Advances in Difference Equations
a SpringerOpen Journal

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Global stability of an epidemic model with stage structure and nonlinear incidence rates in a heterogeneous host population

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Abstract

In this paper, we study an epidemic model with stage structure and latency spreading in a heterogeneous host population. We show that if the disease-free equilibrium exists, then the global dynamics are determined by the basic reproduction number R_0. We prove that the disease-free equilibrium is globally asymptotically stable when $R_0 \leq 1$; and there exists a unique endemic equilibrium which is globally asymptotically stable when $R_0 > 1$. The global stability of the endemic equilibrium is also proved by using a graph-theoretic approach to the method of Lyapunov functions. Finally, numerical simulations are given to illustrate the main theoretical results.

Keywords: heterogeneous host; epidemic model; stage structure; nonlinear incidence rate; Lyapunov function

1 Introduction

A heterogeneous host population can be divided into several homogeneous groups according to models of transmission, contact patterns, or geographic distributions. Multi-group epidemic models have been proposed in the literature of mathematical epidemiology to describe the transmission dynamics of infectious diseases in heterogeneous host populations, such as measles, mumps, gonorrhea, HIV/AIDS, West-Nile virus and vector-borne diseases such as malaria. Various forms of multi-group epidemic models have subsequently been studied to understand the mechanism of disease transmission. One of the most important subjects in this field is to obtain a threshold that determines the persistence or extinction of a disease. Guo *et al.* in [1] developed a graph-theoretic approach to prove the global asymptotic stability of a unique endemic equilibrium of a multi-group epidemic model. By applying the idea, global stability of endemic equilibrium for several classes of multi-group epidemic models was investigated in [1–10].

In the real world, some epidemics, such as malaria, dengue, fever, gonorrhea and bacterial infections, may have a different ability to transmit the infections in different ages. For example, measles and varicella always occur in juveniles, while it is reasonable to consider the transmission of diseases such as typhus, diphtheria in adult population. In recent years, epidemic models with stage structure have been studied in many papers [11–17]. For some disease (for example, tuberculosis, influenza, measles), on adequate contact with an

infective, a susceptible individual becomes exposed, that is, infected but not infective. This individual remains in the exposed class for a certain period before becoming infective (see, for example, [18–22]).

In this paper, we formulate an epidemic model with latency spreading in a heterogeneous host population. Let $S_k^{(1)}$, $S_k^{(2)}$, E_k, I_k and R_k denote the immature susceptible, mature susceptible, infected but non-infectious, infectious and recovered population in the kth group, respectively. The disease incidence in the kth group can be calculated as

$$\sum_{i=1}^{2} S_k^{(i)} \sum_{j=1}^{n} \beta_{kj}^{(i)} G_j(I_j),$$

where the sum takes into account cross-infections from all groups and $\beta_{kj}^{(i)}$ is the transmission coefficient between compartments $S_k^{(i)}$ and I_j. $G_j(I_j)$ includes some special incidence functions in the literature. For instance, $G_j(I_j) = \frac{I_j}{1+\alpha_j I_j}$ (saturation effect). Let $d_k^{(1)}$ and $d_k^{(2)}$ represent death rates of $S_k^{(1)}$ and $S_k^{(2)}$ populations, respectively. Then we obtain the following model for a disease with latency:

$$\begin{cases} \dot{S}_k^{(1)} = \varphi_k(S_k^{(1)}) - \sum_{j=1}^{n} \beta_{kj}^{(1)} S_k^{(1)} G_j(I_j) - a_k S_k^{(1)}, \\ \dot{S}_k^{(2)} = a_k S_k^{(1)} - \sum_{j=1}^{n} \beta_{kj}^{(2)} S_k^{(2)} G_j(I_j) - d_k^{(2)} S_k^{(2)}, \\ \dot{E}_k = \sum_{i=1}^{2} \sum_{j=1}^{n} \beta_{kj}^{(i)} S_k^{(i)} G_j(I_j) - (d_k + \eta_k) E_k, \\ \dot{I}_k = \eta_k E_k - (d_k + \mu_k + \gamma_k) I_k, \\ \dot{R}_k = \gamma_k I_k - d_k R_k, \quad k = 1, 2, \dots, n, \end{cases} \tag{1}$$

where $\varphi_k(S_k^{(1)})$ denotes the net growth rate of the immature susceptible class in the kth group (a typical form of $\varphi_k(S_k^{(1)})$ is $\varphi_k(S_k^{(1)}) = b_k - d_k^{(1)} S_k^{(1)}$ with b_k the recruitment constant and $d_k^{(1)}$ the natural death rate). a_k is the conversion rate from an immature individual to a mature individual in group k. η_k represents the rate of becoming infectious after a latent period in the kth group. d_k, μ_k and γ_k are the natural death rate, the disease-related death rate and the recovery rate in the kth group, respectively. All parameter values are assumed to be nonnegative and $a_k, \eta_k, d_k^{(i)}, d_k > 0$.

Remark The model (1) can be regarded as an SVEIR model such that $S_k^{(1)}$ is unvaccinated and $S_k^{(2)}$ is vaccinated with vaccination rate a_k. References studied on the SVEIR model can be seen in [23, 24] and so on.

Since the variable R_k does not appear in the remaining four equations of (1), if we denote $m_k := d_k + \mu_k + \gamma_k$, then we can obtain the following reduced system:

$$\begin{cases} \dot{S}_k^{(1)} = \varphi_k(S_k^{(1)}) - \sum_{j=1}^{n} \beta_{kj}^{(1)} S_k^{(1)} G_j(I_j) - a_k S_k^{(1)}, \\ \dot{S}_k^{(2)} = a_k S_k^{(1)} - \sum_{j=1}^{n} \beta_{kj}^{(2)} S_k^{(2)} G_j(I_j) - d_k^{(2)} S_k^{(2)}, \\ \dot{E}_k = \sum_{i=1}^{2} \sum_{j=1}^{n} \beta_{kj}^{(i)} S_k^{(i)} G_j(I_j) - (d_k + \eta_k) E_k, \\ \dot{I}_k = \eta_k E_k - m_k I_k, \quad k = 1, 2, \dots, n. \end{cases} \tag{2}$$

The initial conditions for system (2) are

$$S_k^{(1)}(0) > 0, \qquad S_k^{(2)}(0) > 0, \qquad E_k(0) > 0, \qquad I_k(0) > 0, \quad k = 1, 2, \dots, n. \tag{3}$$

The organization of this paper is as follows. In the next section, we prove some preliminary results for system (2). In Section 3, the main theorem of this paper is stated and proved. In the last section, a brief discussion and numerical simulations which support our theoretical analysis are given.

2 Preliminaries

We assume:

(A1) φ_k and G_k are Lipschitz on $[0, +\infty)$;

(A2) φ_k is strictly decreasing on $[0, +\infty)$, and there exists $S_{k0}^{(1)} > 0$ such that

$$\varphi_k\big(S_{k0}^{(1)}\big) - a_k S_{k0}^{(1)} = 0;$$

(A3) $\frac{G_k(x)}{x}$ is nonincreasing on $(0, +\infty)$ and

$$\delta_k = \lim_{x \to 0} \frac{G_k(x)}{x} > 0 \text{ exists}, \quad k = 1, 2, \ldots, n.$$

From our assumptions, it is clear that system (2) has a unique solution for any given initial conditions (3) and the solution remains nonnegative. If (A2) holds, then we see that system (2) has a disease-free equilibrium

$$P_0 = \big(S_{10}^{(1)}, S_{10}^{(2)}, \ldots, S_{n0}^{(1)}, S_{n0}^{(2)}, 0, 0, \ldots, 0\big),$$

where

$$\varphi_k\big(S_{k0}^{(1)}\big) = d_k^{(2)} S_{k0}^{(2)}, \qquad a_k S_{k0}^{(1)} = d_k^{(2)} S_{k0}^{(2)}, \quad k = 1, 2, \ldots, n. \tag{4}$$

For two nonnegative n-square matrices $\mathbf{A} = (a_{kj})$ and $\mathbf{B} = (a_{kj})$, we write $\mathbf{A} \leq \mathbf{B}$ if $a_{kj} \leq b_{kj}$ for all k and j, and $\mathbf{A} < \mathbf{B}$ if $\mathbf{A} \leq \mathbf{B}$ and $\mathbf{A} \neq \mathbf{B}$. Following [25], we set matrices

$$\mathbf{F} := \left(\sum_{i=1}^{2} \beta_{kj}^{(i)} S_{k0}^{(i)} \delta_j \right)_{n \times n}, \qquad \mathbf{V} := \text{diag}\left(\frac{m_1(d_1 + \eta_1)}{\eta_1}, \frac{m_2(d_2 + \eta_2)}{\eta_2}, \ldots, \frac{m_n(d_n + \eta_n)}{\eta_n} \right).$$

The next generation matrix for system (2) is

$$\mathbf{Q} := \mathbf{F}\mathbf{V}^{-1} = \left(\frac{\eta_k \sum_{i=1}^{2} \beta_{kj}^{(i)} S_{k0}^{(i)} \delta_j}{m_k(d_k + \eta_k)} \right)_{n \times n}$$

$$= \begin{bmatrix} \frac{\eta_1 \sum_{i=1}^{2} \beta_{11}^{(i)} S_{10}^{(i)} \delta_1}{m_1(d_1 + \eta_1)} & \cdots & \frac{\eta_1 \sum_{i=1}^{2} \beta_{1n}^{(i)} S_{10}^{(i)} \delta_n}{m_1(d_1 + \eta_1)} \\ \vdots & \ddots & \vdots \\ \frac{\eta_n \sum_{i=1}^{2} \beta_{n1}^{(i)} S_{n0}^{(i)} \delta_1}{m_n(d_n + \eta_n)} & \cdots & \frac{\eta_n \sum_{i=1}^{2} \beta_{nn}^{(i)} S_{n0}^{(i)} \delta_n}{m_n(d_n + \eta_n)} \end{bmatrix}.$$

Thus, we obtain the basic reproduction number R_0 for system (2) as

$$R_0 = \rho(\mathbf{Q}),$$

where ρ denotes the spectral radius.

Let $N_k = S_k^{(1)} + S_k^{(2)} + E_k + I_k$, $\underline{d}_k = \min\{d_k^{(1)}, d_k^{(2)}, d_k, m_k\}$, $k = 1, 2, \ldots, n$. Then from (2) we have

$$\dot{N}_k \le \varphi_k\big(S_k^{(1)}\big) + d_k^{(1)} S_k^{(1)} - \underline{d}_k N_k. \tag{5}$$

We derive from (5) that the region

$$\Gamma = \bigg\{ \big(S_1^{(1)}, S_1^{(2)}, \ldots, S_n^{(1)}, S_n^{(2)}, E_1, \ldots, E_n, I_1, \ldots, I_n\big) \in \mathbb{R}_+^{4n} : S_k^{(1)} \le S_{k0}^{(1)},$$

$$S_k^{(2)} \le S_{k0}^{(2)}, S_k^{(1)} + S_k^{(2)} + E_k + I_k \le \frac{\varphi_k(0) + d_k^{(1)} S_{k0}^{(1)}}{\underline{d}_k}, k = 1, 2, \ldots, n \bigg\}$$

is positively invariant with respect to (2). Let Γ° denote the interior of Γ.

3 Main results

In the section, we study the global stability of equilibria of system (2).

Theorem 3.1 *Assume that* (A1)-(A3) *hold and* $\mathbf{B} = (\sum_{i=1}^2 \beta_{kj}^{(i)})$ *is irreducible.*
  (1) *If $R_0 \le 1$, then P_0 is globally asymptotically stable in Γ;*
  (2) *If $R_0 > 1$, then P_0 is unstable and system* (2) *admits at least one endemic equilibrium in Γ°.*

Proof Let

$$\mathbf{S} = \big(S_1^{(1)}, S_1^{(2)}, \ldots, S_n^{(1)}, S_n^{(2)}\big), \qquad \mathbf{S^0} = \big(S_{10}^{(1)}, S_{10}^{(2)}, \ldots, S_{n0}^{(1)}, S_{n0}^{(2)}\big),$$

$$\mathbf{I} = (I_1, I_2, \ldots, I_n), \qquad \mathbf{Q(S, I)} = \bigg(\frac{\sum_{i=1}^2 \sum_{j=1}^n \eta_k \beta_{kj}^{(i)} S_k^{(i)} G_j(I_j)}{m_k(d_k + \eta_k) I_j} \bigg)_{n \times n}.$$

Notice that $\mathbf{B}$ is irreducible, then $\mathbf{Q(S, I)}$ and $\mathbf{Q}$ are irreducible. By (A3), we have $\mathbf{0} \le \mathbf{Q(S, I)} \le \mathbf{Q}$. Hence $\mathbf{Q(S, I)} + \mathbf{Q}$ is also irreducible. That is, $0 \le \mathbf{Q(S, I)} < \mathbf{Q}$ and $\mathbf{Q(S, I)} + \mathbf{Q}$ is irreducible provided that $\mathbf{S} \ne \mathbf{S^0}$. Thus, by [26], Corollary 1.5, p.27, $\rho(\mathbf{Q(S, I)}) < \rho(\mathbf{Q})$ if $\mathbf{S} \ne \mathbf{S^0}$. Since $\mathbf{Q}$ is irreducible, there exist $\omega_k > 0$, $k = 1, 2, \ldots, n$, such that

$$(\omega_1, \omega_2, \ldots, \omega_n)\rho(\mathbf{Q}) = (\omega_1, \omega_2, \ldots, \omega_n)\mathbf{Q}.$$

Consider a Lyapunov functional

$$L = \sum_{k=1}^n \frac{\omega_k \eta_k}{m_k(d_k + \eta_k)} \bigg[E_k + \frac{d_k + \eta_k}{\eta_k} I_k \bigg].$$

Differentiating L along the solution of system (2), we obtain

$$\dot{L} = \sum_{k=1}^n \frac{\omega_k \eta_k}{m_k(d_k + \eta_k)} \bigg[\sum_{i=1}^2 \sum_{j=1}^n \beta_{kj}^{(i)} S_k^{(i)} G_j(I_j) - \frac{m_k(d_k + \eta_k)}{\eta_k} I_k \bigg]$$

$$= \sum_{k=1}^n \omega_k \bigg[\frac{\sum_{i=1}^2 \sum_{j=1}^n \eta_k \beta_{kj}^{(i)} S_k^{(i)} G_j(I_j)}{m_k(d_k + \eta_k)} - I_k \bigg]$$

$$= (\omega_1, \omega_2, \ldots, \omega_n)\big(\mathbf{Q}(\mathbf{S}, \mathbf{I})\mathbf{I}^T - \mathbf{I}^T\big)$$

$$\leq (\omega_1, \omega_2, \ldots, \omega_n)\big(\mathbf{Q}\mathbf{I}^T - \mathbf{I}^T\big)$$

$$= \big[\rho(\mathbf{Q}) - 1\big](\omega_1, \omega_2, \ldots, \omega_n)\mathbf{I}^T \leq 0.$$

If $R_0 < 1$, then $\dot{L} = 0$ if and only if $\mathbf{I}^T = \mathbf{0}$. If $R_0 = 1$, then $\dot{L} = 0$ implies

$$(\omega_1, \omega_2, \ldots, \omega_n)\big(\mathbf{Q}(\mathbf{S}, \mathbf{I})\mathbf{I}^T - \mathbf{I}^T\big) = 0.$$

Therefore, $\dot{L} = 0$ if and only if $\mathbf{I} = \mathbf{0}$, or $\mathbf{S} = \mathbf{S^0}$. On the other hand, from the last equation in system (2), we see that $\mathbf{I} = \mathbf{0}$ implies that $E_k = 0$ for $k = 1, 2, \ldots, n$. Hence, the largest invariant subset of the set, where $\dot{L} = 0$, is the singleton $\{P_0\}$. By LaSalle's invariance principle, P_0 is globally asymptotically stable for $R_0 \leq 1$.

If $R_0 > 1$ and $\mathbf{I} \neq \mathbf{0}$, then

$$\big[\rho(\mathbf{Q}) - 1\big](\omega_1, \omega_2, \ldots, \omega_n)\mathbf{I}^T > 0.$$

Thus, by continuity, we have $\dot{L} = (\omega_1, \omega_2, \ldots, \omega_n)(\mathbf{Q}(\mathbf{S}, \mathbf{I})\mathbf{I}^T - \mathbf{I}^T) > 0$ in a neighborhood of P_0 in Γ°. This implies that P_0 is unstable. From a uniform persistence result of [27] and a similar argument as in the proof of Proposition 3.3 of [28], we can deduce that the instability of P_0 implies the uniform persistence of system (2) in Γ°. This together with the uniform boundedness of solutions of system (2) in Γ° implies that system (2) has an endemic equilibrium in Γ° (see Theorem 2.8.6 of [29] or Theorem D.3 of [30]). The proof is completed. $\qquad\square$

By Theorem 3.1, we have that if $\mathbf{B} = (\sum_{i=1}^{2} \beta_{kj}^{(i)})$ is irreducible, (A1)-(A3) hold and $R_0 > 1$, then system (2) has an endemic equilibrium P^* in Γ°. Let

$$P^* = \big(S_1^{(1)*}, S_1^{(2)*}, \ldots, S_n^{(1)*}, S_n^{(2)*}, E_1^*, \ldots, E_n^*, I_1^*, \ldots, I_n^*\big),$$

then the components of P^* satisfy

$$\varphi_k\big(S_k^{(1)*}\big) = \sum_{i=1}^{2} S_k^{(i)*} \sum_{j=1}^{n} \beta_{kj}^{(i)} G_j\big(I_j^*\big) + d_k^{(2)} S_k^{(2)*}, \tag{6}$$

$$a_k\big(S_k^{(1)*}\big) = S_k^{(2)*} \sum_{j=1}^{n} \beta_{kj}^{(2)} G_j\big(I_j^*\big) + d_k^{(2)} S_k^{(2)*}, \tag{7}$$

$$\sum_{i=1}^{2} S_k^{(i)*} \sum_{j=1}^{n} \beta_{kj}^{(i)} G_j\big(I_j^*\big) = (d_k + \eta_k)E_k^* = \frac{m_k(d_k + \eta_k)}{\eta_k}I_k^*, \quad k = 1, 2, \ldots, n. \tag{8}$$

Since φ_k is strictly decreasing on $[0, +\infty)$, we have

$$\big[\varphi_k\big(S_k^{(1)}\big) - \varphi_k\big(S_k^{(1)*}\big)\big]\left(1 - \frac{S_k^{(1)*}}{S_k^{(1)}}\right) \leq 0, \tag{9}$$

where equality holds if and only if $S_k^{(1)} = S_k^{(1)*}$, $k = 1, 2, \ldots, n$.

We further make the following assumption:

(A4) G_k is strictly increasing on $[0, +\infty)$, and

$$\frac{G_k(x_k)I_k}{G_k(I_k)x_k} + \frac{G_k(I_k)}{G_k(x_k)} - \frac{I_k}{x_k} - 1 \leq 0, \quad k = 1, 2, \ldots, n, \tag{10}$$

where $x_k > 0$ is chosen in an arbitrary way and equality holds if $I_k = x_k$.

Theorem 3.2 *Assume that* $\mathbf{B} = [\sum_{i=1}^{2} \beta_{kj}^{(i)}]$ *is irreducible. If* $R_0 > 1$, *then* P^* *is globally asymptotically stable.*

Proof Set $\overline{\beta}_{kj} = \sum_{i=1}^{2} \beta_{kj}^{(i)} S_k^{(i)*} G_j(I_j^*)$, $1 \leq k, j \leq n$, and

$$\overline{B} = \begin{bmatrix} \sum_{l \neq 1} \overline{\beta}_{1l} & -\overline{\beta}_{21} & \cdots & -\overline{\beta}_{n1} \\ -\overline{\beta}_{12} & \sum_{l \neq 2} \overline{\beta}_{2l} & \cdots & -\overline{\beta}_{n1} \\ \vdots & \vdots & \ddots & \vdots \\ -\overline{\beta}_{1n} & -\overline{\beta}_{2n} & \cdots & \sum_{l \neq n} \overline{\beta}_{nl} \end{bmatrix}.$$

Then $\overline{B}$ is also irreducible. It follows from Lemma 2.1 of [1] that the solution space of the linear system

$$\overline{B}\mathbf{v} = 0 \tag{11}$$

has dimension 1 with a basis

$$\mathbf{v} := (v_1, v_2, \ldots, v_n)^T = (\xi_1, \xi_2, \ldots, \xi_n)^T, \tag{12}$$

where ξ_k denotes the cofactor of the kth diagonal entry of $\overline{B}$. Note that from (11) we have

$$\sum_{j=1}^{n} \overline{B}_{kj} v_k = \sum_{j=1}^{n} \overline{B}_{jk} v_j, \quad k = 1, 2, \ldots, n. \tag{13}$$

From (13), we have

$$\sum_{k=1}^{n} v_k \sum_{j=1}^{n} \sum_{i=1}^{2} \beta_{kj}^{(i)} S_k^{(i)*} G_j(I_j)$$

$$= \sum_{k,j=1}^{n} \sum_{i=1}^{2} \beta_{jk}^{(i)} S_j^{(i)*} v_j G_k(I_k) = \sum_{k=1}^{n} \left[\sum_{j=1}^{n} \sum_{i=1}^{2} \beta_{jk}^{(i)} S_j^{(i)*} G_k(I_k^*) v_j \right] \frac{G_k(I_k)}{G_k(I_k^*)}$$

$$= \sum_{k=1}^{n} \left[\sum_{j=1}^{n} (\overline{\beta}_{jk} v_j) \right] \frac{G_k(I_k)}{G_k(I_k^*)} = \sum_{k=1}^{n} \left[\sum_{j=1}^{n} (\overline{\beta}_{kj} v_k) \right] \frac{G_k(I_k)}{G_k(I_k^*)}$$

$$= \sum_{k=1}^{n} v_k \sum_{j=1}^{n} \sum_{i=1}^{2} \beta_{kj}^{(i)} S_k^{(i)*} G_j(I_j^*) \frac{G_k(I_k)}{G_k(I_k^*)}. \tag{14}$$

Consider a Lyapunov functional

$$V = \sum_{k=1}^{n} v_k \left[\sum_{i=1}^{2} \left(S_k^{(i)} - S_k^{(i)*} - S_k^{(i)*} \ln \frac{S_k^{(i)}}{S_k^{(i)*}} \right) + E_k - E_k^* \right.$$

$$\left. - E_k^* \ln \frac{E_k}{E_k^*} + \frac{d_k + \eta_k}{\eta_k} \int_{I_k^*}^{I_k} \frac{G_k(x) - G_k(I_k^*)}{G_k(x)} \, dx \right].$$

Differentiating V along the solution of system (2), we obtain

$$\dot{V} = \sum_{k=1}^{n} v_k \left\{ \varphi_k \left(S_k^{(1)} \right) - d_k^{(2)} S_k^{(2)} - \frac{m_k(d_k + \eta_k)}{\eta_k} I_k \right.$$

$$- \frac{S_k^{(1)*}}{S_k^{(1)}} \left[\varphi_k \left(S_k^{(1)} \right) - S_k^{(1)} \sum_{j=1}^{n} \beta_{kj}^{(1)} G_j(I_j) - a_k S_k^{(1)} \right]$$

$$- \frac{S_k^{(2)*}}{S_k^{(2)}} \left[a_k S_k^{(1)} - S_k^{(2)} \sum_{j=1}^{n} \beta_{kj}^{(2)} G_j(I_j) - d_k^{(2)} S_k^{(2)} \right]$$

$$- \frac{E_k^*}{E_k} \left[\sum_{i=1}^{2} \sum_{j=1}^{n} \beta_{kj}^{(i)} S_k^{(i)} G_j(I_j) - (d_k + \eta_k) E_k \right]$$

$$\left. - \frac{G_k(I_k^*)}{G_k(I_k)} \left[(d_k + \eta_k) E_k - \frac{m_k(d_k + \eta_k)}{\eta_k} I_k \right] \right\}$$

$$= \sum_{k=1}^{n} v_k \left\{ \varphi_k \left(S_k^{(1)} \right) \left(1 - \frac{S_k^{(1)*}}{S_k^{(1)}} \right) + d_k^{(2)} S_k^{(2)*} \left(1 - \frac{S_k^{(2)}}{S_k^{(2)*}} \right) \right.$$

$$+ a_k S_k^{(1)*} \left(1 - \frac{S_k^{(1)} S_k^{(2)*}}{S_k^{(1)*} S_k^{(2)}} \right) + \sum_{i=1}^{2} S_k^{(i)*} \sum_{j=1}^{n} \beta_{kj}^{(i)} G_j(I_j)$$

$$- \frac{E_k^*}{E_k} \sum_{i=1}^{2} -S_k^{(i)} \sum_{j=1}^{n} \beta_{kj}^{(i)} G_j(I_j) + (d_k + \eta_k) E_k^* \left(1 - \frac{E_k G_k(I_k^*)}{E_k^* G_k(I_k)} \right)$$

$$\left. + \frac{m_k(d_k + \eta_k)}{\eta_k} I_k^* \frac{I_k G_k(I_k^*)}{I_k^* G_k(I_k)} - \frac{m_k(d_k + \eta_k)}{\eta_k} I_k^* \frac{I_k}{I_k^*} \right\}.$$

From (7) and (8), we have

$$\dot{V} = \sum_{k=1}^{n} v_k \left\{ \varphi_k \left(S_k^{(1)} \right) \left(1 - \frac{S_k^{(1)*}}{S_k^{(1)}} \right) + d_k^{(2)} S_k^{(2)*} \left(1 - \frac{S_k^{(2)}}{S_k^{(2)*}} \right) \right.$$

$$- \frac{E_k^*}{E_k} \sum_{i=1}^{2} S_k^{(i)} \sum_{j=1}^{n} \beta_{kj}^{(i)} G_j(I_j)$$

$$+ \left[S_k^{(2)*} \sum_{j=1}^{n} \beta_{kj}^{(2)} G_j(I_j^*) + d_k^{(2)} S_k^{(2)*} \right] \left(1 - \frac{S_k^{(1)} S_k^{(2)*}}{S_k^{(1)*} S_k^{(2)}} \right)$$

$$\left. + \sum_{i=1}^{2} S_k^{(i)*} \sum_{j=1}^{n} \beta_{kj}^{(i)} G_j(I_j^*) \left[1 - \frac{E_k G_k(I_k^*)}{E_k^* G_k(I_k)} + \frac{G_j(I_j)}{G_j(I_j^*)} + \frac{I_k G_k(I_k^*)}{I_k^* G_k(I_k)} - \frac{I_k}{I_k^*} \right] \right\}.$$

By (10) and (14), we obtain

$$
\dot{V} \le \sum_{k=1}^{n} v_k \Bigg\{ \varphi_k\big(S_k^{(1)}\big)\bigg(1 - \frac{S_k^{(1)*}}{S_k^{(1)}}\bigg) + d_k^{(2)} S_k^{(2)*}\bigg(1 - \frac{S_k^{(2)}}{S_k^{(2)*}}\bigg)
$$

$$
- \frac{E_k^*}{E_k} \sum_{i=1}^{2} S_k^{(i)} \sum_{j=1}^{n} \beta_{kj}^{(i)} G_j(I_j) + \Bigg[S_k^{(2)*} \sum_{j=1}^{n} \beta_{kj}^{(2)} G_j\big(I_j^*\big) + d_k^{(2)} S_k^{(2)*} \Bigg]
$$

$$
\times \bigg(1 - \frac{S_k^{(1)} S_k^{(2)*}}{S_k^{(1)*} S_k^{(2)}}\bigg) + \sum_{i=1}^{2} S_k^{(i)*} \sum_{j=1}^{n} \beta_{kj}^{(i)} G_j\big(I_j^*\big)\bigg[2 - \frac{E_k G_k(I_k^*)}{E_k^* G_k(I_k)} \bigg] \Bigg\} =: B_1. \tag{15}
$$

From (6), we know that

$$
\varphi_k\big(S_k^{(1)*}\big)\bigg(1 - \frac{S_k^{(1)*}}{S_k^{(1)}}\bigg) = \bigg(\sum_{i=1}^{2} S_k^{(i)*} \sum_{j=1}^{n} \beta_{kj}^{(i)} G_j\big(I_j^*\big) + d_k^{(2)} S_k^{(2)*} \bigg)\bigg(1 - \frac{S_k^{(1)*}}{S_k^{(1)}}\bigg). \tag{16}
$$

By (16), we can rewrite B_1 as

$$
\dot{V} = \sum_{k=1}^{n} v_k \Bigg\{ \big[\varphi_k\big(S_k^{(1)}\big) - \varphi_k\big(S_k^{(1)*}\big)\big]\bigg(1 - \frac{S_k^{(1)*}}{S_k^{(1)}}\bigg)
$$

$$
+ d_k^{(2)} S_k^{(2)*}\bigg(3 - \frac{S_k^{(1)*}}{S_k^{(1)}} - \frac{S_k^{(1)} S_k^{(2)*}}{S_k^{(1)*} S_k^{(2)}} - \frac{S_k^{(2)}}{S_k^{(2)*}}\bigg)
$$

$$
- \frac{E_k^*}{E_k} \sum_{i=1}^{2} S_k^{(i)} \sum_{j=1}^{n} \beta_{kj}^{(i)} G_j(I_j) + S_k^{(2)*} \sum_{j=1}^{n} \beta_{kj}^{(2)} G_j\big(I_j^*\big)
$$

$$
\times \bigg(2 - \frac{S_k^{(1)*}}{S_k^{(1)}} - \frac{S_k^{(1)} S_k^{(2)*}}{S_k^{(1)*} S_k^{(2)}}\bigg) + S_k^{(1)*} \sum_{j=1}^{n} \beta_{kj}^{(1)} G_j\big(I_j^*\big)\bigg(1 - \frac{S_k^{(1)*}}{S_k^{(1)}}\bigg)
$$

$$
+ \sum_{i=1}^{2} S_k^{(i)*} \sum_{j=1}^{n} \beta_{kj}^{(i)} G_j\big(I_j^*\big)\bigg[2 - \frac{E_k G_k(I_k^*)}{E_k^* G_k(I_k)} \bigg] \Bigg\}.
$$

By (9) and the arithmetic-geometric mean, we easily see that

$$
B_1 \le \sum_{k=1}^{n} v_k \Bigg\{ -\frac{E_k^*}{E_k} \sum_{i=1}^{2} S_k^{(i)} \sum_{j=1}^{n} \beta_{kj}^{(i)} G_j(I_j)
$$

$$
+ S_k^{(2)*} \sum_{j=1}^{n} \beta_{kj}^{(2)} G_j\big(I_j^*\big)\bigg(2 - \frac{S_k^{(1)*}}{S_k^{(1)}} - \frac{S_k^{(1)} S_k^{(2)*}}{S_k^{(1)*} S_k^{(2)}}\bigg)
$$

$$
+ S_k^{(1)*} \sum_{j=1}^{n} \beta_{kj}^{(1)} G_j\big(I_j^*\big)\bigg(1 - \frac{S_k^{(1)*}}{S_k^{(1)}}\bigg)
$$

$$
+ \sum_{i=1}^{2} S_k^{(i)*} \sum_{j=1}^{n} \beta_{kj}^{(i)} G_j\big(I_j^*\big)\bigg[2 - \frac{E_k G_k(I_k^*)}{E_k^* G_k(I_k)} \bigg] \Bigg\} =: B_2. \tag{17}
$$

We can rewrite B_2 as

$$
\begin{aligned}
B_2 = \sum_{k=1}^{n} \nu_k \Bigg\{ & S_k^{(2)*} \sum_{j=1}^{n} \beta_{kj}^{(2)} G_j\big(I_j^*\big) \bigg[3 - \frac{S_k^{(1)*}}{S_k^{(1)}} - \frac{S_k^{(1)} S_k^{(2)*}}{S_k^{(1)*} S_k^{(2)}} \\
& - \frac{S_k^{(2)} E_k^* G_j(I_j)}{S_k^{(2)*} E_k G_j(I_j^*)} \bigg] + S_k^{(1)*} \sum_{j=1}^{n} \beta_{kj}^{(1)} G_j\big(I_j^*\big) \bigg[2 - \frac{S_k^{(1)*}}{S_k^{(1)}} \\
& - \frac{S_k^{(1)} E_k^* G_j(I_j)}{S_k^{(1)*} E_k G_j(I_j^*)} \bigg] + \sum_{i=1}^{2} S_k^{(i)*} \sum_{j=1}^{n} \beta_{kj}^{(i)} G_j\big(I_j^*\big) \bigg[1 - \frac{E_k G_k(I_k^*)}{E_k^* G_k(I_k)} \bigg] \Bigg\}.
\end{aligned}
$$

By the arithmetic-geometric mean, we have that

$$
\begin{aligned}
B_2 \le \sum_{k=1}^{n} \nu_k \Bigg\{ & 3 S_k^{(2)*} \sum_{j=1}^{n} \beta_{kj}^{(2)} G_j\big(I_j^*\big) \bigg[1 - \bigg(\frac{E_k^* G_j(I_j)}{E_k G_j(I_j^*)} \bigg)^{\frac{1}{3}} \bigg] \\
& + 2 S_k^{(1)*} \sum_{j=1}^{n} \beta_{kj}^{(1)} G_j\big(I_j^*\big) \bigg[1 - \bigg(\frac{E_k^* G_j(I_j)}{E_k G_j(I_j^*)} \bigg)^{\frac{1}{2}} \bigg] \\
& + \sum_{i=1}^{2} S_k^{(i)*} \sum_{j=1}^{n} \beta_{kj}^{(i)} G_j\big(I_j^*\big) \bigg[1 - \frac{E_k G_k(I_k^*)}{E_k^* G_k(I_k)} \bigg] \Bigg\} =: B_3.
\end{aligned}
\tag{18}
$$

We can rewrite B_3 as

$$
\begin{aligned}
B_3 = \sum_{k=1}^{n} \nu_k \Bigg\{ & 3 S_k^{(2)*} \sum_{j=1}^{n} \beta_{kj}^{(2)} G_j\big(I_j^*\big) \bigg[1 - \bigg[\frac{E_k^* G_j(I_j)}{E_k G_j(I_j^*)} \bigg]^{\frac{1}{3}} + \ln \bigg[\frac{E_k^* G_j(I_j)}{E_k G_j(I_j^*)} \bigg]^{\frac{1}{3}} \bigg] \\
& + 2 S_k^{(1)*} \sum_{j=1}^{n} \beta_{kj}^{(1)} G_j\big(I_j^*\big) \bigg[1 - \bigg[\frac{E_k^* G_j(I_j)}{E_k G_j(I_j^*)} \bigg]^{\frac{1}{2}} + \ln \bigg[\frac{E_k^* G_j(I_j)}{E_k G_j(I_j^*)} \bigg]^{\frac{1}{2}} \bigg] \\
& - \sum_{i=1}^{2} S_k^{(i)*} \sum_{j=1}^{n} \beta_{kj}^{(i)} G_j\big(I_j^*\big) \ln \frac{E_k^* G_j(I_j)}{E_k G_j(I_j^*)} \\
& + \sum_{i=1}^{2} S_k^{(i)*} \sum_{j=1}^{n} \beta_{kj}^{(i)} G_j\big(I_j^*\big) \bigg[1 - \frac{E_k G_k(I_k^*)}{E_k^* G_k(I_k)} + \ln \frac{E_k G_k(I_k^*)}{E_k^* G_k(I_k)} \bigg] \\
& - \sum_{i=1}^{2} S_k^{(i)*} \sum_{j=1}^{n} \beta_{kj}^{(i)} G_j\big(I_j^*\big) \ln \frac{E_k G_k(I_k^*)}{E_k^* G_k(I_k)} \Bigg\}.
\end{aligned}
$$

Using the fact that $1 - x + \ln x \le 0$, where equality holds if and only if $x = 1$, we obtain

$$
\begin{aligned}
B_3 &\le \sum_{k=1}^{n} \nu_k \sum_{i=1}^{2} S_k^{(i)*} \sum_{j=1}^{n} \beta_{kj}^{(i)} G_j\big(I_j^*\big) \bigg[-\ln \frac{E_k^* G_j(I_j)}{E_k G_j(I_j^*)} - \ln \frac{E_k G_k(I_k^*)}{E_k^* G_k(I_k)} \bigg] \\
&= \sum_{k=1}^{n} \nu_k \sum_{i=1}^{2} S_k^{(i)*} \sum_{j=1}^{n} \beta_{kj}^{(i)} G_j\big(I_j^*\big) \ln \frac{G_k(I_k) G_j(I_j^*)}{G_k(I_k^*) G_j(I_j)} \\
&= \sum_{k=1}^{n} \nu_k \sum_{j=1}^{n} \overline{\beta}_{kj} \ln \frac{G_k(I_k) G_j(I_j^*)}{G_k(I_k^*) G_j(I_j)}.
\end{aligned}
\tag{19}
$$

In the following, we will show that

$$H_n := \sum_{k=1}^{n} v_k \sum_{j=1}^{n} \overline{\beta}_{kj} \ln \frac{G_k(I_k)G_j(I_j^*)}{G_k(I_k^*)G_j(I_j)} \equiv 0. \tag{20}$$

We first give the proof of (20) for $n = 2$, which would give a reader the basic yet clear ideas without being hidden by the complexity of terms caused by larger values of n. When $n = 2$, we have

$$H_2 = \sum_{k=1}^{2} v_k \sum_{j=1}^{2} \overline{\beta}_{kj} \ln \frac{G_k(I_k)G_j(I_j^*)}{G_k(I_k^*)G_j(I_j)}.$$

Formula (12) gives $v_1 = \overline{\beta}_{21}$ and $v_2 = \overline{\beta}_{12}$ in this case. Expanding H_2 yields

$$\begin{aligned}
H_2 &= \overline{\beta}_{21}\overline{\beta}_{11} \ln \frac{G_1(I_1)G_1(I_1^*)}{G_1(I_1^*)G_1(I_1)} + \overline{\beta}_{12}\overline{\beta}_{22} \ln \frac{G_2(I_2)G_2(I_2^*)}{G_2(I_2^*)G_2(I_2)} \\
&\quad + \overline{\beta}_{21}\overline{\beta}_{12} \ln \frac{G_1(I_1)G_2(I_2^*)}{G_1(I_1^*)G_2(I_2)} + \overline{\beta}_{12}\overline{\beta}_{21} \ln \frac{G_2(I_2)G_1(I_1^*)}{G_2(I_2^*)G_1(I_1)} \\
&= \overline{\beta}_{12}\overline{\beta}_{21} \left[\ln \frac{G_1(I_1)G_2(I_2^*)}{G_1(I_1^*)G_2(I_2)} + \ln \frac{G_2(I_2)G_1(I_1^*)}{G_2(I_2^*)G_1(I_1)} \right] = 0.
\end{aligned}$$

For more general n, by a similar argument as in the proof of $\sum_{k,j=1}^{n} v_k \overline{\beta}_{kj} \ln \frac{E_k^* E_j}{E_k E_j^*} \equiv 0$ in [7], we obtain that

$$\sum_{k=1}^{n} v_k \sum_{j=1}^{n} \overline{\beta}_{kj} \ln \frac{G_k(I_k)G_j(I_j^*)}{G_k(I_k^*)G_j(I_j)} = -\sum_{k,j=1}^{n} v_k \overline{\beta}_{kj} \ln \frac{G_k(I_k^*)G_j(I_j)}{G_k(I_k)G_j(I_j^*)} \equiv 0.$$

From (17)-(19), we see that if $\dot{V} = 0$, then

$$S_k^{(i)} = S_k^{(i)*}, \quad i = 1, 2, k = 1, 2, \dots, n. \tag{21}$$

If (21) holds, it follows from (2) that

$$\begin{cases}
0 = \varphi_k(S_k^{(1)*}) - \sum_{j=1}^{n} \beta_{kj}^{(1)} S_k^{(1)*} G_j(I_j) - a_k S_k^{(1)*}, \\
0 = a_k S_k^{(1)*} - \sum_{j=1}^{n} \beta_{kj}^{(2)} S_k^{(2)*} G_j(I_j) - d_k^{(2)} S_k^{(2)*}.
\end{cases}$$

Then we obtain that

$$\dot{E}_k = \left(\varphi_k(S_k^{(1)*}) - a_k S_k^{(1)*} \right) + \left(a_k S_k^{(1)*} - d_k^{(2)} S_k^{(2)*} \right) - (d_k + \eta_k)E_k.$$

This implies that

$$\lim_{t \to +\infty} E_k = \frac{\left(\varphi_k(S_k^{(1)*}) - a_k S_k^{(1)*} \right) + \left(a_k S_k^{(1)*} - d_k^{(2)} S_k^{(2)*} \right)}{(d_k + \eta_k)} = E_k^*. \tag{22}$$

By (22) and the fourth equation of system (2), we have

$$\lim_{t \to +\infty} I_k = \frac{\eta_k E_k^*}{m_k} = I_k^*. \tag{23}$$

From (21)-(23) and the characteristics of V, we obtain that the largest invariant subset of the set, where $\dot{V} = 0$, is the singleton $\{P^*\}$. By LaSalle's invariance principle, P^* is globally asymptotically stable for $R_0 > 1$. □

4 Numerical examples

For certain sexually transmitted diseases, AIDS/HIV for example, it is natural to consider two groups of people: a group of males and a group of females. Further, it is always assumed that there are two important age stages for the susceptible, a group of immature susceptible $S_k^{(1)}$ who are less than 18 years old, and a group of mature susceptible $S_k^{(2)}$ who are more than 18 years old. Thus, we consider the following model:
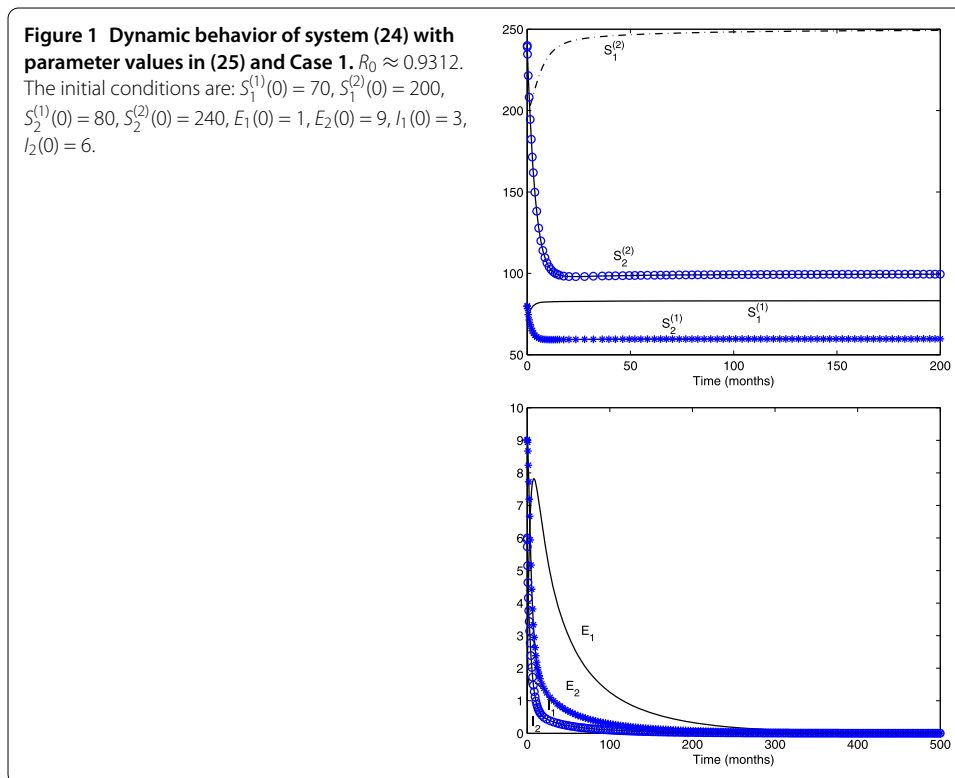
$$\begin{cases} \dot{S}_k^{(1)} = \varphi_k(S_k^{(1)}) - \sum_{j=1}^{2} \beta_{kj}^{(1)} S_k^{(1)} G_j(I_j) - a_k S_k^{(1)}, \\ \dot{S}_k^{(2)} = a_k S_k^{(1)} - \sum_{j=1}^{2} \beta_{kj}^{(2)} S_k^{(2)} G_j(I_j) - d_k^{(2)} S_k^{(2)}, \\ \dot{E}_k = \sum_{i=1}^{2} \sum_{j=1}^{2} \beta_{kj}^{(i)} S_k^{(i)} G_j(I_j) - (d_k + \eta_k)E_k, \\ \dot{I}_k = \eta_k E_k - m_k I_k, \quad k = 1, 2, \end{cases} \tag{24}$$
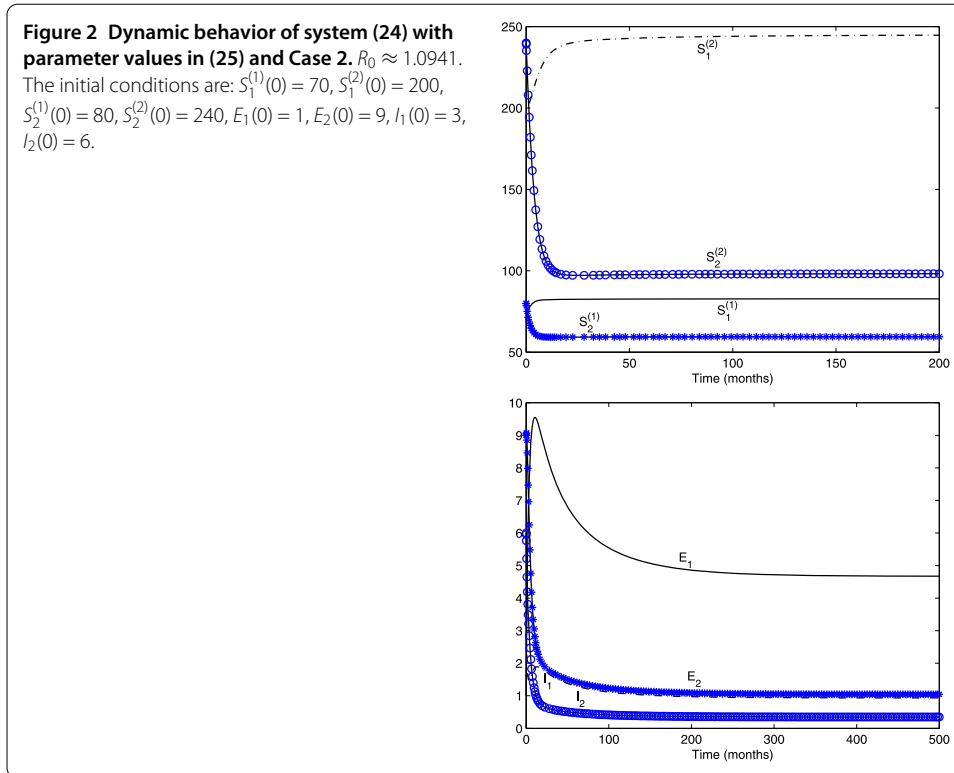
where $\varphi_k(S_k^{(1)}) = b_k - d_k^{(1)} S_k^{(1)}$, $G_j(I_j) = \frac{I_j}{1+\alpha_j I_j}$.

Clearly, (A1)-(A4) hold. We fix the parameters as follows:

$$b_1 = 50, \qquad b_2 = 30, \qquad d_1^{(1)} = 0.001, \qquad d_1^{(2)} = 0.2, \qquad d_2^{(1)} = 0.002,$$

$$d_2^{(2)} = 0.3, \qquad d_1 = 0.1, \qquad d_2 = 0.2, \qquad \eta_1 = 0.1, \qquad \eta_2 = 0.2, \tag{25}$$

$$m_1 = 0.5, \qquad m_2 = 0.6, \qquad a_1 = 0.6, \qquad a_2 = 0.5, \qquad \alpha_1 = \alpha_2 = 0.1.$$

Then we have $P_0 \approx (83.1947, 249.5840, 59.7610, 99.6016, 0, 0, 0, 0)$.

Figure 1 Dynamic behavior of system (24) with parameter values in (25) and Case 1. $R_0 \approx 0.9312$. The initial conditions are: $S_1^{(1)}(0) = 70$, $S_1^{(2)}(0) = 200$, $S_2^{(1)}(0) = 80$, $S_2^{(2)}(0) = 240$, $E_1(0) = 1$, $E_2(0) = 9$, $I_1(0) = 3$, $I_2(0) = 6$.

Figure 2 Dynamic behavior of system (24) with parameter values in (25) and Case 2. $R_0 \approx 1.0941$. The initial conditions are: $S_1^{(1)}(0) = 70$, $S_1^{(2)}(0) = 200$, $S_2^{(1)}(0) = 80$, $S_2^{(2)}(0) = 240$, $E_1(0) = 1$, $E_2(0) = 9$, $I_1(0) = 3$, $I_2(0) = 6$.

Case 1. If $\beta_{1j}^{(1)} = \beta_{k1}^{(2)} = 0.002$, $\beta_{2j}^{(1)} = \beta_{k2}^{(2)} = 0.002$, $k = 1, 2$, $j = 1, 2$, then we obtain

$$\mathbf{Q} \approx \begin{pmatrix} 0.6656 & 0.5546 \\ 0.3187 & 0.2656 \end{pmatrix}, \qquad R_0 \approx 0.9312.$$

By Theorem 3.1, the disease dies out in both groups. Numerical simulation illustrates this fact (see Figure 1).

Case 2. If $\beta_{1j}^{(1)} = \beta_{k1}^{(2)} = 0.0025$, $\beta_{2j}^{(1)} = \beta_{k2}^{(2)} = 0.002$, $k = 1, 2$, $j = 1, 2$, then we have $P^* \approx$ $(82.7845, 244.9127, 59.4787, 98.2113, 4.6734, 1.0441, 0.9347, 0.3480)$ and $R_0 \approx 1.0941$.

By Theorem 3.2, the disease persists in both groups. Numerical simulation illustrates this fact (see Figure 2).

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