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Nonlinear self-adjointness and conservation laws of the variable coefficient combined KdV equation with a forced term

Lihua Zhang[*]

[*]Correspondence:
zzlh100@163.com
School of Mathematics, Dezhou University, Dezhou, Shandong 253023, China

Abstract
In this paper, nonlinear self-adjointness and conservation laws for the variable coefficient combined KdV equation with a forced term are studied. We discuss its self-adjointness and find that the equation is nonlinearly self-adjoint. At the same time, the formal Lagrangian for the equation is obtained. Having performed Lie symmetry analysis for the equation, we derive several nontrivial conservation laws for the equation by using a general theorem on conservation laws, given by Ibragimov.

MSC: 35G20; 35L65; 58J70

Keywords: variable coefficient combined KdV equation; forced term; nonlinear self-adjointness; conservation laws; Lie symmetry analysis

1 Introduction

The notion of conservation laws plays an important role in the study of nonlinear science [1–3]. The existence of a large number of conservation laws of a partial differential equation (system) is a strong indication of its integrability. To search for explicit conservation laws of nonlinear partial differential equations (PDEs), a number of methods have been presented, such as Noether's theorem [4], the multiplier approach [5, 6], the partial Noether approach [7], and so on [8–10]. Among those, the new conservation theorem given by Ibragimov is one of the most frequently used methods [11–25]. Based on the concept of adjoint equation for a given differential equation [9], Ibragimov gives a general conservation theorem by which conservation laws for the system consisting of the given equation and its adjoint equation can be obtained. In fact, we are only interested in the conservation laws for the given equation. Therefore one has to eliminate the nonlocal variable which is introduced in the adjoint equation. For a self-adjoint nonlinear equation, its adjoint equation is equivalent with the original equation after replacing the nonlocal variable with the dependent variable in the original equation. However, many equations, which have remarkable symmetry properties and physical significance, are not self-adjoint. Thus the nonlocal variables of these equations cannot be eliminated easily. To solve this problem, Ibragimov and Gandarias have extended the concept of self-adjoint equations by introducing the definitions of quasi-self-adjoint equations and weak self-adjoint equations [11–16]. Recently, Ibragimov [17] has introduced the concept of nonlinear self-adjointness, which includes the previous two concepts as particular cases and extends the self-adjointness to the most generalized meaning.

The KdV and mKdV equations are the most popular soliton equations and have been extensively investigated. But nonlinear terms of the KdV and mKdV equations often simultaneously exist in practical problems such as fluid physics and quantum field theory and form the so-called combined KdV equation. In many geophysical and marine applications it is necessary to include a forcing term; typical examples are when the waves are generated by moving ships, or by a flow over bottom topography. In this paper, we consider the variable coefficient combined KdV equation with a forced term [26],

$$E_1 \equiv u_t + a(t)uu_x + m(t)u^2u_x + b(t)u_{xxx} - R(t) = 0, \tag{1}$$

where $a(t)$, $m(t)$, $b(t)$, and $R(t)$ are smooth functions.

Equation (1) is the special case of the equation

$$u_t + f(t,u)u_{xxxxx} + r(t,u)u_{xxx} + g(t,u)u_xu_{xx} + h(t,u)u_x^3 + a(t,u)u_x + b(t,u) = 0,$$

nonlinear self-adjointness for the equation has been considered in [19], conservation laws of the time dependent KdV equation,

$$u_t + uu_x + \frac{1}{t}u + \frac{1}{t}u_{xxx} = 0,$$

and the Harry-Dym type equation,

$$u_t + uu_{xxx} = 0,$$

have also been derived in [19]. However, conservation laws for the case of $a(t) \neq 0$, $b(t) \neq 0$, and $m(t) \neq 0$ in Eq. (1) have not been obtained.

Exact solutions including many kinds of solitary wave-like solutions, quasi-periodical solutions and solitary wave solutions of Eq. (1) have been obtained in [26]. When $R(t) = 0$, $a(t)$, $m(t)$, and $b(t)$ are constants, Eq. (1) becomes the constant coefficient combined KdV equation [27, 28]. If $a(t) = a$, $b(t) = b$, $m(t) = 0$, and $R(t) = f(t)$, Eq. (1) becomes the special case of the forced KdV equation [25]. To the best of our knowledge, Lie symmetries and conservation laws of Eq. (1) when $a(t) \neq 0$, $b(t) \neq 0$, and $m(t) \neq 0$ have not been discussed up to now.

The rest of the paper is organized as follows. In Section 2, we introduce the main notations and theorems used in this paper. In Section 3, we first discuss the nonlinear self-adjointness for the combined KdV equation (1) and get its formal Lagrangian. In Section 4, after performing Lie symmetry analysis, nontrivial conservation laws of Eq. (1) are derived making use of the obtained formal Lagrangian and Lie symmetries. A discussion of the results and our conclusion are given in the last section.

2 Preliminaries

We first briefly present the main notation and theorems used in this paper. Consider an sth-order nonlinear evolution equation

$$F(x, u, u_{(1)}, u_{(2)}, \ldots, u_{(s)}) = 0, \tag{2}$$

with n independent variables $x = (x_1, x_2, \ldots, x_n)$ and a dependent variable u, where $u_{(1)}, u_{(2)}, \ldots, u_{(s)}$ denote the collection of all first-order, second-order, …, sth-order partial derivatives. We have $u_i = D_i(u), u_{ij} = D_jD_i(u), \ldots$. Here,

$$D_i = \frac{\partial}{\partial x_i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \cdots, \quad i = 1, 2, \ldots, n,$$

is the total differential operator with respect to x_i.

Definition 1 The adjoint equation of Eq. (2) is defined by

$$F^*(x, u, v, u_{(1)}, v_{(1)}, u_{(2)}, v_{(2)}, \ldots, u_{(s)}, v_{(s)}) = 0, \tag{3}$$

with

$$F^*(x, u, v, u_{(1)}, v_{(1)}, u_{(2)}, v_{(2)}, \ldots, u_{(s)}, v_{(s)}) = \frac{\delta(vF)}{\delta u},$$

where

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} + \sum_{m=1}^{\infty} (-1)^m D_{i_1} \cdots D_{i_m} \frac{\partial}{\partial u_{i_1 i_2 \cdots i_m}}$$

denotes the Euler-Lagrange operator, v is a new dependent variable, $v = v(x)$.

Definition 2 Equation (2) is said to be self-adjoint if the equation obtained from the adjoint equation (3) by the substitution $v = u$:

$$F^*(x, u, u, u_{(1)}, u_{(1)}, u_{(2)}, u_{(2)}, \ldots, u_{(s)}, u_{(s)}) = 0,$$

is identical to the original equation (2). In other words, Eq. (2) is self-adjoint if and only if

$$F^*(x, u, u, u_{(1)}, u_{(1)}, u_{(2)}, u_{(2)}, \ldots, u_{(s)}, u_{(s)}) = \lambda(x, u, u_{(1)}, u_{(2)}, \ldots) F(x, u, u_{(1)}, u_{(2)}, \ldots, u_{(s)}),$$

where λ is an undetermined coefficient.

Definition 3 Equation (2) is said to be nonlinearly self-adjoint if its adjoint equation (3) is satisfied for all solutions u of Eq. (2) upon a substitution

$$v = \phi(x, u), \qquad \phi(x, u) \neq 0.$$

Theorem 1 *The system consisting of Eq.* (2) *and its adjoint equation* (3),

$$\begin{cases} F(x, u, u_{(1)}, u_{(2)}, \ldots, u_{(s)}) = 0, \\ F^*(x, u, v, u_{(1)}, v_{(1)}, u_{(2)}, v_{(2)}, \ldots, u_{(s)}, v_{(s)}) = 0, \end{cases} \tag{4}$$

has a formal Lagrangian, namely

$$L = vF(x, u, u_{(1)}, u_{(2)}, \ldots, u_{(s)}). \tag{5}$$

In the following we recall the ‘new conservation theorem’ given by Ibragimov in [8].

Theorem 2 *Any Lie point, Lie-Bäcklund, and nonlocal symmetry*

$$V = \xi^i \frac{\partial}{\partial x_i} + \eta \frac{\partial}{\partial u} \tag{6}$$

of Eq. (2) *provides a conservation law* $D_i(T^i) = 0$ *for the system* (4). *The conserved vector is given by*

$$\begin{aligned} T^i = {} & \xi^i L + W\left(\frac{\partial L}{\partial u_i} - D_j\left(\frac{\partial L}{\partial u_{ij}}\right) + D_j D_k\left(\frac{\partial L}{\partial u_{ijk}}\right) - D_j D_k D_r\left(\frac{\partial L}{\partial u_{ijkr}}\right) + \cdots\right) \\ & + D_j W\left(\frac{\partial L}{\partial u_{ij}} - D_k\left(\frac{\partial L}{\partial u_{ijk}}\right) + D_k D_r\left(\frac{\partial L}{\partial u_{ijkr}}\right) - \cdots\right) \\ & + D_j D_k W\left(\frac{\partial L}{\partial u_{ijk}} - D_r\left(\frac{\partial L}{\partial u_{ijkr}}\right) + \cdots\right), \end{aligned} \tag{7}$$

where L is determined by Eq. (5), *W is the Lie characteristic function, and*

$$W = \eta - \xi^j u_j.$$

3 Nonlinear self-adjointness of Eq. (1)

To search for conservation laws of Eq. (1) by Theorem 2, adjoint equation and formal Lagrangian of Eq. (1) must be known. We first construct its adjoint equation. According to Definition 1, the adjoint equation of Eq. (1) is

$$E_1^* \equiv v_t + a(t)uv_x + m(t)u^2 v_x + b(t)v_{xxx} = 0, \tag{8}$$

where v is a new dependent variable with respect to x and t. If we replace v by u, Eq. (8) is not identical to Eq. (2), according to Definition 2, we know Eq. (1) is not self-adjoint.

According to Theorem 1, the formal Lagrangian for the system consisting of Eq. (1) and its adjoint equation (8) is

$$L = v\big(u_t + a(t)uu_x + m(t)u^2 u_x + b(t)u_{xxx} - R(t)\big). \tag{9}$$

According to Definition 3, we recall that Eq. (1) is nonlinearly self-adjoint if its adjoint equation (8) becomes equivalent with Eq. (1) after the following substitution:

$$v = \phi(x, t, u). \tag{10}$$

That is to say, Eq. (1) is nonlinearly self-adjoint if and only if

$$E_1^*|_{v=\phi(x,t,u)} = \lambda(x, t, u, u_x, u_t, u_{xx}, \ldots)E_1, \tag{11}$$

where λ is an undetermined function and $\phi(x, t, u) \neq 0$.

The substitution of the expressions of E_1 and E_1^* into (11) results in the following equation:

$$\begin{aligned}&(\phi_u-\lambda)u_t+b(t)(\phi_u-\lambda)u_{xxx}+\phi_t+a(t)u\phi_x+a(t)u\phi_u u_x+m(t)u^2\phi_x\\&\quad+m(t)u^2\phi_u u_x+b(t)\phi_{xxx}+3b(t)\phi_{xxu}u_x+3b(t)\phi_{xuu}u_x^2+3b(t)\phi_{xu}u_{xx}\\&\quad+b(t)\phi_{uuu}u_x^3+3b(t)u_x\phi_{uu}u_{xx}-\lambda a(t)uu_x-\lambda m(t)u^2u_x+\lambda R(t)=0.\end{aligned}\tag{12}$$

Solving the above system with the aid of Maple, the final results read

$$\lambda=M_0,\qquad \phi=M_0u-M_0\int R(t)\,dt,$$

where M_0 is an arbitrary constant.

The result obtained here is a special case of [19] when $f(t,u)=g(t,u)=h(t,u)=0$, $b(t,u)=-R(t)$, $r(t,u)=b(t)$, and $a(t,u)=a(t)u+m(t)u^2$, where $a(t)$, $m(t)$, $b(t)$ and $R(t)$ are the coefficient functions of Eq. (1). We have checked that $\phi=M_0u-M_0\int R(t)\,dt$ satisfies Eqs. (22)-(26) in [19]. However, conservation laws for this special case is not studied in [19]. If $a(t)=a$, $b(t)=b$, $m(t)=0$, and $R(t)=f(t)$, the obtained result is the same as that obtained in [25] when the coefficient $c=0$ in [25].

In summary, we have the following statements.

Theorem 3 *The forced combined KdV equation* (1) *is nonlinearly self-adjoint if and only if*

$$\phi=M_0u-M_0\int R(t)\,dt.$$

Corollary 1 *The formal Lagrangian of Eq.* (1) *reads*

$$L=M_0\left(u-\int R(t)\,dt\right)\bigl(u_t+a(t)uu_x+m(t)u^2u_x+b(t)u_{xxx}-R(t)\bigr).\tag{13}$$

Remark 1 When the formal Lagrangian has the form of (13), the adjoint equation of Eq. (1) expressed by Eq. (8) and Eq. (1) are equivalent.

For simplicity, we take $M_0=1$ in Eq. (13).

4 Lie symmetry analysis and conservation laws of Eq. (1)

In the following, we will first perform Lie symmetry analysis for the forced combined KdV equation (1) using the classical Lie group method. Suppose that the Lie symmetry of Eq. (1) is as follows:

$$V=\xi\frac{\partial}{\partial x}+\tau\frac{\partial}{\partial t}+\eta\frac{\partial}{\partial u},\tag{14}$$

where ξ, τ, and η are undetermined functions with respect to x, t, and u. According to the procedures of Lie group method, the invariant condition that ξ, τ, and η must satisfy is

$$\begin{aligned}&\tau\bigl(a'(t)uu_x+m'(t)u^2u_x+b'(t)u_{xxx}-R'(t)\bigr)+a(t)\bigl(u_x\eta+u\eta^x\bigr)\\&\quad+m(t)\bigl(2uu_x\eta+u^2\eta^x\bigr)+\eta^t+b(t)\eta^{xxx}=0,\end{aligned}\tag{15}$$

where

$$\begin{cases} \eta^x = D_x(\eta - \xi u_x - \tau u_t) + \xi u_{xx} + \tau u_{xt}, \\ \eta^t = D_t(\eta - \xi u_x - \tau u_t) + \xi u_{xt} + \tau u_{tt}, \\ \eta^{xxx} = D_{xxx}(\eta - \xi u_x - \tau u_t) + \xi u_{xxxx} + \tau u_{xxxt}. \end{cases} \tag{16}$$

Here D_x, D_t are the first-order total differential operator with respect to x and t and D_{xxx} is the third-order total differential operator with respect to x. Substituting (16) into (15) with u a solution of Eq. (1), *i.e.*

$$u_t = -a(t)uu_x - m(t)u^2u_x - b(t)u_{xxx} + R(t),$$

we obtain a system of over-determined partial differential equations (PDEs) with respect to ξ, τ, and η:

$$\begin{aligned} &3b^2\tau_x u_{xxxxx} + 3b^2\tau_{xx}u_{xxxx} + u_{xxx}\big(\tau b_t + b^2\tau_{xxx} + b\tau_t + 4mb\tau_x u^2 - 3b\xi_u u_x - 3b\xi_x \\ &\quad + 4ab\tau_x u\big) - 3b\xi_u u_{xx}^2 + u_{xx}\big(-3b\xi_{xx} + 3b\eta_{xu} + 9ab\tau_x u_x + 3ab\tau_{xx}u + 3bm\tau_{xx}u^2 \\ &\quad + 18bm\tau_x uu_x - 6b\xi_{uu}u_x^2 - 9b\xi_{xu}u_x + 3b\eta_{uu}u_x\big) - b\xi_{uuu}u_x^4 \\ &\quad + (b\eta_{uuu} - 3b\xi_{xuu} + 6bm\tau_x)u_x^3 + (3b\eta_{xuu} + 6bm\tau_{xx}u + 3ab\tau_{xx} - 3b\xi_{xxu})u_x^2 \\ &\quad + \big(a\eta + 3b\eta_{xxu} - b\xi_{xxx} - R\xi_u + 2am\tau_x u^3 + ab\tau_{xxx}u + bm\tau_{xxx}u^2 \\ &\quad - \xi_t + m^2\tau_x u^4 + a\tau_t u + m\tau_t u^2 + \tau m_t u^2 + \tau a_t u + 2mu\eta + a^2\tau_x u^2 \\ &\quad - m\xi_x u^2 - a\xi_x u\big)u_x + u(a\eta_x - a\tau_x R) + u^2(m\eta_x - mR\tau_x) \\ &\quad - bR\tau_{xxx} + \eta_t - \tau R_t + \eta_u R + b\eta_{xxx} - \tau_t R = 0. \end{aligned} \tag{17}$$

In the above equation, $a = a(t)$, $b = b(t)$, $m = m(t)$, and $R = R(t)$. If $a(t) = 1$, $m(t) = 0$, and $R(t) = 0$, from the equation we can obtain the same result as that obtained in [29] when the coefficient $a(t) = 0$ in [29]. If $a(t) = 1$, $m(t) = 0$, $b(t) = 0$, and $R(t) = 0$, we can also obtain the same result as that obtained in [22] when the function $a(u) = u$ in [22]. In this paper, we consider symmetries with the coefficients $a(t) \neq 0$, $b(t) \neq 0$, and $m(t) \neq 0$.

Solving Eq. (17) with the aid of Maple, we get the following cases.

Case 1. When a ($\neq 0$), m ($\neq 0$), b ($\neq 0$), and R are all constants, there are two Lie symmetries as follows:

$$V_0 = \frac{\partial}{\partial x}, \qquad V_1 = \frac{\partial}{\partial t}.$$

Case 2. When a, m, and b are nonzero constants, $R(t) = \frac{M_1}{t^{\frac{4}{3}}}$, in addition to the symmetry V_0, there is another symmetry,

$$V_2 = \left(\frac{x}{3} - \frac{a^2t}{6m}\right)\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} - \left(\frac{u}{3} + \frac{a}{6m}\right)\frac{\partial}{\partial u},$$

where M_1 is an arbitrary constant.

Case 3. When b and m are nonzero constants, $a(t) = M_3 t^{\frac{-1}{3}}$, $R(t) = M_2 t^{\frac{-4}{3}}$, in addition to the symmetry V_0, there is another symmetry,

$$V_3 = \frac{x}{3}\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} - \frac{u}{3}\frac{\partial}{\partial u},$$

where M_2 and M_3 ($\neq 0$) are constants.

Case 4. When a and m are nonzero constants, $b(t) = -t$, $R(t) = M_4 t^{\frac{-7}{6}}$, in addition to the symmetry V_0, there is another symmetry,

$$V_4 = \left(\frac{x}{3} - \frac{a^2 t}{24m}\right)\frac{\partial}{\partial x} + \frac{t}{2}\frac{\partial}{\partial t} - \left(\frac{u}{12} + \frac{a}{24m}\right)\frac{\partial}{\partial u},$$

where M_4 is an arbitrary constant.

Case 5. When $a(t) = M_5 A^{-\frac{5}{6}} e^{At}$, $b(t) = e^{At}$, $m(t) = e^{At}A^{-1}$, $R(t) = M_6 A^{-\frac{5}{6}} e^{At}$, in addition to the symmetry V_0, there is another symmetry,

$$V_5 = \frac{1}{e^{At}}\frac{\partial}{\partial t},$$

where M_5 ($\neq 0$), M_6, and A ($\neq 0$) are constants.

Case 6. When $a(t) = M_5 e^{\frac{At}{6}}$, $b(t) = e^{At}$, $m(t) = 1$, $R(t) = M_6 e^{\frac{At}{6}}$, in addition to the symmetry V_0, there is another symmetry,

$$V_6 = \frac{x}{3}\frac{\partial}{\partial x} + \frac{1}{A}\frac{\partial}{\partial t} + \frac{u}{6}\frac{\partial}{\partial u},$$

where M_5, M_6, and A are the same as those in Case 5.

Case 7. When b is a nonzero constant, $a(t) = N_3 e^{\frac{N_1 t}{2}}$, $m(t) = N_2 e^{N_1 t}$, $R(t) = N_4 e^{-\frac{N_1 t}{2}}$, in addition to the symmetry V_0, there is another symmetry,

$$V_7 = \frac{\partial}{\partial t} - \frac{N_1 u}{2}\frac{\partial}{\partial u},$$

where N_1 ($\neq 0$), N_2 ($\neq 0$), N_3 ($\neq 0$), and N_4 are constants.

Through analysis of self-adjointness, the adjoint equation (8) has become equivalent with Eq. (1). Using the formal Lagrangian and Lie symmetries of Eq. (1), conservation laws for Eq. (1) can be obtained by Theorem 2. According to the classifications of Lie symmetries, the conservation laws for Eq. (1) are as follows.

Case 1. For the symmetry V_0, its Lie characteristic function is

$$W = -u_x$$

and the conservation laws for Eq. (1) associated with V_0 are

$$X = uu_t - uR + R^2 t - Rtu_t,$$
$$T = Rtu_x - u_x u.$$

It is easy to check that the conservation laws corresponding to V_0 is trivial since

$$D_x(X) + D_t(T) \equiv 0.$$

Case 2. For the symmetry V_1, its Lie characteristic function is

$$W = -u_t$$

and the conservation laws for Eq. (1) associated with V_1 are

$$X = -au^2u_t + aRtu_tu - mu^3u_t + mRtu^2u_t - bu_tu_{xx} + bu_{tx}u_x - bu_{txx}u + bRtu_{txx},$$
$$T = au^2u_x + mu^3u_x + buu_{xxx} - Ru - aRtuu_x - mRtu^2u_x - bRtu_{xxx} + R^2t.$$

The conservation law corresponding to V_1 is nontrivial because $D_x(X) + D_t(T) = 0$ if and only if u is a solution of Eq. (1).

Case 3. For the Lie symmetry V_2, its Lie characteristic function is

$$W = -\left(\frac{u}{3} + \frac{a}{6m}\right) - \left(\frac{x}{3} - \frac{a^2t}{6m}\right)u_x - tu_t,$$

and the conservation laws for Eq. (1) associated with V_2 are

$$\begin{aligned}X = {}& -btu_{txx}u - 3bM_1t^{\frac{2}{3}}u_{txx} + btu_xu_{tx} - 3bM_1u_{xx}t^{-\frac{1}{3}} - btu_tu_{xx} - \frac{au^3}{2} - \frac{mu^4}{3} - b\frac{au_{xx}}{6m} \\ & - \frac{a^2tuu_t}{6m} - \frac{M_1^2x}{t^{\frac{5}{3}}} + \frac{xuu_t}{3} - \frac{a^2u^2}{6m} - b\frac{4uu_{xx}}{3} + b\frac{2u_x^2}{3} + \frac{a^2M_1^2}{2m}t^{\frac{-2}{3}} - \frac{a^2M_1t^{\frac{2}{3}}u_t}{2m} \\ & - \frac{3aM_1u^2}{2t^{\frac{1}{3}}} - \frac{mM_1u^3}{t^{\frac{1}{3}}} - \frac{M_1xu}{3t^{\frac{4}{3}}} - \frac{a^2M_1u}{3mt^{\frac{1}{3}}} + \frac{M_1xu_t}{t^{\frac{1}{3}}} \\ & - atu_tu^2 - 3aM_1t^{\frac{2}{3}}u_tu - 3mM_1u_tu^2t^{\frac{2}{3}} - mtu_tu^3,\end{aligned}$$

$$\begin{aligned}T = {}& atu^2u_x + mtu^3u_x + btuu_{xxx} - \frac{2M_1u}{t^{\frac{1}{3}}} + 3M_1at^{\frac{2}{3}}uu_x + 3M_1mt^{\frac{2}{3}}u^2u_x + 3bM_1t^{\frac{2}{3}}u_{xxx} \\ & - \frac{3M_1^2}{t^{\frac{2}{3}}} - \frac{u^2}{3} - \frac{au}{6m} - \frac{aM_1}{2mt^{\frac{1}{3}}} - \frac{xu_xu}{3} - \frac{M_1xu_x}{t^{\frac{1}{3}}} + \frac{a^2tu_xu}{6m} + \frac{a^2M_1u_xt^{\frac{2}{3}}}{2m},\end{aligned}$$

where M_1 is an arbitrary constant.

Case 4. For the Lie symmetry V_3, its Lie characteristic function is

$$W = -\frac{u}{3} - \frac{x}{3}u_x - tu_t,$$

and the conservation laws for Eq. (1) associated with V_3 are

$$\begin{aligned}X = {}& -\frac{M_3u^3}{3t^{\frac{1}{3}}} + \frac{2b}{3}u_x^2 - \frac{mu^4}{3} + btu_xu_{tx} + \frac{xuu_t}{3} - \frac{M_2xu}{3t^{\frac{4}{3}}} + \frac{M_2xu_t}{t^{\frac{1}{3}}} - \frac{M_3M_2u^2}{t^{\frac{2}{3}}} \\ & - \frac{mM_2u^3}{t^{\frac{1}{3}}} - \frac{4buu_{xx}}{3} - \frac{3bM_2u_{xx}}{t^{\frac{1}{3}}} - \frac{M_2^2x}{t^{\frac{5}{3}}} - M_3t^{\frac{2}{3}}u_tu^2 - 3M_2M_3t^{\frac{1}{3}}u_tu \\ & - mtu_tu^3 - 3mM_2t^{\frac{2}{3}}u_tu^2 - btu_tu_{xx} - btu_{txx}u - 3bM_2u_{txx}t^{\frac{2}{3}},\end{aligned}$$

$$T = -\frac{1}{3t^{\frac{2}{3}}}\left(ut^{\frac{1}{3}} + 3M_2\right)\left(-3M_3tuu_x - 3mt^{\frac{4}{3}}u^2u_x - 3bt^{\frac{4}{3}}u_{xxx} + 3M_2 + ut^{\frac{1}{3}} + xu_xt^{\frac{1}{3}}\right),$$

where M_2 and M_3 ($\neq 0$) are constants.

Case 5. For the Lie symmetry V_4, its Lie characteristic function is

$$W = -\left(\frac{u}{12} + \frac{a}{24m}\right) - \left(\frac{x}{3} - \frac{a^2 t}{24m}\right)u_x - \frac{t}{2}u_t,$$

and the conservation laws for Eq. (1) associated with V_4 are

$$\begin{aligned} X = {}& -\frac{1}{24mt^{\frac{4}{3}}}\bigl(-6a^2M_4^2t + 48mM_4^2x - 72mM_4t^{\frac{19}{6}}u_{txx} + a^2t^{\frac{4}{3}}u^2 + 2m^2t^{\frac{4}{3}}u^4 - at^{\frac{7}{3}}u_{xx} \\ & + 10mt^{\frac{7}{3}}u_x^2 - 20mt^{\frac{7}{3}}uu_{xx} + 5a^2M_4t^{\frac{7}{6}}u + 12m^2M_4t^{\frac{7}{6}}u^3 + 3mat^{\frac{4}{3}}u^3 + a^2t^{\frac{7}{3}}uu_t \\ & + 6a^2M_4t^{\frac{13}{6}}u_t + 12m^2t^{\frac{7}{3}}u^3u_t - 12mt^{\frac{10}{3}}u_{xx}u_t + 12mt^{\frac{10}{3}}u_xu_{tx} - 108mM_4t^{\frac{16}{3}}u_{xx} \\ & - 12mt^{\frac{10}{3}}u_{txx}u + 8mM_4xut^{\frac{1}{6}} + 12mat^{\frac{7}{3}}u_tu^2 + 72maM_4t^{\frac{13}{6}}u_tu + 72m^2M_4t^{\frac{13}{6}}u_tu^2 \\ & - 48mM_4xu_tt^{\frac{7}{6}} - 8mxut^{\frac{4}{3}}u_t + 18maM_4t^{\frac{7}{6}}u^2\bigr), \\ T = {}& \frac{ut^{\frac{1}{6}} + 6M_4}{24mt^{\frac{1}{3}}}\bigl(12mat^{\frac{7}{6}}uu_x + 12m^2t^{\frac{7}{6}}u^2u_x - 12mt^{\frac{13}{6}}u_{xxx} - 12mM_4 - 2mt^{\frac{1}{6}}u - at^{\frac{1}{6}} \\ & - 8mxu_xt^{\frac{1}{6}} + a^2t^{\frac{7}{6}}u_x\bigr), \end{aligned}$$

where M_4 is an arbitrary constant.

Case 6. For the Lie symmetry V_5, its Lie characteristic function is

$$W = -\frac{1}{e^{At}}u_t,$$

and the conservation laws for Eq. (1) associated with V_5 are

$$\begin{aligned} X = {}& -M_5A^{-\frac{5}{6}}u_tu^2 + M_5M_6A^{-\frac{8}{3}}e^{At}u_tu - A^{-1}u_tu^3 + M_6A^{-\frac{17}{6}}e^{At}u_tu^2 - u_tu_{xx} + u_{tx}u_x \\ & - u_{txx}u + M_6A^{-\frac{11}{6}}u_{txx}e^{At}, \\ T = {}& M_5A^{-\frac{5}{6}}u_xu^2 + A^{-1}u^3u_x + uu_{xxx} - M_6A^{-\frac{5}{6}}u - M_6M_5A^{-\frac{8}{3}}e^{At}uu_x - M_6A^{-\frac{17}{6}}e^{At}u^2u_x \\ & - M_6A^{-\frac{11}{6}}e^{At}u_{xxx} + M_6^2A^{-\frac{8}{3}}e^{At}, \end{aligned}$$

where M_5 ($\neq 0$), M_6, and A ($\neq 0$) are constants.

Case 7. For the Lie symmetry V_6, its Lie characteristic function is

$$W = \frac{u}{6} - \frac{x}{3}u_x - \frac{1}{A}u_t,$$

and the conservation laws for Eq. (1) associated with V_6 are

$$\begin{aligned} X = {}& -A^{-1}u_tu^3 + \frac{1}{6}M_5e^{\frac{At}{6}}u^3 + \frac{1}{3}xuu_t - \frac{1}{3}ue^{At}u_{xx} - M_5M_6A^{-1}e^{\frac{At}{3}}u^2 - \frac{1}{3}M_6xue^{\frac{At}{6}} \\ & + 2M_6^2A^{-1}xe^{\frac{At}{3}} - M_6A^{-1}u^3e^{\frac{At}{6}} - A^{-1}u_te^{At}u_{xx} + A^{-1}e^{At}u_xu_{tx} + 3M_6A^{-1}u_{xx}e^{\frac{7At}{6}} \\ & - A^{-1}e^{At}u_{txx}u + 6M_6A^{-2}u_{txx}e^{\frac{7At}{6}} + \frac{1}{6}e^{At}u_x^2 + \frac{1}{6}u^4 - 2M_6A^{-1}xe^{\frac{At}{6}}u_t - \frac{M_5u_te^{\frac{At}{6}}u^2}{A} \\ & + 6M_5M_6A^{-2}u_te^{\frac{At}{3}}u + 6M_6A^{-2}u_tu^2e^{\frac{At}{6}}, \end{aligned}$$

$$
\begin{aligned}
T ={}& M_5A^{-1}e^{\frac{At}{6}}u^2u_x + A^{-1}u^3u_x + A^{-1}ue^{At}u_{xxx} - 2M_6A^{-1}ue^{\frac{At}{6}} - 6M_6M_5A^{-2}e^{\frac{At}{3}}uu_x \\
&- 6M_6A^{-2}e^{\frac{At}{6}}u^2u_x - 6M_6A^{-2}e^{\frac{7At}{6}}u_{xxx} + 6M_6^2A^{-2}e^{\frac{At}{3}} \\
&+ \frac{1}{6}u^2 - \frac{1}{3}xu_xu + \frac{2M_6xu_xe^{\frac{At}{6}}}{A},
\end{aligned}
$$

where M_5 ($\neq 0$), M_6, and A ($\neq 0$) are constants.

Case 8. For the Lie symmetry V_7, its Lie characteristic function is

$$
W = -\frac{N_1u}{2} - u_t,
$$

and the conservation laws for Eq. (1) associated with V_7 are

$$
\begin{aligned}
X ={}& -\frac{1}{2}N_1N_3e^{\frac{N_1t}{2}}u^3 - N_3N_4u^2 - \frac{1}{2}N_1N_2u^4e^{N_1t} - N_2N_4e^{\frac{N_1t}{2}}u^3 \\
&- bN_1uu_{xx} - N_3e^{\frac{N_1t}{2}}u_tu^2 - \frac{2N_3N_4}{N_1}u_tu - N_2e^{N_1t}u_tu^3 - \frac{2N_2N_4}{N_1}e^{\frac{N_1t}{2}}u_tu^2 \\
&- bu_tu_{xx} + \frac{bN_1}{2}u_x^2 + bu_xu_{tx} - bN_4e^{\frac{-N_1t}{2}}u_{xx} - bu_{txx}u - \frac{2bN_4}{N_1}e^{\frac{-N_1t}{2}}u_{txx}, \\
T ={}& N_3e^{\frac{N_1t}{2}}u^2u_x + N_2e^{N_1t}u^3u_x + buu_{xxx} - 2N_4e^{\frac{-N_1t}{2}}u + \frac{2}{N_1}N_4N_3uu_x - \frac{N_1u^2}{2} \\
&+ \frac{2}{N_1}N_4N_2e^{\frac{N_1t}{2}}u^2u_x + \frac{2b}{N_1}N_4e^{\frac{-N_1t}{2}}u_{xxx} - \frac{2}{N_1}{N_4}^2e^{-N_1t},
\end{aligned}
$$

where N_1 ($\neq 0$), N_2 ($\neq 0$), N_3 ($\neq 0$), and N_4 are constants.

Remark 2 As Eq. (1) does not depend on x explicitly, $V_0 = \frac{\partial}{\partial x}$ is an obvious symmetry for any possible choice of the functions $a(t)$, $m(t)$, $b(t)$, and $R(t)$. In Case 1, we have checked that the conservation laws corresponding to V_0 are trivial. In fact, in the other cases, the conservation laws corresponding to V_0 are also trivial, we omit them for simplicity.

Remark 3 The conservation laws corresponding to V_1-V_7 are nontrivial. The correctness of them has been checked by Maple software.

5 Conclusion

Conservation laws are used for the development of appropriate numerical methods and for mathematical analysis, in particular, existence, uniqueness and stability analysis. For the variable coefficient combined KdV equation (1) with a forced term, the constructing of conservation laws is not easy because of the arbitrariness of the variable coefficients $a(t)$, $b(t)$, $m(t)$, and the forced term $R(t)$. Through analysis of the self-adjointness, we show that Eq. (1) possesses nonlinear self-adjointness. This ensures that we can derive conservation laws of Eq. (1) by Theorem 2. After performing a Lie symmetry analysis, seven cases of Lie symmetries are obtained. Making use of the obtained Lie symmetries, nontrivial conservation laws for Eq. (1) are derived. These conservation laws may be useful for the explanation of some practical physical problems.

Competing interests
The author declares that they have no competing interests.

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