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The fuzzy analogies of some ergodic theorems

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Abstract

In the paper (Markechová in Fuzzy Sets Syst. 48:351-363, 1992), fuzzy dynamical systems have been defined. In this contribution, using the method of F-σ-ideals, we prove analogies of some ergodic theorems for fuzzy dynamical systems.

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Keywords: fuzzy dynamical system; fuzzy measurable space; F-observable; ergodic theorem

1 Introduction

In the classical probability theory, which is based on the Kolmogorov axiomatic system [1], a random event is every element of the σ-algebra S of subsets of a set X. A probability is a normalized measure defined on the σ-algebra S. The notion of a σ-algebra S of random events and the concept of a probability space (X, S, P) are the basis of the classical concept of probability theory. In doing so, the event in the classical probability theory is understood as an exactly defined phenomenon and from a mathematical point of view (as mentioned above) it is a classical set. In real life, however, we often talk about events that carry important information, but they are less exact. For example, 'tomorrow will be nice', 'a large number will be scored', 'the patient's condition improved' are vaguely defined events, so-called fuzzy events. Their probability can be studied using the apparatus of fuzzy sets theory. The first attempts to develop a concept of fuzzy events and their probability came from the founder of fuzzy sets theory, Zadeh [2, 3]. Assuming that the probability space (X, S, P) is given, Zadeh defined a fuzzy event as any S-measurable function $f: X \to \langle 0, 1\rangle$ and the probability $p(f)$ of a fuzzy event f by the formula $p(f) = \int_X f\, dP$. An axiomatic approach to the creation of a probability concept of fuzzy events was devised by Klement [4]. Generally the Klement probability cannot be represented by the Zadeh construction; necessary and sufficient conditions were given by Klement *et al.* in [5]. A different approach was found in [6–8]. The object of our studies in [9–11] was the fuzzy probability space (X, M, m) defined by Polish mathematician K Piasecki. In [12], the concept of a fuzzy dynamical system was introduced. By a fuzzy dynamical system we understand a system (X, M, m, τ), where (X, M, m) is any fuzzy probability space and $\tau: M \to M$ is an m-invariant σ-homomorphism. Fuzzy dynamical systems include the classical dynamical systems; on the other hand they enable one to study more general situations, for example, Markov's operators. Note that the other approaches to a fuzzy generalization of notion of dynamical system can be found in [13] and [14]. In the papers [12, 15] (see also [16]), we defined the entropy of fuzzy dynamical systems, and using the method of F-σ-ideals we proved a fuzzy version of Kolmogorov-Sinai theorem on generators. Our aim in this contribution is to prove analogies of the following Mesiar ergodic theorems for the case of a fuzzy dynamical system (X, M, m, τ).

Theorem 1.1 [17] *Let (X, S, P) be a given probability space, $\xi_1, \xi_2, \dots$ be a sequence of random variables on it. Let $T : X \to X$ be a measure preserving transformation, $K > 0$ a real constant. Suppose that $|\xi_n| \leq K$ for $n = 1, 2, \dots$ and $\xi_n \to 0$ almost everywhere in P (i.e., $P(\bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} \xi_n^{-1}(\langle -\varepsilon, \varepsilon \rangle)) = 1$). Then*

$$\frac{1}{n} \sum_{i=1}^{n} \xi_i \circ T^i \to 0 \quad \textit{almost everywhere in } P.$$

Theorem 1.2 [17] *Let (X, S, P) be a given probability space, $\zeta, \xi_1, \xi_2, \dots$ be random variables on it such that $0 \leq \xi_n \leq \zeta$ for $n = 1, 2, \dots$. Let $T : X \to X$ be a measure preserving transformation and $\xi_n \to 0$ almost everywhere in P. Then*

$$\frac{1}{n} \sum_{i=1}^{n} \xi_i \circ T^i \to 0 \quad \textit{almost everywhere in } P.$$

In Section 3 we give fuzzy analogies of the above results. In the proofs we will use the method of F-σ-ideals. Note that the first authors, who were interested in the ergodic theory on fuzzy measurable spaces, were Harman and Riečan [18]. They proved the validity of Birkhoff's individual ergodic theorem [19] for the compatible case.

2 Fuzzy probability spaces and fuzzy dynamical systems

First, we recall the definitions of basic notions and some facts which will be used in the following.

Definition 2.1 [20] A fuzzy probability space is a triplet (X, M, m), where X is a nonempty set; M is a fuzzy σ-algebra of fuzzy subsets of X (*i.e.*, (i) $1_X \in M$; $(1/2)_X \notin M$; (ii) if $a_n \in M$, $n = 1, 2, \dots$, then $\bigvee_{n=1}^{\infty} a_n \in M$; (iii) if $a \in M$, then $a' \in M$) and the mapping $m : M \to \langle 0, \infty)$ fulfils the following conditions:

(P1) $m(a \vee a') = 1$ for every $a \in M$;

(P2) if $\{a_n\}_{n=1}^{\infty}$ is a sequence of pairwise W-separated fuzzy subsets from M (*i.e.*, $a_i \leq a_j'$ for $i \neq j$), then $m(\bigvee_{n=1}^{\infty} a_n) = \sum_{n=1}^{\infty} m(a_n)$.

The operations with fuzzy sets are defined here by Zadeh [2], *i.e.*, the union of fuzzy subsets a, b of X is a fuzzy set $a \vee b$ defined by $(a \vee b)(x) = \sup(a(x), b(x))$ for all $x \in X$ and the intersection of fuzzy subsets a, b of X is a fuzzy set $a \wedge b$ defined by $(a \wedge b)(x) = \inf(a(x), b(x))$ for all $x \in X$. The complement of a fuzzy subset a of X is the fuzzy set a' defined by $a'(x) = 1 - a(x)$ for all $x \in X$. The difference of fuzzy subsets a, b of X is the fuzzy set $a - b := a \wedge b'$. The partial ordering relation $\leq$ is defined in the following way: for every $a, b \in M$, $a \leq b$ if and only if $a(x) \leq b(x)$ for all $x \in X$. Using the complementation $' : a \to a'$ for every fuzzy subset $a \in M$, we see that the complementation $'$ satisfies two conditions: (i) $(a')' = a$ for every $a \in M$; (ii) if $a \leq b$, then $b' \leq a'$. So, M is a distributive

σ-lattice with the complementation $'$ for which the de Morgan laws hold: $(\bigvee_{n=1}^{\infty} a_n)' = \bigwedge_{n=1}^{\infty} a_n'$ and $(\bigwedge_{n=1}^{\infty} a_n)' = \bigvee_{n=1}^{\infty} a_n'$ for any sequence $\{a_n\}_{n=1}^{\infty} \subset M$. A couple (X, M), where X is a nonempty set and M is a fuzzy σ-algebra of fuzzy subsets of X, is called a fuzzy measurable space. The fuzzy set $(1/2)_X$ is defined by $(1/2)_X = 1/2$ for all $x \in X$. The empty fuzzy set 0_X is defined by $0_X(x) = 0$ for all $x \in X$. The complement of empty fuzzy set is a fuzzy set 1_X defined by the equality $1_X(x) = 1$ for all $x \in X$. It is called a universum. Fuzzy subsets a, b of X such that $a \wedge b = 0_X$ are called separated fuzzy sets. Analogous weak notions (W-notions) were defined by Piasecki in [21] as follows: each fuzzy subset $a \in M$ such that $a \geq a'$ is called a W-universum; each fuzzy subset $a \in M$ such that $a \leq a'$ is called a W-empty set. Fuzzy subsets $a, b \in M$ such that $a \leq b'$ are called W-separated. It can be proved that a fuzzy set $a \in M$ is a W-universum if and only if there exists a fuzzy set $b \in M$ such that $a = b \vee b'$. Each mapping $m : M \to \langle 0, \infty)$ having the properties (P1) and (P2) is called in the terminology of Piasecki a fuzzy P-measure. Any fuzzy P-measure has the properties analogous to the properties of classical probability measure.

Definition 2.2 [12] By a fuzzy dynamical system we shall mean a quadruplet (X, M, m, τ), where (X, M, m) is a fuzzy probability space and $\tau : M \to M$ is an m-invariant σ-homomorphism, *i.e.*, $\tau(a') = (\tau(a))'$, $\tau(\bigvee_{n=1}^{\infty} a_n) = \bigvee_{n=1}^{\infty} \tau(a_n)$ and $m(\tau(a)) = m(a)$, for every $a \in M$ and any sequence $\{a_n\}_{n=1}^{\infty} \subset M$.

An analog of a random variable from the classical probability theory is an F-observable.

Definition 2.3 [22] An F-observable on a fuzzy measurable space (X, M) is a mapping $x : B(R^1) \to M$ such that

(i) $x(E^C) = 1_X - x(E)$ for every $E \in B(R^1)$;

(ii) $x(\bigcup_{n=1}^{\infty} E_n) = \bigvee_{n=1}^{\infty} x(E_n)$ for any sequence $\{E_n\}_{n=1}^{\infty} \subset B(R^1)$,

where $B(R^1)$ is the family of all Borel subsets of the real line R^1 and E^C denotes the complement of a set $E \subset R^1$.

It is easy to see that if x is an F-observable, then the range of the F-observable x, *i.e.*, the set $R(x) := \{x(E); E \in B(R^1)\}$, is a Boolean σ-algebra of (X, M) with a minimal and maximal element $x(\emptyset)$ and $x(R^1)$, respectively. If $\tau : M \to M$ is a σ-homomorphism and x is an F-observable on (X, M), then it is easy to verify that $\tau \circ x : E \to \tau(x(E))$, $E \in B(R^1)$, is an F-observable on (X, M), too. Let any fuzzy probability space (X, M, m) be given. If x is an F-observable on (X, M), then the mapping $m_x : E \mapsto m(x(E))$, $E \in B(R^1)$, is a probability measure on $B(R^1)$. The probability that an F-observable x has a value in $E \in B(R^1)$ is given by $m(x(E))$.

We present some examples of the above notions.

Example 2.1 Let (X, S, P) be a classical probability space and $\xi : X \to R^1$ be a random variable in the sense of classical probability theory. Put $M = \{\chi_A; A \in S\}$, where χ_A is a characteristic function of a set $A \in S$, and define the mapping $m : M \to \langle 0, 1\rangle$ by $m(\chi_A) = P(A)$. Then the triplet (X, M, m) is a fuzzy probability space and the mapping $x : B(R^1) \to M$ defined by $x(E) = \chi_{\xi^{-1}(E)}$, $E \in B(R^1)$, is an F-observable on the fuzzy measurable space (X, M).

Example 2.2 Let (X, M) be any fuzzy measurable space. If $a \in M$, then the mapping x_a defined by putting

$$x_a(E) = \begin{cases} a \vee a', & \text{if } 0, 1 \in E; \\ a', & \text{if } 0 \in E, 1 \notin E; \\ a, & \text{if } 0 \notin E, 1 \in E; \\ a \wedge a', & \text{if } 0, 1 \notin E \end{cases}$$

for any $E \in B(R^1)$, is an F-observable on the fuzzy measurable space (X, M), called the question observable of the fuzzy set a. It is evident that x_a plays the role of an indicator of the fuzzy set a. We denote the question observable of the empty fuzzy set 0_X by σ, *i.e.*, $\sigma = x_{0_X}$. We have

$$\sigma(E) = \begin{cases} 0_X, & \text{if } 0 \notin E; \\ 1_X, & \text{if } 0 \in E \end{cases} \quad \text{for every } E \in B\bigl(R^1\bigr).$$

Lemma 2.1 *Let x be an F-observable on a fuzzy measurable space (X, M) and $f : R^1 \to R^1$ be a Borel measurable function. Then the mapping $f(x) : B(R^1) \to M$ defined, for every $E \in B(R^1)$, by $f(x)(E) = x(f^{-1}(E))$, is an F-observable on (X, M) such that $R(f(x)) \subset R(x)$.*

Proof For every $E \in B(R^1)$,

$$f(x)\bigl(E^C\bigr) = x\bigl(f^{-1}\bigl(E^C\bigr)\bigr) = x\bigl(\bigl(f^{-1}(E)\bigr)^C\bigr) = 1_X - x\bigl(f^{-1}(E)\bigr) = 1_X - f(x)(E)$$

and for any sequence $\{E_n\}_{n=1}^{\infty} \subset B(R^1)$,

$$\begin{aligned} f(x)\left(\bigcup_{n=1}^{\infty} E_n\right) &= x\left(f^{-1}\left(\bigcup_{n=1}^{\infty} E_n\right)\right) = x\left(\bigcup_{n=1}^{\infty} f^{-1}(E_n)\right) \\ &= \bigvee_{n=1}^{\infty} x\bigl(f^{-1}(E_n)\bigr) = \bigvee_{n=1}^{\infty} f(x)(E_n). \end{aligned}$$

This means that the mapping $f(x) : B(R^1) \to M$ is an F-observable on (X, M). If $a \in R(f(x))$, then $a = f(x)(E) = x(f^{-1}(E))$ for some set $E \in B(R^1)$. Since the function $f : R^1 \to R^1$ is a Borel measurable function, $f^{-1}(E) \in B(R^1)$, and hence $a \in R(x)$. The proof is finished. □

In particular, if $f(t) = t^2$, $t \in R^1$, we put $x^2 := f(x)$, *etc.* If $k \in R^1$, then the mapping $kx : B(R^1) \to M$ defined by $(kx)(E) = x(\{t \in R^1; kt \in E\})$, for every $E \in B(R^1)$, is an F-observable on (X, M).

Since there is a one-to-one correspondence between an F-observable x and the system $\{B_x(t) := x((-\infty, t)); t \in R^1\}$ (Dvurečenskij and Tirpáková [23]), the sum of any pair of F-observables x, y on (X, M) has been defined in the following way.

Definition 2.4 [23] Let (X, M) be a fuzzy measurable space. By the sum of any pair of two F-observables x, y on (X, M) we mean an F-observable z such that

$$z\bigl((-\infty, t)\bigr) = \bigvee_{r \in Q} x\bigl((-\infty, r)\bigr) \wedge y\bigl((-\infty, t - r)\bigr), \quad t \in R^1,$$

where Q is the set of all rational numbers in the real line R^1. We write $z = x + y$.

In [23], it has been proved that the sum of any pair of F-observables always exists and it is determined uniquely. Moreover, for any F-observables x, y, z on (X,M) we have $x+y=y+x$; $(x+y)+z=x+(y+z)$. The difference of two F-observables x, y on (X,M) is defined by $x-y=(x+(-y))$, where $(-y)(E)=y(\{t;-t\in E\})$, for every $E\in B(R^1)$. The product $x\cdot y$ of two F-observables x, y on (X,M) is defined as follows:

$$x\cdot y=\bigl((x+y)^2-x^2-y^2\bigr)/2.$$

By the spectrum of an F-observable x we mean the set $\sigma(x):=\bigcap\{C\subset R^1; C \text{ is closed and } x(C)=x(R^1)\}$. An F-observable x is called bounded if $\sigma(x)$ is a bounded set; in this case we define the norm of x via $\|x\|=\sup\{|t|; t\in\sigma(x)\}$. Let x and y be two F-observables on (X,M). We write $x\le y$ if $\sigma(y-x)\subset\langle 0,\infty)$.

Definition 2.5 [24] Let (X,M,m) be a given fuzzy probability space, $x,x_1,x_2,\ldots$ be F-observables on (X,M). We say that the sequence $\{x_n\}_{n=1}^{\infty}$ converges to x almost everywhere in m (and we write $x_n\to x$ m-a.e.), if, for every $\varepsilon>0$,

$$m\left(\bigvee_{k=1}^{\infty}\bigwedge_{n=k}^{\infty}(x_n-x)\bigl(\langle-\varepsilon,\varepsilon\rangle\bigr)\right)=1.$$

3 Main results

In this section, we give analogies of Mesiar's ergodic theorems for the case of a fuzzy dynamical system (X,M,m,τ). In the proofs we will use the method of F-σ-ideals described below as well as the properties of a σ-homomorphism τ. It is noted that these results can be obtained also by factorizing over the σ-ideal of W-empty sets; details of this approach can be found, *e.g.*, in [25].

Let any fuzzy probability space (X,M,m) be given. Put $I_\circ=\{a\in M; \exists c\ge 1/2 \text{ such that } a\wedge c\le 1/2\}$. It is easy to verify that $I_\circ$ is a σ-ideal, *i.e.*, (i) if $a\in M$, $b\in I_\circ$, $a\le b$, then $a\in I_\circ$; (ii) if $\{a_i\}_{i=1}^{\infty}\subset I_\circ$, then $\bigvee_{i=1}^{\infty}a_i\in I_\circ$; (iii) $a\wedge a'\in I_\circ$ for every $a\in M$; (iv) if $a\wedge c\in I_\circ$ for some $c\ge 1/2$, $c\in M$, then $a\in I_\circ$. In the set M we define the relation of equivalence $\sim$ in the following way: for every $a,b\in M$, $a\sim b$ if and only if $a\wedge b', a'\wedge b\in I_\circ$. Put $[a]=\{b\in M; b\sim a\}$ for every $a\in M$. The system $M/I_\circ=\{[a]; a\in M\}$ is a Boolean σ-algebra, where $\bigvee_{n=1}^{\infty}[a_n]=[\bigvee_{n=1}^{\infty}a_n]$ and $[a]'=[a']$. It is easy to verify that if $a_1,a_2\in[a]$, then $m(a_1)=m(a_2)$. If we define the mapping $\mu: M/I_\circ\to\langle 0,1\rangle$ by the equality $\mu([a]):=m(a)$ for every $[a]\in M/I_\circ$, then μ is a probability measure on the Boolean σ-algebra $M/I_\circ$.

Let (X,M,m,τ) be any fuzzy dynamical system. Then the mapping $\bar\tau: M/I_\circ\to M/I_\circ$ defined by $\bar\tau([a])=[\tau(a)]$, $a\in M$, is a σ-homomorphism of the Boolean σ-algebra $M/I_\circ$, *i.e.*, for every $a\in M$, $\bar\tau([a]')=(\bar\tau([a]))'$, and for every sequence $\{a_n\}_{n=1}^{\infty}\subset M$, $\bar\tau(\bigvee_{n=1}^{\infty}[a_n])=\bigvee_{n=1}^{\infty}\bar\tau([a_n])$; moreover, $\bar\tau$ is invariant in μ, *i.e.*, $\mu(\bar\tau([a]))=\mu([a])$, for every $a\in M$.

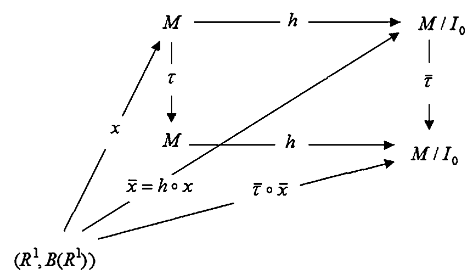
Remark 3.1 Let (X,M) be a given fuzzy measurable space and x be an F-observable on it. Let us define the mapping $h: M\to M/I_\circ$ via $h(a)=[a]$, $a\in M$. Then it is easy to verify that h is a σ-homomorphism from M onto $M/I_\circ$ and $\bar x:=h\circ x$ is an observable on the Boolean σ-algebra $M/I_\circ$, *i.e.*, the following properties hold:

(i) $\bar x(\emptyset)=[0_X]$;

(ii) $\bar x(E^C)=(\bar x(E))'$, for every $E\in B(R^1)$;

(iii) $\bar x(\bigcup_{n=1}^{\infty}E_n)=\bigvee_{n=1}^{\infty}\bar x(E_n)$, for any sequence $\{E_n\}_{n=1}^{\infty}\subset B(R^1)$.

Figure 1 The basic idea of the method of F-σ-ideals for verification of validity of ergodic theorems on a fuzzy dynamical system.



Lemma 3.1 *Let (X, M, m, τ) be a fuzzy dynamical system and let $\{x_n\}_{n=1}^{\infty}$ be a sequence of F-observables on (X, M). If h is the mapping from the preceding remark and $\bar{x}_n := h \circ x_n$, $n = 1, 2, \ldots$, then we have*:

(i) $\bar{\tau}^k \circ \bar{x}_n = h \circ \tau^k \circ x_n$, *for* $k = 1, 2, \ldots$, $n = 1, 2, \ldots$.

(ii) *Let $\mathcal{A}$ be the minimal Boolean sub-σ-algebra of $M/I_\circ$ containing all ranges of $\bar{\tau}^k \circ \bar{x}_n$, $k = 1, 2, \ldots$, $n = 1, 2, \ldots$. Then $\bar{\tau}([a]) \in \mathcal{A}$ for any $[a] \in \mathcal{A}$.*

Proof (i) For every $E \in B(R^1)$ we have

$$\begin{aligned}(h \circ \tau^k \circ x_n)(E) &= h\bigl(\tau^k\bigl(x_n(E)\bigr)\bigr) = \bigl[\tau^k\bigl(x_n(E)\bigr)\bigr] = \bar{\tau}^k\bigl(\bigl[x_n(E)\bigr]\bigr) \\ &= \bar{\tau}^k\bigl(h\bigl(x_n(E)\bigr)\bigr) = \bar{\tau}^k\bigl(\bar{x}_n(E)\bigr) = \bigl(\bar{\tau}^k \circ \bar{x}_n\bigr)(E),\end{aligned}$$

i.e., $\bar{\tau}^k \circ \bar{x}_n = h \circ \tau^k \circ x_n$, $n = 1, 2, \ldots$, $k = 1, 2, \ldots$.

(ii) Put $\mathcal{A}_0 = \{[a] \in \mathcal{A}; \bar{\tau}([a]) \in \mathcal{A}\}$. Since $\mathcal{A}_0 \subset \mathcal{A}$, it is sufficient to show the inclusion $\mathcal{A} \subset \mathcal{A}_0$. We will prove that $\mathcal{A}_0$ is a Boolean σ-algebra. If $[a] \in \mathcal{A}_0$, then $\bar{\tau}([a]) \in \mathcal{A}$, and since $\mathcal{A}$ is a Boolean σ-algebra, $(\bar{\tau}([a]))' \in \mathcal{A}$. The equality $\bar{\tau}([a]') = (\bar{\tau}([a]))'$ implies $[a]' \in \mathcal{A}_0$. Let $[a_n] \in \mathcal{A}_0$ for $n = 1, 2, \ldots$. Then $\bar{\tau}([a_n]) \in \mathcal{A}$ for $n = 1, 2, \ldots$, and since $\mathcal{A}$ is a Boolean σ-algebra, $\bigvee_{n=1}^{\infty} \bar{\tau}([a_n]) \in \mathcal{A}$. From the equality $\bigvee_{n=1}^{\infty} \bar{\tau}([a_n]) = \bar{\tau}(\bigvee_{n=1}^{\infty}[a_n])$ we get $\bigvee_{n=1}^{\infty}[a_n] \in \mathcal{A}_0$. Moreover, $[0_X], [1_X] \in \mathcal{A}_0$. Thus $\mathcal{A}_0$ is a Boolean sub-σ-algebra of $[M]$ containing all ranges of $\bar{\tau}^k \circ \bar{x}_n$, $k = 1, 2, \ldots$, $n = 1, 2, \ldots$, and therefore $\mathcal{A} \subset \mathcal{A}_0$. The proof is finished. □

Figure 1 shows the basic idea of using the method of F-σ-ideals for the verification of extending the possibility of ergodic theory on a fuzzy measurable space.

The following theorem is a fuzzy analogy of Theorem 1.1.

Theorem 3.1 *Let (X, M, m, τ) be a given fuzzy dynamical system and $\{x_n\}_{n=1}^{\infty}$ be a sequence of bounded F-observables on (X, M). Let there exists a real constant $K > 0$ such that $\|x_n\| \le K$ for $n = 1, 2, \ldots$. Suppose $x_n \to \sigma$ almost everywhere in m. Then*

$$\frac{1}{n}\sum_{i=1}^{n} \tau^i \circ x_i \to \sigma \quad \textit{almost everywhere in } m. \tag{3.1}$$

Proof The Boolean sub-σ-algebra $\mathcal{A}$ in Lemma 3.1 has a countable generator, hence, due to Varadarajan [26], there exists an observable

$$z : B\bigl(R^1\bigr) \to M/I_\circ \quad \text{such that } \bigl\{z(E) : E \in B\bigl(R^1\bigr)\bigr\} = \mathcal{A},$$

and a sequence of real-valued Borel functions $\{f_n\}_{n=1}^{\infty}$, such that

$$\bar{x}_n(E) = z\big(f_n^{-1}(E)\big), \quad E \in B\big(R^1\big), n = 1, 2, \ldots.$$

The sequence $\{f_n\}_{n=1}^{\infty}$ is essentially unique in the following sense: if $z(f_n^{-1}(E)) = z(g_n^{-1}(E))$, $E \in B(R^1)$, then $z(\{t : f_n(t) \neq g_n(t)\}) = [0_X]$. From the construction of z it follows that $\bar{\tau}$ is z-measurable, *i.e.*, $\bar{\tau}(z(B(R^1))) \subset z(B(R^1))$. Due to Dvurečenskij and Riečan [22], this is possible iff there is a Borel measurable transformation $T : R^1 \to R^1$ such that

$$\bar{\tau}\big(z(E)\big) = z\big(T^{-1}(E)\big) \quad \text{for every } E \in B\big(R^1\big).$$

Therefore we have

$$\bar{\tau}^k\big(z(E)\big) = z\big(T^{-k}(E)\big), \quad E \in B\big(R^1\big), k = 1, 2, \ldots,$$

and consequently, for every $E \in B(R^1)$,

$$\big(\bar{\tau}^k \circ \bar{x}_n\big)(E) = \bar{\tau}^k\big(\bar{x}_n(E)\big) = \bar{\tau}^k\big(z\big(f_n^{-1}(E)\big)\big) = z\big(T^{-k}\big(f_n^{-1}(E)\big)\big), \quad k = 1, 2, \ldots.$$

Moreover, $\sigma(x_n) \supset \sigma(\bar{x}_n) \supset \sigma(f_n^{-1})$, and therefore $|f_n| \leq \|\bar{x}_n\| \leq \|x_n\| \leq K$.

Take into account the system $(R^1, B(R^1), \mu_z, T)$, where μ_z is the mapping defined by $\mu_z(E) = \mu(z(E))$, for every $E \in B(R^1)$. Then μ_z is a probability measure on $B(R^1)$. Moreover, for every $E \in B(R^1)$,

$$\mu_z\big(T^{-1}(E)\big) = \mu\big(z\big(T^{-1}(E)\big)\big) = \mu\big(\bar{\tau}\big(z(E)\big)\big) = \mu\big(z(E)\big) = \mu_z(E),$$

i.e., the transformation T is μ_z-invariant. This means that the system $(R^1, B(R^1), \mu_z, T)$ is a dynamical system in the classical sense. For observables in a Boolean σ-algebra, there is a well-known way to define their sum [24], and the convergence almost everywhere of observables in $M/I_\circ$ is the same as for F-observables.

By the assumption $x_n \to \sigma$ almost everywhere in m, *i.e.*, for every $\varepsilon > 0$,

$$m\left(\bigvee_{k=1}^{\infty}\bigwedge_{n=k}^{\infty} x_n\big(\langle -\varepsilon, \varepsilon\rangle\big)\right) = 1.$$

But

$$\begin{aligned}
& m\left(\bigvee_{k=1}^{\infty}\bigwedge_{n=k}^{\infty} x_n\big(\langle -\varepsilon, \varepsilon\rangle\big)\right) = 1 \\
&\quad \Leftrightarrow \quad \mu\left(\bigvee_{k=1}^{\infty}\bigwedge_{n=k}^{\infty} \bar{x}_n\big(\langle -\varepsilon, \varepsilon\rangle\big)\right) = 1 \\
&\quad \Leftrightarrow \quad \mu\left(\bigvee_{k=1}^{\infty}\bigwedge_{n=k}^{\infty} z\big(f_n^{-1}\big(\langle -\varepsilon, \varepsilon\rangle\big)\big)\right) = 1 \\
&\quad \Leftrightarrow \quad \mu_z\left(\bigcup_{k=1}^{\infty}\bigcap_{n=k}^{\infty} f_n^{-1}\big(\langle -\varepsilon, \varepsilon\rangle\big)\right) = 1 \\
&\quad \Leftrightarrow \quad f_n \to 0 \quad \text{almost everywhere in } \mu_z.
\end{aligned}$$

Hence, due to Theorem 1.1, we have

$$\frac{1}{n}\sum_{i=1}^{n} f_i \circ T^i \to 0 \quad \text{almost everywhere in } \mu_z.$$

On the other hand,

$$\frac{1}{n}\sum_{i=1}^{n} \tau^i \circ x_i \to \sigma \quad m\text{-a.e.} \quad \text{iff} \quad \frac{1}{n}\sum_{i=1}^{n} \bar{\tau}^i \circ \bar{x}_i \to \bar{\sigma} \quad \mu\text{-a.e.},$$

where $\bar{\sigma}(E) = [0_X]$ if $0 \notin E$ and $\bar{\sigma}(E) = [1_X]$ in the other cases. The previous convergence is true if and only if

$$\frac{1}{n}\sum_{i=1}^{n} \left(f_i \circ T^i\right)(z) \to \bar{\sigma} \quad \mu\text{-a.e.},$$

which is possible if and only if

$$\frac{1}{n}\sum_{i=1}^{n} f_i \circ T^i \to 0 \quad \mu_z\text{-a.e.}$$

Therefore, (3.1) is proved. □

Theorem 3.2 *Let (X, M, m, τ) be any fuzzy dynamical system and $\{x_n\}_{n=1}^{\infty}$ be a sequence of F-observables on (X, M). Let y be an F-observable on (X, M) such that $\sigma \le x_n \le y$ for $n = 1, 2, \ldots$. Suppose $x_n \to \sigma$ almost everywhere in m. Then*

$$\frac{1}{n}\sum_{i=1}^{n} \tau^i \circ x_i \to \sigma \quad \textit{almost everywhere in } m.$$

Proof We will use similar arguments to above. Let $\mathcal{A}_1$ be the minimal Boolean sub-σ-algebra of $M/I_\circ$ containing all ranges of $\bar{\tau}^k \circ \bar{x}_n$ and $\bar{\tau}^k \circ \bar{y}$ for $k = 1, 2, \ldots, n = 1, 2, \ldots$. Then $\bar{\tau}\mathcal{A}_1 \subset \mathcal{A}_1$ and $\mathcal{A}_1$ has a countable generator. In view of Varadarajan [26], there is an observable $z : B(R^1) \to M/I_\circ$ such that

$$\left\{z(E) : E \in B\left(R^1\right)\right\} = \mathcal{A}_1.$$

Moreover, there are real-valued Borel functions f, f_n, $n = 1, 2, \ldots$, such that

$$\bar{x}_n(E) = z\left(f_n^{-1}(E)\right), \quad n = 1, 2, \ldots \quad \text{and} \quad \bar{y}(E) = z\left(f^{-1}(E)\right) \quad \text{for any } E \in B\left(R^1\right).$$

Denote $g_n = \max(0, f_n)$, $n = 1, 2, \ldots$, and $g = \max(0, f)$. Then, for every $E \in B(R^1)$, we have $\bar{x}_n(E) = z(g_n^{-1}(E))$, $n = 1, 2, \ldots$, and $\bar{y}(E) = z(g^{-1}(E))$. Let $h_n = \min(f_n, f)$, $n = 1, 2, \ldots$, and $h = f$. Then $\bar{x}_n(E) = z(h_n^{-1}(E))$, $\bar{y}(E) = z(h^{-1}(E))$. Moreover, $0 \le \|\bar{x}_n\| \le \|\bar{y}\|$ and $0 \le h_n \le h$. Similar to the proof of the preceding theorem, we take into account the dynamical system $(R^1, B(R^1), \mu_z, T)$, where $\mu_z : E \to \mu(z(E))$, $E \in B(R^1)$, is a probability measure on $B(R^1)$ such that $\mu_z(T^{-1}(E)) = \mu_z(E)$, for every $E \in B(R^1)$, and $\bar{\tau}(z(E)) = z(T^{-1}(E))$, $E \in B(R^1)$. Thus we have

$$\frac{1}{n}\sum_{i=1}^{n}\tau^i \circ x_i \to \sigma \quad m\text{-a.e.} \quad \text{iff} \quad \frac{1}{n}\sum_{i=1}^{n}\bar{\tau}^i \circ \bar{x}_i \to \bar{\sigma} \quad \mu\text{-a.e.} \quad \text{iff}$$

$$\frac{1}{n}\sum_{i=1}^{n} h_i \circ T^i \to 0 \quad \mu_z\text{-a.e.}$$

But the last assertion follows from Theorem 1.2. The proof is finished. □

Corollary 3.1 *Let (X, M, m, τ) be any fuzzy dynamical system and $\{x_n\}_{n=1}^{\infty}$ be a sequence of F-observables on (X, M). Let y be an F-observable on (X, M) such that $|x_n| \leq y$ for $n = 1, 2, \ldots$. If $x_n \to \sigma$ almost everywhere in m, then*

$$\frac{1}{n}\sum_{i=1}^{n}\tau^i \circ x_i \to \sigma \quad \textit{almost everywhere in } m.$$

Proof We put $x_n = x_n^+ - x_n^-$, where $x_n^+ = f^+ \circ x_n$, $x_n^- = f^- \circ x_n$, $f^+(t) = \max(0, t)$ and $f^-(t) = -\min(0, t)$, $t \in R^1$. Then $|x_n| = x_n^+ + x_n^-$. Applying Theorem 3.2 to both sequences $\{x_n^+\}_{n=1}^{\infty}$ and $\{x_n^-\}_{n=1}^{\infty}$ we get what was claimed. □

4 Conclusions

In this paper we have presented generalizations of some ergodic theorems from the classical ergodic theory to the fuzzy case. In the proofs the method of F-σ-ideals was used.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
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