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Necessary and sufficient condition for the existence of positive solution to singular fractional differential equations

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Abstract

In this paper, we discuss the existence of positive solution to singular fractional differential equations involving Caputo fractional derivative. Necessary and sufficient condition for the existence of $C^2[0,1]$ positive solution is obtained by means of the fixed point theorems on cones. In addition, the uniqueness results and the iterative sequences of the solution are also given.

MSC: 34B10; 34B15

Keywords: fractional differential equation; singular boundary value problem; positive solution; necessary and sufficient condition

1 Introduction

In this paper, we consider the following singular fractional differential equation:

$$\begin{cases} {}^cD_{0+}^{\alpha}u(t) + f(t,u(t)) = 0, & 0 < t < 1, \\ u(0) = u'(1) = u''(0) = 0, \end{cases} \tag{1.1}$$

where $2 < \alpha \leq 3$ is a real number, ${}^cD_{0+}^{\alpha}$ is the Caputo fractional derivative and f may be singular at $t = 0, 1$.

Singular differential equation boundary value problems (BVP for short) arise from many branches of applied mathematics and physics. The theory of singular boundary value problems has become an important area of investigation in recent years. Differential equations of fractional order arise from many engineering and scientific disciplines as the mathematical modeling of systems and processes in the fields of physics, chemistry, control theory, *etc.*; see [1–4] and the references therein. Recently, much attention has been paid to the existence results of solutions for fractional differential equations, for example [5–17].

In [5], Bai and Qiu considered the existence of positive solution to problem (1.1), where $2 < \alpha \leq 3$ is a real number, ${}^cD_{0+}^{\alpha}$ is the Caputo fractional derivative, $f : (0,1] \times [0,\infty) \to [0,\infty)$ is continuous and singular at $t = 0$. The sufficient conditions for the existence of positive solution to (1.1) were obtained by using the Krasnosel'skii fixed-point theorem and the Leray-Schauder nonlinear alternative.

In [9], the authors investigated the existence of positive solution to the following boundary value problem:

$$\begin{cases} {}^{c}D_{0+}^{\alpha}u(t) + \lambda f(t,u(t)) = 0, \quad 0 < t < 1, \\ u(0) = u'(1) = u''(0) = 0, \end{cases} \tag{1.2}$$

where $2 < \alpha \leq 3$ is a real number, ${}^{c}D_{0+}^{\alpha}$ is the Caputo fractional derivative, λ is a positive parameter, f may change sign and may be singular at $t = 0, 1$.

In recent years, many results dealing with necessary and sufficient conditions for the existence of positive solutions to integer-order differential equations were obtained (for example, [18–23]) with one of the following conditions:

(A1) $f \in C((0,1) \times [0,\infty), [0,\infty))$, $f(t, e(t)) > 0$, $t \in (0,1)$, here $e \in C([0,1], [0,\infty))$; there exist constants $0 < \lambda_1 \leq \lambda_2 < 1$ such that for $(t,x) \in (0,1) \times [0,\infty)$,

$$c^{\lambda_2} f(t,x) \leq f(t,cx) \leq c^{\lambda_1} f(t,x), \quad \forall c \in (0,1). \tag{1.3}$$

(A2) $f \in C((0,1) \times [0,\infty), [0,\infty))$; for each fixed $t \in (0,1)$, $f(t,x)$ is increasing in x; there exists $0 < \alpha < 1$ such that

$$f(t,rx) \geq r^{\alpha} f(t,x), \quad \forall 0 < r < 1, (t,x) \in (0,1) \times [0,\infty). \tag{1.4}$$

(A3) $f \in C((0,1) \times [0,\infty), [0,\infty))$; for each fixed $t \in (0,1)$, $f(t,x)$ is increasing in x; for all $0 < r < 1$, there exists $g(r) = m(r^{-\alpha} - 1)$ such that

$$f(t,rx) \geq r\bigl(1 + g(r)\bigr) f(t,x), \quad \forall (t,x) \in (0,1) \times [0,\infty), 0 < m \leq 1, 0 < \alpha < 1. \tag{1.5}$$

While there are a lot of works dealing with necessary and sufficient conditions for integer-order differential equations, the results of fractional differential equations are relatively scarce due to the difficulties caused by the singularity of nonlinearity. In [7], the authors considered the necessary and sufficient condition for the existence of $C^3[0,1]$ positive solution of singular sub-linear boundary value problems for a fractional differential equation with condition (A2).

Inspired by the previous works, in this paper we aim to establish necessary and sufficient condition for the existence of $C^2[0,1]$ positive solutions to BVP (1.1). In this paper, by a $C^2[0,1]$ positive solution to BVP (1.1), we mean a function $u \in C'[0,1] \cap C^2[0,1)$ which satisfies $u''(1^-)$ exists, is positive on $(0,1]$ and satisfies (1.1).

Throughout this paper, we assume that the following condition holds.

(H) $f \in C((0,1) \times [0,\infty), [0,\infty))$, $f(t,x)$ is increasing in x; there exists a function $\eta : [0,1] \to [0,+\infty)$ satisfying $\eta(r) > r$ $(0 < r < 1)$ such that

$$f(t,rx) \geq \eta(r) f(t,x), \quad \forall 0 < r < 1, (t,x) \in (0,1) \times [0,\infty). \tag{1.6}$$

Remark 1.1 Inequality (1.6) is equivalent to

$$f(t,rx) \leq \frac{f(t,x)}{\eta(r^{-1})}, \quad \forall r > 1, (t,x) \in (0,1) \times [0,\infty). \tag{1.7}$$

Remark 1.2 Condition (H) includes conditions (A1), (A2) and (A3) as special cases.

Remark 1.3 The function η defined in (H) satisfies $\eta(1) = 1$, and $\eta(r) \leq 1$, $\forall r \in (0,1)$.

Remark 1.4 If condition (H) holds, then there exists a strictly increasing function φ satisfying $\varphi(r) > r$ $(0 < r < 1)$ such that

$$f(t,rx) \geq \varphi(r)f(t,x), \quad \forall 0 < r < 1, (t,x) \in (0,1) \times [0,\infty), \tag{1.8}$$

without loss of generality, we may suppose that η is strictly increasing on (0.1].

Proof If there exist $t_0 \in (0,1)$, $x_0 > 0$ such that $f(t_0,x_0) = 0$. By the monotonicity of f and (1.7), we have $f(t_0,x) \equiv 0$, $\forall x \in [0,+\infty)$. Set

$$\Omega = \bigl\{t \in (0,1) : \exists x_1 > 0 \text{ such that } f(t,x_1) = 0\bigr\}. \tag{1.9}$$

For any $r \in (0,1)$, denote

$$D_r = \bigl\{c : f(t,rx) \geq cf(t,x), (t,x) \in \bigl((0,1)\backslash\Omega\bigr) \times (0,\infty)\bigr\}. \tag{1.10}$$

It is clear that $\sup D_r$ exists. Let $\psi(r) = \sup D_r$, then

$$f(t,rx) \geq \psi(r)f(t,x), \quad \forall (t,x) \in (0,1) \times [0,\infty), \tag{1.11}$$

and $r < \eta(r) \leq \psi(r) \leq 1$. For any $0 < r_2 < r_1 < 1$ and $x \in [0,\infty)$, we have

$$f(t,r_1x) = f\left(t, r_2 \cdot \frac{r_1}{r_2}x\right) \geq \psi(r_2)f\left(t, \frac{r_1}{r_2}x\right) \geq \psi(r_2)f(t,x). \tag{1.12}$$

By the definition of ψ, we get $\psi(r_1) \geq \psi(r_2)$, therefore ψ is nondecreasing. Let $\varphi(r) = \frac{\psi(r)+r}{2}$. It is clear that φ is strictly increasing on $(0,1)$, satisfies $\varphi(r) > r$ and

$$f(t,rx) \geq \varphi(r)f(t,x), \quad \forall (t,x) \in (0,1) \times [0,\infty), r \in (0,1). \tag{1.13}$$

The proof is completed. □

2 Basic definitions and preliminaries

In this section, we present some preliminaries and lemmas that are useful to the proof of the main results, we also present here some necessary definitions.

Definition 2.1 The Riemann-Liouville fractional integral I_{0+}^{α} and derivative D_{0+}^{α} are defined by

$$I_{0+}^{\alpha}u(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}u(s)\,ds \tag{2.1}$$

and

$$D_{0+}^{\alpha}u(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^{(n)}\int_0^t (t-s)^{n-\alpha-1}u(s)\,ds, \tag{2.2}$$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of number α, provided that the right-hand side is defined pointwise on $(0, +\infty)$.

Definition 2.2 (see [2]) The Caputo fractional derivative of order $\alpha > 0$ on $[0,1]$ is defined via the above Riemann-Liouville fractional derivative by

$$ {}^{c}D_{0+}^{\alpha}u(t) = D_{0+}^{\alpha}\left[u(t) - \sum_{k=0}^{n-1} u^{(k)}(0)t^{k}\right], \tag{2.3}$$

where $n = [\alpha] + 1$.

Remark 2.1 (see Theorem 2.1 of [2]) If $u(t) \in AC^{n}[0,1]$, then the Caputo fractional derivative of order $\alpha > 0$ exists almost everywhere on $[0,1]$ and can be represented by

$$ {}^{c}D_{0+}^{\alpha}u(t) = \frac{1}{\Gamma(n-\alpha)}\int_{0}^{t}(t-s)^{n-\alpha-1}u^{(n)}(s)\,ds, \tag{2.4}$$

where $n = [\alpha] + 1$, and

$$AC^{n}[0,1] = \left\{y : [0,1] \to \mathbb{R} \text{ and } \frac{d^{n-1}y}{dt^{n-1}} \text{ is absolutely continuous on } [0,1]\right\}.$$

Lemma 2.1 (see Lemma 2.5 of [2]) *Let* $\alpha > 0$, $u \in L[0,1]$ *and* $D_{0+}^{\alpha}u \in L[0,1]$, *then the following equality holds*:

$$I_{0+}^{\alpha}D_{0+}^{\alpha}u(t) = u(t) + c_{1}t^{\alpha-1} + c_{2}t^{\alpha-2} + \cdots + c_{n}t^{\alpha-n},$$

where $c_i \in R$, $i = 1, 2, \ldots, n$, $n = [\alpha] + 1$.

Lemma 2.2 (see Lemma 2.4 of [2]) *If* $\alpha > 0$ *and* $y(t) \in L[0,1]$, *then the equality*

$$D_{0+}^{\alpha}I_{0+}^{\alpha}y(t) = y(t)$$

holds almost everywhere on $[0,1]$.

Lemma 2.3 (see Property 2.8 of [2]) *Let* $\alpha > \beta > 0$. *If* $y(t) \in L[0,1]$, *then*

$$D_{0+}^{\beta}I_{0+}^{\alpha}y(t) = I_{0+}^{\alpha-\beta}y(t).$$

Lemma 2.4 *If* $2 < \alpha \leq 3$, $y \in L[0,1] \cap C(0,1)$ *and*

$$\lim_{t\to 0+} t^{\alpha-2}\int_{0}^{1}(1-s)^{\alpha-2}y(ts)\,ds = 0, \tag{2.5}$$

then the problem

$$\begin{cases} {}^{c}D_{0+}^{\alpha}u(t) + y(t) = 0, & 0 < t < 1, \\ u(0) = u'(1) = u''(0) = 0, \end{cases} \tag{2.6}$$

has a unique solution

$$u(t) = \int_0^1 G(t,s)y(s)\,ds, \tag{2.7}$$

where

$$G(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} (\alpha-1)t(1-s)^{\alpha-2}, & 0 \le t \le s \le 1, \\ (\alpha-1)t(1-s)^{\alpha-2} - (t-s)^{\alpha-1}, & 0 \le s \le t \le 1. \end{cases} \tag{2.8}$$

Proof Deduced from Lemma 2.1, the solution of (2.6) satisfies

$$u(t) = u'(0)t - I_{0+}^{\alpha} y(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + c_3 t^{\alpha-3}.$$

By direct calculation of $u(0)$, $u'(0)$ and $u''(0)$, there is $c_1 = c_2 = c_3 = 0$. Consequently,

$$u(t) = u'(0)t - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)\,ds \tag{2.9}$$

and

$$u'(t) = u'(0) - \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} y(s)\,ds. \tag{2.10}$$

By $u'(1) = 0$, we have

$$u'(0) = \frac{1}{\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} y(s)\,ds. \tag{2.11}$$

Therefore,

$$u(t) = \int_0^1 G(t,s)y(s)\,ds.$$

On the other hand, for

$$u(t) = \int_0^1 G(t,s)y(s)\,ds = \frac{t}{\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} y(s)\,ds - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)\,ds,$$

we have $u(0) = u'(1) = 0$ and $u'(0) = \frac{1}{\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} y(s)\,ds$. From (2.5), we get

$$\begin{aligned} u''(0) &= \lim_{t\to 0+} \frac{u'(t) - u'(0)}{t} = \lim_{t\to 0+} \frac{\int_0^t (t-s)^{\alpha-2} y(s)\,ds}{\Gamma(\alpha-1)t} \\ &= \lim_{t\to 0+} \frac{t^{\alpha-2}}{\Gamma(\alpha-1)} \int_0^1 (1-x)^{\alpha-2} y(tx)\,dx = 0. \end{aligned} \tag{2.12}$$

By Definition 2.2 and Lemma 2.2, we have u is a solution of problem (2.6). The proof is completed. □

Remark 2.2 If $\alpha = 3$ and $y \in L[0,1]$, then condition (2.5) holds naturally.

Lemma 2.5 (see [9]) *The function $G(t,s)$ has the following properties*:

(1) $G(t,s) \leq \frac{1}{\Gamma(\alpha-1)} t(1-s)^{\alpha-2}$, $\forall t,s \in [0,1]$;
(2) $G(t,s) \leq \frac{1}{\Gamma(\alpha-1)} (\alpha-2+s)(1-s)^{\alpha-2}$, $\forall t,s \in [0,1]$;
(3) $G(t,s) \geq \frac{1}{\Gamma(\alpha)} (\alpha-2+s)t(1-s)^{\alpha-2}$, $\forall t,s \in [0,1]$.

Lemma 2.6 *Suppose that u is a positive solution of BVP* (1.1), *then there exist $L_u, l_u > 0$ such that*

$$l_u t \leq u(t) \leq L_u t, \quad \forall t \in [0,1]. \tag{2.13}$$

Proof By Lemma 2.4, u can be expressed by

$$u(t) = \int_0^1 G(t,s) f\bigl(s,u(s)\bigr)\, ds. \tag{2.14}$$

From (1) of Lemma 2.5, we have

$$u(t) \leq \frac{t}{\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} f\bigl(s,u(s)\bigr)\, ds. \tag{2.15}$$

By (2), (3) of Lemma 2.5, we get

$$u(t) \geq \frac{t}{\Gamma(\alpha)} \int_0^1 (\alpha-2+s)(1-s)^{\alpha-2} f\bigl(s,u(s)\bigr)\, ds \tag{2.16}$$

and

$$u(t) \leq \frac{1}{\Gamma(\alpha-1)} \int_0^1 (\alpha-2+s)(1-s)^{\alpha-2} f\bigl(s,u(s)\bigr)\, ds. \tag{2.17}$$

Inequalities (2.16) and (2.17) imply $u(t) \geq \frac{t}{\alpha-1}\|u(t)\|$.

Let

$$l_u = \frac{\|u(t)\|}{\alpha-1}, \qquad L_u = \frac{\int_0^1 (1-s)^{\alpha-2} f(s,u(s))\, ds}{\Gamma(\alpha-1)}. \tag{2.18}$$

Then (2.13) holds. The proof is completed. □

Lemma 2.7 *Assume that $g(x)$, $\{g_n(x)\}$, $h(x)$, $\{h_n(x)\}$ are Lebesgue integrable on* $[0,1]$, *satisfy*

$$\bigl|g_n(x)\bigr| \leq h_n(x), \qquad \lim_{n\to\infty} g_n(x) = g(x), \qquad \lim_{n\to\infty} h_n(x) = h(x), \quad a.e.\ [0,1] \tag{2.19}$$

and

$$\lim_{n\to\infty} \int_0^1 h_n(x)\, dx = \int_0^1 h(x)\, dx, \tag{2.20}$$

then

$$\lim_{n\to\infty} \int_0^1 g_n(x)\, dx = \int_0^1 g(x)\, dx. \tag{2.21}$$

Proof By $|g_n(x)| \leq h_n(x)$, a.e. $[0,1]$, we have

$$\bigl|g(x)\bigr| \leq h(x), \quad \text{a.e. } [0,1]. \tag{2.22}$$

Set

$$k_n(x) = h_n(x) + h(x) - \bigl|g_n(x) - g(x)\bigr|, \tag{2.23}$$

then $k_n(x) \to 2h(x)$ ($n \to \infty$) a.e. $[0,1]$. By the Fatou lemma, we get

$$\begin{aligned}\int_0^1 2h(x)\,dx &= \int_0^1 \lim_{n\to\infty} k_n(x)\,dx \leq \liminf_{n\to\infty} \int_0^1 k_n(x)\,dx \\ &= 2\int_0^1 h(x)\,dx - \limsup_{n\to\infty} \int_0^1 \bigl|g_n(x) - g(x)\bigr|\,dx,\end{aligned} \tag{2.24}$$

which implies

$$\lim_{n\to\infty} \int_0^1 \bigl|g_n(x) - g(x)\bigr|\,dx = 0. \tag{2.25}$$

Thus

$$\lim_{n\to\infty} \int_0^1 g_n(x)\,dx = \int_0^1 g(x)\,dx. \tag{2.26}$$

The proof is completed. □

3 Main result

Theorem 3.1 *Suppose that* (H) *holds. Then the necessary and sufficient condition for BVP* (1.1) *to have a* $C^2[0,1]$ *positive solution is*

$$0 < \int_0^1 (1-s)^{\alpha-3} f(s,s)\,ds < +\infty, \tag{3.1a}$$

$$\lim_{t\to 0+} t^{\alpha-2} \int_0^1 (1-s)^{\alpha-3} f(ts,ts)\,ds = 0, \tag{3.1b}$$

$$\lim_{t\to 1-} \int_0^1 (1-s)^{\alpha-3} f(ts,ts)\,ds = \int_0^1 (1-s)^{\alpha-3} f(s,s)\,ds. \tag{3.1c}$$

Proof (i) *Necessity*. Assume that u is a $C^2[0,1]$ positive solution of BVP (1.1). In the following, we will divide the rather long proof into three steps.

Step 1: By Lemma 2.4, u can be expressed by

$$u(t) = \frac{t}{\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} f\bigl(s,u(s)\bigr)\,ds - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f\bigl(s,u(s)\bigr)\,ds. \tag{3.2}$$

For any $t \in (0,1)$, Lemma 2.3 implies

$$u'(t) = \frac{1}{\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} f\bigl(s,u(s)\bigr)\,ds - \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} f\bigl(s,u(s)\bigr)\,ds \tag{3.3}$$

and

$$\begin{aligned} u''(t) &= -\frac{1}{\Gamma(\alpha-2)}\int_0^t (t-s)^{\alpha-3} f\big(s,u(s)\big)\,ds \\ &= -\frac{t^{\alpha-2}}{\Gamma(\alpha-2)}\int_0^1 (1-s)^{\alpha-3} f\big(ts,u(ts)\big)\,ds. \end{aligned} \tag{3.4}$$

It is clear that $u'(t) \geq 0$, and $u''(t) \leq 0$, $\forall t \in (0,1)$.

$\forall \varepsilon \in (0,\frac{1}{2})$, $t \in (0,1)$, we deduce that

$$\int_\varepsilon^{1-\varepsilon} (1-s)^{\alpha-3} f\big(ts,u(ts)\big)\,ds \leq -\Gamma(\alpha-2)t^{2-\alpha}u''(t). \tag{3.5}$$

Let $t \to 1$, noticing (H) and the existence of $u''(1^-)$, we have

$$\int_\varepsilon^{1-\varepsilon} (1-s)^{\alpha-3} f\big(s,u(s)\big)\,ds \leq -\Gamma(\alpha-2)u''\big(1^-\big), \quad \forall \varepsilon \in \left(0,\frac{1}{2}\right). \tag{3.6}$$

Thus $\int_0^1 (1-s)^{\alpha-3} f(s,u(s))\,ds$ is well defined, that is, $u''(1)$ is well defined. By Lemma 2.6, we have

$$\begin{aligned} \int_0^1 (1-s)^{\alpha-3} f\big(s,u(s)\big)\,ds &\geq \int_0^1 (1-s)^{\alpha-3} f(s,l_u s)\,ds \\ &\geq \int_0^1 (1-s)^{\alpha-3} f\big(s,\min\{1,l_u\}s\big)\,ds \\ &\geq \eta\big(\min\{1,l_u\}\big)\int_0^1 (1-s)^{\alpha-3} f(s,s)\,ds \\ &\geq \min\{1,l_u\}\int_0^1 (1-s)^{\alpha-3} f(s,s)\,ds, \end{aligned} \tag{3.7}$$

which implies

$$\int_0^1 (1-s)^{\alpha-3} f(s,s)\,ds < +\infty. \tag{3.8}$$

On the other hand, we have

$$\begin{aligned} u(t) = \int_0^1 G(t,s) f\big(s,u(s)\big)\,ds &\leq \frac{t}{\Gamma(\alpha-1)}\int_0^1 (1-s)^{\alpha-2} f(s,L_u s)\,ds \\ &\leq \frac{t}{\Gamma(\alpha-1)}\int_0^1 (1-s)^{\alpha-2} f\big(s,(1+L_u)s\big)\,ds \\ &\leq \frac{t}{\Gamma(\alpha-1)\eta([1+L_u]^{-1})}\int_0^1 (1-s)^{\alpha-3} f(s,s)\,ds. \end{aligned} \tag{3.9}$$

Since u is a positive solution, then

$$\int_0^1 (1-s)^{\alpha-3} f(s,s)\,ds > 0. \tag{3.10}$$

Inequalities (3.8) and (3.10) yield (3.1a) holds.

Step 2: From u is a $C^2[0,1]$ positive solution, we get

$$\lim_{t\to 0+} t^{\alpha-2}\int_0^1 (1-s)^{\alpha-3} f\bigl(ts,u(ts)\bigr)\,ds = -\Gamma(\alpha-2)u''(0) = 0 \tag{3.11}$$

and

$$\lim_{t\to 1-}\int_0^1 (1-s)^{\alpha-3} f\bigl(ts,u(ts)\bigr)\,ds = \int_0^1 (1-s)^{\alpha-3} f\bigl(s,u(s)\bigr)\,ds. \tag{3.12}$$

Similar to (3.7) and (3.9), we have

$$f\bigl(s,u(s)\bigr) \ge \min\{1,l_u\} f(s,s), \quad \forall s\in(0,1) \tag{3.13}$$

and

$$f\bigl(s,u(s)\bigr) \le \frac{1}{\eta([1+L_u]^{-1})} f(s,s), \quad \forall s\in(0,1). \tag{3.14}$$

Then, for any $t\in(0,1)$, we have

$$\int_0^1 (1-s)^{\alpha-3} f\bigl(ts,u(ts)\bigr)\,ds \ge \min\{1,l_u\}\int_0^1 (1-s)^{\alpha-3} f(ts,ts)\,ds \tag{3.15}$$

and

$$\int_0^1 (1-s)^{\alpha-3} f\bigl(ts,u(ts)\bigr)\,ds \le \frac{1}{\eta([1+L_u]^{-1})}\int_0^1 (1-s)^{\alpha-3} f(ts,ts)\,ds. \tag{3.16}$$

Combining (3.11) with (3.15), we obtain (3.1b) holds.

Step 3: $\forall\{t_n\}\subset(0,1)$ satisfies $t_n\to 1$ ($n\to\infty$). Set

$$\begin{aligned} &g_n(s) = \min\{1,l_u\}(1-s)^{\alpha-3} f(t_ns,t_ns), \qquad g(s) = \min\{1,l_u\}(1-s)^{\alpha-3} f(s,s),\\ &h_n(s) = (1-s)^{\alpha-3} f\bigl(t_ns,u(t_ns)\bigr), \qquad h(s) = (1-s)^{\alpha-3} f\bigl(s,u(s)\bigr). \end{aligned} \tag{3.17}$$

It is clear that $\{g_n(s)\}$, $g(s)$, $\{h_n(x)\}$, $h(x)$ are Lebesgue integrable on $[0,1]$, and

$$\lim_{n\to\infty} g_n(x) = g(x), \qquad \lim_{n\to\infty} h_n(x) = h(x), \quad \text{a.e. } [0,1]. \tag{3.18}$$

From (3.13), we get

$$0 \le g_n(x) \le h_n(x).$$

Equation (3.12) yields

$$\lim_{n\to\infty}\int_0^1 h_n(s)\,ds = \int_0^1 h(s)\,ds. \tag{3.19}$$

By Lemma 2.7, we have

$$\lim_{n\to\infty}\int_0^1 (1-s)^{\alpha-3} f(t_ns,t_ns)\,ds = \int_0^1 (1-s)^{\alpha-3} f(s,s)\,ds. \tag{3.20}$$

Then (3.1c) holds.

(ii) *Sufficiency*. Let $P = \{u \in C[0,1] : u \geq 0\}$. Clearly P is a normal cone of $C[0,1]$. Denote $e(t) = t$, and

$$P_e = \{u \in P : \exists L_u, l_u > 0 \text{ such that } l_u e \leq u \leq L_u e\}.$$

Set

$$Au(t) = \int_0^1 G(t,s) f\big(s, u(s)\big)\, ds. \tag{3.21}$$

For any $u \in P_e$, by (3.1a), (3.13), (3.14) and Lemma 2.5, we have

$$\begin{aligned} Au(t) &\leq \frac{t}{\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} f\big(s, u(s)\big)\, ds \\ &\leq \frac{t}{\Gamma(\alpha-1)\eta((1+L_u)^{-1})} \int_0^1 (1-s)^{\alpha-3} f(s,s)\, ds \end{aligned} \tag{3.22}$$

and

$$\begin{aligned} Au(t) &\geq \frac{t}{\Gamma(\alpha)} \int_0^1 (\alpha-2+s)(1-s)^{\alpha-2} f\big(s, u(s)\big)\, ds \\ &\geq \frac{t}{\Gamma(\alpha)} \min\{1, l_u\} \int_0^1 (\alpha-2)(1-s)^{\alpha-2} f(s,s)\, ds, \end{aligned} \tag{3.23}$$

which implies $A : P_e \to P_e$ is well defined.

It is clear that $e \in P_e$, so there exist positive numbers $L_e > 1 > l_e > 0$ such that $l_e e \leq Ae \leq L_e e$. Noticing $\eta(r) > r$ on $(0,1)$, we can choose a positive integer m large enough such that

$$\left(\frac{\eta(l_e)}{l_e}\right)^m > \frac{1}{l_e}, \qquad \left(\frac{\eta(L_e^{-1})}{L_e^{-1}}\right)^m > L_e. \tag{3.24}$$

Let

$$u_0 = l_e^m e, \qquad v_0 = L_e^m e, \qquad u_{n+1} = Au_n, \qquad v_{n+1} = Av_n, \quad n = 0,1,2,\ldots. \tag{3.25}$$

It is easy to see that

$$\left(\frac{l_e}{L_e}\right)^m v_0 = u_0 \leq v_0 = \left(\frac{L_e}{l_e}\right)^m u_0 \tag{3.26}$$

and

$$\begin{aligned} u_1 = Au_0 &= \int_0^1 G(t,s) f\big(s, l_e^m e(s)\big)\, ds \geq \eta(l_e) \int_0^1 G(t,s) f\big(s, l_e^{m-1} e(s)\big)\, ds \\ &\geq \cdots \geq \eta^m(l_e) \int_0^1 G(t,s) f\big(s, e(s)\big)\, ds = \eta^m(l_e) Ae \geq \eta^m(l_e) l_e e \geq l_e^m e = u_0. \end{aligned} \tag{3.27}$$

In a similar way, we can get $v_0 \geq v_1$. It follows from the increasing property of A that

$$u_0 \leq u_1 \leq \cdots \leq u_n \leq \cdots \leq v_n \leq \cdots \leq v_1 \leq v_0. \tag{3.28}$$

Therefore, $u_n \geq u_0 = (\frac{l_e}{L_e})^m v_0 \geq (\frac{l_e}{L_e})^m v_n$. Let

$$c_n = \sup\{c > 0 | u_n \geq c v_n\}, \quad n = 1, 2, \ldots. \tag{3.29}$$

Then $u_n \geq c_n v_n$. Noticing (3.28), we have $1 \geq c_{n+1} \geq c_n$. Thus, we can suppose that $\{c_n\}$ converges to c^*. It is clear that $0 < c^* \leq 1$, we now prove that $c^* = 1$. In fact, if $0 < c^* < 1$, then

$$u_{n+1} = Au_n \geq A(c_n v_n) \geq A\left(\frac{c_n}{c^*} c^* v_n\right) \geq \eta\left(\frac{c_n}{c^*}\right)\eta\left(c^*\right)A(v_n) \geq \frac{c_n}{c^*}\eta\left(c^*\right)v_{n+1}. \tag{3.30}$$

Therefore $c_{n+1} \geq \frac{c_n}{c^*}\eta(c^*)$. Let $n \to \infty$, we have $c^* \geq \eta(c^*)$, which contradicts (H). Hence $c^* = 1$.

For each natural number p, we have

$$\begin{aligned} &0 \leq u_{n+p} - u_n \leq v_n - u_n \leq v_n - c_n v_n \leq (1 - c_n)v_0, \\ &0 \leq v_n - v_{n+p} \leq v_n - u_n \leq (1 - c_n)v_0. \end{aligned} \tag{3.31}$$

Since P is normal, then

$$\|u_{n+p} - u_n\| \to 0, \qquad \|v_n - v_{n+p}\| \to 0, \quad n \to \infty, \tag{3.32}$$

which implies $\{u_n\}$, $\{v_n\}$ are Cauchy sequences. There exist u_*, v_* such that $u_n \to u_*$, $v_n \to v_*$. From (3.28), we get $u_n \leq u_* \leq v_* \leq v_n$. By (3.31), we have $\|u_* - v_*\| \to 0$. Then $u_* = v_*$ is a fixed point of A.

Equations (3.1b) and (3.16) yield

$$\lim_{t\to 0+} t^{\alpha-2}\int_0^1 (1-s)^{\alpha-3} f\bigl(ts, u_*(ts)\bigr)\, ds = 0. \tag{3.33}$$

Then

$$\lim_{t\to 0+} t^{\alpha-2}\int_0^1 (1-s)^{\alpha-2} f\bigl(ts, u_*(ts)\bigr)\, ds = 0. \tag{3.34}$$

By Lemma 2.4, u_* is a positive solution of BVP (1.1).

Noticing (3.13), (3.14) and (3.1a), it is clear that $u_* \in C'[0,1] \cap C^2(0,1)$. Again from (3.1b) and (3.16), we get $u_*''(0+) = u_*''(0) = 0$. By (3.1c), (3.14) and Lemma 2.7, we have (3.12) holds, which implies $u_*''(1^-)$ exists. Therefore, u_* is a $C^2[0,1]$ positive solution of BVP (1.1). The proof is completed. □

Theorem 3.2 *Suppose that* (3.1a), (3.1b), (3.1c) *and* (H) *hold. Then*:

(i) *BVP* (1.1) *has a unique* $C^2[0,1]$ *positive solution* $u_* \in P_e$.

(ii) *For any initial value* $\omega_0 \in P_e$, *the sequence of functions defined by*

$$\omega_n = \int_0^1 G(t,s) f\bigl(s, \omega_{n-1}(s)\bigr)\, ds, \quad n = 1, 2, \ldots \tag{3.35}$$

converges uniformly to u_* *on* $[0,1]$.

Proof (i) It follows from Theorem 3.1 that BVP (1.1) has a $C^2[0,1]$ positive solution $u_* \in P_e$. Let v be another $C^2[0,1]$ positive solution of BVP (1.1). Lemma 2.6 implies $v \in P_e$. So there exist two positive numbers $0 < l_v < 1 < L_v$ such that

$$l_v e(t) \le v(t) \le L_v e(t), \quad t \in [0,1]. \tag{3.36}$$

Let m defined by (3.24) be large enough such that $l_v > l_e^m$ and $L_v < L_e^m$. Then

$$u_0 \le v \le v_0. \tag{3.37}$$

It is clear that A is an increasing operator and $Av = v$, therefore

$$u_n \le v \le v_n, \quad n = 1, 2, \ldots. \tag{3.38}$$

Let $n \to \infty$, we get $v = u_*$. So the $C^2[0,1]$ positive solution of BVP (1.1) is unique.

(ii) For any initial value $\omega_0 \in P_e$, there exist two positive numbers $0 < l_{\omega_0} < 1 < L_{\omega_0}$ such that

$$l_{\omega_0} e(t) \le \omega_0(t) \le L_{\omega_0} e(t), \quad t \in [0,1]. \tag{3.39}$$

Let m defined by (3.24) be large enough such that $l_{\omega_0} > l_e^m$ and $L_{\omega_0} < L_e^m$. Then

$$u_0 \le \omega_0 \le v_0. \tag{3.40}$$

Notice that A is an increasing operator, we have

$$u_n \le \omega_n \le v_n, \quad n = 1, 2, \ldots. \tag{3.41}$$

Let $n \to \infty$, then $\omega_n = u_*$. It follows from (3.28), (3.31) and (3.41) that ω_n converges uniformly to the unique positive solution u_* on $[0,1]$. The proof is completed. □

4 Example

Example 4.1 Consider the following problem:

$$\begin{cases} D_{0+}^{\alpha} u(t) + f(t, u(t)) = 0, & t \in (0,1), 2 < \alpha < 3, \\ u(0) = u'(1) = u''(0) = 0, \end{cases} \tag{4.1}$$

where

$$f(t,x) = t^{-\sigma} x^{\beta}, \quad \sigma, \beta \in (0,1).$$

Obviously, assumption (H) holds. By Theorem 3.1, we have that the necessary and sufficient condition for the existence of a $C^2[0,1]$ positive solution to BVP (4.1) is

$$\beta - \sigma + \alpha > 2.$$

Example 4.2 Consider the following problem:

$$\begin{cases} D_{0+}^{\alpha}u(t) + a(t)f(u(t)) = 0, & t \in (0,1), 2 < \alpha < 3, \\ u(0) = u'(1) = u''(0) = 0, \end{cases} \tag{4.2}$$

where $a \in C((0,1),[0,\infty))$,

$$f(x) = \begin{cases} x^{\beta} + x, & x \in [0,1], 0 < \beta < 1, \\ 2x^{\beta}, & x \in (1,+\infty). \end{cases}$$

Let

$$\eta(r) = \frac{r^{\beta} + r}{2},$$

then assumption (H) holds. Noticing

$$x^{\beta} \le f(x) \le 2x^{\beta}, \quad x \in [0,+\infty),$$

by Theorem 3.1 and Lemma 2.7, we have that the necessary and sufficient condition for the existence of a $C^2[0,1]$ positive solution of BVP (4.2) is

$$0 < \int_0^1 a(s)s^{\beta}(1-s)^{\alpha-3}\,ds < +\infty,$$
$$\lim_{t\to 0+} t^{\alpha+\beta-2}\int_0^1 a(ts)s^{\beta}(1-s)^{\alpha-3}\,ds = 0,$$
$$\lim_{t\to 1-} \int_0^1 a(ts)s^{\beta}(1-s)^{\alpha-3}\,ds = \int_0^1 a(s)s^{\beta}(1-s)^{\alpha-3}\,ds.$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors declare that the study was realized in collaboration with the same responsibility. All authors read and approved the final manuscript.

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