin{aligned}
&\frac{\mathrm{Li}_k(1-e^{-t})}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{x}{\lambda}} \\
&\quad = \frac{\mathrm{Li}_k(1-e^{-t})}{(1+\lambda t)^{\frac{d}{\lambda}}-1} \sum_{a=0}^{d-1} (1+\lambda t)^{\frac{l+x}{\lambda}} \\
&\quad = \left(\frac{\mathrm{Li}_k(1-e^{-t})}{t} \right) \frac{1}{d} \sum_{a=0}^{d-1} \frac{dt}{(1+\lambda t)^{\frac{d}{\lambda}}-1} (1+\lambda t)^{\frac{l+x}{\lambda}}
\end{aligned}$$

$$\begin{aligned}
&= \sum_{l=0}^{\infty}\left(\sum_{p=1}^{l+1}\frac{(-1)^{p+l+1}}{p^k}p!\frac{S_2(l+1,p)}{l+1}\right)\frac{t^l}{l!} \\
&\quad\times \sum_{a=0}^{d-1}\sum_{m=0}^{\infty}\beta_m\left(\frac{l+x}{d}\,\Big|\,\frac{\lambda}{d}\right)d^{m-1}\frac{t^m}{m!} \\
&= \sum_{a=0}^{d-1}\left(\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\sum_{p=1}^{l+1}\binom{n}{l}\frac{(-1)^{p+l+1}}{p^k}p!\frac{S_2(l+1,p)}{l+1}\beta_{n-l}\left(\frac{l+x}{d}\,\Big|\,\frac{\lambda}{d}\right)d^{n-l-1}\right)\frac{t^n}{n!}\right) \\
&= \sum_{n=0}^{\infty}\left\{\sum_{a=0}^{d-1}\sum_{l=0}^{n}\sum_{p=1}^{l+1}\binom{n}{l}\frac{(-1)^{p+l+1}}{p^k}p!\frac{S_2(l+1,p)}{l+1}\beta_{n-l}\left(\frac{l+x}{d}\,\Big|\,\frac{\lambda}{d}\right)d^{n-l-1}\right\}\frac{t^n}{n!}, \qquad (2.11)
\end{aligned}$$

where d is a fixed positive integer.

On the other hand,

$$\begin{aligned}
&\frac{\mathrm{Li}_k(1-e^{-t})}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{x}{\lambda}} \\
&\quad= \sum_{n=0}^{\infty}\beta_n^{(k)}(x\mid\lambda)\frac{t^n}{n!}. \qquad (2.12)
\end{aligned}$$

Therefore, by (2.11) and (2.12), we obtain the following theorem.

Theorem 2.4 *For $n \geq 0$, $d \in \mathbb{N}$ and $k \in \mathbb{Z}$, we have*

$$\begin{aligned}
&\beta_n^{(k)}(x\mid\lambda) \\
&\quad= \sum_{a=0}^{d-1}\sum_{l=0}^{n}\sum_{p=1}^{l+1}\binom{n}{l}\frac{(-1)^{p+l+1}}{p^k}p!\frac{S_2(l+1,p)}{l+1}\beta_{n-l}\left(\frac{l+x}{d}\,\Big|\,\frac{\lambda}{d}\right)d^{n-l-1}.
\end{aligned}$$

From (2.4), we can derive the following equation:

$$\begin{aligned}
&\sum_{n=0}^{\infty}\beta_n^{(k)}(x+y\mid\lambda)\frac{t^n}{n!} \\
&\quad= \frac{\mathrm{Li}_k(1-e^{-t})}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{x+y}{\lambda}} \\
&\quad= \left(\frac{\mathrm{Li}_k(1-e^{-t})}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+t\lambda)^{\frac{x}{\lambda}}\right)(1+\lambda t)^{\frac{y}{\lambda}} \\
&\quad= \left(\sum_{l=0}^{\infty}\beta_l^{(k)}(x\mid\lambda)\frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty}(y\mid\lambda)_m\frac{t^m}{m!}\right) \\
&\quad= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}\beta_l^{(k)}(x\mid\lambda)(y\mid\lambda)_{n-l}\right)\frac{t^n}{n!}. \qquad (2.13)
\end{aligned}$$

Therefore, by (2.13), we obtain the following theorem.

Theorem 2.5 *For $n \geq 0$, we have*

$$\beta_n^{(k)}(x+y \mid \lambda) = \sum_{l=0}^{n} \binom{n}{l} \beta_l^{(k)}(x \mid \lambda)(y \mid \lambda)_{n-l}.$$

Remark

$$\begin{aligned}
&\frac{d}{dx}\beta_n^{(k)}(x \mid \lambda) \\
&\quad = \frac{d}{dx}\sum_{l=0}^{n} \binom{n}{l} \beta_{n-l}^{(k)}(\lambda)(x \mid \lambda)_l \\
&\quad = \sum_{l=0}^{n} \binom{n}{l} \beta_{n-l}^{(k)}(\lambda) \sum_{j=0}^{l-1} \frac{1}{x-\lambda j} \prod_{i=0}^{l-1}(x-\lambda i) \\
&\quad = \sum_{l=0}^{n} \binom{n}{l} \beta_{n-l}^{(k)}(\lambda) \sum_{j=0}^{l-1} \prod_{\substack{i=0 \\ i \neq j}}^{l-1}(x-\lambda i).
\end{aligned}$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to this work. All authors read and approved the final manuscript.

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References
1. Carlitz, L: Degenerate Stirling, Bernoulli and Eulerian numbers. Util. Math. **15**, 51-88 (1979)
2. Kim, DS, Kim, T, Dolgy, DV, Komatsu, T: Barnes-type degenerate Bernoulli polynomials. Adv. Stud. Contemp. Math. **25**(1), 121-146 (2015)
3. Kim, T: Barnes' type multiple degenerate Bernoulli and Euler polynomials. Appl. Math. Comput. **258**, 556-564 (2015)
4. Jolany, H, Mohebbi, H: Some results on Generalized multi poly-Bernoulli and Euler polynomials. Int. J. Math. Comb. **2**, 117-129 (2011)
5. Kim, DS, Kim, T, Mansour, T, Dolgy, DV: On poly-Bernoulli polynomials of the second kind with umbral calculus viewpoint. Adv. Differ. Equ. **2015**(1), 27 (2015)
6. Kim, DS, Kim, T: Some identities of degenerate Euler polynomials arising from p-adic fermionic integrals on $\mathbb{Z}_p$. Integral Transforms Spec. Funct. **26**(4), 295-302 (2015)
7. Kim, DS, Kim, T: Higher-order Frobenius-Euler and poly-Bernoulli mixed-type polynomials. Adv. Differ. Equ. **2013**, 251 (2013)
8. Kim, DS, Kim, T: Hermite and poly-Bernoulli mixed-type polynomials. Adv. Differ. Equ. **2013**(343), 12 (2013)
9. Kim, DS, Kim, T: A note on poly-Bernoulli and higher-order poly-Bernoulli polynomials. Russ. J. Math. Phys. **22**(1), 26-33 (2015)
10. Kim, DS, Kim, T, Lee, SH: A note on poly-Bernoulli polynomials arising umbral calculus. Adv. Stud. Theor. Phys. **7**(15), 731-744 (2013)
11. Kim, T, Jang, YS, Seo, JJ: Poly-Bernoulli polynomials and their applications. Int. J. Math. Anal. **8**(30), 1495-1503 (2014)
12. Acikgoz, M, Erdal, D, Araci, S: A new approach to q-Bernoulli numbers and q-Bernoulli polynomials related to q-Bernstein polynomials. Adv. Differ. Equ. **9**, Article ID 951764 (2010)
13. Araci, S, Acikgoz, M, Jolany, H: On the families of q-Euler polynomials and their applications. J. Egypt. Math. Soc. **23**(1), 1-5 (2015)
14. Bayad, A, Hamahata, Y: Multiple polylogarithms and multi-poly-Bernoulli polynomials. Funct. Approx. Comment. Math. **46**, 45-61 (2012)
15. Bayad, A, Gaboury, S: Generalized Dirichlet L-function of arbitrary order with applications. Adv. Stud. Contemp. Math. (Kyungshang) **23**(4), 607-619 (2013)
16. Gaboury, S, Tremblay, R, Fugère, B-J: Some explicit formulas for certain new classes of Bernoulli, Euler and Genocchi polynomials. Proc. Jangjeon Math. Soc. **17**(1), 115-123 (2014)

17. Carlitz, L: A degenerate Staudt-Clausen theorem. Arch. Math. (Basel) **7**, 28-33 (1956)
18. Dere, R, Simsek, Y: Applications of umbral algebra to some special polynomials. Adv. Stud. Contemp. Math. (Kyungshang) **22**(3), 433-438 (2012)
19. Ding, D, Yang, J: Some identities related to the Apostol-Euler and Apostol-Bernoulli polynomials. Adv. Stud. Contemp. Math. (Kyungshang) **20**(1), 7-21 (2010)
20. Kim, T, Kwon, HK, Lee, SH, Seo, JJ: A note on poly-Bernoulli numbers and polynomials of the second kind. Adv. Differ. Equ. **2014**, 219 (2014)
21. Kim, T: Some identities on the q-Euler polynomials of higher order and q-Stirling numbers by the fermionic p-adic integral on $\mathbb{Z}_p$. Russ. J. Math. Phys. **16**(4), 484-491 (2009)
22. Luo, Q-M, Guo, B-N, Qi, F: On evaluation of Riemann zeta function $\zeta(s)$. Adv. Stud. Contemp. Math. (Kyungshang) **7**(2), 135-144 (2003)
23. Luo, Q-M, Qi, F: Relationships between generalized Bernoulli numbers and polynomials and generalized Euler numbers and polynomials. Adv. Stud. Contemp. Math. (Kyungshang) **7**(1), 11-18 (2003)
24. Park, J-W, Rim, S-H: On the modified q-Bernoulli polynomials with weight. Proc. Jangjeon Math. Soc. **17**(2), 231-236 (2014)
25. Şen, E: Theorems on Apostol-Euler polynomials of higher order arising from Euler basis. Adv. Stud. Contemp. Math. (Kyungshang) **23**(2), 337-345 (2013)
26. Srivastava, HM, Kim, T, Simsek, Y: q-Bernoulli numbers and polynomials associated with multiple q-zeta functions and basic L-series. Russ. J. Math. Phys. **12**(2), 241-268 (2005)
27. Zhang, Z, Yang, J: On sums of products of the degenerate Bernoulli numbers. Integral Transforms Spec. Funct. **20**(9-10), 751-755 (2009)