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Solvability for some boundary value problems with discrete ϕ-Laplacian operators

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Abstract

In this paper, we study the existence of solutions of the discrete ϕ-Laplacian equation $\nabla[\phi(\Delta u_k)] = \lambda f(k, u_k, \Delta u_k)$, $k \in [2, n-1]_{\mathbb{Z}}$, with Dirichlet or mixed boundary conditions. Under general conditions, an explicit estimate of λ_0 is given such that the problem possesses a solution for any $|\lambda| < \lambda_0$.

Keywords: ϕ-Laplacian; existence; Schauder's fixed point theorem; mixed boundary value problem; Dirichlet boundary value problem

1 Introduction

The study of difference equations represents a very important field in mathematical research. Different mathematical models coupled with the basic theory of this type of equation can be found in the classical monograph by Goldberg [1] and in the book by Lakshmikantham and Trigiante [2]. Besides, they are also natural consequences of the discretization of differential problems.

In this paper, we consider the existence of solutions of discrete ϕ-Laplacian problems

$$\nabla\left[\phi(\Delta u_k)\right] = \lambda f(k, u_k, \Delta u_k), \quad k \in [2, n-1]_{\mathbb{Z}}, \tag{1}$$

with Dirichlet boundary conditions

$$u_1 = 0 = u_n \tag{2}$$

or mixed boundary conditions

$$u_1 = 0 = \Delta u_{n-1} \tag{3}$$

under the following assumptions:

(H1) $\phi : (-a, a) \to (-b, b)$ is an increasing homeomorphism with $\phi(0) = 0$ and $0 < a, b \le \infty$;

(H2) for all $k \in [2, n-1]_{\mathbb{Z}}, f(k, \cdot, \cdot) : \mathbb{R} \times (-a, a) \to \mathbb{R}$ with $\mathbf{f} = (f(2, u, v), \dots, f(n-1, u, v))$, and for each compact set $A \subset \mathbb{R} \times (-a, a)$, there exists a bounded function $\mathbf{h}_A = (h_A(2), \dots, h_A(n-1))$ defined from $\mathbb{R}$ to $\mathbb{R}^{n-2}$ such that

$$\left|f(k, u, v)\right| \le h_A(k), \quad \text{for } k \in [2, n-1]_{\mathbb{Z}} \text{ and all } (u, v) \in A.$$

The study of the ϕ-Laplacian equation is a classical topic that has attracted the attention of many researchers because of its interest for applications. Usually, a ϕ-Laplacian operator is said to be *singular* when the domain of ϕ is finite (that is, $a < +\infty$); in the contrary case, the operator is said to be *regular*. On the other hand we say that ϕ is *bounded* if its range is finite (that is, $b < +\infty$) and *unbounded* in the other case. There are three paradigmatic models in this context:

(i) $a = b = +\infty$ (regular unbounded): we have the p-Laplacian operator

$$\phi_1(x) = |x|^{p-2}x \quad \text{with } p > 1.$$

(ii) $a < +\infty$, $b = +\infty$ (singular unbounded): we have the relativistic operator

$$\phi_2(x) = \frac{x}{\sqrt{1-x^2}}.$$

(iii) $a = +\infty$, $b < +\infty$ (regular bounded): we have the one-dimensional mean curvature operator

$$\phi_3(x) = \frac{x}{\sqrt{1+x^2}}.$$

Among them, the p-Laplacian operator has received a lot of attention and the number of related references is huge (we only mention [3–6] and references therein). For the relativistic operator, it has recently been proved in [7–9] that the Dirichlet problem is always solvable. This is a striking result closely related to the '*a priori*' bound of the derivatives of the solutions. For the curvature operator, this is no longer true, but other results as regards the existence of solutions of differential problems can be found in [10, 11]. To the best of our knowledge, the discrete problem has received almost no attention. In this article, we will discuss it in detail.

The purpose of this paper is to show that analogs of the existence results of solutions for differential problems proved in [10] hold for the corresponding difference equations. However, some basic ideas from differential calculus are not necessarily available in the field of difference equations, such as Rolle's theorem and symmetry of the domain of solutions. Thus, new challenges are faced and innovation is required. In addition, we extend some results of Bereanu and Mawhin in [7]; see Remark 1. The proof is elementary and relies on Schauder's fixed point theorem after a suitable reduction of the problem to a first-order summing-difference equation.

We end this section with some notations. For $\mathbf{x} \in \mathbb{R}^p$ set

$$|\mathbf{x}|_\infty = \max_{1\le k\le p} |x_k|, \qquad |\mathbf{x}|_1 = \sum_{k=1}^{p} |x_k|.$$

For every $l, m \in \mathbb{N}$ with $m > l$, we set $\sum_{k=m}^{l} x_k = 0$, and $[l, m]_{\mathbb{Z}} := \{l, l+1, \dots, m\}$.

Let $n \in \mathbb{N}$, $n \ge 4$ be fixed and $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$. Define

$$\Delta\mathbf{x} = (\Delta x_1, \dots, \Delta x_{n-1}) \in \mathbb{R}^{n-1},$$

as

$$\Delta x_k = x_{k+1} - x_k \quad (1 \le k \le n-1),$$

and if $|\Delta \mathbf{x}|_\infty < a$, define

$$\nabla\big[\phi(\Delta \mathbf{x})\big] = \big(\nabla\big[\phi(\Delta x_2)\big], \ldots, \nabla\big[\phi(\Delta x_{n-1})\big]\big) \in \mathbb{R}^{n-2},$$

as

$$\nabla\big[\phi(\Delta x_k)\big] = \phi(\Delta x_k) - \phi(\Delta x_{k-1}) \quad (2 \le k \le n-1).$$

For convenience, for each $0 < r < b$, let M_r be defined as

$$M_r := |\mathbf{h}_{A_r}|_1, \tag{4}$$

where $A_r = [(n-1)\phi^{-1}(-r), (n-1)\phi^{-1}(r)] \times [\phi^{-1}(-r), \phi^{-1}(r)]$.

2 The Dirichlet boundary value problem

Let us consider the boundary value problem

$$\nabla\big[\phi(\Delta u_k)\big] = \lambda f(k, u_k, \Delta u_k), \quad k \in [2, n-1]_{\mathbb{Z}}, \qquad u_1 = 0 = u_n, \tag{5}$$

where $\lambda \in \mathbb{R}$ is a parameter and (H1) and (H2) hold.

Let us define the space

$$H = \left\{ \mathbf{y} \in \mathbb{R}^{n-1} \;\middle|\; \mathbf{y} = (y_1, \ldots, y_{n-1}) \text{ with } |\mathbf{y}|_\infty < \frac{b}{2} \right\},$$

of course, $\frac{b}{2}$ must be understood as $+\infty$ when $b = +\infty$. The following result is a slight modification of Lemma 2 in [7], but we include the proof for the sake of completeness.

Lemma 2.1 *For any $\mathbf{y} \in H$, there exists a unique constant $\gamma := P_\phi[\mathbf{y}]$ such that*

$$\sum_{k=1}^{n-1} \phi^{-1}(\gamma + y_k) = 0. \tag{6}$$

Besides, $|P_\phi[\mathbf{y}]| \le |\mathbf{y}|_\infty$, and the function $P_\phi : H \to (-\frac{b}{2}, \frac{b}{2})$ is continuous.

Proof From the properties of ϕ, it is clear that

$$\sum_{k=1}^{n-1} \phi^{-1}\big(-|\mathbf{y}|_\infty + y_k\big) \le 0 \le \sum_{k=1}^{n-1} \phi^{-1}\big(|\mathbf{y}|_\infty + y_k\big). \tag{7}$$

Because ϕ is a homeomorphism such that $\phi(0) = 0$, it follows that ϕ^{-1} is strictly monotone and satisfies (7), so there exists a unique $\gamma := P_\phi[\mathbf{y}]$ verifying (6) with $\gamma \le |\mathbf{y}|_\infty$. Note that P_ϕ takes bounded sets into bounded sets. So the continuity of P_ϕ follows easily. □

By means of a suitable change of variable, we relate the problem (5) with the nonlocal first-order equation

$$\nabla y_k = \lambda f\left(k, \sum_{i=1}^{k-1} \phi^{-1}\big(P_\phi[\mathbf{y}] + y_i\big), \phi^{-1}\big(P_\phi[\mathbf{y}] + y_k\big)\right), \quad k \in [2, n-1]_{\mathbb{Z}}. \tag{8}$$

By a simple computation, we can get the following result.

Lemma 2.2 *If* $\mathbf{y} = (y_1, \ldots, y_{n-1})$ *is a solution of the problem* (8) *with* $|\mathbf{y}|_\infty < \frac{b}{2}$, *then*

$$u_k = \sum_{i=1}^{k-1} \phi^{-1}\bigl(P_\phi[\mathbf{y}] + y_i\bigr)$$

is a solution of (5).

Now we are in a position to prove the main result of this section: the solvability of problem (5) for small λ.

Theorem 2.3 *For each* $0 < r < \frac{b}{2}$, *let* M_r *be defined by* (4). *If*

$$|\lambda| < \lambda_0 := \sup_{0<r<\frac{b}{2}} \frac{r}{M_{2r}},$$

then problem (5) *has a solution.*

Proof Let $0 < r_1 < \frac{b}{2}$ be such that $|\lambda| \le \frac{r_1}{M_{2r_1}}$ and consider the ball

$$B_{r_1} = \bigl\{\mathbf{y} \in \mathbb{R}^n \mid \mathbf{y} = (y_1, \ldots, y_n) \text{ with } |\mathbf{y}|_\infty \le r_1\bigr\}.$$

For each $\mathbf{y} \in B_{r_1}$, we define the operator

$$Ty_k = \lambda \sum_{j=2}^{k} f\left(j, \sum_{i=1}^{j-1} \phi^{-1}\bigl(P_\phi[\mathbf{y}] + y_i\bigr), \phi^{-1}\bigl(P_\phi[\mathbf{y}] + y_j\bigr)\right).$$

It is easy to show that T is completely continuous. Moreover, by our assumptions and the choice of r_1 we have

$$|T\mathbf{y}|_\infty \le |\lambda| M_{2r_1} \le r_1,$$

which implies that $T(B_{r_1}) \subset B_{r_1}$. Thus Schauder's fixed point theorem yields a fixed point for T which is a solution of (8), and therefore by Lemma 2.2 it is also a solution of problem (5). □

Now we are going to apply Theorem 2.3 to study the solvability of the Dirichlet problem

$$\nabla\bigl[\phi(\Delta u_k)\bigr] = f(k, u_k, \Delta u_k), \quad k \in [2, n-1]_{\mathbb{Z}}, \qquad u_1 = 0 = u_n. \tag{9}$$

We point out that problem (9) presents interesting different features depending on the bounded or unbounded behavior of ϕ.

2.1 Unbounded ϕ-Laplacian ($b = +\infty$)

A consequence of Theorem 2.3 is that, whenever ϕ is unbounded, then (9) is solvable.

Corollary 1 *Assume that ϕ is unbounded (that is, $b = +\infty$) and there exists a bounded function $\mathbf{h} = (h(2), \dots, h(n-1))$ such that*

$$\bigl|f(k,u,v)\bigr| \le h(k), \quad k \in [2, n-1]_{\mathbb{Z}}, u \in \bigl(-(n-1)a, (n-1)a\bigr), v \in (-a, a). \tag{10}$$

Then the Dirichlet problem (9) *has at least one solution.*

Proof By conditions (10) it is clear that

$$M_r = |\mathbf{h}|_1, \quad \text{for each } r > 0.$$

Therefore

$$\lambda_0 = \sup_{0<r<+\infty} \frac{r}{M_{2r}} = +\infty,$$

and then Theorem 2.3 ensures us that the problem (9) has a solution for each $\lambda \in \mathbb{R}$, and in particular for $\lambda = 1$. □

Remark 1 Corollary 1 applies in particular if ϕ is also singular $a < +\infty$ and for each $k \in [2, n-1]_{\mathbb{Z}}, f(k, \cdot, \cdot)$ is continuous on $\mathbb{R}^2$. In this way Corollary 1 improves Theorem 1 in [7].

Example 1 The assumptions of Corollary 1 are satisfied for $\phi(v) = \frac{v}{\sqrt{1-v^2}}$ and for all $k \in [2, n-1]_{\mathbb{Z}}$,

$$f(k,u,v) = \begin{cases} 1, & u \in [0, (n-1)a), v \in (-a, a), \\ -1, & u \in (-(n-1)a, 0), v \in (-a, a). \end{cases}$$

Thus, by Corollary 1, the problem (9) has a solution.

2.2 Bounded ϕ-Laplacian ($b < +\infty$)

In the case of a bounded ϕ-Laplacian, the 'universal' solvability of (9) is no longer true even for a constant nonlinearity $f(t,u,v) \equiv M$ as we show in the following result.

Proposition 1 *Assume that ϕ is bounded (that is, $b < +\infty$), let $M \in \mathbb{R}$ and consider the Dirichlet problem*

$$\nabla\bigl[\phi(\Delta u_k)\bigr] = M, \quad k \in [2, n-1]_{\mathbb{Z}}, \qquad u_1 = 0 = u_n. \tag{11}$$

Then the following claims hold:

(i) *If $|M| \ge \frac{b}{n-2}$, then* (11) *has no solution.*
(ii) *If $|M| < \frac{b}{2(n-2)}$, then* (11) *has a solution.*

Before proving the Proposition 1, we first give the following result, which is a slight modification of Lemma 2 in [7].

Lemma 2.4 *For any $\mathbf{y} \in E$,*

$$E := \left\{ \mathbf{y} \in \mathbb{R}^{n-1} \;\middle|\; \mathbf{y} = (y_1, \dots, y_{n-1}) \textit{ with } |\mathbf{y}|_1 < \frac{b}{2} \right\},$$

there exists a unique constant $\gamma := Q_\phi[\mathbf{y}]$ *such that*

$$\phi^{-1}(\gamma) + \sum_{k=2}^{n-1} \phi^{-1}\left(\gamma + \sum_{j=2}^{k} y_j\right) = 0.$$

Besides, $|Q_\phi[\mathbf{y}]| \leq |\mathbf{y}|_1$, *and the function* $Q_\phi : E \to (-\frac{b}{2}, \frac{b}{2})$ *is continuous.*

Proof The proof of Lemma 2.4 is similar to Lemma 2.1 and we omit it. □

Proof of the Proposition 1 (i) If $\mathbf{u} = (u_1, \ldots, u_n)$ is a solution of (11), then there exists some $\tau_0 \in [2, n]_{\mathbb{Z}}$ such that $\Delta u_{\tau_0 - 1} \geq 0$, and therefore from (11) we have

$$\phi(\Delta u_k) = \phi(\Delta u_{\tau_0 - 1}) + \sum_{j=\tau_0}^{k} M$$

and

$$u_k = \sum_{j=1}^{k-1} \phi^{-1}\bigl[M(j - \tau_0 + 1) + \phi(\Delta u_{\tau_0 - 1})\bigr]. \tag{12}$$

Observe that $\phi(\Delta u_{\tau_0-1}) \geq 0$ and suppose that $|M| \geq \frac{b}{n-2}$, then

$$\max_{j \in [1, n-1]_{\mathbb{Z}}, \tau_0 \in [2,n]_{\mathbb{Z}}} \bigl|M(j - \tau_0 + 1) + \phi(\Delta u_{\tau_0 - 1})\bigr| \geq b.$$

In this case $\mathbf{u}$ given by (12) would not be well defined since the domain of ϕ^{-1} is the interval $(-b, b)$, and thus a solution of (11) cannot exist.

(ii) If $|M| < \frac{b}{2(n-2)}$, then the function $\mathbf{u}$ given by (12) is well defined for $\tau_0 = 2$, and satisfies

$$\nabla\bigl[\phi(\Delta u_k)\bigr] = M, \quad k \in [2, n-1]_{\mathbb{Z}} \quad \text{and} \quad u_1 = 0,$$

and hence

$$u_k = \phi^{-1}(\gamma) + \sum_{l=2}^{k-1} \phi^{-1}\left(\gamma + \sum_{j=2}^{l} M\right), \quad k = 2, \ldots, n.$$

Using Lemma 2.4, we can show that $u_n = 0$ (in fact $\gamma = \phi(\Delta u_1)$). Thus $\mathbf{u}$ is a solution of (11). □

Remark 2 Regrettably our method does not determine whether or not there exists a solution of (11) when $\frac{b}{2(n-2)} \leq |M| < \frac{b}{(n-2)}$, since we cannot guarantee $u_n = 0$ (in fact, the difference equation loses the symmetric property about the domain of solution). However, in [10], Proposition 1, the authors prove that the differential problem corresponding to (11) has the complete result.

As a consequence of Theorem 2.3 we obtain the following sufficient condition for the solvability of the Dirichlet problem.

Corollary 2 *Assume that ϕ is bounded (that is, $b < +\infty$) and there exists a bounded function $\mathbf{h} = (h(2), \dots, h(n-1))$ such that*

$$\left|f(k,u,v)\right| \le h(k), \quad k \in [2, n-1]_{\mathbb{Z}}, u \in \bigl(-(n-1)a, (n-1)a\bigr), v \in (-a, a),$$

with $|\mathbf{h}|_1 < \frac{b}{2}$. Then the Dirichlet problem (9) *has at least one solution.*

Proof Now, for each $0 < r < \frac{b}{2}$ we have $M_{2r} = |\mathbf{h}|_1 < \frac{b}{2}$. Therefore

$$\lambda_0 = \sup_{0<r<\frac{b}{2}} \frac{r}{M_{2r}} > 1,$$

and thus Theorem 2.3 implies the existence of a solution for problem (9) with $\lambda = 1$. $\square$

Example 2 Let $n \in \mathbb{N}$, $n \ge 4$ be fixed. We take $\phi(v) = \frac{v}{\sqrt{1+v^2}}$ and $f(k,u,v) = \frac{1}{4n}\sin(k+u-v)\pi$. Thus, by a simple computation, we show by Corollary 2 that problem (9) has a solution.

3 The mixed boundary value problem

Compared with the Dirichlet problem, the mixed boundary value problem

$$\nabla\bigl[\phi(\Delta u_k)\bigr] = \lambda f(k, u_k, \Delta u_k), \quad k \in [2, n-1]_{\mathbb{Z}}, \qquad u_1 = 0 = \Delta u_{n-1} \tag{13}$$

is less studied in the related literature. In this case, by means of the change of variable $y_k = \phi(\Delta u_k)$, $k = 1, \dots, n-1$, we see that a solution $\mathbf{u} = (u_1, \dots, u_n)$ of (13) is equivalent to a solution $\mathbf{y} = (y_1, \dots, y_{n-1})$ of the following nonlocal first-order problem:

$$\nabla y_k = \lambda f\left(k, \sum_{j=1}^{k-1} \phi^{-1}(y_j), \phi^{-1}(y_k)\right), \quad k \in [2, n-1]_{\mathbb{Z}}, \qquad y_{n-1} = 0. \tag{14}$$

By using the same idea as in Theorem 2.3, we can prove the following result.

Theorem 3.1 *If*

$$|\lambda| < \tilde{\lambda}_0 := \sup_{0<r<b} \frac{r}{M_r},$$

then problem (13) *has a solution.*

Proof Let $0 < r_1 < b$ be such that $\lambda \le \frac{r_1}{M_{r_1}}$, define

$$B_{r_1} = \bigl\{\mathbf{y} \mid \mathbf{y} = (y_1, \dots, y_{n-1}), \text{for all } k \in [1, n-1]_{\mathbb{Z}}, y_k \in (-b, b) \text{ with } |\mathbf{y}|_\infty \le r_1\bigr\}$$

and apply Schauder's fixed point theorem to the completely continuous operator $T : B_{r_1} \to B_{r_1}$ defined as

$$Ty_k = \lambda \sum_{j=k+1}^{n-1} f\left(j, \sum_{l=1}^{j-1} \phi^{-1}(y_l), \phi^{-1}(y_j)\right) \le \lambda \sum_{j=k+1}^{n-1} h(j) \le \lambda M_{r_1} \le r_1.$$

Thus (14) has a solution $\mathbf{y}$, which implies that $\mathbf{u}$ is a solution of (13). $\square$

Remark 3 The preceding theorem is sharp in the following sense: when $\phi(x) = \frac{x}{\sqrt{1+x^2}}$ and $f(k,u,v) \equiv M$, then it is easy to show that problem (13) has a solution if and only if $|\lambda| < \tilde{\lambda}_0 = \sup_{0<r<1} \frac{r}{M_r} = \frac{1}{M}$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors have achieved equal contributions to each part of this paper. All the authors read and approved the final manuscript.

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