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Random attractors for the stochastic damped Klein-Gordon-Schrödinger system

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Abstract
This paper is concerned with random attractors for the stochastic Klein-Gordon-Schrödinger system. By using the tail-estimates method, we prove the asymptotic compactness of random dynamical systems and obtain the existence of random attractor on unbounded domain in $\mathbb{R}^n$.

1 Introduction

In recent decades, much attention has been paid to the existence of random attractor for stochastic partial differential equations. To the authors' knowledge, Crauel and Flandoli [1] and Flandoli and Schmalfuß [2] first introduced a corresponding generalization of the attractor (random attractor) to the stochastic partial differential equations. After that, the study of random attractors has gained considerable attention, see [3–6] for a comprehensive survey. There are also many other papers concerning the existence of random attractor for some SDEs on bounded domains, see [7–9] for stochastic damped sine-Gordon equation, [10] for Ladyzhenskaya model. In 2009, Bates *et al.* [11] introduced a new method to study the existence of random attractors for stochastic reaction-diffusion equations in unbounded domains, and they proved the asymptotic compactness of the solutions and then obtained random attractor in $\mathbb{L}^2(\mathbb{R}^n)$ by approaching $\mathbb{R}^n$ with a sequence of bounded domains Q_k, and combining the tail estimates in spatial variables with the compactness of Sobolev embedding in the bounded domain Q_k. Wang [12] used this method to study the existence of random attractors for the Benjamin-Bona-Mahony equation.

In this paper, we consider the damped Klein-Gordon-Schrödinger system perturbed by an ϵ-small random term

$$idu + (\triangle u + i\alpha u + uv)\,dt = f\,dt + \epsilon u\,dW_1, \quad x \in \mathbb{R}^n, t > 0,$$
$$dv_t + \left(\nu v_t - \triangle v + \mu v - \beta|u|^2\right)dt = g\,dt + \epsilon\delta\,dW_2, \quad x \in \mathbb{R}^n, t > 0$$

with the initial value conditions

$$u(x,0) = u_0(x), \qquad v(x,0) = v_0(x), \qquad v_t(x,0) = v_1(x), \quad x \in \mathbb{R}^n,$$

where α, ϵ, ν, μ and β are positive constants, $n \le 3$, $f, g \in \mathbb{L}^2$, $\delta \in \mathbb{H}^1$, W_1 and W_2 are independent two-side real-valued Wiener processes on a probability space which will be specified later.

The coupled Klein-Gordon-Schrödinger (KGS) system is an important model in nonlinear science. It is encountered in several diverse branches of physics, for example, in the description of the interaction between high frequency electron waves and low frequency ion plasma waves in a homogeneous magnetic field, adapted to model the UHH plasma heating scheme. The system also focuses on the vital role of collisions, by considering the non-homogeneous polarization drift for the low frequency coupling [13], as the following nonautonomous KGS equation:

$$i(u_t + \alpha u) + \triangle v + uv = f,$$
$$v_{tt} - \triangle v + \nu v_t + \mu v - \beta |u|^2 = g,$$

where u denotes a complex scalar nucleon field and v represents a real meson field, the complex-valued function f and the real-valued function g are external sources. By using Galerkin's method, Fukuda and Tsutsumi [14] first studied the coupled KGS system and obtained the existence of global strong solutions. Biler [15] obtained the existence of weak global attractors for the KGS system in a bounded domain. Wang and Lange [16] pointed out that the weak global attractor is actually a strong one. Recently, Guo and Li [17] proved the existence of a global attractor in $\mathbb{H}^2(\mathbb{R}^3) \times \mathbb{H}^2(\mathbb{R}^3)$ which attracts bounded sets of space $\mathbb{H}^3(\mathbb{R}^3) \times \mathbb{H}^3(\mathbb{R}^3)$ in the topology of $\mathbb{H}^2(\mathbb{R}^3) \times \mathbb{H}^2(\mathbb{R}^3)$. Lu and Wang [18] improved their results and obtained global attractors in space $\mathbb{H}^k(\mathbb{R}^3) \times \mathbb{H}^k(\mathbb{R}^3)$ for $k \geq 1$ which attracts all bounded sets of $\mathbb{H}^k(\mathbb{R}^3) \times \mathbb{H}^k(\mathbb{R}^3)$ in norm topology.

However, a system in reality is usually affected by external perturbations which in many cases are of great uncertainty or random influence. These random effects are not only introduced to compensate for the defects in some deterministic models, but also to explain the intrinsic phenomena. In [19], we have investigated the dynamical behavior for stochastic Klein-Gordon-Schrödinger lattice system in one dimension, which can be regarded as an approximation to a stochastic continuous case. Yan *et al.* [20, 21] studied the existence of random attractors for a class of first order stochastic lattice dynamical systems with delay.

Throughout this paper, we denote by $L^2(\mathbb{R}^n)$ both the standard real and complex Hilbert spaces and equip $L^2(\mathbb{R}^n)$ with the inner product and norm as

$$(u, v) = \int_{\mathbb{R}^n} u(x)\bar{v}(x)\,dx, \qquad \|u\|^2 = (u, u).$$

Hereafter, we denote by C any positive constants which may change from line to line.

This paper is organized as follows. In Section 2, we recall the basic concepts and some known results related to random dynamical systems and the pullback random attractors. In Section 3, we first derive the uniform estimates of solutions, and then prove the existence of the pullback random attractor for the stochastic KGS system.

2 Preliminaries

First we recall the definitions of the random dynamical systems and random attractors which are taken from [5, 6, 11, 12, 22].

Let $(H, \|\cdot\|_H)$ be a separable Hilbert space with Borel σ-algebra $\mathbb{B}(H)$, and let $(\Omega, \mathbb{F}, \mathbb{P})$ be a probability space.

Definition 2.1 $(\Omega, \mathbb{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$ is called a metric dynamical system if $\theta : \mathbb{R} \times \Omega \to \Omega$ is $(\mathbb{B}(\mathbb{R}) \times \mathbb{F}, \mathbb{F})$ measurable, $\theta_0 = id$, $\theta_{t+s} = \theta_t \circ \theta_s$ for all $t, s \in \mathbb{R}$, and $\theta_t \mathbb{P} = \mathbb{P}$ for all $t \in \mathbb{R}$.

Definition 2.2 A stochastic process $\phi(t,\omega)$ is called a random dynamical system (RDS) over $(\Omega, \mathbb{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$ if ϕ is $(\mathbb{B}(\mathbb{R}^+) \times \mathbb{F} \times \mathbb{B}(H), \mathbb{B}(H))$-measurable, and for all $\omega \in \Omega$

- the mapping $\phi : \mathbb{R}^+ \times \Omega \times H \to H$ is continuous;
- $\phi(0,\omega,\cdot) = id$ on H;
- $\phi(t+s,\omega,x) = \phi(t,\theta_s\omega,\phi(s,\omega,x))$ for all $t, s \geq 0$ and $x \in H$ (cocycle property).

Definition 2.3 A random bounded set $B(\omega) \subset H$ is called tempered with respect to $(\theta_t)_{t\in\mathbb{R}}$ if for a.e. $\omega \in \Omega$ and all $\epsilon > 0$,

$$\lim_{t\to\infty} e^{-\epsilon t} d\bigl(B(\theta_{-t}\omega)\bigr) = 0,$$

where $d(B) = \sup_{x\in B} \|x\|_H$.

Definition 2.4 Let $\mathbb{D}$ be a collection of random subsets of H. Then $\mathbb{D}$ is called inclusion-closed if $D = \{D(\omega)\}_{\omega\in\Omega} \in \mathbb{D}$ and $\tilde{D} = \{\tilde{D}(\omega) \subseteq H : \omega \in \Omega\}$ with $\tilde{D} \subseteq D(\omega)$ for all $\omega \in \Omega$ imply that $\tilde{D} \in \mathbb{D}$.

Definition 2.5 A random set $\mathbb{K}(\omega)$ is called an absorbing set in $\mathbb{D}$ if for all $B \in \mathbb{D}$ and P-a.e. $\omega \in \Omega$, there exists $t_B(\omega) > 0$ such that

$$\phi\bigl(t, \theta_{-t}\omega, B(\theta_{-t}\omega)\bigr) \subset \mathbb{K}(\omega), \quad t \geq t_B(\omega).$$

Definition 2.6 A random set $\mathbb{A}(\omega)$ is a random $\mathbb{D}$-attractor for RDS ϕ if

- $\mathbb{A}(\omega)$ is a random compact set, *i.e.*, $\omega \to d(x, \mathbb{A}(\omega))$ is measurable for every $x \in H$ and $\mathbb{A}(\omega)$ is compact for a.e. $\omega \in \Omega$;
- $\mathbb{A}(\omega)$ is strictly invariant, *i.e.*, $\phi(t,\omega,\mathbb{A}(\omega)) = \mathbb{A}(\theta_t\omega)$, $\forall t \geq 0$ and for a.e. $\omega \in \Omega$;
- $\mathbb{A}(\omega)$ attracts all sets in $\mathbb{D}$, *i.e.*, for all $B \in \mathbb{D}$ and a.e. $\omega \in \Omega$ we have

$$\lim_{t\to\infty} d\bigl(\phi\bigl(t, \theta_{-t}\omega, B(\theta_{-t}\omega)\bigr), \mathbb{A}(\omega)\bigr) = 0,$$

where $d(X,Y) = \sup_{x\in X} \inf_{y\in Y} \|x-y\|_H$, $X, Y \subset H$.

The collection $\mathbb{D}$ is called the domain of attraction of $\mathbb{A}$.

Definition 2.7 Let ϕ be an RDS on a Hilbert space H. ϕ is called $\mathbb{D}$ pullback asymptotically compact in H if for P-a.e. $\omega \in \Omega$, $\{\phi(t_n, \theta_{-t_n}\omega, x_n)\}_{n=1}^{\infty}$ has a convergent subsequence in H whenever $t_n \to \infty$, and $x_n \in B(\theta_{t_n}\omega)$ with $\{B(\omega)\}_{\omega\in\Omega} \in \mathbb{D}$.

From [11, 12, 16] we have the following result.

Proposition 2.8 *Let $\mathbb{D}$ be an inclusion-closed collection of random subsets of H, and let ϕ be a continuous RDS on H over $(\Omega, \mathbb{F}, P, (\theta_t)_{t\mathbb{R}})$. Suppose that $\{\mathbb{K}(\omega)\}_{\omega\in\Omega}$ is a closed absorbing set of ϕ in $\mathbb{D}$ and ϕ is $\mathbb{D}$-pullback asymptotic compact in H. Then ϕ has a unique*

$\mathbb{D}$*-random attractor* $\{\mathbb{A}(\omega)\}_{\omega\in\Omega}$ *which is given by*

$$\mathbb{A}(\omega) = \bigcap_{\kappa\geq 0}\overline{\bigcup_{t\geq\kappa}\phi\bigl(t,\theta_{-t}\omega,\mathbb{K}(\theta_{-t}\omega)\bigr)}.$$

3 The existence of random attractors

In this section, we will apply Proposition 2.8 to prove the existence of random attractors for the Klein-Gordon-Schrödinger system with ϵ-small random perturbation. The main tool is the tail-estimates method which is extensively used to prove the existence of random attractor, see [11, 12].

Consider the Klein-Gordon-Schrödinger system with ϵ-small random perturbation

$$\begin{aligned} &i\,du + (\triangle u + i\alpha u + uv)\,dt = f\,dt + \epsilon u\,dW_1, \quad x\in\mathbb{R}^n, t>0,\\ &dv_t + \bigl(\nu v_t - \triangle v + \mu v - \beta|u|^2\bigr)\,dt = g\,dt + \epsilon\delta\,dW_2, \quad x\in\mathbb{R}^n, t>0, \end{aligned} \tag{3.1}$$

with the initial value conditions

$$u(x,0)=u_0(x), \qquad v(x,0)=v_0(x), \qquad v_t(x,0)=v_1(x), \quad x\in\mathbb{R}^n,$$

where α, ϵ, ν, μ and β are positive constants, $n\leq 3$, $f,g\in\mathbb{L}^2$, $\delta\in\mathbb{H}^1$, W_1 and W_2 are independent two-side real-valued Wiener processes on a probability space $(\Omega,\mathbb{F},\mathbb{P})$.

For our purpose, we introduce the probability space

$$\Omega = \bigl\{\omega=(\omega_1,\omega_2)\in\mathbb{C}\bigl(\mathbb{R},\mathbb{R}^2\bigr):\omega(0)=0\bigr\}$$

endowed with the compact open topology [5]. Then we have $(W_1(t,\omega),W_2(t,\omega))=\omega(t)$, $t\in\mathbb{R}$. Let $\mathbb{P}$ be the corresponding Wiener measure, $\mathbb{F}$ be the $\mathbb{P}$-completion of the Borel σ-algebra on Ω, and $\theta_t\omega(\cdot)=\omega(\cdot+t)-\omega(t)$, $t\in\mathbb{R}$. Then $(\Omega,\mathbb{F},\mathbb{P},(\theta_t)_{t\in\mathbb{R}})$ is a metric dynamical system with the filtration $\mathbb{F}_t:=\bigvee_{s\leq t}\mathbb{F}_s^t$, $t\in\mathbb{R}$, where $\mathbb{F}_s^t=\sigma\{W(t_2)-W(t_1):s\leq t_1\leq t_2\leq t\}$ is the smallest σ-algebra generated by the random variable $W(t_2)-W(t_1)$ for all t_1, t_2 such that $s\leq t_1\leq t_2\leq t$, see [5] for more details.

We introduce an Ornstein-Uhlenbeck process $(\Omega,\mathbb{F},\mathbb{P},\theta_t)$ given by the Wiener process:

$$\begin{aligned} &y_1(\theta_t\omega_1) = -\nu\int_{-\infty}^{0} e^{\nu h}(\theta_t\omega_1)(h)\,dh, \quad t\in\mathbb{R},\\ &y_2(\theta_t\omega_2) = -\lambda\int_{-\infty}^{0} e^{\lambda h}(\theta_t\omega_2)(h)\,dh, \quad t\in\mathbb{R}, \end{aligned}$$

where ν and λ are positive. The above integral exists in the sense that for any path ω with a subexponential growth, y_1, y_2 solve the following Itô equations:

$$\begin{aligned} &dy_1 + \nu y_1\,dt = dW_1(t), \quad t\in\mathbb{R},\\ &dy_2 + \lambda y_2\,dt = dW_2(t), \quad t\in\mathbb{R}. \end{aligned}$$

Furthermore, there exists a θ_t invariant set $\Omega'\subset\Omega$ of full $\mathbb{P}$ measure such that:

(1) the mappings $t\to y_i(\theta_t\omega_i)$, $i=1,2$, are continuous for each $\omega\in\Omega'$;

(2) the random variables $\|y_i(\omega_i)\|$, $i = 1, 2$, are tempered (for more details, see [11, 12]).

Let $z_1(\theta_t\omega) = y_1(\theta_t\omega_1)$ and $z_2(\theta_t\omega) = \delta y_2(\theta_t\omega_2)$. Then we have

$$dz_1 + \nu z_1\,dt = dW_1(t), \quad t \in \mathbb{R},$$
$$dz_2 + \lambda z_2\,dt = \delta\,dW_2(t), \quad t \in \mathbb{R}.$$

Lemma 3.1 ([23]) *There exists a $(\theta_t)_{t\in\mathbb{R}}$-invariant set $\tilde{\Omega} \subset \Omega$ of full measure with sublinear growth*

$$\lim_{t\to\infty} \frac{\|W_i(t)\|}{t} = 0$$

of $\mathbb{P}$-measure one. In addition,

$$\lim_{t\to\infty} \frac{|y_i(\theta_t\omega_i)|}{|t|} = 0 \quad \textit{and} \quad \lim_{t\to\infty} \frac{\int_0^t y_i(\theta_s\omega_i)\,ds}{t} = 0,$$

where $\omega \in \tilde{\Omega}$, $i = 1, 2$.

By introducing the transformation $\psi = \frac{dv}{dt} + \rho v$ with ρ a positive constant which satisfies $\rho < \nu$, system (3.1) becomes

$$i\frac{du}{dt} + \triangle u + i\alpha u + uv = f + \epsilon u\frac{dW_1}{dt}, \quad x \in \mathbb{R}^n, \tag{3.2}$$

$$\frac{dv}{dt} = \psi - \rho v, \quad x \in \mathbb{R}^n, \tag{3.3}$$

$$\frac{d\psi}{dt} + (\nu - \rho)\psi + \bigl[\mu - \rho(\nu - \rho) - \triangle\bigr]v - \beta|u|^2 = g + \epsilon\delta\frac{dW_2}{dt}, \quad x \in \mathbb{R}^n. \tag{3.4}$$

In order to prove the existence of global solutions of (3.1), we introduce the processes

$$\tilde{u}(t) = z(t)u(t)$$

and

$$\tilde{\psi}(t) = \psi(t) - \epsilon\delta z_2(\theta_t\omega),$$

where $z(t) = e^{i\epsilon z_1(\theta_t\omega)}$ satisfies the stochastic differential equation

$$dz(t) = -\frac{\epsilon^2}{2}z(t)\,dt + i\epsilon z(t)\,dz_1. \tag{3.5}$$

Then system (3.2)-(3.4) can be changed into the following system:

$$\begin{cases} i\frac{d\tilde{u}}{dt} + \triangle\tilde{u} + (i\alpha + \frac{\epsilon^2}{2})\tilde{u} + \tilde{u}v - fz = 0, \\ \frac{dv}{dt} + \rho v = \tilde{\psi} + \epsilon\delta z_2, \\ \frac{d\tilde{\psi}}{dt} + (\nu - \rho)\tilde{\psi} + [\mu - \rho(\nu - \rho) - \triangle]v - \beta|\tilde{u}z^{-1}|^2 + \epsilon(\nu - \rho)\delta z_2 = g, \end{cases} \tag{3.6}$$

with the initial data $\tilde{u}_0 = u_0$, $v_0 = v_0$ and $\tilde{\psi}_0 = \psi_0 - \epsilon\delta z_2(\omega)$.

Lemma 3.2 *Let $f \in \mathbb{L}^2$. Then the solution of the first equation in* (3.6) *satisfies*

$$\|\tilde{u}\|^2 \leq e^{-\alpha t}\|\tilde{u}_0\|^2 + \frac{4}{\alpha^2}\|f\|^2.$$

Proof Taking the imaginary part of the inner product of (3.6) with $\tilde{u}$, we obtain

$$\frac{d}{dt}\|\tilde{u}\|^2 + 2\alpha\|\tilde{u}\|^2 = 2\operatorname{Im}\int_{\mathbb{R}^n}(fz, \tilde{u})\,dx. \tag{3.7}$$

Obviously, the right-hand side of (3.7) is bounded by

$$2\operatorname{Im}\int_{\mathbb{R}^n}(fz, \tilde{u})\,dx \leq \alpha\|\tilde{u}\|^2 + \frac{4}{\alpha}\|f\|^2\|z\|^2.$$

Thus we have

$$\frac{d}{dt}\|\tilde{u}\|^2 + \alpha\|\tilde{u}\|^2 \leq \frac{4}{\alpha}\|f\|^2\|z\|^2 \leq \frac{4}{\alpha}\|f\|^2.$$

By Gronwall's lemma we get

$$\|\tilde{u}\|^2 \leq e^{-\alpha t}\|\tilde{u}_0\|^2 + \frac{4}{\alpha^2}\|f\|^2.$$

The proof is completed. □

Remark 3.1 By Lemma 3.2, we know that there is $T_1 > 0$ such that $\|\tilde{u}\|$ is bounded for $t > T_1$, *i.e.*, $\|\tilde{u}\| \leq M_1$, $t > T_1$.

Lemma 3.3 *Let $f \in \mathbb{L}^4$. Then, for any $m \geq 0$, the solution of the first equation in* (3.1) *satisfies*

$$\int_{|x|\geq m}|\tilde{u}|^4\,dx \leq e^{-\alpha t}\int_{|x|\geq m}|\tilde{u}_0|^4\,dx + \frac{64}{\alpha^4}\int_{|x|\geq m}|f|^4\,dx$$

for all $\omega \in \Omega$.

Proof The proof is similar to the proof in Lemma 3.2, so we omit it here. □

Here and after, $\mathbb{I}$ denotes the space $\mathbb{L}^2(\mathbb{R}^n) \times \mathbb{H}^1(\mathbb{R}^n) \times \mathbb{L}^2(\mathbb{R}^n)$. By Galerkin's method, it is easy to prove that, for $\mathbb{P}$-a.e. $\omega \in \Omega$ and for all $(\tilde{u}_0, v_0, \tilde{\psi}_0) \in \mathbb{I}$, system (3.6) has a unique solution $(\tilde{u}(\cdot,\omega,\tilde{u}_0), v(\cdot,\omega,v_0), \tilde{\psi}(\cdot,\omega,\tilde{\psi}_0)) \in \mathbb{C}([0,\infty),\mathbb{I})$ with $\tilde{u}(0,\omega,\tilde{u}_0) = \tilde{u}_0$, $v(0,\omega,v_0) = v_0$ and $\tilde{\psi}(0,\omega,\tilde{\psi}_0) = \tilde{\psi}_0$. Furthermore, the solution is continuous with respect to $(\tilde{u}_0, v_0, \tilde{\psi}_0) \in \mathbb{I}$. To indicate the dependence of (u, v, ψ) on the initial data (u_0, v_0, ψ_0), we define a mapping

$$\phi_\epsilon : \mathbb{R}^+ \times \Omega \times \mathbb{I} \to \mathbb{I}$$

by

$$\begin{aligned}\phi_\epsilon\big(t,\omega,(u_0,v_0,\psi_0)\big) &= \big(u(t,\omega,u_0), v(t,\omega,v_0), \psi(t,\omega,\psi_0)\big)\\ &= \big(\tilde{u}(t,\omega,\tilde{u}_0)z^{-1}(t), v(t,\omega,v_0), \tilde{\psi}(t,\omega,\tilde{\psi}_0) + \epsilon\delta z_2(\theta_t\omega)\big)\end{aligned}$$

for all $(t,\omega,(u_0,v_0,\psi_0)) \in \mathbb{R}^+ \times \Omega \times \mathbb{I}$.

It is obvious that ϕ_ϵ satisfies all conditions in Definition 2.2. Therefore, ϕ_ϵ is a continuous random dynamical system associated with (3.1). It is easy to verify that ϕ_ϵ satisfies

$$\phi_\epsilon\big(t,\theta_{-t}\omega,(u_0,v_0,\psi_0)\big) = \big(u(t,\theta_{-t}\omega,u_0), v(t,\theta_{-t}\omega,v_0), \psi(t,\theta_{-t}\omega,\psi_0)\big)$$

for $\mathbb{P}$-a.e. $\omega \in \Omega$ and $t \geq 0$.

The following lemma shows that ϕ_ϵ has a closed bounded random absorbing set in $\mathbb{D}$, which verifies the first condition (I) in Proposition 2.8.

Lemma 3.4 *Let $f,g \in \mathbb{L}^2$, $\delta \in \mathbb{H}^1$. Assume that $\mu - \rho(\nu - \rho) > 0$. Then there exists $\{\mathbb{K}(\omega)\}_{\omega\in\Omega} \in \mathbb{D}$ such that $\{\mathbb{K}(\omega)\}_{\omega\in\Omega}$ is a random absorbing set for ϕ in $\mathbb{D}$, that is, for any $\mathbb{B} = \{\mathbb{B}(\omega)\}_{\omega\in\Omega} \in \mathbb{D}$ and $\mathbb{P}$-a.e. $\omega \in \Omega$, there is $T_{\mathbb{B}}(\omega) > 0$ such that*

$$\phi_\epsilon\big(t,\theta_{-t}\omega,\mathbb{B}(\theta_{-t}\omega)\big) \subseteq \mathbb{K}(\omega) \quad \textit{for all } t \geq T_{\mathbb{B}}(\omega).$$

Proof Taking the imaginary part of the inner product of the first equation of (3.6) with $\tilde{u}$, we have

$$\frac{d}{dt}\|\tilde{u}\|^2 + 2\alpha\|\tilde{u}\|^2 = 2\,\mathrm{Im}(fz,\tilde{u}). \tag{3.8}$$

Taking the inner product of the third equation of (3.6) with $\tilde{\psi}$, we get

$$\begin{aligned}&\frac{d}{dt}\|\tilde{\psi}\|^2 + 2(\nu-\rho)\|\tilde{\psi}\|^2 + 2\big[\mu - \rho(\nu-\rho)\big](v,\tilde{\psi}) - 2(\triangle v,\tilde{\psi})\\ &\quad - 2\beta\big(\big|\tilde{u}z^{-1}\big|^2,\tilde{\psi}\big) + 2\epsilon(\nu-\rho)\big(\delta z_2(\theta_t\omega),\tilde{\psi}\big) = 2(g,\tilde{\psi}).\end{aligned} \tag{3.9}$$

Note that

$$2(v,\tilde{\psi}) = 2\left(v, \frac{dv}{dt} + \rho v - \epsilon\delta z_2(\theta_t\omega)\right) = \frac{d}{dt}\|v\|^2 + 2\rho\|v\|^2 - 2\epsilon\big(v,\delta z_2(\theta_t\omega)\big) \tag{3.10}$$

and

$$\begin{aligned}-2(\triangle v,\tilde{\psi}) &= \frac{d}{dt}\|\nabla v\|^2 + 2\rho\|\nabla v\|^2 + 2\epsilon\big(\nabla v, z_2(\theta_t\omega)\nabla\delta\big)\\ &\quad + 2\epsilon\big(\nabla v,\delta\nabla z_2(\theta_t\omega)\big).\end{aligned} \tag{3.11}$$

Summing up (3.8)-(3.11), we have

$$\begin{aligned}&\frac{d}{dt}\big(\|\tilde{u}\|^2 + \big[\mu-\rho(\nu-\rho)\big]\|v\|^2 + \|\nabla v\|^2 + \|\tilde{\psi}\|^2\big) + 2\alpha\|\tilde{u}\|^2\\ &\qquad + 2\rho\big[\mu-\rho(\nu-\rho)\big]\|v\|^2 + 2\rho\|\nabla v\|^2 + 2(\nu-\rho)\|\tilde{\psi}\|^2\\ &\quad = 2\,\mathrm{Im}(fz,\tilde{u}) - 2\epsilon\big(\nabla v, z_2(\theta_t\omega)\nabla\delta\big) - 2\epsilon\big(\nabla v,\delta\nabla z_2(\theta_t\omega)\big)\\ &\qquad + 2\epsilon\big[\mu-\rho(\nu-\rho)\big]\big(v,\delta z_2(\theta_t\omega)\big) + 2\beta\big(\big|\tilde{u}z^{-1}\big|^2,\tilde{\psi}\big)\\ &\qquad - 2\epsilon(\nu-\rho)\big(\delta z_2(\theta_t\omega),\tilde{\psi}\big) + 2(g,\tilde{\psi}).\end{aligned} \tag{3.12}$$

Now, we estimate each term on the right-hand side of (3.12). By Young's inequality, we have

$$
\begin{aligned}
&2\left|\mathrm{Im}(fz,\tilde{u})\right| \le \alpha\|\tilde{u}\|^2 + \frac{4}{\alpha}\|f\|^2,\\
&2\left|\epsilon\left(\nabla\nu, z_2(\theta_t\omega)\nabla\delta\right)\right| \le \frac{\rho}{2}\|\nabla\nu\|^2 + \frac{8\epsilon^2}{\rho}\|\nabla\delta\|^2\left\|z_2(\theta_t\omega)\right\|^2,\\
&2\left|\epsilon\left(\nabla\nu, \delta\nabla z_2(\theta_t\omega)\right)\right| \le \frac{\rho}{2}\|\nabla\nu\|^2 + \frac{8\epsilon^2}{\rho}\|\delta\|^2\left\|\nabla z_2(\theta_t\omega)\right\|^2,\\
&2\left|\epsilon\left[\mu-\rho(\nu-\rho)\right]\left(\nu, \delta z_2(\theta_t\omega)\right)\right|\\
&\quad\le \rho\left(\mu-\rho(\nu-\rho)\right)\|\nu\|^2 + \frac{4\epsilon^2}{\rho(\mu-\rho(\nu-\rho))}\|\delta\|^2\left\|z_2(\theta_t\omega)\right\|^2,\\
&2\beta\left|\left(\left|\tilde{u}z^{-1}\right|^2, \tilde{\psi}\right)\right| \le \frac{\nu-\rho}{3}\|\tilde{\psi}\|^2 + \frac{12\beta^2}{\nu-\rho}\|\tilde{u}\|^4,\\
&2\left|\epsilon(\nu-\rho)\left(\delta z_2(\theta_t\omega), \tilde{\psi}\right)\right| \le \frac{\nu-\rho}{3}\|\tilde{\psi}\|^2 + 12\epsilon^2(\nu-\rho)\|\delta\|^2\left\|z_2(\theta_t\omega)\right\|^2,\\
&2\left|(g,\tilde{\psi})\right| \le \frac{\nu-\rho}{3}\|\tilde{\psi}\|^2 + \frac{12}{\nu-\rho}\|g\|^2.
\end{aligned}
$$

By combining the above estimates with (3.12), we obtain

$$
\begin{aligned}
&\frac{d}{dt}\left(\|\tilde{u}\|^2 + \left[\mu-\rho(\rho-\mu)\right]\|\nu\|^2 + \|\nabla\nu\|^2 + \|\tilde{\psi}\|^2\right) + \alpha\|\tilde{u}\|^2\\
&\qquad + \rho\left[\mu-\rho(\rho-\mu)\right]\|\nu\|^2 + \rho\|\nabla\nu\|^2 + (\nu-\rho)\|\tilde{\psi}\|^2\\
&\quad\le \frac{4}{\alpha}\|f\|^2 + \frac{8\epsilon^2}{\rho}\|\delta\|^2\left\|\nabla z_2(\theta_t\omega)\right\|^2 + \frac{12\beta^2}{\nu-\rho}\|\tilde{u}\|^4 + \frac{12}{\nu-\rho}\|g\|^2\\
&\qquad + \left[\frac{8\epsilon^2}{\rho}\|\nabla\delta\|^2 + \frac{4\epsilon^2}{\rho(\mu-\rho(\nu-\rho))}\|\delta\|^2 + 12\epsilon^2(\nu-\rho)\|\delta\|^2\right]\left\|z_2(\theta_t\omega)\right\|^2. \qquad (3.13)
\end{aligned}
$$

Furthermore, by Lemma 3.2, for $t \ge 0$

$$
\|\tilde{u}\|^4 \le e^{-2\alpha t}\|\tilde{u}_0\|^4 + \frac{8}{\alpha^2}e^{-\alpha t}\|u_0\|^2\|f\|^2 + \frac{16}{\alpha^4}\|f\|^4. \qquad (3.14)
$$

Therefore, by (3.13) and (3.14), we have

$$
\begin{aligned}
&\frac{d}{dt}\left(\|\tilde{u}\|^2 + \left[\mu-\rho(\rho-\mu)\right]\|\nu\|^2 + \|\nabla\nu\|^2 + \|\tilde{\psi}\|^2\right) + \alpha\|\tilde{u}\|^2\\
&\qquad + \rho\left[\mu-\rho(\rho-\mu)\right]\|\nu\|^2 + \rho\|\nabla\nu\|^2 + (\nu-\rho)\|\tilde{\psi}\|^2\\
&\quad\le \frac{4}{\alpha}\|f\|^2 + \frac{8\epsilon^2}{\rho}\|\delta\|^2\left\|\nabla z_2(\theta_t\omega)\right\|^2 + \frac{12\beta^2}{\nu-\rho}\left[e^{-2\alpha t}\|\tilde{u}_0\|^4\right.\\
&\qquad \left. + \frac{8}{\alpha^2}e^{-\alpha t}\|u_0\|^2\|f\|^2 + \frac{16}{\alpha^4}\|f\|^4\right] + \frac{12}{\nu-\rho}\|g\|^2\\
&\qquad + \left[\frac{8\epsilon^2}{\rho}\|\nabla\delta\|^2 + \frac{4\epsilon^2}{\rho(\mu-\rho(\nu-\rho))}\|\delta\|^2 + 12\epsilon^2(\nu-\rho)\|\delta\|^2\right]\left\|z_2(\theta_t\omega)\right\|^2. \qquad (3.15)
\end{aligned}
$$

Let $C_1 := \min\{\alpha, \rho, (\nu - \rho)\} > 0$. Then, by (3.15), we have

$$\frac{d}{dt}E_1(\tilde{u}, \nu, \tilde{\psi}) + C_1 E_1(\tilde{u}, \nu, \tilde{\psi}) \le F\bigl(f, g, z_2(\theta_t\omega), \nabla z_2(\theta_t\omega)\bigr), \tag{3.16}$$

where

$$E_1(\tilde{u}, \nu, \tilde{\psi}) = \|\tilde{u}\|^2 + \bigl[\mu - \rho(\rho - \mu)\bigr]\|\nu\|^2 + \|\nabla \nu\|^2 + \|\tilde{\psi}\|^2,$$

and

$$\begin{aligned} &F\bigl(f, g, z_2(\theta_t\omega), \nabla z_2(\theta_t\omega)\bigr) \\ &\quad = \frac{4}{\alpha}\|f\|^2 + \frac{8\epsilon^2}{\rho}\|\delta\|^2\bigl\|\nabla z_2(\theta_t\omega)\bigr\|^2 + \frac{12}{\nu - \rho}\|g\|^2 \\ &\qquad + \frac{12\beta^2}{\nu - \rho}\left[e^{-2\alpha t}\|\tilde{u}_0\|^4 + \frac{8}{\alpha^2}e^{-\alpha t}\|u_0\|^2\|f\|^2 + \frac{16}{\alpha^4}\|f\|^4\right] \\ &\qquad + \left[\frac{8\epsilon^2}{\rho}\|\nabla\delta\|^2 + \frac{4\epsilon^2}{\rho(\mu - \rho(\nu - \rho))}\|\delta\|^2 + 12\epsilon^2(\nu - \rho)\|\delta\|^2\right]\bigl\|z_2(\theta_t\omega)\bigr\|^2. \end{aligned}$$

Note that $z_1(\theta_t\omega) = y_1(\theta_t\omega_1)$ and $z_2(\theta_t\omega) = \delta y_2(\theta_t\omega_2)$, $\delta \in \mathbb{H}^1(\mathbb{R}^n)$, therefore, by Lemma 3.1, F is bounded by

$$C\sum_{i=1}^{2}\bigl(\bigl|y_i(\theta_t\omega_i)\bigr|^2 + \bigl|y_i(\theta_t\omega_i)\bigr|^p\bigr) + C = P_1(\theta_t\omega) + C. \tag{3.17}$$

By Proposition 4.3.3 in [5], there exists a tempered function $r(\omega) > 0$ such that $r(\theta_t\omega) \le e^{\frac{C}{2}|t|}r(\omega)$. Therefore, by (3.17), we find that for $\mathbb{P}$-a.e. $\omega \in \Omega$,

$$P_1(\theta_s\omega) \le Ce^{\frac{C}{2}|s|}r(\omega), \quad \forall s \in \mathbb{R}.$$

By replacing ω by $\theta_{-t}\omega$, we get from (3.16) and (3.17)

$$\begin{aligned} &E_1\bigl(\tilde{u}\bigl(t, \theta_{-t}\omega, \tilde{u}_0(\theta_{-t}\omega)\bigr), \nu\bigl(t, \theta_{-t}\omega, \nu_0(\theta_{-t}\omega)\bigr), \tilde{\psi}\bigl(t, \theta_{-t}\omega, \tilde{\psi}_0(\theta_{-t}\omega)\bigr)\bigr) \\ &\quad \le e^{-C_1 t}E_1\bigl(\tilde{u}_0(\theta_{-t}\omega), \nu_0(\theta_{-t}\omega), \tilde{\psi}_0(\theta_{-t}\omega)\bigr) + \epsilon\int_0^t e^{C_1(s-t)}P_1(\theta_{s-t}\omega)\,ds + C_2 \\ &\quad \le e^{-C_1 t}E_1\bigl(\tilde{u}_0(\theta_{-t}\omega), \nu_0(\theta_{-t}\omega), \tilde{\psi}_0(\theta_{-t}\omega)\bigr) + \epsilon\int_{-t}^0 e^{C_1 s}P_1(\theta_s\omega)\,ds + C_2 \\ &\quad \le e^{-C_1 t}E_1\bigl(\tilde{u}_0(\theta_{-t}\omega), \nu_0(\theta_{-t}\omega), \tilde{\psi}_0(\theta_{-t}\omega)\bigr) + \epsilon C_1\int_{-t}^0 e^{\frac{C_1}{2}s}r(\omega)\,ds + C_2 \\ &\quad \le e^{-C_1 t}E_1\bigl(\tilde{u}_0(\theta_{-t}\omega), \nu_0(\theta_{-t}\omega), \tilde{\psi}_0(\theta_{-t}\omega)\bigr) + \epsilon C_1 r(\omega) + C_2. \end{aligned} \tag{3.18}$$

Note that

$$\begin{aligned} &\phi\bigl(t, \theta_{-t}\omega, (u_0, \nu_0, \psi_0)\bigr) \\ &\quad = \bigl(\tilde{u}\bigl(t, \theta_{-t}\omega, u_0 e^{-i\epsilon z_1(\omega)}\bigr)z^{-1}, \nu(t, \theta_{-t}\omega, \nu_0), \tilde{\psi}\bigl(t, \theta_{-t}\omega, \tilde{\psi}_0 + \epsilon\delta z_2(\omega)\bigr) + \epsilon\delta z_2(\theta_{-t}\omega)\bigr). \end{aligned}$$

Therefore, by (3.18) we know that there exists a positive constant C_3 such that, for all $t \geq 0$,

$$\begin{aligned} &\big\|\phi\big(t,\theta_{-t}\omega,(u_0,v_0,\psi_0)(\theta_{-t}\omega)\big)\big\|_{\mathbb{I}}^2 \\ &\quad = \big\|\tilde{u}\big(t,\theta_{-t}\omega,u_0(\theta_{-t}\omega)e^{-i\epsilon z_1(\theta_{-t}\omega)}\big)z^{-1}\big\|^2 \\ &\qquad + \big\|v\big(t,\theta_{-t}\omega,v_0(\theta_{-t}\omega)\big)\big\|_{\mathbb{H}^1}^2 + \big\|\tilde{\psi}\big(t,\theta_{-t}\omega,\psi_0(\theta_{-t}\omega)+\epsilon\delta z_2(\omega)\big)+\epsilon\delta z_2(\omega)\big\|^2 \\ &\quad \leq \big\|\tilde{u}\big(t,\theta_{-t}\omega,u_0(\theta_{-t}\omega)e^{-i\epsilon z_1(\theta_{-t}\omega)}\big)\big\|^2 + \big\|v\big(t,\theta_{-t}\omega,v_0(\theta_{-t}\omega)\big)\big\|_{\mathbb{H}^1}^2 \\ &\qquad + 2\big\|\tilde{\psi}\big(t,\theta_{-t}\omega,\psi_0(\theta_{-t}\omega)+\epsilon\delta z_2(\theta_{-t}\omega)\big)\big\|^2 + 2\epsilon^2\|\delta\|^2\big\|z_2(\omega)\big\|^2 \\ &\quad \leq C_3e^{-C_1t}E_1\big(u_0(\theta_{-t}\omega),v_0(\theta_{-t}\omega),\psi_0(\theta_{-t}\omega)\big) + C_3e^{-C_1t}\epsilon^2\|\delta\|^2\big\|z_2(\theta_{-t}\omega)\big\|^2 \\ &\qquad + 2\epsilon^2\|\delta\|^2\big\|z_2(\omega)\big\|^2 + \epsilon C_1r(\omega) + C_2. \end{aligned} \tag{3.19}$$

By definition of $z_2(\omega)$, it is easy to see that $\|z_2(\omega)\|^2$ is tempered. In addition, $\{\mathbb{B}(\omega)\}_{\omega\in\Omega} \subset \mathbb{D}$ is also tempered by assumption. Therefore, for $(u_0(\theta_{-t}\omega), v_0(\theta_{-t}\omega), \psi_0(\theta_{-t}\omega)) \in \mathbb{B}(\theta_{-t}\omega)$, there is $T_{\mathbb{B}}(\omega) > 0$ such that for all $t \geq T_{\mathbb{B}}(\omega)$,

$$e^{-C_1t}E_1\big(u_0(\theta_{-t}\omega),v_0(\theta_{-t}\omega),\psi_0(\theta_{-t}\omega)\big) + C_3e^{-C_1t}\epsilon^2\|\delta\|^2\big\|z_2(\theta_{-t}\omega)\big\|^2 \leq \epsilon C_4r(\omega) + C_4,$$

which along with (3.19) leads to

$$\big\|\phi\big(t,\theta_{-t}\omega,(u_0,v_0,\psi_0)(\theta_{-t}\omega)\big)\big\|_{\mathbb{I}}^2 \leq C_5 + \epsilon C_5r(\omega) + 2\epsilon^2\|\delta\|^2\big\|z_2(\omega)\big\|^2.$$

Given $\omega \in \Omega$, define

$$\mathbb{K}(\omega) = \big\{(u,v,\psi)\in\mathbb{I} : \|u\|^2 + \|v\|_{\mathbb{H}^1}^2 + \|\psi\|^2 \leq C + \epsilon Cr(\omega) + 2\epsilon^2\|\delta\|^2\big\|z_2(\omega)\big\|^2\big\}.$$

Then $\{\mathbb{K}(\omega)\}_{\omega\in\Omega}$ is an absorbing set for ϕ in $\mathbb{D}$, which completes the proof. □

Lemma 3.5 *Let $f,g \in \mathbb{H}^1$, $\delta \in \mathbb{H}^2$, $\mathbb{B} = \{\mathbb{B}(\omega)\}_{\omega\in\Omega} \in \mathbb{D}$ and $(u_0(\omega), v_0(\omega), \psi_0(\omega)) \in \mathbb{B}(\omega)$. Assume that $\alpha(\nu-\rho) > 12M_1\beta^2$, $\alpha(\mu-\rho(\nu-\rho)) > 8M_1$. Then, for $\mathbb{P}$-a.e. $\omega\in\Omega$, there exists $T'_{\mathbb{B}}(\omega) > 0$ such that for all $t \geq T'_{\mathbb{B}}$,*

$$\big\|u\big(t,\theta_{-t}\omega,u_0(\theta_{-t}\omega)\big)\big\|_{\mathbb{H}^1}^2 + \big\|v\big(t,\theta_{-t}\omega,v_0(\theta_{-t}\omega)\big)\big\|_{\mathbb{H}^2}^2 + \big\|\psi\big(t,\theta_{-t}\omega,\psi_0(\theta_{-t}\omega)\big)\big\|_{\mathbb{H}^1}^2 \leq R(\omega),$$

where $R(\omega)$ is a positive tempered random function.

Proof Taking the real part of the inner product of the first equation in system (3.6) with $-\triangle\tilde{u}$ in $\mathbb{L}^2(\mathbb{R}^n)$, we have that

$$\frac{d}{dt}\|\nabla\tilde{u}\|^2 + 2\alpha\|\nabla\tilde{u}\|^2 + 2(\nabla\tilde{u},\tilde{u}\nabla v) + 2\operatorname{Im}\big(\nabla\tilde{u},\nabla(fz)\big) = 0. \tag{3.20}$$

Taking the inner product of the third equation in system (3.6) with $-\triangle\tilde{\psi}$ in $\mathbb{L}^2(\mathbb{R}^n)$, we find

$$\begin{aligned} &\frac{d}{dt}\|\nabla\tilde{\psi}\|^2 + 2(\nu-\rho)\|\nabla\tilde{\psi}\|^2 + 2(\triangle\tilde{\psi},\triangle v) - 2\big[\mu-\rho(\nu-\rho)\big](\triangle\tilde{\psi},v) \\ &\quad + 2\beta\big(\triangle\tilde{\psi},\big|\tilde{u}z^{-1}\big|^2\big) - 2\epsilon(\nu-\rho)\big(\triangle\tilde{\psi},\delta z_2(\theta_{-t}\omega)\big) + 2(\triangle\tilde{\psi},g) = 0. \end{aligned} \tag{3.21}$$

By the second equation in system (3.6), we have

$$\begin{aligned}
&2(\triangle\tilde{\psi},\triangle v)-2\bigl[\mu-\rho(\nu-\rho)\bigr](\triangle\tilde{\psi},v)\\
&\quad=2\bigl(\triangle v_t+\rho\triangle v-\epsilon\triangle\bigl(z_2(\theta_t\omega)\delta\bigr),\triangle v\bigr)\\
&\qquad-2\bigl[\mu-\rho(\nu-\rho)\bigr]\bigl(\triangle v_t+\rho\triangle v-\epsilon\triangle\bigl(z_2(\theta_t\omega)\delta\bigr),v\bigr)\\
&\quad=\frac{d}{dt}\bigl(\|\triangle v\|^2+\bigl(\mu-\rho(\nu-\rho)\bigr)\|\nabla v\|^2\bigr)+2\bigl(\rho\|\triangle v\|^2+\bigl(\mu-\rho(\nu-\rho)\bigr)\|\nabla v\|^2\bigr)\\
&\qquad-2\epsilon\bigl(z_2(\theta_t\omega)\triangle\delta,\triangle v\bigr)-2\epsilon\bigl(\delta\triangle z_2(\theta_t\omega),\triangle v\bigr)\\
&\qquad-2\epsilon\bigl[\mu-\rho(\nu-\rho)\bigr]\bigl(z_2(\theta_t\omega)\nabla\delta,\nabla v\bigr)\\
&\qquad-2\epsilon\bigl[\mu-\rho(\nu-\rho)\bigr]\bigl(\delta\nabla z_2(\theta_t\omega),\nabla v\bigr).
\end{aligned}\tag{3.22}$$

Then it follows from (3.20)-(3.22) that

$$\begin{aligned}
&\frac{d}{dt}\bigl(\|\nabla\tilde{u}\|^2+\|\nabla\tilde{\psi}\|^2+\|\triangle v\|^2+\bigl(\mu-\rho(\nu-\rho)\bigr)\|\nabla v\|^2\bigr)\\
&\qquad+2\alpha\|\nabla\tilde{u}\|^2+2(\nu-\rho)\|\nabla\tilde{\psi}\|^2+2\rho\|\triangle v\|^2+2\bigl(\mu-\rho(\nu-\rho)\bigr)\|\nabla v\|^2\\
&\quad=-2\operatorname{Im}\bigl(\nabla\tilde{u},\nabla(fz)\bigr)-2(\nabla\tilde{u},\tilde{u}\nabla v)-2\beta\bigl(\triangle\tilde{\psi},\bigl|\tilde{u}z^{-1}\bigr|^2\bigr)\\
&\qquad+2\epsilon(\nu-\rho)\bigl(\triangle\tilde{\psi},z_2(\theta_t\omega)\delta\bigr)-2(\triangle\tilde{\psi},g)+2\epsilon\bigl(z_2(\theta_t\omega)\triangle\delta,\triangle v\bigr)\\
&\qquad+2\epsilon\bigl(\delta\triangle z_2(\theta_t\omega),\triangle v\bigr)+2\epsilon\bigl[\mu-\rho(\nu-\rho)\bigr]\bigl(z_2(\theta_t\omega)\nabla\delta,\nabla v\bigr)\\
&\qquad+2\epsilon\bigl[\mu-\rho(\nu-\rho)\bigr]\bigl(\delta\nabla z_2(\theta_t\omega),\nabla v\bigr).
\end{aligned}\tag{3.23}$$

Now, we estimate each term on the right-hand side of (3.23). By Young's inequality, we get

$$\begin{aligned}
&\bigl|-2\operatorname{Im}\bigl(\nabla\tilde{u},\nabla(fz)\bigr)\bigr|=2\bigl|\operatorname{Im}(\nabla\tilde{u},z\nabla f)+\operatorname{Im}(\nabla\tilde{u},f\nabla z)\bigr|\\
&\qquad\qquad\leq\frac{\alpha}{2}\|\nabla\tilde{u}\|^2+\frac{8}{\alpha}\bigl(\|\nabla f\|^2+\epsilon^2\|f\|^2\|\nabla z_1\|^2\bigr),\\
&\bigl|-2(\nabla\tilde{u},\tilde{u}\nabla v)\bigr|\leq\frac{\alpha}{2}\|\nabla\tilde{u}\|^2+\frac{8}{\alpha}\|\tilde{u}\|^2\|\nabla v\|^2,\\
&\bigl|-2\beta\bigl(\triangle\tilde{\psi},\bigl|\tilde{u}z^{-1}\bigr|^2\bigr)\bigr|\leq\frac{\nu-\rho}{3}\|\nabla\tilde{\psi}\|^2+\frac{12\beta^2}{\nu-\rho}\|\tilde{u}\|^2\|\nabla\tilde{u}\|^2,\\
&\bigl|2\epsilon(\nu-\rho)\bigl(\triangle\tilde{\psi},z_2(\theta_t\omega)\delta\bigr)\bigr|\\
&\quad\leq\frac{\nu-\rho}{3}\|\nabla\tilde{\psi}\|^2+12\epsilon^2(\nu-\rho)\bigl(\|\delta\|^2\bigl\|\nabla z_2(\theta_t\omega)\bigr\|^2+\bigl\|z_2(\theta_t\omega)\bigr\|^2\|\nabla\delta\|^2\bigr),\\
&\bigl|-2(\triangle\tilde{\psi},g)\bigr|\leq\frac{\nu-\rho}{3}\|\nabla\tilde{\psi}\|^2+\frac{12}{\nu-\rho}\|\nabla g\|^2,\\
&\bigl|2\epsilon\bigl(z_2(\theta_t\omega)\triangle\delta,\triangle v\bigr)\bigr|\leq\frac{\rho}{2}\|\triangle v\|^2+\frac{8\epsilon^2}{\rho}\|\triangle\delta\|^2\bigl\|z_2(\theta_t\omega)\bigr\|^2,\\
&\bigl|2\epsilon\bigl(\delta\triangle z_2(\theta_t\omega),\triangle v\bigr)\bigr|\leq\frac{\rho}{2}\|\triangle v\|^2+\frac{8\epsilon^2}{\rho}\|\delta\|^2\bigl\|\triangle z_2(\theta_t\omega)\bigr\|^2,\\
&\bigl|2\epsilon\bigl[\mu-\rho(\nu-\rho)\bigr]\bigl(z_2(\theta_t\omega)\nabla\delta,\nabla v\bigr)\bigr|\\
&\quad\leq\frac{\mu-\rho(\nu-\rho)}{2}\|\nabla v\|^2+8\epsilon^2\bigl[\mu-\rho(\nu-\rho)\bigr]\bigl\|z_2(\theta_t\omega)\bigr\|^2\|\nabla\delta\|^2,
\end{aligned}$$

$$\begin{aligned}&\big|2\epsilon\big[\mu-\rho(\nu-\rho)\big]\big(\delta\nabla z_2(\theta_t\omega),\nabla v\big)\big|\\&\quad\le\frac{\mu-\rho(\nu-\rho)}{2}\|\nabla v\|^2+8\epsilon^2\big[\mu-\rho(\nu-\rho)\big]\big\|\nabla z_2(\theta_t\omega)\big\|^2\|\delta\|^2,\end{aligned}$$

which along with (3.23), Remark 3.1 and assumption gives that there exists a positive constant

$$C_6=\min\left\{\alpha-\frac{12M_1\beta^2}{\nu-\rho},\nu-\rho,\rho,\mu-\rho(\nu-\rho)-\frac{8M_1}{\alpha}\right\}>0$$

such that for $t>T_1$

$$\frac{d}{dt}E_2(\tilde{u},v,\tilde{\psi})+C_6E_2(\tilde{u},v,\tilde{\psi})\le F\big(z_1(\theta_t\omega),z_2(\theta_t\omega)\big),\tag{3.24}$$

where

$$E_2(\tilde{u},v,\tilde{\psi})=\|\nabla\tilde{u}\|^2+\|\nabla\tilde{\psi}\|^2+\|\triangle v\|^2+\big(\mu-\rho(\nu-\rho)\big)\|\nabla v\|^2,$$

and

$$\begin{aligned}&F\big(z_1(\theta_t\omega),z_2(\theta_t\omega)\big)\\&\quad=\frac{8}{\alpha}\big(\|\nabla f\|^2+\epsilon^2\|f\|^2\big\|\nabla z_1(\theta_t\omega)\big\|^2\big)\\&\qquad+24\epsilon^2(\nu-\rho)\big(\|\delta\|^2\big\|\nabla z_2(\theta_t\omega)\big\|^2+\big\|z_2(\theta_t\omega)\big\|^2\|\nabla\delta\|^2\big)+\frac{12}{\nu-\rho}\|\nabla g\|^2\\&\qquad+\frac{12\epsilon^2}{\rho}\big(\|\triangle\delta\|^2\big\|z_2(\theta_t\omega)\big\|^2+\|\delta\|^2\big\|\triangle z_2(\theta_t\omega)\big\|^2\big)\\&\qquad+8\epsilon^2\big[\mu-\rho(\nu-\rho)\big]\big(\big\|z_2(\theta_t\omega)\big\|^2\|\nabla\delta\|^2+\big\|\nabla z_2(\theta_t\omega)\big\|^2\|\delta\|^2\big).\end{aligned}$$

Let

$$P_2(\theta_t\omega)=C\big(\|\nabla z_1\|^4+\|z_2\|^2+\|\nabla z_2\|^4+\|\triangle z_2\|^4\big).$$

Since $z_1(\theta_t\omega)=y_1(\theta_t\omega_1)$ and $z_2(\theta_t\omega)=\delta y_2(\theta_t\omega_2)$, $\delta\in\mathbb{H}^2(\mathbb{R}^n)$. Therefore, by Lemma 3.1, we have

$$P_2(\theta_t\omega)\le C\sum_{i=1}^{2}\big(\big|y_i(\theta_t\omega_i)\big|^2+\big|y_i(\theta_t\omega_i)\big|_p^p\big)+C,$$

and

$$P_2(\theta_t\omega)\le Ce^{\frac{C}{2}|t|}r(\omega)+C\quad\text{for all }t\in\mathbb{R}.$$

By replacing ω by $\theta_{-t}\omega$, it follows from Gronwall's lemma that

$$\begin{aligned}&E_2\big(\tilde{u}\big(t,\theta_{-t}\omega,\tilde{u}_0(\theta_{-t}\omega)\big),v\big(t,\theta_{-t}\omega,v_0(\theta_{-t}\omega)\big),\tilde{\psi}\big(t,\theta_{-t}\omega,\tilde{\psi}_0(\theta_{-t}\omega)\big)\big)\\&\quad\le e^{-C_6t}E_2\big(\tilde{u}_0(\theta_{-t}\omega),v_0(\theta_{-t}\omega),\tilde{\psi}_0(\theta_{-t}\omega)\big)+\epsilon\int_0^t e^{C_6(s-t)}P_2(\theta_{s-t}\omega)\,ds+C_7\end{aligned}$$

$$
\begin{aligned}
&\leq e^{-C_6 t} E_2\big(\tilde{u}_0(\theta_{-t}\omega), v_0(\theta_{-t}\omega), \tilde{\psi}_0(\theta_{-t}\omega)\big) + \epsilon \int_{-t}^{0} e^{C_6 s} P_2(\theta_s\omega)\, ds + C_7 \\
&\leq e^{-C_6 t} E_2\big(\tilde{u}_0(\theta_{-t}\omega), v_0(\theta_{-t}\omega), \tilde{\psi}_0(\theta_{-t}\omega)\big) + \epsilon C_6 \int_{-t}^{0} e^{\frac{C_6}{2} s} r(\omega)\, ds + C_7 \\
&\leq e^{-C_6 t} E_2\big(\tilde{u}_0(\theta_{-t}\omega), v_0(\theta_{-t}\omega), \tilde{\psi}_0(\theta_{-t}\omega)\big) + \epsilon C_6 r(\omega) + C_7,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&\big\|\tilde{u}\big(t, \theta_{-t}\omega, \tilde{u}_0(\theta_{-t}\omega)\big)\big\|_{\mathbb{H}^1}^2 + \big\|v\big(t, \theta_{-t}\omega, v_0(\theta_{-t}\omega)\big)\big\|_{\mathbb{H}^2}^2 + \big\|\tilde{\psi}\big(t, \theta_{-t}\omega, \tilde{\psi}_0(\theta_{-t}\omega)\big)\big\|_{\mathbb{H}^1}^2 \\
&\quad\leq e^{-C_6 t} E_2\big(\tilde{u}_0(\theta_{-t}\omega), v_0(\theta_{-t}\omega), \tilde{\psi}_0(\theta_{-t}\omega)\big) + \epsilon C_6 r(\omega) + C_7,
\end{aligned}
\tag{3.25}
$$

where $r(\omega)$ is a tempered function.

Note that $\tilde{u} = uz$, $\tilde{\psi} = \psi - \epsilon\delta z_2(\theta_t\omega)$, by (3.25) we obtain

$$
\begin{aligned}
&\big\|u\big(t, \theta_{-t}\omega, u_0(\theta_{-t}\omega)\big)\big\|_{\mathbb{H}^1}^2 + \big\|v\big(t, \theta_{-t}\omega, v_0(\theta_{-t}\omega)\big)\big\|_{\mathbb{H}^2}^2 + \big\|\psi\big(t, \theta_{-t}\omega, \psi_0(\theta_{-t}\omega)\big)\big\|_{\mathbb{H}^1}^2 \\
&\quad= \big\|\tilde{u}\big(t, \theta_{-t}\omega, \tilde{u}_0(\theta_{-t}\omega)e^{-i\epsilon z_1(\theta_{-t}\omega)}\big)e^{-i\epsilon z_1(\omega)}\big\|_{\mathbb{H}^1}^2 + \big\|v\big(t, \theta_{-t}\omega, v_0(\theta_{-t}\omega)\big)\big\|_{\mathbb{H}^2}^2 \\
&\qquad+ \big\|\tilde{\psi}\big(t, \theta_{-t}\omega, \tilde{\psi}_0(\theta_{-t}\omega) + \epsilon\delta z_2(\theta_{-t}\omega)\big) + \epsilon\delta z_2(\omega)\big\|_{\mathbb{H}^1}^2 \\
&\quad\leq \big\|\tilde{u}\big(t, \theta_{-t}\omega, \tilde{u}_0(\theta_{-t}\omega)e^{-i\epsilon z_1(\theta_{-t}\omega)}\big)\big\|_{\mathbb{H}^1}^2 + \big\|\nabla v\big(t, \theta_{-t}\omega, v_0(\theta_{-t}\omega)\big)\big\|_{\mathbb{H}^2}^2 \\
&\qquad+ \big\|\tilde{\psi}\big(t, \theta_{-t}\omega, \tilde{\psi}_0(\theta_{-t}\omega) + \epsilon\delta z_2(\theta_{-t}\omega)\big)\big\|_{\mathbb{H}^1}^2 + 2\epsilon^2\|\delta\|_{\mathbb{H}^1}^2\big\|z_2(\omega)\big\|_{\mathbb{H}^1}^2 \\
&\quad\leq e^{-C_6 t} E_2\big(\tilde{u}_0(\theta_{-t}\omega), v_0(\theta_{-t}\omega), \tilde{\psi}_0(\theta_{-t}\omega)\big) + 2\epsilon^2\|\delta\|_{\mathbb{H}^1}^2\big\|z_2(\theta_{-t}\omega)\big\|_{\mathbb{H}^1}^2 \\
&\qquad+ \epsilon C_6 r(\omega) + C_7 + 2\epsilon^2\|\delta\|_{\mathbb{H}^1}^2\big\|z_2(\omega)\big\|_{\mathbb{H}^1}^2.
\end{aligned}
$$

Since $\|\nabla z_2(\omega)\|^2$ is tempered, there exists $T'_{\mathbb{B}}(\omega) > T_1 > 0$, for all $t \geq T'_{\mathbb{B}}$,

$$
\big\|u\big(t, \theta_{-t}\omega, u_0(\theta_{-t}\omega)\big)\big\|_{\mathbb{H}^1}^2 + \big\|v\big(t, \theta_{-t}\omega, v_0(\theta_{-t}\omega)\big)\big\|_{\mathbb{H}^2}^2 + \big\|\psi\big(t, \theta_{-t}\omega, \psi_0(\theta_{-t}\omega)\big)\big\|_{\mathbb{H}^1}^2 \leq R(\omega),
$$

where $R(\omega) = \epsilon C_6 r(\omega) + C_7 + 2\epsilon^2\|\delta\|_{\mathbb{H}^1}^2\|z_2(\omega)\|_{\mathbb{H}^1}^2$. This completes the proof. □

In what follows, we will use the method of [11, 16] to derive uniform estimates on the tails of solution when x exists on unbounded domains.

Lemma 3.6 *Let $f \in \mathbb{L}^4$, $g \in \mathbb{L}^2$, $\delta \in \mathbb{H}^1 \cap \mathbb{W}^{1,4}$, $\mathbb{B} = \{B(\omega)\}_{\omega\in\Omega} \in \mathbb{D}$ and $(u_0, v_0, \psi_0) \in \mathbb{B}(\omega)$. Assume that $\mu - \rho(\nu - \rho) > 0$. Then, for every $\varepsilon > 0$ and P-a.e. $\omega \in \Omega$, there exist $T' = T'(\omega, \varepsilon) < 0$ and $m = m(\omega, \varepsilon) > 0$ such that the solution of system* (3.1) *satisfies, for all $t \geq T'(\omega, \varepsilon)$,*

$$
\int_{|x|\geq m} \big|\phi\big(t, \theta_{-t}\omega, (u_0, v_0, \phi_0)(\theta_{-t}\omega)\big)\big|^2\, dx \leq \varepsilon.
$$

Proof Let $\eta(x) \in \mathbb{C}(\mathbb{R}^+, [0,1])$ be a cut-off function satisfying

$$
\eta(x) = 0, \quad \text{for all } x \in [0,1]; \qquad \eta(x) = 1, \quad \text{for all } x \in [2, +\infty),
$$

and $|\eta'(x)| \leq \eta_0$ (a positive constant).

Taking the imaginary part of the inner product of the first equation of (3.6) with $\eta(\frac{|x|^2}{m^2})\tilde{u}$ in $\mathbb{L}^2(\mathbb{R}^n)$, we have

$$\frac{d}{dt}\int_{\mathbb{R}^n}\eta\left(\frac{|x|^2}{m^2}\right)|\tilde{u}|^2\,dx+2\alpha\int_{\mathbb{R}^n}\eta\left(\frac{|x|^2}{m^2}\right)|\tilde{u}|^2\,dx=2\operatorname{Im}\int_{\mathbb{R}^n}\eta\left(\frac{|x|^2}{m^2}\right)fz\tilde{u}\,dx. \tag{3.26}$$

Taking the inner product of the third equation of (3.6) with $\eta(\frac{|x|^2}{m^2})\tilde{\psi}$ in $\mathbb{L}^2(\mathbb{R}^n)$, we get

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^n}\eta\left(\frac{|x|^2}{m^2}\right)|\tilde{\psi}|^2\,dx+2(\nu-\rho)\int_{\mathbb{R}^n}\eta\left(\frac{|x|^2}{m^2}\right)|\tilde{\psi}|^2\,dx\\
&\quad=-2\big[\mu-\rho(\nu-\rho)\big]\int_{\mathbb{R}^n}\eta\left(\frac{|x|^2}{m^2}\right)v\tilde{\psi}\,dx+2\int_{\mathbb{R}^n}\eta\left(\frac{|x|^2}{m^2}\right)(\triangle v)\tilde{\psi}\,dx\\
&\qquad+2\beta\int_{\mathbb{R}^n}\eta\left(\frac{|x|^2}{m^2}\right)\big|\tilde{u}z^{-1}\big|^2\tilde{\psi}\,dx-2\epsilon(\nu-\rho)\int_{\mathbb{R}^n}\eta\left(\frac{|x|^2}{m^2}\right)\delta z_2(\theta_t\omega)\tilde{\psi}\,dx\\
&\qquad+2\int_{\mathbb{R}^n}\eta\left(\frac{|x|^2}{m^2}\right)g\tilde{\psi}\,dx.
\end{aligned}\tag{3.27}$$

Note that

$$\begin{aligned}
&-2\big[\mu-\rho(\nu-\rho)\big]\int_{\mathbb{R}^n}\eta\left(\frac{|x|^2}{m^2}\right)v\tilde{\psi}\,dx\\
&\quad=-\big[\mu-\rho(\nu-\rho)\big]\frac{d}{dt}\int_{\mathbb{R}^n}\eta\left(\frac{|x|^2}{m^2}\right)|v|^2\,dx-2\rho\big[\mu-\rho(\nu-\rho)\big]\int_{\mathbb{R}^n}\eta\left(\frac{|x|^2}{m^2}\right)|v|^2\,dx\\
&\qquad+2\epsilon\big[\mu-\rho(\nu-\rho)\big]\int_{\mathbb{R}^n}\eta\left(\frac{|x|^2}{m^2}\right)\delta z_2(\theta_t\omega)v\,dx,
\end{aligned}\tag{3.28}$$

and

$$\begin{aligned}
&2\int_{\mathbb{R}^n}\eta\left(\frac{|x|^2}{m^2}\right)(\triangle v)\tilde{\psi}\,dx\\
&\quad=-2\int_{\mathbb{R}^n}\eta'\left(\frac{|x|^2}{m^2}\right)\left(\frac{2x}{m^2}\right)(\nabla v)\tilde{\psi}\,dx-\frac{d}{dt}\int_{\mathbb{R}^n}\eta\left(\frac{|x|^2}{m^2}\right)|\nabla v|^2\,dx\\
&\qquad-2\rho\int_{\mathbb{R}^n}\eta\left(\frac{|x|^2}{m^2}\right)|\nabla v|^2\,dx-2\rho\int_{\mathbb{R}^n}\eta'\left(\frac{|x|^2}{m^2}\right)\left(\frac{2x}{m^2}\right)v\nabla v\,dx\\
&\qquad+2\epsilon\int_{\mathbb{R}^n}\eta\left(\frac{|x|^2}{m^2}\right)z_2(\theta_t\omega)\nabla v\nabla\delta\,dx+2\epsilon\int_{\mathbb{R}^n}\eta\left(\frac{|x|^2}{m^2}\right)\delta\nabla v\nabla z_2(\theta_t\omega)\,dx\\
&\qquad+2\epsilon\int_{\mathbb{R}^n}\eta'\left(\frac{|x|^2}{m^2}\right)\left(\frac{2x}{m^2}\right)\delta z_2(\theta_{-t}\omega)\nabla v.
\end{aligned}\tag{3.29}$$

Summing up (3.26)-(3.29), we have

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^n}\eta\left(\frac{|x|^2}{m^2}\right)\left(|\tilde{u}|^2+\big(\mu-\rho(\nu-\rho)\big)|v|^2+|\nabla v|^2+|\tilde{\psi}|^2\right)dx\\
&\qquad+2\alpha\int_{\mathbb{R}^n}\eta\left(\frac{|x|^2}{m^2}\right)|\tilde{u}|^2\,dx+2\rho\big(\mu-\rho(\nu-\rho)\big)\int_{\mathbb{R}^n}\eta\left(\frac{|x|^2}{m^2}\right)|v|^2\,dx\\
&\qquad+2\rho\int_{\mathbb{R}^n}\eta\left(\frac{|x|^2}{m^2}\right)|\nabla v|^2\,dx+2(\nu-\rho)\int_{\mathbb{R}^n}\eta\left(\frac{|x|^2}{m^2}\right)|\tilde{\psi}|^2\,dx
\end{aligned}$$

$$
\begin{aligned}
&= 2\,\mathrm{Im}\int_{\mathbb{R}^n}\eta\left(\frac{|x|^2}{m^2}\right)fz\tilde{u}\,dx + 2\epsilon\big(\mu-\rho(\nu-\rho)\big)\int_{\mathbb{R}^n}\eta\left(\frac{|x|^2}{m^2}\right)\delta\nu z_2(\theta_t\omega)\,dx \\
&\quad - 2\int_{\mathbb{R}^n}\eta'\left(\frac{|x|^2}{m^2}\right)\left(\frac{2x}{m^2}\right)(\nabla\nu)\tilde{\psi}\,dx + 2\epsilon\int_{\mathbb{R}^n}\eta\left(\frac{|x|^2}{m^2}\right)z_2(\theta_t\omega)\nabla\nu\nabla\delta\,dx \\
&\quad + 2\epsilon\int_{\mathbb{R}^n}\eta\left(\frac{|x|^2}{m^2}\right)\delta\nabla\nu\nabla z_2(\theta_t\omega)\,dx - 2\rho\int_{\mathbb{R}^n}\eta'\left(\frac{|x|^2}{m^2}\right)\left(\frac{2x}{m^2}\right)\nu\nabla\nu\,dx \\
&\quad + 2\epsilon\int_{\mathbb{R}^n}\eta'\left(\frac{|x|^2}{m^2}\right)\left(\frac{2x}{m^2}\right)\delta z_2(\theta_{-t}\omega)\nabla\nu\,dx + 2\beta\int_{\mathbb{R}^n}\eta\left(\frac{|x|^2}{m^2}\right)\big|\tilde{u}z^{-1}\big|^2\tilde{\psi}\,dx \\
&\quad + 2\int_{\mathbb{R}^n}\eta\left(\frac{|x|^2}{m^2}\right)g\tilde{\psi}\,dx - 2\epsilon(\nu-\rho)\int_{\mathbb{R}^n}\eta\left(\frac{|x|^2}{m^2}\right)\delta z_2(\theta_t\omega)\tilde{\psi}\,dx.
\end{aligned}
\tag{3.30}
$$

We now estimate the right-hand side term in (3.30) as follows. Firstly, by the definition of η, we have

$$
\begin{aligned}
-2\int_{\mathbb{R}^n}\eta'\left(\frac{|x|^2}{m^2}\right)\left(\frac{2x}{m^2}\right)(\nabla\nu)\tilde{\psi}\,dx &\le 4\eta_0\int_{m\le x\le\sqrt{2}m}\left(\frac{|x|}{m^2}\right)|\nabla\nu||\tilde{\psi}|\,dx \\
&\le \frac{4\sqrt{2}\eta_0}{m}\big(\|\nabla\nu\|^2+\|\tilde{\psi}\|^2\big),
\end{aligned}
\tag{3.31}
$$

$$
\begin{aligned}
-2\rho\int_{\mathbb{R}^n}\eta'\left(\frac{|x|^2}{m^2}\right)\left(\frac{2x}{m^2}\right)\nu\nabla\nu\,dx &\le 4\rho\eta_0\int_{m\le x\le\sqrt{2}m}\left(\frac{|x|}{m^2}\right)|\nu||\nabla\nu|\,dx \\
&\le \frac{4\rho\sqrt{2}\eta_0}{m}\big(\|\nabla\nu\|^2+\|\nu\|^2\big),
\end{aligned}
\tag{3.32}
$$

$$
\begin{aligned}
&2\epsilon\int_{\mathbb{R}^n}\eta'\left(\frac{|x|^2}{m^2}\right)\left(\frac{2x}{m^2}\right)\delta z_2(\theta_t\omega)\nabla\nu\,dx \\
&\quad\le 4\epsilon\eta_0\int_{m\le x\le\sqrt{2}m}\left(\frac{|x|}{m^2}\right)|\delta|\big|z_2(\theta_t\omega)\big||\nabla\nu|\,dx \\
&\quad\le \frac{4\epsilon\sqrt{2}\eta_0}{m}\big(\|\nabla\nu\|^2+\|\delta\|^2\big\|z_2(\theta_{-t}\omega)\big\|^2\big).
\end{aligned}
\tag{3.33}
$$

Then, by Young's inequality, we get

$$
2\,\mathrm{Im}\int_{\mathbb{R}^n}\eta\left(\frac{|x|^2}{m^2}\right)fz\tilde{u}\,dx \le \alpha\int_{\mathbb{R}^n}\eta\left(\frac{|x|^2}{m^2}\right)|\tilde{u}|^2\,dx + \frac{4}{\alpha}\int_{\mathbb{R}^n}\eta\left(\frac{|x|^2}{m^2}\right)|f|^2\,dx;
\tag{3.34}
$$

$$
\begin{aligned}
&2\epsilon\big(\mu-\rho(\nu-\rho)\big)\int_{\mathbb{R}^n}\eta\left(\frac{|x|^2}{m^2}\right)\delta\nu z_2(\theta_t\omega)\,dx \\
&\quad\le \rho\big(\mu-\rho(\nu-\rho)\big)\int_{\mathbb{R}^n}\eta\left(\frac{|x|^2}{m^2}\right)|\nu|^2\,dx \\
&\qquad + \frac{4\epsilon^2}{\rho(\mu-\rho(\nu-\rho))}\int_{\mathbb{R}^n}\eta\left(\frac{|x|^2}{m^2}\right)|\delta|^2\big|z_2(\theta_t\omega)\big|^2\,dx;
\end{aligned}
\tag{3.35}
$$

$$
\begin{aligned}
&2\epsilon\int_{\mathbb{R}^n}\eta\left(\frac{|x|^2}{m^2}\right)z_2(\theta_t\omega)\nabla\nu\nabla\delta\,dx \\
&\quad\le \frac{\rho}{2}\int_{\mathbb{R}^n}\eta\left(\frac{|x|^2}{m^2}\right)|\nabla\nu|^2\,dx + \frac{8\epsilon^2}{\rho}\int_{\mathbb{R}^n}\eta\left(\frac{|x|^2}{m^2}\right)\big|z_2(\theta_t\omega)\big|^2|\nabla\delta|^2\,dx;
\end{aligned}
\tag{3.36}
$$

$$\begin{aligned}
&2\epsilon \int_{\mathbb{R}^n} \eta\left(\frac{|x|^2}{m^2}\right)\delta \nabla z_2(\theta_t\omega)\nabla v\, dx \\
&\quad\leq \frac{\rho}{2}\int_{\mathbb{R}^n} \eta\left(\frac{|x|^2}{m^2}\right)|\nabla v|^2\, dx + \frac{8\epsilon^2}{\rho}\int_{\mathbb{R}^n} \eta\left(\frac{|x|^2}{m^2}\right)\left|\nabla z_2(\theta_t\omega)\right|^2|\delta|^2\, dx; \qquad (3.37)\\
&2\beta \int_{\mathbb{R}^n} \eta\left(\frac{|x|^2}{m^2}\right)\left|\tilde{u}z^{-1}\right|^2\tilde{\psi}\, dx \\
&\quad\leq \frac{\nu-\rho}{2}\int_{\mathbb{R}^n} \eta\left(\frac{|x|^2}{m^2}\right)|\tilde{\psi}|^2\, dx + \frac{8\beta^2}{\nu-\rho}\int_{\mathbb{R}^n} \eta\left(\frac{|x|^2}{m^2}\right)|\tilde{u}|^4\, dx; \qquad (3.38)\\
&-2\epsilon(\nu-\rho) \int_{\mathbb{R}^n} \eta\left(\frac{|x|^2}{m^2}\right)\delta z_2(\theta_t\omega)\tilde{\psi}\, dx \\
&\quad\leq \frac{\nu-\rho}{2}\int_{\mathbb{R}^n} \eta\left(\frac{|x|^2}{m^2}\right)|\tilde{\psi}|^2\, dx + 8\epsilon^2(\nu-\rho)\int_{\mathbb{R}^n} \eta\left(\frac{|x|^2}{m^2}\right)\left|z_2(\theta_t\omega)\right|^2|\delta|^2\, dx; \qquad (3.39)\\
&2 \int_{\mathbb{R}^n} \eta\left(\frac{|x|^2}{m^2}\right)g\tilde{\psi}\, dx \\
&\quad\leq \frac{\nu-\rho}{2}\int_{\mathbb{R}^n} \eta\left(\frac{|x|^2}{m^2}\right)|\tilde{\psi}|^2\, dx + \frac{8}{\nu-\rho}\int_{\mathbb{R}^n} \eta\left(\frac{|x|^2}{m^2}\right)|g|^2\, dx. \qquad (3.40)
\end{aligned}$$

Finally, by (3.30)-(3.40), we obtain

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^n} \eta\left(\frac{|x|^2}{m^2}\right)\left(|\tilde{u}|^2 + \left(\mu-\rho(\nu-\rho)\right)|v|^2 + |\nabla v|^2 + |\tilde{\psi}|^2\right) dx \\
&\qquad + \alpha\int_{\mathbb{R}^n} \eta\left(\frac{|x|^2}{m^2}\right)|\tilde{u}|^2\, dx + \rho\left(\mu-\rho(\nu-\rho)\right)\int_{\mathbb{R}^n} \eta\left(\frac{|x|^2}{m^2}\right)|v|^2\, dx \\
&\qquad + \rho\int_{\mathbb{R}^n} \eta\left(\frac{|x|^2}{m^2}\right)|\nabla v|^2\, dx + \frac{\nu-\rho}{2}\int_{\mathbb{R}^n} \eta\left(\frac{|x|^2}{m^2}\right)|\tilde{\psi}|^2\, dx \\
&\quad\leq \frac{4}{\alpha}\int_{\mathbb{R}^n} \eta\left(\frac{|x|^2}{m^2}\right)|f|^2\, dx + \frac{4\epsilon^2}{\rho(\mu-\rho(\nu-\rho))}\int_{\mathbb{R}^n} \eta\left(\frac{|x|^2}{m^2}\right)|\delta|^2\left|z_2(\theta_t\omega)\right|^2\, dx \\
&\qquad + \frac{8\epsilon^2}{\rho}\int_{\mathbb{R}^n} \eta\left(\frac{|x|^2}{m^2}\right)\left(\left|z_2(\theta_t\omega)\right|^2|\nabla\delta|^2 + \left|\nabla z_2(\theta_t\omega)\right|^2|\delta|^2\right) dx \\
&\qquad + \frac{8\beta^2}{\nu-\rho}\int_{\mathbb{R}^n} \eta\left(\frac{|x|^2}{m^2}\right)|\tilde{u}|^4\, dx + 8\epsilon^2(\nu-\rho)\int_{\mathbb{R}^n} \eta\left(\frac{|x|^2}{m^2}\right)\left|z_2(\theta_t\omega)\right|^2|\delta|^2\, dx \\
&\qquad + \frac{8}{\nu-\rho}\int_{\mathbb{R}^n} \eta\left(\frac{|x|^2}{m^2}\right)|g|^2\, dx + \frac{4\sqrt{2}\eta_0}{m}\left(\|\nabla v\|^2 + \|\tilde{\psi}\|^2\right) \\
&\qquad + \frac{4\rho\sqrt{2}\eta_0}{m}\left(\|\nabla v\|^2 + \|v\|^2\right) + \frac{4\epsilon\sqrt{2}\eta_0}{m}\left(\|\nabla v\|^2 + \|\delta\|^2\left\|z_2(\theta_{-t}\omega)\right\|^2\right). \qquad (3.41)
\end{aligned}$$

By (3.41) and assumption, there exist positive constants $C_8 := \min\{\alpha, \rho, \frac{\nu-\rho}{2}\}$ and $C_9 := C_9(\alpha, \beta, \mu, \nu, \rho)$ such that

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^n} \eta\left(\frac{|x|^2}{m^2}\right)\left(|\tilde{u}|^2 + \left(\mu-\rho(\nu-\rho)\right)|v|^2 + |\nabla v|^2 + |\tilde{\psi}|^2\right) dx \\
&\qquad + C_8\int_{\mathbb{R}^n} \eta\left(\frac{|x|^2}{m^2}\right)\left(|\tilde{u}|^2 + \rho\left(\mu-\rho(\nu-\rho)\right)|v|^2 + |\nabla v|^2 + |\tilde{\psi}|^2\right) dx \\
&\quad\leq \epsilon C_9\int_{\mathbb{R}^n} \eta\left(\frac{|x|^2}{m^2}\right)\left(\left|z_2(\theta_t\omega)\right|^4 + \left|\nabla z_2(\theta_t\omega)\right|^4\right) dx
\end{aligned}$$

$$
\begin{aligned}
&+ C_9 \int_{\mathbb{R}^n} \eta\left(\frac{|x|^2}{m^2}\right)\left(|f|^4 + |g|^2 + |\delta|^4 + |\nabla\delta|^4\right) dx + C_9 \int_{\mathbb{R}^n} \eta\left(\frac{|x|^2}{m^2}\right)|\tilde{u}|^4\, dx \\
&+ \frac{4\sqrt{2}\eta_0}{m}\left(\|\nabla v\|^2 + \|\tilde{\psi}\|^2\right) + \frac{4\epsilon\sqrt{2}\eta_0}{m}\left(\|\nabla v\|^2 + \|\delta\|^2 \left\|z_2(\theta_{-t}\omega)\right\|^2\right).
\end{aligned} \tag{3.42}
$$

Now, replacing ω by $\theta_{-t}\omega$, and then integrating (3.42) over (T_2, t) with $t \geq T_2$, we have

$$
\begin{aligned}
&\int_{\mathbb{R}^n} \eta\left(\frac{|x|^2}{m^2}\right)\Big(\left|\tilde{u}\big(t,\theta_{-t}\omega,\tilde{u}_0(\theta_{-t}\omega)\big)\right|^2 + \big(\mu - \rho(\nu-\rho)\big)\left|v\big(t,\theta_{-t}\omega,v_0(\theta_{-t}\omega)\big)\right|^2 \\
&\quad + \left|\nabla v\big(t,\theta_{-t}\omega,v_0(\theta_{-t}\omega)\big)\right|^2 + \left|\tilde{\psi}\big(t,\theta_{-t}\omega,\tilde{\psi}_0(\theta_{-t}\omega)\big)\right|^2\Big)\, dx \\
&\leq e^{-C_8(t-T_2)} \int_{\mathbb{R}^n} \eta\left(\frac{|x|^2}{m^2}\right)\Big(\left|\tilde{u}\big(T_2,\theta_{-t}\omega,\tilde{u}_0(\theta_{-t}\omega)\big)\right|^2 + \left|\nabla v\big(T_2,\theta_{-t}\omega,v_0(\theta_{-t}\omega)\big)\right|^2 \\
&\quad + \rho\big(\mu - \rho(\nu-\rho)\big)\left|v\big(T_2,\theta_{-t}\omega,v_0(\theta_{-t}\omega)\big)\right|^2 + \left|\tilde{\psi}\big(T_2,\theta_{-t}\omega,\tilde{\psi}_0(\theta_{-t}\omega)\big)\right|^2\Big)\, dx \\
&\quad + \epsilon C_9 \int_{T_2}^t e^{C_8(s-t)} \int_{\mathbb{R}^n} \eta\left(\frac{|x|^2}{m^2}\right)\left(\left|z_2(\theta_{s-t}\omega)\right|^4 + \left|\nabla z_2(\theta_{s-t}\omega)\right|^4\right) dx\, ds \\
&\quad + C_9 \int_{T_2}^t e^{C_8(s-t)} \int_{\mathbb{R}^n} \eta\left(\frac{|x|^2}{m^2}\right)\left(|f|^4 + |g|^2 + |\delta|^4 + |\nabla\delta|^4 + |\tilde{u}|^4\right) dx\, ds \\
&\quad + \frac{C^*}{m}\int_{T_2}^t e^{C_8(s-t)}\Big(\left\|v\big(s,\theta_{-t}\omega,v_0(\theta_{-t}\omega)\big)\right\|^2 + \left\|\nabla v\big(s,\theta_{-t}\omega,v_0(\theta_{-t}\omega)\big)\right\|^2 \\
&\quad + \left\|\tilde{\psi}\big(s,\theta_{-t}\omega,\tilde{\psi}_0(\theta_{-t}\omega)\big)\right\|^2\Big)\, ds + \frac{4\epsilon\sqrt{2}\eta_0\|\delta\|^2}{m}\int_{T_2}^t e^{C_8(s-t)}\left\|z_2(\theta_{-t}\omega)\right\|^2\, ds,
\end{aligned} \tag{3.43}
$$

where C^* is a fixed constant.

In what follows, we estimate the terms in (3.43). First replacing t by T_2 and then replacing ω by $\theta_{-t}\omega$ in (3.18), we have the following bounds for the first term on the right-hand side of (3.43):

$$
\begin{aligned}
&e^{-C_8(t-T_2)} \int_{\mathbb{R}^n} \eta\left(\frac{|x|^2}{m^2}\right)\Big(\left|\tilde{u}\big(T_2,\theta_{-t}\omega,\tilde{u}_0(\theta_{-t}\omega)\big)\right|^2 + \left|\nabla v\big(T_2,\theta_{-t}\omega,v_0(\theta_{-t}\omega)\big)\right|^2 \\
&\quad + \rho\big(\mu - \rho(\nu-\rho)\big)\left|v\big(T_2,\theta_{-t}\omega,v_0(\theta_{-t}\omega)\big)\right|^2 + \left|\tilde{\psi}\big(T_2,\theta_{-t}\omega,\tilde{\psi}_0(\theta_{-t}\omega)\big)\right|^2\Big)\, dx \\
&\leq e^{-C_8(t-T_2)}\Big(e^{-C_8 T_2}\left\|\tilde{u}_0(\theta_{-t}\omega)\right\|^2 + \rho\big(\mu - \rho(\nu-\rho)\big)\left\|v_0(\theta_{-t}\omega)\right\|^2 \\
&\quad + \left\|\tilde{\psi}_0(\theta_{-t}\omega)\right\|^2\Big) + e^{-C_8(t-T_2)}\int_0^{T_2} e^{C_8(s-T_2)}\big(P_1(\theta_{s-t}\omega) + C_9\big)\, ds \\
&\leq e^{-C_8 t}\Big(\left\|\tilde{u}_0(\theta_{-t}\omega)\right\|^2 + \rho\big(\mu - \rho(\nu-\rho)\big)\left\|v_0(\theta_{-t}\omega)\right\|^2 + \left\|\tilde{\psi}_0(\theta_{-t}\omega)\right\|^2\Big) \\
&\quad + \int_{-t}^{T_2-t} e^{C_8 s}P_1(\theta_s\omega)\, ds + C_9 e^{C_8(T_2-t)} \\
&\leq e^{-C_8 t}\Big(\left\|\tilde{u}_0(\theta_{-t}\omega)\right\|^2 + \rho\big(\mu - \rho(\nu-\rho)\big)\left\|v_0(\theta_{-t}\omega)\right\|^2 + \left\|\tilde{\psi}_0(\theta_{-t}\omega)\right\|^2\Big) \\
&\quad + C_9\int_{-t}^{T_2-t} e^{\frac{C_8}{2}s} r(\omega)\, ds + C_9 e^{C_8(T_2-t)} \\
&\leq e^{-C_8 t}\Big(\left\|\tilde{u}_0(\theta_{-t}\omega)\right\|^2 + \rho\big(\mu - \rho(\nu-\rho)\big)\left\|v_0(\theta_{-t}\omega)\right\|^2 + \left\|\tilde{\psi}_0(\theta_{-t}\omega)\right\|^2\Big) \\
&\quad + \frac{2C_9}{C_8} e^{\frac{C_8}{2}(T_2-t)} r(\omega) + C_9 e^{C_8(T_2-t)}.
\end{aligned} \tag{3.44}
$$

By (3.44) we find that, given $\varepsilon > 0$, there is $T_3 = T_3(\mathbb{B}, \omega, \varepsilon) > T_2$ such that for all $t > T_3$,

$$\begin{aligned} & e^{-C(t-T_2)} \int_{\mathbb{R}^n} \eta\left(\frac{|x|^2}{m^2}\right)\left(\left|\tilde{u}\left(T_2, \theta_{-t}\omega, \tilde{u}_0(\theta_{-t}\omega)\right)\right|^2 + \left|\nabla v\left(T_2, \theta_{-t}\omega, v_0(\theta_{-t}\omega)\right)\right|^2 \right. \\ & \qquad \left. + \rho\big(\mu - \rho(\nu - \rho)\big)\left|v\left(T_2, \theta_{-t}\omega, v_0(\theta_{-t}\omega)\right)\right|^2 + \left|\tilde{\psi}\left(T_2, \theta_{-t}\omega, \tilde{\psi}_0(\theta_{-t}\omega)\right)\right|^2\right) dx \\ & \quad \le \varepsilon. \end{aligned} \tag{3.45}$$

Note that $\delta \in \mathbb{H}^1 \cap \mathbb{W}^{1,4}$, hence there is $R_1 = R_1(\omega, \varepsilon)$ such that for all $m \ge R_1$,

$$\int_{|x| \ge m} \left(\left|\nabla \delta(x)\right|^4 + \left|\delta(x)\right|^4\right) dx \le \frac{\varepsilon}{r(\omega)}, \tag{3.46}$$

where $r(\omega)$ is a tempered function. By (3.46), we have the following estimate:

$$\begin{aligned} & C_9 \int_{T_2}^t e^{C_8(s-t)} \int_{\mathbb{R}^n} \eta\left(\frac{|x|^2}{m^2}\right)\left(\left|z_2(\theta_{s-t}\omega)\right|^4 + \left|\nabla z_2(\theta_{s-t}\omega)\right|^4\right) dx\, ds \\ & \quad \le C_9 \int_{T_2}^t e^{C_8(s-t)} \int_{|x| \ge m} \left(\left|z_2(\theta_{s-t}\omega)\right|^4 + \left|\nabla z_2(\theta_{s-t}\omega)\right|^4\right) dx\, ds \\ & \quad \le C_9 \int_{T_2}^t e^{C_8(s-t)} \int_{|x| \ge m} \left(|\delta|^4 \left|y_2(\theta_{s-t}\omega_2)\right|^4 + |\nabla\delta|^4 \left|\nabla y_2(\theta_{s-t}\omega_2)\right|^4\right) dx\, ds \\ & \quad \le \frac{C_9\varepsilon}{r(\omega)} \int_{T_2}^t e^{C_8(s-t)} \sum_{i=1}^{2}\left(\left|y_i(\theta_{s-t}\omega_i)\right|^2 + \left|y_i(\theta_{s-t}\omega_i)\right|^4\right) ds \\ & \quad \le \frac{C_9\varepsilon}{r(\omega)} \int_{T_2}^t e^{C_8(s-t)} r(\theta_{s-t}\omega)\, ds \le \frac{C_9\varepsilon}{r(\omega)} \int_{T_2-t}^0 e^{C_8 s} r(\theta_s\omega)\, ds \\ & \quad \le \frac{C_9\varepsilon}{r(\omega)} \int_0^{T_2-t} e^{\frac{C_8}{2}s} r(\omega)\, ds \le \varepsilon. \end{aligned} \tag{3.47}$$

By Lemma 3.3, we have

$$\begin{aligned} & C_9 \int_{T_2}^t e^{C_8(s-t)} \int_{\mathbb{R}^n} \eta\left(\frac{|x|^2}{m^2}\right) |\tilde{u}|^4\, dx\, ds \\ & \quad \le C_9 \int_{T_2}^t e^{C_8(s-t)} e^{-\alpha s} \int_{|x| \ge m} |\tilde{u}_0|^4\, dx\, ds + \frac{64 C_9}{\alpha^4} \int_{T_2}^t e^{C_8(s-t)} \int_{|x| \ge m} |f|^4\, dx\, ds \\ & \quad \le \frac{C_9}{C_9 + \alpha} e^{-\alpha t} \int_{|x| \ge m} |\tilde{u}_0|^4\, dx\, ds + \frac{64 C_9}{\alpha^4} \int_{T_2}^t e^{C_8(s-t)} \int_{|x| \ge m} |f|^4\, dx\, ds. \end{aligned} \tag{3.48}$$

Note that $f \in \mathbb{L}^4$, $g \in \mathbb{L}^2$ and $\delta \in \mathbb{H}^1 \cap \mathbb{W}^{1,4}$, there is $R_2 = R_2(\varepsilon)$ such that for all $m \ge R_2$,

$$\int_{|x| \ge m} \left(|f|^4 + |g|^2 + |\delta|^4 + |\nabla\delta|^2\right) dx \le \varepsilon. \tag{3.49}$$

Then, by (3.48) and (3.49), there is $T_4 = T_4(\mathbb{B}, \omega, \varepsilon) > 0$ such that for all $t > T_4$,

$$\begin{aligned} & C_9 \int_{T_1}^t e^{C_8(s-t)} \int_{\mathbb{R}^n} \eta\left(\frac{|x|^2}{m^2}\right)\left(|f|^4 + |g|^2 + |\delta|^2 + |\nabla\delta|^4 + |\tilde{u}|^4\right) dx\, ds \\ & \quad \le C_9 \int_{T_1}^t e^{C_8(s-t)} \int_{|x| \ge m} \left(|f|^4 + |g|^2 + |\delta|^2 + |\nabla\delta|^4\right) dx\, ds \end{aligned}$$

$$+ C_9 e^{-\alpha t} \int_{|x|\geq m} |\tilde{u}_0|^4 \, dx \, ds$$
$$\leq C_9 \varepsilon \int_{T_1}^t e^{C_8(s-t)} \, ds + C_9 e^{-\alpha t} \int_{|x|\geq m} |\tilde{u}_0|^4 \, dx \, ds \leq \varepsilon. \tag{3.50}$$

Now, we estimate the last term on the right-hand side of (3.43). Denote $E_3(v,\tilde{\psi}) = \|v\|^2 + \|\nabla v\|^2 + \|\tilde{\psi}\|^2$. Replacing t by s and then replacing ω by $\theta_{-t}\omega$ in (3.18), we have the following estimate:

$$\begin{aligned}
&\frac{C^*}{m} \int_{T_2}^t e^{C_8(s-t)} E_3\bigl(v\bigl(s,\theta_{-t}\omega, v_0(\theta_{-t}\omega)\bigr), \tilde{\psi}\bigl(s,\theta_{-t}\omega, \tilde{\psi}_0(\theta_{-t}\omega)\bigr)\bigr) \, ds \\
&\quad\leq \frac{C^*}{m} \int_{T_2}^t e^{-C_8 t} E_1\bigl(\tilde{u}_0(\theta_{-t}\omega), v_0(\theta_{-t}\omega), \tilde{\psi}_0(\theta_{-t}\omega)\bigr) \, ds \\
&\qquad + \frac{C^*}{m} \int_{T_2}^t e^{C_8(s-t)} \int_0^s e^{C_8(\tau-s)} P_1(\theta_{\tau-t}\omega) \, d\tau \, ds + \frac{C^*}{m} \int_{T_2}^t e^{C_8(s-t)} \, ds \\
&\quad\leq \frac{C^*}{m} e^{-C_8 t}(t-T_2) E_1\bigl(\tilde{u}_0(\theta_{-t}\omega), v_0(\theta_{-t}\omega), \tilde{\psi}_0(\theta_{-t}\omega)\bigr) \\
&\qquad + \frac{C^*}{m} \int_{T_2}^t \int_0^s e^{C_8(\tau-t)} P_1(\theta_{\tau-t}\omega) \, d\tau \, ds + \frac{C^*}{m} \\
&\quad\leq \frac{C^*}{m} e^{-C_8 t}(t-T_2) E_1\bigl(\tilde{u}_0(\theta_{-t}\omega), v_0(\theta_{-t}\omega), \tilde{\psi}_0(\theta_{-t}\omega)\bigr) \\
&\qquad + \frac{C^*}{m} \int_{T_2}^t \int_{-t}^{s-t} e^{C_8 \tau} P_1(\theta_\tau\omega) \, d\tau \, ds + \frac{C^*}{m} \\
&\quad\leq \frac{C^*}{m} e^{-C_8 t}(t-T_2) E_1\bigl(\tilde{u}_0(\theta_{-t}\omega), v_0(\theta_{-t}\omega), \tilde{\psi}_0(\theta_{-t}\omega)\bigr) \\
&\qquad + \frac{C^*}{m} r(\omega) \int_{T_2}^t \int_{-t}^{s-t} e^{\frac{C_8}{2}\tau} \, d\tau \, ds + \frac{C^*}{m} \\
&\quad\leq \frac{C^*}{m} e^{-C_8 t}(t-T_2) E_1\bigl(\tilde{u}_0(\theta_{-t}\omega), v_0(\theta_{-t}\omega), \tilde{\psi}_0(\theta_{-t}\omega)\bigr) + \frac{C^*}{m} r(\omega) + \frac{C^*}{m},
\end{aligned} \tag{3.51}$$

which implies that there exist $T_5 = T_5(\mathbb{B},\omega,\varepsilon) > T_2$ and $R_3 = R_3(\omega,\varepsilon)$ such that for all $t \geq T_5$ and $m \geq R_3$,

$$\frac{C^*}{m} \int_{T_2}^t e^{C_8(s-t)} E_3\bigl(v\bigl(s,\theta_{-t}\omega, v_0(\theta_{-t}\omega)\bigr), \tilde{\psi}\bigl(s,\theta_{-t}\omega, \tilde{\psi}_0(\theta_{-t}\omega)\bigr)\bigr) \, ds \leq \varepsilon. \tag{3.52}$$

Obviously, there exists $R_4 = R_4(\omega,\epsilon)$ such that for all $m > R_4$, we have

$$\frac{4\epsilon\sqrt{2}\eta_0\|\delta\|^2}{m} \int_{T_2}^t e^{C_8(s-t)} \bigl\|z_2(\theta_{-t}\omega)\bigr\|^2 \, ds \leq \varepsilon. \tag{3.53}$$

By (3.43), (3.45), (3.47), (3.50), (3.52) and (3.53), there exist $R = \max\{R_1,R_2,R_3,R_4\}$ and $T = \min\{T_2,T_3,T_4,T_5\}$ such that for all $m \geq R$ and $t \geq T$,

$$\begin{aligned}
&\int_{\mathbb{R}^n} \eta\biggl(\frac{|x|^2}{m^2}\biggr) \bigl(\bigl|\tilde{u}\bigl(t,\theta_{-t}\omega, \tilde{u}_0(\theta_{-t}\omega)\bigr)\bigr|^2 + \bigl(\mu - \rho(\nu-\rho)\bigr)\bigl|v\bigl(t,\theta_{-t}\omega, v_0(\theta_{-t}\omega)\bigr)\bigr|^2 \\
&\quad + \bigl|\tilde{\psi}\bigl(t,\theta_{-t}\omega, \tilde{\psi}_0(\theta_{-t}\omega)\bigr)\bigr|^2\bigr) \leq 5\varepsilon,
\end{aligned}$$

which shows that for all $m \geq R$ and $t \geq T$,

$$\int_{|x|\geq\sqrt{2}m} \big(\big|\tilde{u}\big(t,\theta_{-t}\omega,\tilde{u}_0(\theta_{-t}\omega)\big)\big|^2 + \big|v\big(t,\theta_{-t}\omega,v_0(\theta_{-t}\omega)\big)\big|^2 + \big|\tilde{\psi}\big(t,\theta_{-t}\omega,\tilde{\psi}_0(\theta_{-t}\omega)\big)\big|^2\big) \leq 5\varepsilon.$$

Note that

$$\int_{|x|\geq\sqrt{2}m} \big|z_2(\omega)\big|^2\,dx = \int_{|x|\geq\sqrt{2}m} |\delta|^2\big|y_2(\omega_2)\big|^2\,dx \leq \frac{\varepsilon}{r(\omega)}\big|y_2(\omega_2)\big|^2 \leq \varepsilon.$$

Therefore

$$\begin{aligned}
&\int_{|x|\geq\sqrt{2}m} \big|\phi\big(t,\theta_{-t}\omega,(u_0,v_0,\phi_0)(\theta_{-t}\omega)\big)\big|^2\,dx \\
&\quad= \int_{|x|\geq\sqrt{2}m} \big(\big|u\big(t,\theta_{-t}\omega,u_0(\theta_{-t}\omega)\big)\big|^2 + \big|v\big(t,\theta_{-t}\omega,v_0(\theta_{-t}\omega)\big)\big|^2 \\
&\qquad+ \big|\psi\big(t,\theta_{-t}\omega,\psi_0(\theta_{-t}\omega)\big)\big|^2\big)\,dx \\
&\quad= \int_{|x|\geq\sqrt{2}m} \big(\big|\tilde{u}\big(t,\theta_{-t}\omega,\tilde{u}_0(\theta_{-t}\omega)\big)e^{i\epsilon z_1(\omega)}\big|^2 + \big|v\big(t,\theta_{-t}\omega,v_0(\theta_{-t}\omega)\big)\big|^2 \\
&\qquad+ \big|\tilde{\psi}\big(t,\theta_{-t}\omega,\tilde{\psi}_0(\theta_{-t}\omega)\big) + \epsilon\delta z_2(\omega)\big|^2\big) \\
&\quad\leq 2\int_{|x|\geq\sqrt{2}m} \big(\big|\tilde{u}\big(t,\theta_{-t}\omega,\tilde{u}_0(\theta_{-t}\omega)\big)\big|^2 + \big|v\big(t,\theta_{-t}\omega,v_0(\theta_{-t}\omega)\big)\big|^2 \\
&\qquad+ \big|\tilde{\psi}\big(t,\theta_{-t}\omega,\tilde{\psi}_0(\theta_{-t}\omega)\big)\big|^2 + 2\epsilon^2|\delta|^2\big|z_2(\omega)\big|^2\big) \\
&\quad\leq C'\varepsilon,
\end{aligned}$$

where C' is a fixed positive constant. This completes the proof. □

Theorem 3.7 *Let $f \in \mathbb{H}^2$, $g \in \mathbb{L}^2$ and $\delta \in \mathbb{H} \cap \mathbb{W}^{2,4}$. Assume that $\alpha(\nu - \rho) > 12M_1\beta^2$, $\alpha(\mu - \rho(\nu - \rho)) > 8M_1$ and $\mu - \rho(\nu - \rho) > 0$. Then the random dynamical system ϕ_ϵ possesses a unique $\mathbb{D}$-random attractor in $\mathbb{I}$.*

Proof From Proposition 2.8, we only need to prove the asymptotic compactness of RDS ϕ. By Lemma 3.4, for any sequence $t_n \to \infty$ and $x_n(\theta_{-t_n}\omega) \in \mathbb{K}(\omega)$, we have

$$\big\{\phi(t_n,\theta_{-t_n}\omega,x_n)\big\}_{n=1}^{\infty} \text{ is bounded in } \mathbb{I}. \tag{3.54}$$

Hence, by (3.54), there exists $\phi_0 \in L^2(\mathbb{R}^n)$ such that

$$\big\{\phi\big(t_n,\theta_{-t_n}\omega,x_n(\theta_{-t_n}\omega)\big)\big\}_{n=1}^{\infty} \to \phi_0 \text{ weakly in } \mathbb{I}. \tag{3.55}$$

Next, we prove that (3.55) is actually strong convergence. Define the set S by $S = \{x \in \mathbb{R}^n : |x| \leq m\}$, where m will be specified latter. Notice the compactness of embedding $H^1(S), H^2(S) \hookrightarrow L^2(S)$, by Lemma 3.5 it follows that

$$\phi\big(t_n,\theta_{-t_n}\omega,x_n(\theta_{-t_n}\omega)\big) \to \phi_0 \text{ strongly in } \mathbb{I}(S),$$

which shows that for any given $\varepsilon > 0$, there exists $N_1 = N_1(\mathbb{K}(\omega), \omega, \varepsilon)$ such that for all $n \geq N_1$,

$$\left\| \phi\left(t_n, \theta_{-t_n}\omega, x_n(\theta_{-t_n}\omega)\right) - \phi_0 \right\|^2_{\mathbb{I}(S)} \leq \varepsilon. \tag{3.56}$$

On the other hand, by Lemma 3.6, there exist $T_1 = T_1(\mathbb{K}(\omega), \omega, \varepsilon)$ and $N_2 = N_2(\omega, \varepsilon)$ (large enough) such that $t_n \geq T_1$ for every $n \geq N_2$, we have

$$\int_{|x|>m_1} \left| \phi\left(t_n, \theta_{-t_n}\omega, x_n(\theta_{-t_n}\omega)\right)(x) \right|^2 dx \leq \varepsilon. \tag{3.57}$$

Since $\delta \in \mathbb{H} \cap \mathbb{W}^{2,4}$, there exists $m_2 = m_2(\varepsilon)$ such that

$$\int_{|x|>m_2} \left| \phi_0(x) \right|^2 dx \leq \varepsilon. \tag{3.58}$$

Let $m = \max\{m_1, m_2\}$ and $N = \max\{N_1, N_2\}$. By (3.56)-(3.58) we find that for all $n \geq N$,

$$\begin{aligned} &\left\| \phi\left(t_n, \theta_{-t_n}\omega, x_n(\theta_{-t_n}\omega)\right)(x) - \phi_0(x) \right\|^2_{\mathbb{I}} \\ &\quad \leq \int_{|x|\leq m} \left| \phi\left(t_n, \theta_{-t_n}\omega, x_n(\theta_{-t_n}\omega)\right)(x) - \phi_0(x) \right|^2 dx \\ &\qquad + \int_{|x|>m} \left| \phi\left(t_n, \theta_{-t_n}\omega, x_n(\theta_{-t_n}\omega)\right)(x) - \phi_0(x) \right|^2 dx \\ &\quad \leq 3\varepsilon. \end{aligned}$$

This completes the proof. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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