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Approximate controllability of fractional impulsive evolution systems involving nonlocal initial conditions

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Abstract

This work is concerned with the approximate controllability of a nonlinear fractional impulsive evolution system under the assumption that the corresponding linear system is approximate controllable. Using the fractional calculus, the Krasnoselskii fixed point theorem, and the technique of controllability theory, some new sufficient conditions for approximate controllability of fractional impulsive evolution equations are obtained. The results in this paper are generalizations and continuations of the recent results on this issue. At the end, an example is given to illustrate the effectiveness of the main results.

MSC: 93B05; 34A08; 34A37

Keywords: approximate controllability; evolution equations; mild solution; nonlocal conditions

1 Introduction

In recent years, fractional differential systems have provided us with an excellent tool in electrochemistry, physics, porous media, control theory, engineering, *etc.*, due to the descriptions of memory and hereditary properties of various materials and processes. The research as regards the fractional systems has received more and more attention very recently.

Our interest is the following fractional impulsive evolution control system involving nonlocal conditions:

$$\begin{cases} {}^{c}D_t^q x(t) = -Ax(t) + f(t, x(t), Gx(t)) + Bu(t), \quad t \in J := [0, T], t \neq t_k, \\ \Delta x(t_k) = I_k(x(t_k^-)), \quad k = 1, 2, \ldots, n, \\ x(0) = x_0 + h(x), \end{cases} \tag{1.1}$$

where ${}^{c}D_t^q$ is the Caputo fractional derivative of order $0 < q < 1$, the state variable x takes values in a Hilbert space X, $\Delta x(t_k) := x(t_k^+) - x(t_k^-)$ represents the jump in the state x at time t_k, $x(t_k^+)$ and $x(t_k^-)$ denote, respectively, the right and left limits of $x(t)$ at $t = t_k$ with $0 < t_1 < t_2 < \cdots < t_n < T$ and $x(t_k^-) = x(t_k)$, and $I_k : X_\alpha \to X_\alpha$ is the jump operator. $-A$ is the infinitesimal generator of an analytic semigroup $\{S(t), t \geq 0\}$ of a bounded operator on the Hilbert space X, the control function u is given in $L^2([0, T], U)$, U is a Hilbert space,

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$B : U \to X_\alpha$ is a bounded linear operator, $f : J \times X_\alpha \times X_\alpha \to X$, $h : PC([0, T], X_\alpha) \to X_\alpha$ are given functions which will be specified later in Section 3, and

$$Gx(t) = \int_0^t K(t,s)x(s)\,ds$$

is a Volterra integral operator with integral kernel $K \in C(\Delta, [0, +\infty))$, $\Delta = \{(t,s) : 0 \leq s \leq t \leq T\}$.

Controllability is one of the important concepts both in mathematics and in control theory. Generally speaking, controllability enables one to steer the control system from an arbitrary initial state to an arbitrary final state using the set of admissible controls. The controllability problem is a mathematical description of many physical systems such as fluid mechanic systems, quantum systems, and so forth. Controllability of deterministic and stochastic dynamical control systems is well developed by using different kinds of methods which can be found in [1–5]. But the question is that the concept of exact controllability is usually too strong if we consider the problem in the infinite dimensional spaces. Therefore, approximate controllability, the weaker concept of controllability, has gained much attention recently, which steers the system to an arbitrary small neighborhood of a final state (see, for example, [6–10]). Mahmudov and Zorlu [7] researched the approximate controllability of fractional evolution equations involving the Caputo fractional derivative, the sufficient conditions are established under the assumption that the corresponding linear system is approximate controllable, by using the theory of fractional calculus and semigroup and the Schauder fixed point theorem. Ganesh *et al.* [6] derived a set of sufficient conditions for the approximate controllability of a class of fractional integro-differential evolution equations.

A strong motivation for investigating the fractional evolution equation comes from the fact that it provides an excellent tool for the modeling of various phenomena in many fields of physics, engineering, economics, *etc.* The existence and uniqueness of the fractional evolution equation have been studied by several authors (see [11–16] and references therein) with the help of various fixed point theorem and operator theory. Zhou and Jiao [15, 16] discussed the fractional evolution equations and defined the mild solution by means of the probability density function, which has been developed by Wang *et al.* [13]. On the other hand, there are significant developments in the theory of impulses especially in the area of impulsive differential equations with fixed moments, which provide a natural description of observed evolution processes, regarding these as an important tool for better understanding several real-world phenomena in applied sciences. Very recently, we [17] considered the fractional impulsive differential equations with delay, and the resonance case in [18]. For more details as regards impulsive differential equations, the reader can refer to the monograph of Lakshmikantham *et al.* [19] and [1, 20–22].

However, there are limited works considering the approximate controllability of the fractional impulsive evolution system with nonlocal conditions [22, 23]. So, in this work, the main objective is to provide the sufficient conditions for the approximate controllability of the control system (1.1). The nonlocal boundary condition, initiated by Byszewski [24], is studied in [8, 13]. It is claimed there that it may be used in some physical problems successfully. The technique we use is the Krasnoselskii fixed point theorem and the semigroup theory. More precisely, by using the constructive control function, we transfer the approximate controllability problem for control system (1.1) into the fixed point problem for operator Λ. Furthermore, the results on the approximate controllability of fractional control systems are derived.

The outline of this paper is as follows. In Section 2, we recall some essential results on the fractional powers of the generator of a compact analytic semigroup and introduce the mild solution for the system (1.1). In Section 3, we study the existence of a mild solution for the system (1.1) under the feedback control $u_\epsilon(x)$ defined in (3.4). We show that the control system (1.1) is approximately controllable on $[0,T]$ provided that the corresponding linear system is approximate controllable. Finally, an example is given to illustrate the effectiveness of the main results.

2 Background materials and preliminaries

We assume that X is a Hilbert space with the norm $\|\cdot\| = \sqrt{\langle\cdot,\cdot\rangle}$. Throughout this paper, we assume that $-A : D(A) \subset X \to X$ is the infinitesimal generator of a compact analytic semigroup $\{S(t), t \geq 0\}$ of uniformly bounded linear operators in X, *i.e.*, there exists $M > 1$ such that $\|S(t)\| \leq M$ for $t \geq 0$ (see [25, 26]). We assume without loss of generality that $0 \in \rho(A)$, where $\rho(A)$ is the resolvent set of A. Then for any $\alpha > 0$, we can define $A^{-\alpha}$ as

$$A^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} S(t)\,dt,$$

which shows that $A^{-\alpha}$ is an injective continuous endomorphism of X. Therefore, we can define $A^\alpha = (A^{-\alpha})^{-1}$, it is a closed bijective linear operator in X.

Denote X_α by the Hilbert space $D(A^\alpha)$ equipped with the norm $\|x\|_\alpha = \|A^\alpha x\|$ for $x \in D(A^\alpha)$, which is equivalent to the graph norm of A^α. Moreover, the fractional power A^α has the following basic properties.

Lemma 2.1 ([25]) *A^α and $S(t)$ have the following properties*:

(1) $S(t) : X \to X_\alpha$ *for each* $t > 0$ *and* $\alpha \geq 0$.

(2) $A^\alpha S(t)x = S(t)A^\alpha x$ *for each* $x \in D(A^\alpha)$ *and* $t \geq 0$.

(3) *For every* $t > 0$, $A^\alpha S(t)$ *is bounded in* X *and there exists* $M_\alpha > 0$ *such that*

$$\left\|A^\alpha S(t)\right\| \leq M_\alpha t^{-\alpha}.$$

(4) $A^{-\alpha}$ *is bounded linear operator for* $0 \leq \alpha \leq 1$, *there exists* $C_\alpha > 0$ *such that* $\|A^{-\alpha}\| \leq C_\alpha$.

Next, we present some basic knowledge and definitions as regards fractional calculus theory, which can be found in the monographs of Podlubny [27], Miller and Ross [28], and Kilbas *et al.* [29].

Definition 2.1 The fractional integral of order $\alpha > 0$ with the lower limit 0 for a function y is defined as

$$I^\alpha y(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\,ds, \quad t > 0, \alpha > 0,$$

provided the right-hand side is pointwise defined on $[0,\infty)$, where Γ is the gamma function.

Definition 2.2 The Caputo fractional derivative of order $\alpha > 0$ with the lower limit 0 for a function y is defined as

$$({}^{c}D^{\alpha}y)(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1}y^{(n)}(s)\,ds = I^{n-\alpha}y^{(n)}, \quad t > 0, 0 \le n-1 < \alpha < n.$$

Remark 2.1

(1) The Caputo derivative of a constant is equal to zero.

(2) If y is an abstract function with values in X, then the integrals that appear in Definitions 2.1 and 2.2 are taken in the Bochner sense.

According to Definitions 2.1 and 2.2, it is suitable to rewrite the problem (1.1) in the following equivalent integral equation:

$$\begin{aligned} x(t) &= x_0 + h(x) + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\bigl[-Ax(s) + f\bigl(s, x(s), Gx(s)\bigr) + Bu(s)\bigr]\,ds \\ &\quad + \sum_{0<t_k<t} I_k\bigl(x(t_k^-)\bigr), \quad t \in [0,T], \end{aligned} \tag{2.1}$$

provided that the integral above exists. Applying the Laplace transform

$$\begin{aligned} &X(\lambda) = \int_0^\infty e^{-\lambda s}x(s)\,ds, \qquad F(\lambda) = \int_0^\infty e^{-\lambda s}f\bigl(s, x(s), Gx(s)\bigr)\,ds, \\ &b(\lambda) = \int_0^\infty e^{-\lambda s}Bu(s)\,ds \end{aligned}$$

to (2.1) and applying the method similar to [7], we get

$$\begin{aligned} X(\lambda) &= \frac{1}{\lambda}\bigl(x_0 + h(x)\bigr) + \frac{1}{\lambda^q}\bigl[-AX(\lambda) + F(\lambda) + b(\lambda)\bigr] + \frac{1}{\lambda}\sum_{0<t_k<t} I_k\bigl(x(t_k^-)\bigr) \\ &= \bigl(\lambda^q E + A\bigr)^{-1}\Bigl[\lambda^{q-1}\bigl(x_0 + h(x)\bigr) + F(\lambda) + b(\lambda) + \sum_{0<t_k<t}\lambda^{q-1}I_k\bigl(x(t_k^-)\bigr)\Bigr] \\ &= \lambda^{q-1}\int_0^\infty e^{-\lambda^q r}S(r)\bigl[x_0 + h(x)\bigr]\,dr + \int_0^\infty e^{-\lambda^q r}S(r)\bigl[F(\lambda) + b(\lambda)\bigr]\,dr \\ &\quad + \sum_{0<t_k<t}\lambda^{q-1}\int_0^\infty e^{-\lambda^q r}S(r)I_k\bigl(x(t_k^-)\bigr)\,dr, \end{aligned} \tag{2.2}$$

provided the integrals above exist, where E is the identity operator defined on X.

Consider the one-sided stable probability density [30]

$$\omega_q(\theta) = \frac{1}{\pi}\sum_{n=1}^\infty (-1)^{n-1}\theta^{-qn-1}\frac{\Gamma(nq+1)}{n!}\sin(n\pi q), \quad \theta \in (0,\infty),$$

whose Laplace transform is given by

$$\int_0^\infty e^{-\lambda\theta}\omega_q(\theta)\,d\theta = e^{-\lambda^q}, \quad q \in (0,1). \tag{2.3}$$

Using (2.2) and (2.3), it means that

$$
\begin{aligned}
X(\lambda) = \int_0^\infty e^{-\lambda t}\bigg\{&\int_0^\infty \omega_q(\theta)S\left(\frac{t^q}{\theta^q}\right)\big(x_0 + h(x)\big)\,d\theta \\
&+ \sum_{0<t_k<t}\int_0^\infty \omega_q(\theta)S\left(\frac{t^q}{\theta^q}\right)I_k\big(x\big(t_k^-\big)\big)\,d\theta \\
&+ q\int_0^t\int_0^\infty \omega_q(\theta)S\left(\frac{(t-s)^q}{\theta^q}\right)\big[f\big(s,x(s),Gx(s)\big) + Bu(s)\big]\frac{(t-s)^{q-1}}{\theta^q}\,d\theta\,ds\bigg\}\,dt.
\end{aligned}
$$

Now we can invert the last Laplace transform to get

$$
\begin{aligned}
x(t) &= \int_0^\infty \omega_q(\theta)S\left(\frac{t^q}{\theta^q}\right)\big(x_0 + h(x)\big)\,d\theta + \sum_{0<t_k<t}\int_0^\infty \omega_q(\theta)S\left(\frac{t^q}{\theta^q}\right)I_k\big(x\big(t_k^-\big)\big)\,d\theta \\
&\quad + q\int_0^t\int_0^\infty \omega_q(\theta)S\left(\frac{(t-s)^q}{\theta^q}\right)\big[f\big(s,x(s),Gx(s)\big) + Bu(s)\big]\frac{(t-s)^{q-1}}{\theta^q}\,d\theta\,ds \\
&= \int_0^\infty \xi_q(\theta)S\big(t^q\theta\big)\big(x_0 + h(x)\big)\,d\theta + \sum_{0<t_k<t}\int_0^\infty \xi_q(\theta)S\big((t-t_k)^q\theta\big)I_kx\big(t_k^-\big)\,d\theta \\
&\quad + q\int_0^t\int_0^\infty \theta(t-s)^{q-1}\xi_q(\theta)S\big((t-s)^q\theta\big)\big[f\big(s,x(s),Gx(s)\big) + Bu(s)\big]\,d\theta\,ds,
\end{aligned}
$$

where $\xi_q(\theta) = \frac{1}{q}\theta^{-1-\frac{1}{q}}\omega_q(\theta^{-\frac{1}{q}})$ is the probability density function on $(0,\infty)$, that is, $\xi_q(\theta) \geq 0$ and $\int_0^\infty \xi_q(\theta)\,d\theta = 1$.

For $x \in X$ and $0 < q < 1$, let us define the following two sets $\{U(t) : t \geq 0\}$ and $\{V(t) : t \geq 0\}$ of operators:

$$
U(t) = \int_0^\infty \xi_q(\theta)S\big(t^q\theta\big)\,d\theta, \qquad V(t) = q\int_0^\infty \theta\xi_q(\theta)S\big(t^q\theta\big)\,d\theta.
$$

We introduce the space $PC(J,X_\alpha)$ formed by all piecewise continuous functions $x : J \to X_\alpha$ such that $x(\cdot)$ is continuous at $t \neq t_k$, $x(t_k^-) = x(t_k)$ and $x(t_k^+)$ exist, for $k = 1,\ldots,n$. It is clear that $PC(J,X_\alpha)$ endowed with the norm $\|x\|_\alpha = \sup_{t\in[0,T]}\|x(t)\|_\alpha$ is a Banach space. Then a mild solution of system (1.1) can be defined as follows.

Definition 2.3 A solution $x \in PC(J,X_\alpha)$ is said to be a mild of (1.1) if for any $u \in L^2([0,T],U)$, the integral equation

$$
\begin{aligned}
x(t) = U(t)\big[x_0 + h(x)\big] &+ \int_0^t (t-s)^{q-1}V(t-s)\big[Bu(s) + f\big(s,x(s),Gx(s)\big)\big]\,ds \\
&+ \sum_{0<t_k<t} U(t-t_k)I_k\big(x\big(t_k^-\big)\big)
\end{aligned}
$$

is satisfied.

We now state the following lemmas which will be used in the sequel.

Lemma 2.2 ([13]) *The operators U and V have the following properties*:

(i) *For any fixed $t \geq 0$ and $x \in X_\alpha$, we find that the operators $U(t)$ and $V(t)$ are linear and bounded operators, i.e., for any $x \in X$,*

$$\left\|U(t)x\right\|_\alpha \leq M\|x\|_\alpha \quad \text{and} \quad \left\|V(t)x\right\|_\alpha \leq \frac{M}{\Gamma(q)}\|x\|_\alpha.$$

(ii) *The operators $U(t)$ and $V(t)$ are strongly continuous for all $t \geq 0$.*
(iii) *$U(t)$ and $V(t)$ are compact operators in X for all $t > 0$.*
(iv) *For every $t > 0$, the restriction of $U(t)$ to X_α and the restriction of $V(t)$ to X_α are compact operators in X_α.*
(v) *For all $x \in X$ and $t \in (0, \infty)$,*

$$\left\|A^\alpha V(t)x\right\| \leq C_\alpha t^{-\alpha q}\|x\|, \quad \text{where } C_\alpha := \frac{M_\alpha q\Gamma(2-\alpha)}{\Gamma(1+q(1-\alpha))}.$$

Lemma 2.3 (Krasnoselskii fixed point theorem [31, 32]) *Let M be a closed, convex, and nonempty subset of a Banach space X. Let A, B be the operators such that*
(i) *$Ax + By \in M$, wherever $x, y \in M$;*
(ii) *A is completely continuous;*
(iii) *B is a contraction mapping.*
Then there exists $z \in M$ such that $z = Az + Bz$.

3 Main results

Let $x(T; x_0, u)$ be the state value of (1.1) at terminal time T corresponding to the control u and the initial value x_0. Introduce the set $\mathfrak{R}(T, x_0) = \{x(T; x_0, u) : u \in L^2([0, T], U)\}$, which is called the reachable set of system (1.1) at terminal time T, its closure in X_α is denoted by $\overline{\mathfrak{R}(T, x_0)}$.

Definition 3.1 ([7]) The system (1.1) is said to be approximately controllable on $[0, T]$ if $\overline{\mathfrak{R}(T, x_0)} = X_\alpha$, that is, given an arbitrary $\epsilon > 0$ it is possible to steer from the point x_0 to within a distance ϵ from all points in the state space X_α at time T.

In order to study the approximate controllability for the fractional control system (1.1), we introduce the following linear fractional differential system:

$$\begin{cases} {}^cD_t^q x(t) = -Ax(t) + Bu(t), & t \in J := [0, T], \\ x(0) = x_0. \end{cases} \tag{3.1}$$

The approximate controllability for linear fractional system (3.1) is a natural generalization of approximate controllability of a linear first order control system [33]. Definite the operator $L_0^T := \int_0^T (T-s)^{q-1}V(T-s)Bu(s)\,ds : L^2([0, T], U) \to PC([0, T], X_\alpha)$. Let us introduce the controllability and resolvent operators associated with (3.1) as

$$\Gamma_0^T = \int_0^T (T-s)^{q-1}V(T-s)BB^*V^*(T-s)\,ds : X_\alpha \to X_\alpha, \tag{3.2}$$

$$R\left(\epsilon, \Gamma_0^T\right) = \left(\epsilon E + \Gamma_0^T\right)^{-1} : X_\alpha \to X_\alpha, \quad \epsilon > 0, \tag{3.3}$$

respectively, where B^* and $V^*(t)$ are the adjoint of B and $V(t)$, respectively. It is not difficult to see that Γ_0^T is a linear bounded operator.

Lemma 3.1 ([8]) *The linear fractional control system* (3.1) *is approximately controllable on* $[0,T]$ *if and only if* $\epsilon R(\epsilon,\Gamma_0^T)\to 0$ *as* $\epsilon\to 0^+$ *in the strong operator topology.*

Before proving the approximately controllable results, we need the following basic assumptions.

(H_f) $f:[0,T]\times X_\alpha\times X_\alpha\to X$ is continuous, and for any $r\in\mathbb{R}^+$, there exist a constant $\gamma\in(0,(1-\alpha)q)$ and functions $\varphi_r\in L^{\frac{1}{\gamma}}([0,T],\mathbb{R}^+)$ such that

$$\sup\bigl\{\bigl\|f(t,x,Gx)\bigr\|:\|x\|_\alpha\le r\bigr\}\le\varphi_r(t)\quad\text{and}\quad\liminf_{r\to\infty}\frac{\|\varphi_r\|_L^{\frac{1}{\gamma}}}{r}=\sigma<\infty.$$

(H_h) $h:PC([0,T],X_\alpha)\to X_\alpha$ is a Lipschitz function with Lipschitz constant L_h.
(H_I) $I:PC([0,T],X_\alpha)\to X_\alpha$ is a Lipschitz function with Lipschitz constant L_I.
(H_c) The linear system associated with (1.1) is approximately controllable on $[0,T]$.

For an arbitrary function $x\in PC([0,T],X_\alpha)$, considering the form of a mild solution of (1.1) in Definition 2.3, as well as the controllability and resolvent operators in (3.2)-(3.3), we choose the feedback control function associated with the nonlinear system (1.1) as follows:

$$\begin{aligned}u_\epsilon(t,x)={}&B^*V^*(T-t)R\bigl(\epsilon,\Gamma_0^T\bigr)\biggl[x_T-U(T)\bigl(x_0+h(x)\bigr)-\sum_{0<t_k<t}U(T-t_k)I_kx(t_k)\\&-\int_0^T(T-s)^{q-1}V(T-s)f\bigl(s,x(s),Gx(s)\bigr)\,ds\biggr].\end{aligned}\tag{3.4}$$

Let $B_r=\{x\in PC([0,T],X_\alpha):\|x\|_\alpha\le r\}$, where r is a positive constant. Then B_r is clearly a bounded, closed, and convex subset in $PC([0,T],X_\alpha)$. For any $\epsilon>0$, using the above control function in (3.4), define the operator $\Lambda:B_r\to B_r$ as follows:

$$(\Lambda x)(t)=(\Phi x)(t)+(\Pi x)(t),\quad t\in[0,T],\tag{3.5}$$

where

$$(\Phi x)(t)=U(t)\bigl(x_0+h(x)\bigr)+\sum_{0<t_k<t}U(t-t_k)I_k\bigl(x(t_k)\bigr),\tag{3.6}$$

$$\begin{aligned}(\Pi x)(t)={}&\int_0^t(t-s)^{q-1}V(t-s)f\bigl(s,x(s),Gx(s)\bigr)\,ds\\&+\int_0^t(t-s)^{q-1}V(t-s)Bu_\epsilon\bigl(s,x(s)\bigr)\,ds.\end{aligned}\tag{3.7}$$

Theorem 3.1 *If the hypotheses* (H_f), (H_h), *and* (H_I) *are satisfied, then the fractional Cauchy problem* (1.1) *with* $u=u_\epsilon(t,x)$ *has at least one mild solution provided that*

$$L_c+\frac{C_\alpha ML_B^2T^{(1-\alpha)q}}{\epsilon q(1-\alpha)\Gamma(q)}L_c<1,\tag{3.8}$$

where

$$L_c:=ML_h+MnL_I+C_\alpha\biggl[\frac{1-\gamma}{(1-\alpha)q-\gamma}\biggr]^{1-\gamma}T^{(1-\alpha)q-\gamma}\sigma.$$

Proof Obviously, the fractional Cauchy problem (1.1) with the control in (3.4) has a mild solution if and only if the operator Λ has a fixed point on B_r. According to the requirements of the Krasnoselskii theorem, our proof will be divided into several steps.

Step 1. For any $x, y \in B_r$, we claim that $\Phi x + \Pi y \subset B_r$. At first, using the assumption yields the following estimation:

$$\begin{aligned}
\|u_\epsilon(t,x)\| &\le \frac{ML_B}{\epsilon\Gamma(q)}\Bigg[\big\|x_T - U(T)\big(x_0 + h(x)\big)\big\|_\alpha + \bigg\|\sum_{0<t_k<t} U(T-t_k)I_k\big(x(t_k)\big)\bigg\|_\alpha \\
&\quad + \bigg\|\int_0^T (T-s)^{q-1}V(T-s)f\big(s,x(s),Gx(s)\big)\,ds\bigg\|_\alpha\Bigg] \\
&\le \frac{ML_B}{\epsilon\Gamma(q)}\Bigg[\|x_T\|_\alpha + M\big(\|x_0\|_\alpha + \big\|h(0)\big\|_\alpha + L_h r\big) + ML_I\Bigg(nr + \sum_{k=1}^n \big\|I_k(0)\big\|_\alpha\Bigg) \\
&\quad + C_\alpha\bigg\|\int_0^T (T-s)^{(1-\alpha)q-1}f\big(s,x(s),Gx(s)\big)\,ds\bigg\|\Bigg] \\
&\le \frac{ML_B}{\epsilon\Gamma(q)}L_u(r),
\end{aligned}$$

where

$$\begin{aligned}
L_u(r) &:= \|x_T\|_\alpha + M\big(\|x_0\|_\alpha + \big\|h(0)\big\|_\alpha + L_h r\big) + ML_I\Bigg(nr + \sum_{k=1}^n \big\|I_k(0)\big\|_\alpha\Bigg) \\
&\quad + C_\alpha\bigg(\frac{1-\gamma}{(1-\alpha)q-\gamma}\bigg)^{1-\gamma} T^{(1-\alpha)q-\gamma}\|\varphi_r\|_{L^{\frac{1}{\gamma}}}.
\end{aligned}$$

Then, for any $x, y \in B_r$,

$$\begin{aligned}
\|\Phi x + \Pi y\|_\alpha &\le \big\|U(t)\big(x_0 + h(x)\big)\big\|_\alpha + \bigg\|\int_0^t (t-s)^{q-1}V(t-s)f\big(s,y(s),Gy(s)\big)\,ds\bigg\|_\alpha \\
&\quad + \bigg\|\sum_{0<t_k<t} U(t-t_k)I_k\big(x(t_k)\big)\bigg\|_\alpha + \bigg\|\int_0^t (t-s)^{q-1}V(t-s)Bu_\epsilon(s,y)\,ds\bigg\|_\alpha \\
&\le M\big(\|x_0\|_\alpha + \big\|h(0)\big\|_\alpha + L_h r\big) + C_\alpha\bigg(\frac{1-\gamma}{(1-\alpha)q-\gamma}\bigg)^{1-\gamma} T^{(1-\alpha)q-\gamma}\|\varphi_r\|_{L^{\frac{1}{\gamma}}} \\
&\quad + ML_I\Bigg(nr + \sum_{k=1}^n \big\|I_k(0)\big\|_\alpha\Bigg) + C_\alpha\int_0^t (t-s)^{(1-\alpha)q-1}\big\|Bu_\epsilon(s,y)\big\|\,ds \\
&\le M\big(\|x_0\|_\alpha + \big\|h(0)\big\|_\alpha + L_h r\big) + C_\alpha\bigg(\frac{1-\gamma}{(1-\alpha)q-\gamma}\bigg)^{1-\gamma} T^{(1-\alpha)q-\gamma}\|\varphi_r\|_{L^{\frac{1}{\gamma}}} \\
&\quad + ML_I\Bigg(nr + \sum_{k=1}^n \big\|I_k(0)\big\|_\alpha\Bigg) + \frac{C_\alpha L_B^2 M T^{(1-\alpha)q}}{(1-\alpha)q\epsilon\Gamma(q)}L_u(r).
\end{aligned}$$

Thus, for any $\epsilon > 0$, from (3.8), there exists $r(\epsilon) > 0$ such that $\|\Phi x + \Pi_y\|_\alpha \in B_{r(\epsilon)}$.

Step 2. We show that Π in (3.7) is completely continuous on $B_{r(\epsilon)}$; here we divide the argument into two substeps.

Step 2.1. We first show that Π is continuous on $B_{r(\epsilon)}$. Let $\{x_n\}_{n=1}^{\infty} \subset B_{r(\epsilon)}$ be a sequence such that $x_n \to x$ as $n \to \infty$ in $PC([0,T], X_\alpha)$. For any $t \in [0,T]$, with the help of the continuity of f, u_ϵ, we can see that

$$f\bigl(s, x_n(s), Gx_n(s)\bigr) \to f\bigl(s, x(s), Gx(s)\bigr), \qquad u_\epsilon(s, x_n) \to u_\epsilon(s, x).$$

Since

$$\begin{aligned}
\bigl\| \Pi x_n(t) - \Pi x(t) \bigr\|_\alpha &= \biggl\| \int_0^t (t-s)^{q-1} V(t-s) \bigl[f\bigl(s, x_n(s), Gx_n(s)\bigr) - f\bigl(s, x(s), Gx(s)\bigr) \bigr] \, ds \biggr\|_\alpha \\
&\quad + \biggl\| \int_0^t (t-s)^{q-1} V(t-s) \bigl[Bu_\epsilon(s, x_n) - Bu_\epsilon(s, x) \bigr] \, ds \biggr\|_\alpha \\
&\le C_\alpha \int_0^t (t-s)^{(1-\alpha)q-1} \bigl\| f\bigl(s, x_n(s), Gx_n(s)\bigr) - f\bigl(s, x(s), Gx(s)\bigr) \bigr\| \, ds \\
&\quad + C_\alpha \int_0^t (t-s)^{(1-\alpha)q-1} \bigl\| Bu_\epsilon(s, x_n) - Bu_\epsilon(s, x) \bigr\| \, ds
\end{aligned}$$

and applying the Lebesgue dominated convergence theorem, for all $t \in [0,T]$, we conclude that

$$\bigl\| \Pi x_n(t) - \Pi x(t) \bigr\|_\alpha \to 0 \quad \text{as } n \to \infty,$$

which implies that Π is continuous on $B_{r(\epsilon)}$.

Step 2.2. Π is compact on $B_{r(\epsilon)}$. For the sake of convenience, we define $Nx(t) := f(t, x(t), Gx(t)) + Bu_\epsilon(t, x(t))$. Let $t \in [0,T]$ be fixed and $\delta, \eta > 0$ be small enough. For $x \in B_{r(\epsilon)}$, define the operator

$$\begin{aligned}
\Pi^{\delta,\eta} x(t) &= \int_0^{t-\delta} \int_\eta^\infty q\theta (t-s)^{q-1} \xi_q(\theta) S\bigl((t-s)^q \theta\bigr) Nx(s) \, d\theta \, ds \\
&= S\bigl(\delta^q \eta\bigr) \int_0^{t-\delta} \int_\eta^\infty q\theta (t-s)^{q-1} \xi_q(\theta) S\bigl((t-s)^q \theta - \delta^q \eta\bigr) Nx(s) \, d\theta \, ds \\
&:= S\bigl(\delta^q \eta\bigr) y(t, \delta).
\end{aligned}$$

By means of the compactness of $S(\delta^q \eta)$ and the boundedness of $y(t, \delta)$, for any $t \in [0,T]$, the set $\{\Pi^{\delta,\eta} x : x \in B_{r(\epsilon)}\}$ is relatively compact in X_α.

On the other hand,

$$\begin{aligned}
&\bigl\| \Pi x(t) - \Pi^{\delta,\eta} x(t) \bigr\|_\alpha \\
&\quad \le \biggl\| \int_0^t \int_0^\eta q\theta (t-s)^{q-1} \xi_q(\theta) S\bigl((t-s)^q \theta\bigr) Nx(s) \, d\theta \, ds \biggr\|_\alpha \\
&\qquad + \biggl\| \int_{t-\delta}^t \int_\eta^\infty q\theta (t-s)^{q-1} \xi_q(\theta) S\bigl((t-s)^q \theta\bigr) Nx(s) \, d\theta \, ds \biggr\|_\alpha \\
&\quad \le qM_\alpha \int_0^t (t-s)^{(1-\alpha)q-1} \bigl\| Nx(s) \bigr\| \, ds \cdot \int_0^\eta \theta^{1-\alpha} \xi_q(\theta) \, d\theta \\
&\qquad + qM_\alpha \int_{t-\delta}^t (t-s)^{(1-\alpha)q-1} \bigl\| Nx(s) \bigr\| \, ds \cdot \int_\eta^\infty \theta^{1-\alpha} \xi_q(\theta) \, d\theta
\end{aligned}$$

$$\begin{aligned}&\leq qM_\alpha\left[\left(\frac{1-\gamma}{(1-\alpha)q-\gamma}\right)^{1-\gamma}T^{(1-\alpha)q-\gamma}\|\varphi_r\|_{L^{\frac{1}{\gamma}}}+\frac{ML_B^2L_u(r)T^{(1-\alpha)q}}{\epsilon(1-\alpha)q\Gamma(q)}\right]\int_0^\eta\theta^{1-\alpha}\xi_q(\theta)\,d\theta\\&\quad+\frac{qM_\alpha\Gamma(2-\alpha)}{\Gamma((1+(1-\alpha)q))}\left[\left(\frac{1-\gamma}{(1-\alpha)q-\gamma}\right)^{1-\gamma}\delta^{(1-\alpha)q-\gamma}\|\varphi_r\|_{L^{\frac{1}{\gamma}}}+\frac{ML_B^2L_u(r)\delta^{(1-\alpha)q}}{\epsilon(1-\alpha)q\Gamma(q)}\right]\end{aligned}$$

approaches zero as $\delta\to 0$, $\eta\to 0$. This implies that there are relatively compact sets arbitrarily close to the set $\{\Pi : x\in B_{r(\epsilon)}\}$ for $t\in(0,T]$, then $\{\Pi : x\in B_{r(\epsilon)}\}$ is relatively compact in X_α. Since it is compact at $t=0$, we conclude that the set $\{\Pi : x\in B_{r(\epsilon)}\}$ is relatively compact in X_α for $t\in[0,T]$.

Next, for $0<t_1<t_2\leq T$ and $\delta>0$ small enough,

$$\begin{aligned}&\|\Pi x(t_1)-\Pi x(t_2)\|_\alpha\\&=\left\|\int_0^{t_2}(t_2-s)^{q-1}V(t_2-s)Nx(s)\,ds-\int_0^{t_1}(t_1-s)^{q-1}V(t_1-s)Nx(s)\,ds\right\|_\alpha\\&\leq\left\|\int_{t_1}^{t_2}(t_2-s)^{q-1}V(t_2-s)Nx(s)\,ds\right\|_\alpha\\&\quad+\left\|\int_0^{t_1}\left[(t_2-s)^{q-1}-(t_1-s)^{q-1}\right]V(t_2-s)Nx(s)\,ds\right\|_\alpha\\&\quad+\left\|\int_0^{t_1}(t_1-s)^{q-1}\left[V(t_2-s)-V(t_1-s)\right]Nx(s)\,ds\right\|_\alpha\\&:=I_1+I_2+I_3.\end{aligned}$$

By the Hölder inequality, we can see that

$$\begin{aligned}I_1&\leq C_\alpha\int_{t_1}^{t_2}(t_2-s)^{(1-\alpha)q-1}\left(\varphi_r(s)+\frac{ML_B^2}{\epsilon\Gamma(q)}L_u(r)\right)ds\\&\leq C_\alpha\left(\frac{1-\gamma}{(1-\alpha)q-\gamma}\right)^{1-\gamma}(t_2-t_1)^{(1-\alpha)q-\gamma}\|\varphi_r\|_{L^{\frac{1}{\gamma}}}+\frac{C_\alpha ML_B^2L_u(r)(t_2-t_1)^{(1-\alpha)q}}{\epsilon(1-\alpha)q\Gamma(q)}\end{aligned}$$

and

$$\begin{aligned}I_2&\leq C_\alpha\left\|\int_0^{t_1}\left[(t_2-s)^{(1-\alpha)q-1}-(t_1-s)^{q-1}(t_2-s)^{-\alpha q}\right]\left(\varphi_r(s)+\frac{ML_B^2L_u(r)}{\epsilon\Gamma(q)}\right)ds\right\|\\&\leq\left(1-\frac{\alpha q}{(1-\gamma)^2}\right)^{(1-\gamma)^2}\left[t_2^{1-\frac{\alpha q}{(1-\gamma)^2}}-(t_2-t_1)^{1-\frac{\alpha q}{(1-\gamma)^2}}\right]^{(1-\gamma)^2}\left[\int_0^T\left((t_2-s)^{q-1}\right.\right.\\&\quad\left.\left.-(t_1-s)^{q-1}\right)^{\frac{1}{\gamma(1-\gamma)}}ds\right]^{\gamma(1-\gamma)}\|\varphi_r\|_L^{\frac{1}{\gamma}}+\frac{C_\alpha ML_B^2L_u(r)}{\epsilon\Gamma(q)}\left(\frac{1-\gamma}{1-\gamma-\alpha q}\right)^{1-\gamma}\\&\quad\times\left[t_2^{\frac{1-\gamma-\alpha q}{1-\gamma}}-(t_2-t_1)^{\frac{1-\gamma-\alpha q}{1-\gamma}}\right]^{1-\gamma}\left(\int_0^T\left((t_2-s)^{q-1}-(t_1-s)^{q-1}\right)^{\frac{1}{\gamma}}ds\right)^\gamma.\end{aligned}$$

For $t_1=0$, $I_3=0$. For $t_1>0$ and $\eta>0$ small enough, we obtain

$$\begin{aligned}I_3&\leq\int_0^{t_1-\eta}(t_1-s)^{q-1}\left[\varphi_r(s)+\frac{\varphi ML_B^2L_u(r)}{\epsilon\Gamma(q)}\right]ds\cdot\sup_{0\leq s\leq t_1-\delta}\left\|A^\alpha V(t_2-s)-A^\alpha V(t_1-s)\right\|\\&\quad+2C_\alpha\int_{t_1-\eta}^{t_1}(t_1-s)^{(1-\alpha)q-1}\left[\varphi_r(s)+\frac{\varphi ML_B^2L_u(r)}{\epsilon\Gamma(q)}\right]ds\end{aligned}$$

$$\begin{aligned}
&\leq \sup_{0\leq s\leq t_1-\delta}\big\|A^\alpha V(t_2-s)-A^\alpha V(t_1-s)\big\|\bigg[\bigg(\frac{1-\gamma}{q-\gamma}\bigg)^{1-\gamma}\big(t_1^{\frac{q-\gamma}{1-\gamma}}-\eta^{\frac{q-\gamma}{1-\gamma}}\big)^{1-\gamma}\|\varphi_r\|_L^{\frac{1}{\gamma}} \\
&\quad+\frac{ML_B^2L_u(r)}{\epsilon q\Gamma(q)}\big(t_1^q-\eta^q\big)\bigg]+2C_\alpha\bigg[\bigg(\frac{1-\gamma}{(1-\alpha)q-\gamma}\bigg)^{1-\gamma}\eta^{(1-\alpha)q-\gamma}\|\varphi_r\|_L^{\frac{1}{\gamma}} \\
&\quad+\frac{ML_B^2L_u(r)}{\epsilon(1-\alpha)q\Gamma(q)}\eta^{(1-\alpha)q}\bigg],
\end{aligned}$$

thus, I_1, I_2, I_3 tend to zero independent of $x\in B_r$ as $t_2-t_1\to 0$ and $\eta\to 0$. So, we conclude that

$$\big\|\Pi x(t_2)-\Pi x(t_1)\big\|_\alpha\to 0\quad\text{as } t_2-t_1\to 0.$$

Consequently, Π is equicontinuous. Applying the Arzela-Ascoli theorem, Π is compact on $B_{r(\epsilon)}$.

Step 3. Φ is a contraction operator. From the assumptions (H_h) and (H_I), it is easy to see that

$$\begin{aligned}
\|\Phi x-\Phi y\|_\alpha &\leq M\big\|h(x)-h(y)\big\|_\alpha+M\sum_{0<t_k<t}\big\|I_k\big(x(t_k)\big)-I_k\big(y(t_k)\big)\big\|_\alpha \\
&\leq M(L_h+nL_I)\|x-y\|_\alpha,
\end{aligned}$$

thus, from (3.8), Φ is contraction.

Therefore, by means of the Krasnoselskii fixed point theorem, we conclude that Λ in (3.5) has a fixed point, which gives rise to a mild solution of Cauchy problem (1.1) with the control given in (3.4). The proof is completed. □

Theorem 3.2 *If the hypothesis* (H_c) *is satisfied, and the conditions of Theorem* 3.1 *hold, and, further,* f, I, h *are bounded in* X_α, *then the fractional Cauchy problem* (1.1) *is approximately controllable.*

Proof Let x_ϵ be a fixed point of Λ in B_r. Any fixed point of Λ is a mild solution of (1.1) under the control

$$\begin{aligned}
u_\epsilon(t,x_\epsilon)&=B^*V^*(T-t)R\big(\epsilon,\Gamma_0^T\big) \\
&\quad\times\bigg[x_T-U(T)\big(x_0+h(x_\epsilon)\big)-\int_0^T(T-s)^{q-1}V(T-s)f\big(s,x_\epsilon(s), \\
&\quad Gx_\epsilon(s)\big)\,ds-\sum_{0<t_k<t}U(T-t_k)I_k\big(x_\epsilon(t_k)^-\big)\bigg],
\end{aligned}$$

and it satisfies the equality

$$x_\epsilon(T)=x_T-\epsilon R\big(\epsilon,\Gamma_0^T\big)p(x_\epsilon),\tag{3.9}$$

where

$$\begin{aligned}
p(x_\epsilon)&=x_T-U(T)\big(x_0+h(x_\epsilon)\big)-\int_0^T(T-s)^{q-1}V(T-s)f\big(s,x_\epsilon(s),Gx_\epsilon(s)\big)\,ds \\
&\quad-\sum_{0<t_k<t}U(T-t_k)I_k(x_\epsilon).
\end{aligned}$$

With the help of boundedness of the functions f and I, and the Dunford-Pettis theorem [34], stating that a class of random variables $x_n \in L^1(\mu)$ is uniformly integrable if and only if it is relatively weakly compact, the sequences $\{f(s,x_\epsilon(s),Gx_\epsilon(s))\}$ and $\{I_k(x_\epsilon)\}$ are relatively weakly compact in $L^2([0,T],X_\alpha)$. There are subsequences still denoted by $\{f(s,x_\epsilon(s),Gx_\epsilon(s))\}$ and $\{I_k(x_\epsilon)\}$ weakly converging to $f(s)$ and $\tilde{I}_k$. Meanwhile, there exists $\tilde{h} \in X_\alpha$ such that $h(x_\epsilon)$ converges to $\tilde{h}$ weakly in X_α.

Denote

$$\omega = x_T - U(T)(x_0 + \tilde{h}) - \int_0^T (T-s)^{q-1} V(T-s) f(s)\,ds - \sum_{0<t_k<t} U(T-t_k)\tilde{I}_k.$$

It follows that

$$\begin{aligned}\|p(x_\epsilon) - \omega\|_\alpha \le{}& \|U(T)h(x_\epsilon) - U(T)\tilde{h}\|_\alpha \\ &+ \left\| \int_0^T (T-s)^{q-1} V(T-s)\bigl[f\bigl(s,x_\epsilon(s),Gx_\epsilon(s)\bigr) - f(s)\bigr]\,ds \right\|_\alpha \\ &+ \left\| \sum_{0<t_k<t} U(T-t_k)\bigl[I_k(x_\epsilon) - \tilde{I}_k\bigr] \right\|_\alpha \to 0\end{aligned}$$

as $\epsilon \to 0^+$ because of the compactness of $U(t)$ and the operator

$$l(\cdot) \to \int_0^{\cdot} (\cdot - s)^{q-1} V(\cdot - s) l(s)\,ds : L^2\bigl([0,T],X_\alpha\bigr) \to C\bigl([0,T],X_\alpha\bigr).$$

Hence, from (3.9), one obtains

$$\|x_\epsilon(T) - x_T\|_\alpha \le \|\epsilon R(\epsilon,\Gamma_0^T)(\omega)\|_\alpha + \|\epsilon R(\epsilon,\Gamma_0^T)\| \cdot \|p(x_\epsilon) - \omega\|_\alpha \to 0$$

as $\epsilon \to 0^+$. This proves the approximate controllability of (1.1). □

4 An example

Example 4.1 As a simple application, we consider a control system governed by a fractional partial differential equation of the form:

$$\begin{cases} {}^{c}\partial_t^q x(t,z) = \partial_z^2 x(t,z) + u(t,z) + F(t,z,x(t,z),\int_0^t K(t,s)x(s,z)\,dz), \\ \quad t \in [0,T], z \in [0,\pi], t \neq t_k, \\ x(t,0) = x(t,\pi) = 0, \\ x(0,z) = x_0(z) + \int_0^\pi k(z,r)\sin(x(t,r))\,dr, \\ \Delta x(t_k) = \int_0^{t_k} p(t_k - s)x(s,z)\,dz, \quad k = 1,2,\ldots,n, \end{cases} \tag{4.1}$$

where $\frac{1}{2} < q < 1$, F is a given continuous and uniformly bounded function, K, k, p are measurable and continuous on $\Delta = \{(t,s) : 0 \le s \le t \le T\}$.

Let $X = L^2[0,\pi]$ and define the operator A by $A\omega = -\omega''$ with the domain $D(A) = \{\omega(\cdot) \in L^2[0,\pi], \omega, \omega'$ are absolutely continuous, $\omega'' \in L^2[0,\pi], \omega(0) = \omega(\pi)\}$. Then

$$A\omega = \sum_{n=1}^{\infty} n^2 \langle \omega, e_n \rangle e_n, \quad \omega \in D(A),$$

is the orthogonal set of eigenvectors of A, where $e_n(z) = \sqrt{\frac{2}{\pi}} \sin nz$, $0 \le z \le \pi$, $n = 1, 2, \ldots$. Clearly, $-A$ generates a compact analytic semigroup $\{S(t), t > 0\}$ in X and it can be written

$$S(t)\omega = \sum_{n=1}^{\infty} e^{-n^2 t} \langle \omega, e_n \rangle e_n, \quad \omega \in X.$$

Taking $\alpha = \frac{1}{2}$, the operator $A^{\frac{1}{2}}$ is given by

$$A^{\frac{1}{2}}\omega = \sum_{n=1}^{\infty} n \langle \omega, e_n \rangle e_n, \quad \omega \in D\bigl(A^{\frac{1}{2}}\bigr).$$

Let $X_{\frac{1}{2}} = (D(A^{\frac{1}{2}}), \| \cdot \|_{\frac{1}{2}})$, $B = E$ (identity operator), and $U = X_{\frac{1}{2}}$, where $\|x\|_{\frac{1}{2}} = \|A^{\frac{1}{2}} x\|$ for $x \in D(A^{\frac{1}{2}})$.

Define the functions

$$\begin{aligned} f\bigl(t, x(t), Gx(t)\bigr)(z) &= F\left(t, z, x(t,z), \int_0^t K(t,s) x(s,z)\, dz\right), \\ h(x)(z) &= \int_0^{\pi} k(z,r) \sin\bigl(x(t,r)\bigr)\, dr, \\ I_k(x)(z) &= \int_0^{t_k} p(t_k - s) x(s,z)\, dz. \end{aligned}$$

With the above choice of f, h, and I, the system (4.1) can be written in the abstract form of system (1.1). Moreover, f, h, and I are bounded linear operators and satisfy the Lipschitz condition. All the conditions of Theorem 3.2 are fulfilled, so we can claim that the system (4.1) is approximately controllable.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Each of the authors, XZ, CZ, and CY, contributed to each part of this work equally. All authors read and approved the final manuscript.

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