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Global asymptotic stability for quadratic fractional difference equation

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Abstract

Consider the difference equation

$$x_{n+1}=\frac{\alpha+\sum_{i=0}^{k}a_ix_{n-i}+\sum_{i=0}^{k}\sum_{j=i}^{k}a_{ij}x_{n-i}x_{n-j}}{\beta+\sum_{i=0}^{k}b_ix_{n-i}+\sum_{i=0}^{k}\sum_{j=i}^{k}b_{ij}x_{n-i}x_{n-j}},\quad n=0,1,\ldots,$$

where all parameters α, β, a_i, b_i, a_{ij}, b_{ij}, $i,j=0,1,\ldots,k$, and the initial conditions x_i, $i\in\{-k,\ldots,0\}$, are nonnegative. We investigate the asymptotic behavior of the solutions of the considered equation. We give simple explicit conditions for the global stability and global asymptotic stability of the zero or positive equilibrium of this equation.

MSC: 39A10; 39A30; 65L20

Keywords: attractivity; difference equations; rational; stability

1 Introduction

Consider the difference equation

$$x_{n+1}=\frac{\alpha+\sum_{i=0}^{k}a_ix_{n-i}+\sum_{i=0}^{k}\sum_{j=i}^{k}a_{ij}x_{n-i}x_{n-j}}{\beta+\sum_{i=0}^{k}b_ix_{n-i}+\sum_{i=0}^{k}\sum_{j=i}^{k}b_{ij}x_{n-i}x_{n-j}},\quad n=0,1,\ldots,\tag{1}$$

where $k\in\{0,1,\ldots\}$, the parameters α, β, a_i, b_i, a_{ij}, b_{ij}, $i,j=0,1,\ldots,k$, and the initial conditions x_i, $i\in\{-k,\ldots,0\}$, are nonnegative and such that the denominator of Eq. (1) is always positive. The important special cases of Eq. (1) are the linear fractional equations such as the well-known Riccati equation

$$x_{n+1}=\frac{\alpha+a_0x_n}{\beta+b_0x_n},\quad n=0,1,\ldots,\tag{2}$$

the second order linear fractional difference equation

$$x_{n+1}=\frac{\alpha+\sum_{i=0}^{1}a_ix_{n-i}}{\beta+\sum_{i=0}^{1}b_ix_{n-i}},\quad n=0,1,\ldots,\tag{3}$$

and the third order linear fractional difference equation that we get from Eq. (1) for $k=2$ and $a_{ij}=b_{ij}=0$ for all i,j. The global behavior and the exact solutions of Eq. (2) even for real parameters were found in [1]. The global behavior of solutions of Eq. (3), in many subcases when one or more parameters are zero, was established in [1]. There is still one conjecture left whose answer will complete the global picture of the asymptotic behavior of solutions of Eq. (3). As far as the third order linear fractional difference equation is concerned, there is a large number of sporadic results that are systemized in a book [2]. The characterization of the global asymptotic behavior of solutions of Eq. (1) for $k = 2$ seems to be much harder than for the second order Eq. (3). Consequently an attempt at giving the characterization of the global asymptotic behavior of solutions of Eq. (1) seems to be a formidable task at this time. However, by using some known global attractivity results, we can describe the global asymptotic behavior of solutions of Eq. (1) in some subspaces of the parametric space and the space of initial conditions. See [2–5] for a complete description of the behavior of some special cases of Eq. (1), in particular for the cases known as periodic trichotomies. See [6] where the difference in global behavior between the second and third order linear fractional difference equation is emphasized. The results on the global periodicity, that is, the results which describe all special cases of Eq. (1) where all solutions are periodic of the same period were obtained in [7, 8]. Most results in [2–5, 9, 10] are based on known global attractivity or global asymptotic stability results obtained in [1, 2, 11–14].

The special case of Eq. (1) with quadratic terms such as

$$x_{n+1} = \frac{ax_n^2}{x_n + bx_{n-1}^2}, \quad n = 0, 1, \ldots, \tag{4}$$

where $a, b > 0$ and the initial conditions $x_{-1}, x_0 \geq 0$, $x_{-1} + x_0 > 0$, has only a negative equilibrium point for $a \leq 1$ and yet all solutions of Eq. (4) satisfy

$$\lim_{n\to\infty} x_n = 0.$$

A kth order generalization of Eq. (4) with the same property is

$$x_{n+1} = \frac{ax_n^2}{x_n + \sum_{i=1}^{k-1} b_i x_{n-i}^2}, \quad n = 0, 1, \ldots, \tag{5}$$

where $a, b_i > 0$ and the initial conditions $x_{-k+1}, \ldots, x_0 \geq 0$, $x_{-k+1} + \cdots + x_0 > 0$, when $a \leq 1$.

Another special case of Eq. (1) with quadratic terms is

$$x_{n+1} = \frac{Bx_n x_{n-1}}{dx_n + ex_{n-1}}, \quad n = 0, 1, \ldots, \tag{6}$$

where $B, d, e > 0$ and the initial conditions $x_{-1}, x_0 \geq 0$, $x_{-1} + x_0 > 0$, it does not have an equilibrium point for $B \neq d + e$ and yet all solutions of Eq. (6) satisfy

$$\lim_{n\to\infty} x_n = 0, \quad \text{when } B < d + e,$$

and

$$\lim_{n\to\infty} x_n = \infty, \quad \text{when } B > d + e.$$

Finally, when $B = d + e$, then Eq. (6) has an infinite number of the equilibrium points, each with its basin of attraction, see [15].

Another interesting case of Eq. (1) with quadratic terms is the equation

$$x_{n+1} = \frac{ax_n^2}{1 + x_n^2}, \quad n = 0, 1, \ldots, \tag{7}$$

where $a > 0$ and $x_0 \in \mathbb{R}$. When $a > 2$ every solution of Eq. (7) converges either to the zero equilibrium or to the bigger positive equilibrium $x_+ = \frac{a+\sqrt{a^2-4}}{2}$, with basins of attraction being $\mathcal{B}(0) = [0, x_-)$ and $\mathcal{B}(x_+) = (x_-, \infty)$, where $x_- = \frac{a-\sqrt{a^2-4}}{2}$ is the smaller positive equilibrium.

None of these asymptotic behaviors which are present in the cases of Eqs. (4)-(7) are possible in the case of the linear fractional equation

$$x_{n+1} = \frac{\alpha + \sum_{i=0}^{k} a_i x_{n-i}}{\beta + \sum_{i=0}^{k} b_i x_{n-i}}, \quad n = 0, 1, \ldots,$$

and the appearance of these behaviors is caused by the presence of the quadratic terms. The results presented here have been successfully applied to some special classes of Eq. (1), see [16, 17] and can be applied to equations considered in [18–22].

This paper is an attempt at establishing some global stability results for the equilibrium solution(s) of Eq. (1). Our results give effective conditions for global asymptotic stability of the equilibrium solution(s) of Eq. (1) expressed in terms of the inequalities on the parameters.

2 Preliminaries

The following general global results will be applied to Eq. (1), see [23].

Consider the difference equation

$$x_{n+1} = f(x_n, \ldots, x_{n-k}), \quad n = 0, 1, \ldots, \tag{8}$$

where $k \in \{0, 1, \ldots\}$. Sometimes it is more advantageous to investigate Eq. (8) by embedding Eq. (8) into a higher iteration of the form

$$x_{n+l} = F(x_{n+l-1}, \ldots, x_{n-k}), \quad n = 0, 1, \ldots, \tag{9}$$

where $l \in \{1, 2, \ldots\}$, see [23, 24] and then linearizing (rearranging) Eq. (9) so that it has the form

$$x_{n+l} = \sum_{i=1-l}^{k} g_i x_{n-i}, \quad n = 0, 1, \ldots, \tag{10}$$

where the functions $g_i : \mathbb{R}^{k+l} \to \mathbb{R}$.

Theorem 1 *Let $l \in \{1, 2, \ldots\}$ and let $a \in \mathbb{R}$. Suppose that Eq.* (8) *has the linearization* (10) *where the functions $g_i : \mathbb{R}^{k+l} \to \mathbb{R}$ are such that*

$$\sum_{i=1-l}^{k} |g_i| \leq a < 1, \quad n = 0, 1, \ldots. \tag{11}$$

Then

$$\lim_{n\to\infty} x_n = 0.$$

As we have observed in [23], condition (11) is actually a contraction condition in the Banach contraction principle.

In addition, we will need the following stability result from [25].

Theorem 2 *Suppose that Eq.* (8) *can be linearized into the form*

$$x_{n+1} - \bar{x} = \sum_{i=0}^{k} g_i(x_{n-i} - \bar{x}), \quad n = 0, 1, \ldots, \tag{12}$$

where $\bar{x}$ *is an equilibrium of Eq.* (8) *and the functions* $g_i : \mathbb{R}^{k+1} \to \mathbb{R}$. *If* $\sum_{i=0}^{k} |g_i| \le 1$, $n \ge 0$, *then the equilibrium* $\bar{x}$ *of Eq.* (8) *is stable.*

The next result follows from Theorems 1 and 2.

Corollary 1 *Let* $a \in \mathbb{R}$. *Suppose that Eq.* (8) *has the linearization* (12), *where* $\bar{x}$ *is a unique nonnegative equilibrium of Eq.* (8) *on the interval I and the functions* $g_i : \mathbb{R}^{k+1} \to \mathbb{R}$ *are such that*

$$\sum_{i=0}^{k} |g_i| \le a < 1, \quad n = 0, 1, \ldots,$$

then the unique nonnegative equilibrium Eq. (8) *is globally asymptotically stable on the interval* I.

The next result is an analogue of the result obtained in [23].

Lemma 1 *Let* $m \le \bar{x} \le M$ *and* $m \le x_{N-i} \le M$, $i = 0, 1, \ldots, k$, *for some* $N \in \{0, 1, \ldots\}$. *Suppose that*

$$x_{n+1} - \bar{x} = \sum_{i=0}^{k} h_i(x_{n-i} - \bar{x}), \quad n = 0, 1, \ldots,$$

where the nonnegative functions $h_i : [0,\infty)^{k+1} \to [0,\infty)$. *Assume that for this N*

$$\sum_{i=0}^{k} h_i \le 1.$$

Then $m \le x_{N+1} \le M$.

3 Main results

In this section we investigate the stability of the unique positive equilibrium $\bar{x}$ of Eq. (1) by using Theorems 1 and 2. Observe that Eq. (1) has a zero equilibrium if and only if $\alpha = 0$

and $\beta > 0$, in which case Eq. (1) becomes

$$x_{n+1} = \frac{\sum_{i=0}^{k} a_i x_{n-i} + \sum_{i=0}^{k}\sum_{j=i}^{k} a_{ij}x_{n-i}x_{n-j}}{\beta + \sum_{i=0}^{k} b_i x_{n-i} + \sum_{i=0}^{k}\sum_{j=i}^{k} b_{ij}x_{n-i}x_{n-j}}, \quad n = 0,1,\ldots. \tag{13}$$

Equation (13) yields the nonzero equilibrium points

$$\bar{x} = \frac{-(\sum_{i=0}^{k} b_i - \sum_{i=0}^{k}\sum_{j=i}^{k} a_{ij}) \pm \sqrt{(\sum_{i=0}^{k} b_i - \sum_{i=0}^{k}\sum_{j=i}^{k} a_{ij})^2 - 4(\beta - \sum_{i=0}^{k} a_i)\sum_{i=0}^{k}\sum_{j=i}^{k} b_{ij}}}{2\sum_{i=0}^{k}\sum_{j=i}^{k} b_{ij}}.$$

Thus if $\beta \ge \sum_{i=0}^{k} a_i$ and $\sum_{i=0}^{k} b_i > \sum_{i=0}^{k}\sum_{j=i}^{k} a_{ij}$, then there is no positive equilibrium. The following result shows that there is an interval in which every solution of Eq. (13) converges to the zero equilibrium. For convenience of notation, let Q denote the denominator of Eq. (1), that is,

$$Q = \beta + \sum_{i=0}^{k} b_i x_{n-i} + \sum_{i=0}^{k}\sum_{j=i}^{k} b_{ij}x_{n-i}x_{n-j}.$$

Theorem 3 *Let $M \in (0,\infty)$ be such that $M < \frac{\beta - \sum_{i=0}^{k} a_i}{\sum_{i=0}^{k}\sum_{j=i}^{k} a_{ij}}$. Assume that there is no positive equilibrium. Then the zero equilibrium of Eq.* (13) *is globally asymptotically stable on the interval* $[0,M)$.

Proof Observe that Eq. (13) can be linearized into the form (10) where $l = 1$ as follows for $n \ge 0$

$$x_{n+1} = \frac{a_0 + \sum_{j=0}^{k} a_{0j}x_{n-j}}{Q}x_n + \frac{a_1 + \sum_{j=1}^{k} a_{1j}x_{n-j}}{Q}x_{n-1} + \cdots + \frac{a_k + a_{kk}x_{n-k}}{Q}x_{n-k}.$$

Then, for $i = 0,\ldots,k$,

$$|g_i| = \frac{a_i + \sum_{j=i}^{k} a_{ij}x_{n-j}}{Q}, \quad n = 0,1,\ldots.$$

Let $\{x_n\}$ be a solution of Eq. (13) where $\max\{x_0,\ldots,x_{-k}\} \le M$. Then for $n = 0$

$$\sum_{i=0}^{k} |g_i| = \frac{\sum_{i=0}^{k} a_i + \sum_{i=0}^{k}\sum_{j=i}^{k} a_{ij}x_{-j}}{\beta + \sum_{i=0}^{k} b_i x_{-i} + \sum_{i=0}^{k}\sum_{j=i}^{k} b_{ij}x_{-i}x_{-j}} \le \frac{\sum_{i=0}^{k} a_i + M\sum_{i=0}^{k}\sum_{j=i}^{k} a_{ij}}{\beta} < 1.$$

By Lemma 1 with $\bar{x} = 0$, $h_i = |g_i|$, $i = 0,\ldots,k$, and $N = 0$, we get that $x_1 \le M$. Hence $x_i \le M$ for $i = 1,0,\ldots,-k$. Then for $n = 1$

$$\sum_{i=0}^{k} |g_i| = \frac{\sum_{i=0}^{k} a_i + \sum_{i=0}^{k}\sum_{j=i}^{k} a_{ij}x_{1-j}}{\beta + \sum_{i=0}^{k} b_i x_{1-i} + \sum_{i=0}^{k}\sum_{j=i}^{k} b_{ij}x_{1-i}x_{1-j}} \le \frac{\sum_{i=0}^{k} a_i + M\sum_{i=0}^{k}\sum_{j=i}^{k} a_{ij}}{\beta} < 1.$$

Again using Lemma 1 with $\bar{x} = 0$, $h_i = |g_i|$, $i = 0, \dots, k$, and $N = 1$, we get that $x_2 \le M$. Hence $x_i \le M$ for $i = 2, 1, \dots, -k$. Then for $n = 2$

$$\sum_{i=0}^{k} |g_i| = \frac{\sum_{i=0}^{k} a_i + \sum_{i=0}^{k} \sum_{j=i}^{k} a_{ij} x_{2-j}}{\beta + \sum_{i=0}^{k} b_i x_{2-i} + \sum_{i=0}^{k} \sum_{j=i}^{k} b_{ij} x_{2-i} x_{2-j}} \le \frac{\sum_{i=0}^{k} a_i + M \sum_{i=0}^{k} \sum_{j=i}^{k} a_{ij}}{\beta} < 1.$$

By induction we get that for $n \ge 0$

$$\sum_{i=0}^{k} |g_i| = \frac{\sum_{i=0}^{k} a_i + \sum_{i=0}^{k} \sum_{j=i}^{k} a_{ij} x_{n-j}}{\beta + \sum_{i=0}^{k} b_i x_{n-i} + \sum_{i=0}^{k} \sum_{j=i}^{k} b_{ij} x_{n-i} x_{n-j}} \le \frac{\sum_{i=0}^{k} a_i + M \sum_{i=0}^{k} \sum_{j=i}^{k} a_{ij}}{\beta} < 1,$$

and so the result follows from Corollary 1 where $\bar{x} = 0$ and the interval is $[0, M)$. □

All equilibrium solutions of Eq. (1) satisfy the equilibrium equation

$$\sum_{i=0}^{k} \sum_{j=i}^{k} b_{ij} x^3 + \left(\sum_{i=0}^{k} b_i - \sum_{i=0}^{k} \sum_{j=i}^{k} a_{ij} \right) x^2 + \left(\beta - \sum_{i=0}^{k} a_i \right) x - \alpha = 0, \tag{14}$$

which can be rewritten as

$$\alpha - \beta x + x \sum_{i=0}^{k} (a_i - x b_i) + x^2 \sum_{i=0}^{k} \sum_{j=i}^{k} (a_{ij} - x b_{ij}) = 0. \tag{15}$$

Equilibrium Eq. (14) has at least one nonnegative zero and it may have between 0 and 3 positive zeros. When $\alpha > 0$ and either $\sum_{i=0}^{k} b_i - \sum_{i=0}^{k} \sum_{j=i}^{k} a_{ij} \ge 0$, $\beta - \sum_{i=0}^{k} a_i \ge 0$ or $\sum_{i=0}^{k} b_i - \sum_{i=0}^{k} \sum_{j=i}^{k} a_{ij} \ge 0$, $\beta - \sum_{i=0}^{k} a_i \le 0$ or $\sum_{i=0}^{k} b_i - \sum_{i=0}^{k} \sum_{j=i}^{k} a_{ij} \le 0$, $\beta - \sum_{i=0}^{k} a_i \le 0$, Descartes rule of sign implies that there is a unique positive equilibrium of Eq. (1).

If $\bar{x} > 0$ is an equilibrium, then for $n \ge 0$

$$\begin{aligned}
x_{n+1} - \bar{x} &= \frac{\alpha + \sum_{i=0}^{k} a_i x_{n-i} + \sum_{i=0}^{k} \sum_{j=i}^{k} a_{ij} x_{n-i} x_{n-j}}{\beta + \sum_{i=0}^{k} b_i x_{n-i} + \sum_{i=0}^{k} \sum_{j=i}^{k} b_{ij} x_{n-i} x_{n-j}} - \bar{x} \\
&= \frac{\alpha - \beta \bar{x} + \sum_{i=0}^{k} (a_i - b_i \bar{x}) x_{n-i} + \sum_{i=0}^{k} \sum_{j=i}^{k} (a_{ij} - b_{ij} \bar{x}) x_{n-i} x_{n-j}}{Q}, \\
&= \frac{\sum_{i=0}^{k} (a_i - b_i \bar{x})(x_{n-i} - \bar{x}) + \sum_{i=0}^{k} \sum_{j=i}^{k} (a_{ij} - b_{ij} \bar{x})(x_{n-i} x_{n-j} - \bar{x}^2) + R}{Q},
\end{aligned}$$

where in view of Eqs. (14), (15)

$$R = \alpha - \beta \bar{x} + \bar{x} \sum_{i=0}^{k} (a_i - b_i \bar{x}) + \bar{x}^2 \sum_{i=0}^{k} \sum_{j=i}^{k} (a_{ij} - b_{ij} \bar{x}) = 0.$$

Now applying the identity

$$x_{n-i} x_{n-j} - \bar{x}^2 = x_{n-i}(x_{n-j} - \bar{x}) + \bar{x}(x_{n-i} - \bar{x}), \quad i, j = 0, 1, \dots,$$

we get that for $n \geq 0$

$$x_{n+1} - \bar{x} = \frac{\sum_{i=0}^{k}(a_i - b_i\bar{x})(x_{n-i} - \bar{x}) + \sum_{i=0}^{k}\sum_{j=i}^{k}(a_{ij} - b_{ij}\bar{x})\bar{x}(x_{n-i} - \bar{x}) + \sum_{i=0}^{k}\sum_{j=i}^{k}(a_{ij} - b_{ij}\bar{x})x_{n-i}(x_{n-j} - \bar{x})}{Q}.$$

Observe that for $n \geq 0$

$$\sum_{i=0}^{k}\sum_{j=i}^{k}(a_{ij} - b_{ij}\bar{x})x_{n-i}(x_{n-j} - \bar{x}) = \sum_{i=0}^{k}\sum_{j=0}^{i}(a_{ji} - b_{ji}\bar{x})x_{n-j}(x_{n-i} - \bar{x}).$$

Thus for $n \geq 0$

$$\begin{aligned} x_{n+1} - \bar{x} &= \frac{\sum_{i=0}^{k}(a_i - b_i\bar{x})(x_{n-i} - \bar{x}) + \sum_{i=0}^{k}\sum_{j=i}^{k}(a_{ij} - b_{ij}\bar{x})\bar{x}(x_{n-i} - \bar{x}) + \sum_{i=0}^{k}\sum_{j=0}^{i}(a_{ji} - b_{ji}\bar{x})x_{n-j}(x_{n-i} - \bar{x})}{Q} \\ &= \frac{\sum_{i=0}^{k}(a_i - b_i\bar{x})(x_{n-i} - \bar{x}) + \sum_{i=0}^{k}(\sum_{j=i}^{k}(a_{ij} - b_{ij}\bar{x})\bar{x} + \sum_{j=0}^{i}(a_{ji} - b_{ji}\bar{x})x_{n-j})(x_{n-i} - \bar{x})}{Q}. \end{aligned}$$

Therefore for $n \geq 0$

$$x_{n+1} - \bar{x} = \sum_{i=0}^{k} \frac{(a_i - b_i\bar{x}) + \sum_{j=i}^{k}(a_{ij} - b_{ij}\bar{x})\bar{x} + \sum_{j=0}^{i}(a_{ji} - b_{ji}\bar{x})x_{n-j}}{\beta + \sum_{i=0}^{k} b_i x_{n-i} + \sum_{i=0}^{k}\sum_{j=i}^{k} b_{ij}x_{n-i}x_{n-j}}(x_{n-i} - \bar{x}). \tag{16}$$

Equation (16) is the linearized equation of Eq. (1) of the form (12) where for $i = 0, \dots, k$

$$g_i = \frac{(a_i - b_i\bar{x}) + \sum_{j=i}^{k}(a_{ij} - b_{ij}\bar{x})\bar{x} + \sum_{j=0}^{i}(a_{ji} - b_{ji}\bar{x})x_{n-j}}{\beta + \sum_{i=0}^{k} b_i x_{n-i} + \sum_{i=0}^{k}\sum_{j=i}^{k} b_{ij}x_{n-i}x_{n-j}}, \quad n = 0, 1, \dots. \tag{17}$$

We can now obtain easy-to-check conditions which show when the positive equilibrium of Eq. (1) is globally asymptotically stable. We will then apply these conditions to various cases of Eq. (1).

Theorem 4 *Assume that Eq.* (1) *has a unique positive equilibrium* $\bar{x}$ *and there exist* $L \geq 0$ *and* $U, N > 0$ *such that for every solution* $\{x_n\}$ *of Eq.* (1) $L \leq x_n \leq U$ *for all* $n \geq N$ *and*

$$\sum_{i=0}^{k} |a_i - b_i\bar{x}| + (U + \bar{x})\sum_{i=0}^{k}\sum_{j=i}^{k} |a_{ij} - b_{ij}\bar{x}| < \beta + L\sum_{i=0}^{k} b_i + L^2\sum_{i=0}^{k}\sum_{j=i}^{k} b_{ij}, \tag{18}$$

where $\beta + L > 0$. *Then the unique positive equilibrium* $\bar{x}$ *of Eq.* (1) *is globally asymptotically stable on the interval* $[0, \infty)$.

Proof As we have seen Eq. (1) can be written in the form of the linearized Eq. (16), where the coefficients g_i are given as (17).

We have for $n \geq 0$

$$Q = \beta + \sum_{i=0}^{k} b_i x_{n-i} + \sum_{i=0}^{k}\sum_{j=i}^{k} b_{ij}x_{n-i}x_{n-j} \geq \beta + L\sum_{i=0}^{k} b_i + L^2\sum_{i=0}^{k}\sum_{j=i}^{k} b_{ij}, \tag{19}$$

and so for $i = 0, \dots, k$ and $n \geq 0$

$$|g_i| \leq \frac{|a_i - b_i\bar{x}| + \sum_{j=i}^{k} |a_{ij} - b_{ij}\bar{x}|\bar{x} + \sum_{j=0}^{i} |a_{ji} - b_{ji}\bar{x}|x_{n-j}}{Q}.$$

Thus for $n \geq 0$

$$\sum_{i=0}^{k} |g_i| \leq \frac{\sum_{i=0}^{k} |a_i - b_i\bar{x}| + \sum_{i=0}^{k} \sum_{j=i}^{k} |a_{ij} - b_{ij}\bar{x}|\bar{x} + \sum_{i=0}^{k} \sum_{j=0}^{i} |a_{ji} - b_{ji}\bar{x}|x_{n-j}}{Q}.$$

By rearranging the terms we can show that for $n \geq 0$

$$\sum_{i=0}^{k} \sum_{j=0}^{i} |a_{ji} - b_{ji}\bar{x}|x_{n-j} = \sum_{i=0}^{k} \sum_{j=i}^{k} |a_{ij} - b_{ij}\bar{x}|x_{n-i}.$$

Thus for $n \geq 0$

$$\sum_{i=0}^{k} |g_i| \leq \frac{\sum_{i=0}^{k} |a_i - b_i\bar{x}| + \sum_{i=0}^{k} \sum_{j=i}^{k} |a_{ij} - b_{ij}\bar{x}|\bar{x} + \sum_{i=0}^{k} \sum_{j=i}^{k} |a_{ij} - b_{ij}\bar{x}|x_{n-i}}{Q},$$

and so

$$\sum_{i=0}^{k} |g_i| \leq \frac{\sum_{i=0}^{k} |a_i - b_i\bar{x}| + \sum_{i=0}^{k} \sum_{j=i}^{k} |a_{ij} - b_{ij}\bar{x}|(\bar{x} + x_{n-i})}{Q}.$$

In view of (18) and (19), we obtain

$$\sum_{i=0}^{k} |g_i| \leq \frac{\sum_{i=0}^{k} |a_i - b_i\bar{x}| + \sum_{i=0}^{k} \sum_{j=i}^{k} |a_{ij} - b_{ij}\bar{x}|(\bar{x} + U)}{\beta + L\sum_{i=0}^{k} b_i + L^2 \sum_{i=0}^{k} \sum_{j=i}^{k} b_{ij}} < 1$$

for $n \geq 0$. So by Corollary 1 $\bar{x}$ is globally asymptotically stable on the interval $[L, U]$. By assumption every solution of Eq. (1) enters the interval $[L, U]$ and so the result follows. □

From (18) we see that establishing a lower bound for all the solutions of Eq. (1) will give us a better result. We present some of these cases.

Remark 1 The results on boundedness of all solutions of Eq. (1) are well known, see [2, 24]. For instance, if for every $i, j \in \{0, \dots, k\}$ such that $b_i > 0$, $b_{ij} > 0$ we have $a_i > 0$, $a_{ij} > 0$, then the uniform lower bound L for all solutions $\{x_n\}$ of Eq. (1) for $n \geq 1$ is

$$L = \frac{\min\{\alpha, a_i, a_{ij} | a_i > 0, a_{ij} > 0\}}{\max\{\beta, b_i, b_{ij} > 0 | b_i > 0, b_{ij} > 0\}}.$$

On the other hand, if for every $i, j \in \{0, \dots, k\}$ such that $a_i, a_{ij} > 0$ we have $b_i, b_{ij} > 0$, then the uniform lower bound L for all solutions of Eq. (1) for $n \geq 1$ is

$$L = \frac{\min\{\alpha, a_i, a_{ij} | a_i, a_{ij} > 0\}}{\max\{U \sum_{j, a_j = 0} b_j, U \sum_{i,j, a_{ij} = 0} b_{ij}, \beta, b_i, b_{ij} | b_i > 0\}},$$

where

$$U = \frac{\max\{\alpha, a_i, a_{ij} | a_i, a_{ij} > 0\}}{\min\{\beta, b_i, b_{ij} | b_i, b_{ij} > 0\}}.$$

The next result follows from Lemma 1 and can be used to find the part of the basin of attraction of a positive equilibrium in the case when there are several positive equilibrium points. The proof of this result is similar to the proof of Theorem 3 and it will be omitted.

Theorem 5 *Let* $M = \max\{\bar{x}, x_{-i} : i = 0, \dots, k\}$ *be such that*

$$\sum_{i=0}^{k} |a_i - b_i\bar{x}| + (M + \bar{x}) \sum_{i=0}^{k} \sum_{j=i}^{k} |a_{ij} - b_{ij}\bar{x}| < \beta + m \sum_{i=0}^{k} b_i + m^2 \sum_{i=0}^{k} \sum_{j=i}^{k} b_{ij}, \tag{20}$$

where $0 \le m = \min\{\bar{x}, x_{-i} : i = 0, \dots, k\}$ *is such that* $\beta + m > 0$. *Then the equilibrium* $\bar{x}$ *of Eq.* (1) *is globally asymptotically stable on the interval* $[m, M]$.

The following result is a consequence of Theorem 4 in some special cases when the unique positive equilibrium satisfies some specific conditions.

Theorem 6 *Assume that* $\beta > 0$ *and* $\bar{x}$ *is the unique positive equilibrium of Eq.* (1). *Suppose there exist* $L \ge 0$ *and* $U, N > 0$ *such that for every solution* $\{x_n\}$ *of Eq.* (1) $L \le x_n \le U$ *for all* $n \ge N$. *Then the positive equilibrium* $\bar{x}$ *of Eq.* (1) *is globally asymptotically stable on the interval* $[0, \infty)$ *provided one of the following holds*:

(1) $a_i = \bar{x}b_i$, $a_{ij} = \bar{x}b_{ij}$ *for all* $i, j \in \{0, \dots, k\}$;

(2) $a_i \ge \bar{x}b_i$, $a_{ij} \ge \bar{x}b_{ij}$ *for all* $i, j \in \{0, \dots, k\}$ *and* $\alpha \ge 0$ *and*

$$U \sum_{i=0}^{k} \sum_{j=i}^{k} (a_{ij} - \bar{x}b_{ij}) < \frac{\alpha}{\bar{x}} + L \sum_{i=0}^{k} b_i + L^2 \sum_{i=0}^{k} \sum_{j=i}^{k} b_{ij}; \tag{21}$$

(3) $a_i \le \bar{x}b_i$, $a_{ij} \le \bar{x}b_{ij}$ *for all* $i, j \in \{0, \dots, k\}$ *and*

$$U \sum_{i=0}^{k} \sum_{j=i}^{k} (\bar{x}b_{ij} - a_{ij}) < 2\beta - \frac{\alpha}{\bar{x}} + L \sum_{i=0}^{k} b_i + L^2 \sum_{i=0}^{k} \sum_{j=i}^{k} b_{ij}. \tag{22}$$

Proof The positive equilibrium $\bar{x}$ of Eq. (1) satisfies

$$\beta - \frac{\alpha}{\bar{x}} = \sum_{i=0}^{k} a_i - \bar{x}^2 \sum_{i=0}^{k} \sum_{j=i}^{k} b_{ij} - \bar{x} \left(\sum_{i=0}^{k} b_i - \sum_{i=0}^{k} \sum_{j=i}^{k} a_{ij} \right). \tag{23}$$

(1) Let $a_i = \bar{x}b_i$, $a_{ij} = \bar{x}b_{ij}$ for all $i, j \in \{0, \dots, k\}$. Then $\sum_{i=0}^{k} a_i = \bar{x} \sum_{i=0}^{k} b_i$ and $\sum_{i=0}^{k} \sum_{j=i}^{k} a_{ij} = \bar{x} \sum_{i=0}^{k} \sum_{j=i}^{k} b_{ij}$, which by (23) implies $\beta = \frac{\alpha}{\bar{x}}$. Then Eq. (1) becomes for $n \ge 0$

$$\begin{aligned} x_{n+1} &= \frac{\alpha + \sum_{i=0}^{k} a_i x_{n-i} + \sum_{i=0}^{k} \sum_{j=i}^{k} a_{ij} x_{n-i} x_{n-j}}{\beta + \sum_{i=0}^{k} b_i x_{n-i} + \sum_{i=0}^{k} \sum_{j=i}^{k} b_{ij} x_{n-i} x_{n-j}} \\ &= \frac{\beta\bar{x} + \bar{x} \sum_{i=0}^{k} b_i x_{n-i} + \bar{x} \sum_{i=0}^{k} \sum_{j=i}^{k} b_{ij} x_{n-i} x_{n-j}}{\beta + \sum_{i=0}^{k} b_i x_{n-i} + \sum_{i=0}^{k} \sum_{j=i}^{k} b_{ij} x_{n-i} x_{n-j}} = \bar{x}. \end{aligned}$$

(2) In view of our assumption for $i, j \in \{0, \dots, k\}$, we have $|a_i - \bar{x}b_i| = a_i - \bar{x}b_i$, $|a_{ij} - \bar{x}b_{ij}| = a_{ij} - \bar{x}b_{ij}$. By using (23) we obtain

$$\begin{aligned}\sum_{i=0}^{k}|a_i - \bar{x}b_i| + (U + \bar{x})\sum_{i=0}^{k}\sum_{j=i}^{k}|a_{ij} - \bar{x}b_{ij}| &= \sum_{i=0}^{k}a_i - \bar{x}\sum_{i=0}^{k}b_i + (U + \bar{x})\sum_{i=0}^{k}\sum_{j=i}^{k}(a_{ij} - \bar{x}b_{ij}) \\ &= \beta - \frac{\alpha}{\bar{x}} + U\sum_{i=0}^{k}\sum_{j=i}^{k}(a_{ij} - \bar{x}b_{ij}).\end{aligned}$$

Now condition (18) is simplified to

$$U\sum_{i=0}^{k}\sum_{j=i}^{k}(a_{ij} - \bar{x}b_{ij}) < \frac{\alpha}{\bar{x}} + L\sum_{i=0}^{k}b_i + L^2\sum_{i=0}^{k}\sum_{j=i}^{k}b_{ij}$$

and the result follows from Theorem 4.

(3) In this case we have

$$\begin{aligned}\sum_{i=0}^{k}|a_i - \bar{x}b_i| + (U + \bar{x})\sum_{i=0}^{k}\sum_{j=i}^{k}|a_{ij} - \bar{x}b_{ij}| &= \bar{x}\sum_{i=0}^{k}b_i - \sum_{i=0}^{k}a_i + (U + \bar{x})\sum_{i=0}^{k}\sum_{j=i}^{k}(\bar{x}b_{ij} - a_{ij}) \\ &= \frac{\alpha}{\bar{x}} - \beta + U\sum_{i=0}^{k}\sum_{j=i}^{k}(\bar{x}b_{ij} - a_{ij}).\end{aligned}$$

In view of our assumption,

$$\frac{\alpha}{\bar{x}} - \beta + U\sum_{i=0}^{k}\sum_{j=i}^{k}(\bar{x}b_{ij} - a_{ij}) < \beta + L\sum_{i=0}^{k}b_i + L^2\sum_{i=0}^{k}\sum_{j=i}^{k}b_{ij},$$

and so the result follows from Theorem 4. □

Many cases of Eq. (1) have some combination of $a_i < \bar{x}b_i$, $a_i > \bar{x}b_i$ and $a_i = \bar{x}b_i$. In view of this we will adopt the following notations, where $I_> = \{i|$ such that $a_i > \bar{x}b_i\}$, $I_= = \{i|$ such that $a_i = \bar{x}b_i\}$, $I_< = \{i|$ such that $a_i < \bar{x}b_i\}$:

$$\left\{\begin{array}{l}A_S = \sum_{i\in I_>} a_i = \text{the sum of all the } a_i\text{'s} \\ B_S = \sum_{i\in I_>} b_i = \text{the sum of all the } b_i\text{'s}\end{array}\right\} \text{ such that } a_i > \bar{x}b_i,$$

$$\left\{\begin{array}{l}A_N = \sum_{i\in I_=} a_i = \text{the sum of all the } a_i\text{'s} \\ B_N = \sum_{i\in I_=} b_i = \text{the sum of all the } b_i\text{'s}\end{array}\right\} \text{ such that } a_i = \bar{x}b_i,$$

$$\left\{\begin{array}{l}A_R = \sum_{i\in I_<} a_i = \text{the sum of all the } a_i\text{'s} \\ B_R = \sum_{i\in I_<} b_i = \text{the sum of all the } b_i\text{'s}\end{array}\right\} \text{ such that } a_i < \bar{x}b_i.$$

Then $A_S + A_N + A_R = \sum_{i=0}^{k} a_i$ and $B_S + B_N + B_R = \sum_{i=0}^{k} b_i$. Also $A_S > \bar{x}B_S$, $A_N = \bar{x}B_N$ and $A_R < \bar{x}B_R$.

Similarly define

$$\left\{\begin{array}{l}\bar{A}_S = \sum_{i=0}^{k}\sum_{j=i}^{k} a_{ij} = \text{the sum of all the } a_{ij}\text{'s} \\ \bar{B}_S = \sum_{i=0}^{k}\sum_{j=i}^{k} b_{ij} = \text{the sum of all the } b_{ij}\text{'s}\end{array}\right\} \text{ such that } a_{ij} > \bar{x}b_{ij},$$

$$\left\{\begin{array}{l}\bar{A}_N = \sum_{i=0}^k \sum_{j=i}^k a_{ij} = \text{the sum of all the } a_{ij}\text{'s} \\ \bar{B}_N = \sum_{i=0}^k \sum_{j=i}^k b_{ij} = \text{the sum of all the } b_{ij}\text{'s}\end{array}\right\} \text{ such that } a_{ij} = \bar{x}b_{ij},$$

$$\left\{\begin{array}{l}\bar{A}_R = \sum_{i=0}^k \sum_{j=i}^k a_{ij} = \text{the sum of all the } a_{ij}\text{'s} \\ \bar{B}_R = \sum_{i=0}^k \sum_{j=i}^k b_{ij} = \text{the sum of all the } b_{ij}\text{'s}\end{array}\right\} \text{ such that } a_{ij} < \bar{x}b_{ij}.$$

Then $\bar{A}_S + \bar{A}_N + \bar{A}_R = \sum_{i=0}^k \sum_{j=i}^k a_{ij}$ and $\bar{B}_S + \bar{B}_N + \bar{B}_R = \sum_{i=0}^k \sum_{j=i}^k b_{ij}$. Also $\bar{A}_S > \bar{x}\bar{B}_S$, $\bar{A}_N = \bar{x}\bar{B}_N$, $\bar{A}_R < \bar{x}\bar{B}_R$ and

$$\begin{aligned}\sum_{i=0}^k \sum_{j=i}^k |a_{ij} - \bar{x}b_{ij}| &= (\bar{A}_S - \bar{x}\bar{B}_S) + (\bar{x}\bar{B}_N - \bar{A}_N) + (\bar{x}\bar{B}_R - \bar{A}_R) \\ &= \sum_{i=0}^k \sum_{j=i}^k \bar{x}b_{ij} - \sum_{i=0}^k \sum_{j=i}^k a_{ij} + 2(\bar{A}_S - \bar{x}\bar{B}_S).\end{aligned}$$

Corollary 2 *Suppose that the assumptions of Theorem* 6 *are satisfied. Let* $i, j \in \{0, \ldots, k\}$. *Assume that*

(a) *For some* i, j*'s* $a_{ij} > \bar{x}b_{ij}$ *and for other* i, j*'s* $a_{ij} < \bar{x}b_{ij}$.

(b) *For some* i*'s* $a_i > \bar{x}b_i$ *and for other* i*'s* $a_i < \bar{x}b_i$.

(c)
$$\begin{aligned}&\frac{\alpha}{\bar{x}} + 2(A_S - \bar{x}B_S) + (U + 2\bar{x})(\bar{A}_S - \bar{x}\bar{B}_S) + U(\bar{x}B_R - \bar{A}_R) \\ &\quad < 2\beta + L\sum_{i=0}^k b_i + L^2 \sum_{i=0}^k \sum_{j=i}^k b_{ij}.\end{aligned} \tag{24}$$

Then the positive equilibrium $\bar{x}$ *of Eq.* (1) *is globally asymptotically stable on the interval* $[0, \infty)$.

Proof In view of the equilibrium Eq. (15) and by assumption (c), we have that

$$\begin{aligned}&\sum_{i=0}^k |a_i - \bar{x}b_i| + (U + \bar{x}) \sum_{i=0}^k \sum_{j=i}^k |a_{ij} - \bar{x}b_{ij}| \\ &\quad = \bar{x}\sum_{i=0}^k b_i - \sum_{i=0}^k a_i + 2(A_S - \bar{x}B_S) + 2(U + \bar{x})(\bar{A}_S - \bar{x}\bar{B}_S) + (U + \bar{x}) \sum_{i=0}^k \sum_{j=i}^k \bar{x}b_{ij} \\ &\qquad - (U + \bar{x}) \sum_{i=0}^k \sum_{j=i}^k a_{ij} \\ &\quad = \sum_{i=0}^k (\bar{x}b_i - a_i) + \bar{x}\sum_{i=0}^k \sum_{j=i}^k (\bar{x}b_{ij} - a_{ij}) + 2(A_S - \bar{x}B_S) + (2U + 2\bar{x})(\bar{A}_S - \bar{x}\bar{B}_S) \\ &\qquad + U\sum_{i=0}^k \sum_{j=i}^k (\bar{x}b_{ij} - a_{ij}) \\ &\quad = \frac{\alpha}{\bar{x}} - \beta + 2(A_S - \bar{x}B_S) + (2U + 2\bar{x})(\bar{A}_S - \bar{x}\bar{B}_S) + U\bar{x}(\bar{B}_S + \bar{B}_N + \bar{B}_R) \\ &\qquad - U(\bar{A}_S + \bar{A}_N + \bar{A}_R)\end{aligned}$$

$$= \frac{\alpha}{\bar{x}} - \beta + 2(A_S - \bar{x}B_S) + (U + 2\bar{x})(\bar{A}_S - \bar{x}\bar{B}_S) + U(\bar{x}\bar{B}_R - \bar{A}_R)$$
$$< \beta + L\sum_{i=0}^{k} b_i + L^2 \sum_{i=0}^{k}\sum_{j=i}^{k} b_{ij}.$$

Now, the conclusion follows from Theorem 4. □

In the case of general second order quadratic fractional equation of the form

$$x_{n+1} = \frac{Ax_n^2 + Bx_nx_{n-1} + Cx_{n-1}^2 + Dx_n + Ex_{n-1} + F}{ax_n^2 + bx_nx_{n-1} + cx_{n-1}^2 + dx_n + ex_{n-1} + f}, \quad n = 0, 1, \ldots, \tag{25}$$

with nonnegative parameters and initial conditions such that $A + B + C > 0$, and $ax_n^2 + bx_nx_{n-1} + cx_{n-1}^2 + dx_n + ex_{n-1} + f > 0$, $n = 0, 1, \ldots$, the obtained results take the following form.

Corollary 3 *Assume that Eq.* (25) *has the unique positive equilibrium* $\bar{x}$. *If the following condition holds*:

$$\frac{(|A - a\bar{x}| + |B - b\bar{x}| + |C - c\bar{x}|)(U + \bar{x}) + |D - d\bar{x}| + |E - e\bar{x}|}{(a + b + c)L^2 + (d + e)L + f} < 1, \tag{26}$$

where L *and* U *are lower and upper bounds of all solutions of Eq.* (25) *and* $f + L > 0$, *then* $\bar{x}$ *is globally asymptotically stable on the interval* $[0, \infty)$.

In the special case of second order equation with quadratic terms only, we obtain the following result.

Corollary 4 *Consider the following equation*:

$$x_{n+1} = \frac{Ax_n^2 + Bx_nx_{n-1} + Cx_{n-1}^2}{ax_n^2 + bx_nx_{n-1} + cx_{n-1}^2}, \quad n = 0, 1, \ldots, \tag{27}$$

with all positive parameters and nonnegative initial conditions such that $ax_n^2 + bx_nx_{n-1} + cx_{n-1}^2 > 0$ *for all* $n \geq 0$. *If the following condition holds*:

$$\frac{(|A - a\bar{x}| + |B - b\bar{x}| + |C - c\bar{x}|)(U + \bar{x})}{(a + b + c)L^2} < 1,$$

where

$$\bar{x} = \frac{A + B + C}{a + b + c}, \qquad L = \frac{\min\{A, B, C\}}{\max\{a, b, c\}}, \qquad U = \frac{\max\{A, B, C\}}{\min\{a, b, c\}},$$

then the unique equilibrium $\bar{x}$ *of Eq.* (27) *is globally asymptotically stable on the interval* $[0, \infty)$.

Remark 2 If the strict inequality in conditions (18), (20), (21), (22), and (24) is replaced by equality, then the conclusions of Theorems 4, 5, 6, and Corollary 2 should be changed from global asymptotic stability to stability.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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