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# Nontrivial homoclinic solutions for prescribed mean curvature Rayleigh equations

Meilin Zheng<sup>1</sup> and Jin Li<sup>2\*</sup>

\*Correspondence:  
lijin7912@gmail.com  
<sup>2</sup>School of Science, Jiujiang  
University, Jiujiang, 332005, China  
Full list of author information is  
available at the end of the article

## Abstract

In this paper, some sufficient conditions on the existence of  $2k\pi$ -periodic solutions for a kind of prescribed mean curvature Rayleigh equations are given. Then the existence of nontrivial homoclinic solutions for prescribed mean curvature Rayleigh equations is obtained.

**Keywords:** prescribed mean curvature Rayleigh equation; homoclinic solution; Mawhin's continuation theorem

## 1 Introduction

In this paper, we are concerned with the existence of homoclinic solutions for the following prescribed mean curvature Rayleigh equation:

$$\left( \frac{x'(t)}{\sqrt{1+x'^2(t)}} \right)' + f(x'(t)) + g(x(t)) = e(t), \quad (1.1)$$

where  $f, g, e \in C(\mathbb{R}, \mathbb{R})$  with  $f(0) = 0$  and  $g(0) = 0$ .

A solution  $x(t)$  of (1.1) is named homoclinic (to 0) if  $x(t) \rightarrow 0$  and  $x'(t) \rightarrow 0$  as  $|t| \rightarrow \infty$ . Furthermore,  $x(t)$  is called a nontrivial homoclinic solution (1.1) if  $x(t)$  is a homoclinic solution of (1.1) and  $x \neq 0$ .

Various types of prescribed mean equations have been studied widely by some authors in many papers (see [1–8]) because of their having appeared in some scientific fields, such as differential geometry and physics. Recently, some results on the existence of solutions for prescribed mean equations were obtained (see [9–12] and references cited therein). Feng in [10] discussed a delay prescribed mean curvature Liénard equation of the form

$$\left( \frac{x'(t)}{\sqrt{1+x'^2(t)}} \right)' + f(x(t))x'(t) + g(t, x(t - \tau(t))) = e(t), \quad (1.2)$$

estimated *a priori* bounds by eliminating the nonlinear term  $\left( \frac{x'(t)}{\sqrt{1+x'^2(t)}} \right)'$  and established sufficient conditions on the existence of periodic solutions for (1.2) by using Mawhin's continuation theorem. The main difficulty overcome by Feng in [10] lies in the nonlinear term  $\left( \frac{x'(t)}{\sqrt{1+x'^2(t)}} \right)'$ , the existence of which obstructs the usual method of finding *a priori*

bounds. He transformed (1.2) into the following equivalent system:

$$\begin{cases} x_1'(t) = \psi(x_2(t)) = \frac{x_2(t)}{\sqrt{1-x_2^2(t)}}, \\ x_2'(t) = -f(x_1(t))\psi(x_2(t)) - g(t, x_1(t)) + e(t), \end{cases} \tag{1.3}$$

and used Mawhin’s continuation theorem to prove the existence result. From then on, similar approaches were used by Li and Wang [13] and Li *et al.* [14]. Clearly,  $|x_2| < c < 1$  ( $c$  is a constant) is necessarily satisfied when Mawhin’s continuation theorem is applied to the system (1.3). But, as pointed out by Liang and Lu in [15], the proof of  $|x_2| < c < 1$  was not done in [10, 13, 14]. To do it, Li and Wang in [16] gave a complementary proof.

In [15], Liang and Lu investigated the following prescribed mean curvature Duffing equation:

$$\left( \frac{x'(t)}{\sqrt{1+x^2(t)}} \right)' + cx'(t) + f(x(t)) = p(t), \tag{1.4}$$

where  $f \in C^1(\mathbb{R}, \mathbb{R})$ ,  $c > 0$ . Assume

(A<sub>1</sub>) there exist constants  $m_0 > 0$ ,  $\alpha > 1$  such that  $xf(x) \leq -m_0|x|^\alpha$  and  $f'(x) < 0$ ,  $\forall x \in \mathbb{R}$ ,

and

(A<sub>2</sub>)  $p \in C(\mathbb{R}, \mathbb{R})$  is a bounded function with  $p(t) \neq 0$  and

$$B := \max \left\{ \left( \int_{\mathbb{R}} |p(t)|^2 dt \right)^{\frac{1}{2}}, \left( \int_{\mathbb{R}} |p(t)|^\beta dt \right)^{\frac{1}{\beta}} \right\} + \sup_{t \in \mathbb{R}} |p(t)| < +\infty,$$

where  $\frac{1}{\alpha} + \frac{1}{\beta} = 1$ .

If (A<sub>1</sub>), (A<sub>2</sub>), and condition

$$B^{\frac{\alpha}{2\alpha-2}} + TBm_0^{\frac{1}{2\alpha-2}} < \sqrt{2T}m_0^{\frac{1}{2\alpha-2}} \tag{1.5}$$

hold, they obtained the existence of homoclinic solutions for (1.4). For the method of obtaining homoclinic solution, we also see the papers [17–20]. It is not difficult to see that the condition (1.5) is complex and strong. In this paper, we will consider more general equation (1.1) and obtain the existence of homoclinic solutions under the more simple and reasonable conditions. To find a homoclinic solution for (1.1), we seek a limit of a certain sequence of  $2kT$ -periodic solutions  $x_k(t)$  for the following equations:

$$\left( \frac{x'(t)}{\sqrt{1+x^2(t)}} \right)' + f(x'(t)) + g(x(t)) = e_k(t), \tag{1.6}$$

where  $k \in \mathbb{N}$ ,  $e_k : \mathbb{R} \rightarrow \mathbb{R}$  is a  $2kT$ -periodic function ( $T > 0$  is a constant) defined by

$$e_k(t) = \begin{cases} e(t), & t \in [-kT, kT - \varepsilon_0), \\ e(kT - \varepsilon_0) + \frac{e(-kT) - e(kT - \varepsilon_0)}{\varepsilon_0}(t - kT + \varepsilon_0), & t \in [kT - \varepsilon_0, kT], \end{cases} \tag{1.7}$$

$\varepsilon_0 \in (0, T)$  is a constant independent of  $k$ .

## 2 Preliminaries

Let  $X$  and  $Y$  be real Banach spaces and  $L : X \supset \text{Dom } L \rightarrow Y$  be a linear operator.  $L$  is said to be a Fredholm operator with index zero provided that

- (i)  $\text{Im } L$  is closed subset of  $Y$ ,
- (ii)  $\dim \ker L = \text{codim } \text{Im } L < +\infty$ .

Set  $X = \ker L \oplus X_1, Y = \text{Im } L \oplus Y_1$ . Let  $P : X \rightarrow \ker L$  and  $Q : Y \rightarrow Y_1$  be the nature projections. It is easy to see that  $\ker L \cap (\text{Dom } L \cap X_1) = 0$ . Thus the restriction  $L_P := L|_{\text{Dom } L \cap X_1}$  is invertible. We denote by  $K$  the inverse of  $L_P$ .

Let  $\Omega$  be an open bounded subset of  $X$  with  $\text{Dom } L \cap \Omega \neq \emptyset$ . A map  $N : \overline{\Omega} \rightarrow Y$  is said to be  $L$ -compact in  $\overline{\Omega}$  if  $QN : \overline{\Omega} \rightarrow Y$  and  $K(I - Q)N : \overline{\Omega} \rightarrow X$  are compact.

The following lemma due to Mawhin (see [21]) is a fundamental tool to prove the existence of  $2kT$ -periodic solutions for (1.6).

**Lemma 2.1** *Let  $L$  be a Fredholm operator of index zero and Let  $N$  be  $L$ -compact on  $\overline{\Omega}$ . If the following conditions hold.*

- (h<sub>1</sub>)  $Lx \neq \lambda Nx, \forall (x, \lambda) \in [(D(L) \setminus \ker L) \cap \partial\Omega] \times (0, 1)$ ;
- (h<sub>2</sub>)  $Nx \notin \text{Im } L, \forall x \in \ker L \cap \partial\Omega$ ;
- (h<sub>3</sub>)  $\deg(JQN|_{\ker L}, \Omega \cap \ker L, 0) \neq 0$ , where  $J : \text{Im } Q \rightarrow \ker L$  is an isomorphism.

Then  $Lx = Nx$  has at least one solution in  $D(L) \cap \overline{\Omega}$ .

The following lemma is a special case of Lemma 2.1 in [22].

**Lemma 2.2** *If  $x : \mathbb{R} \rightarrow \mathbb{R}$  is continuously differentiable on  $\mathbb{R}, a > 0$ , then the following inequality holds:*

$$|x(t)| \leq (2a)^{-\frac{1}{2}} \left( \int_{t-a}^{t+a} |x(s)|^2 ds \right)^{\frac{1}{2}} + a(2a)^{-\frac{1}{p}} \left( \int_{t-a}^{t+a} |x'(s)|^2 ds \right)^{\frac{1}{2}}. \tag{2.1}$$

**Lemma 2.3** [22] *Let  $x_k \in C_{2kT}^2$  be  $2kT$ -periodic function for each  $k \in \mathbb{N}$  with*

$$\|x_k\|_\infty \leq A_0, \quad \|x'_k\|_\infty \leq A_1, \quad \|x''_k\|_\infty \leq A_2,$$

where  $A_0, A_1,$  and  $A_2$  are constants independent of  $k \in \mathbb{N}$ . Then there exists a function  $x_0 \in C^1(\mathbb{R}, \mathbb{R})$  such that for each interval  $[c, d] \subset \mathbb{R}$ , there is a subsequence  $\{x_{k_j}\}$  of  $\{x_k\}_{k \in \mathbb{N}}$  with  $x'_{k_j}(t) \rightarrow x'_0(t)$  uniformly on  $[c, d]$ .

In order to apply Mawhin’s continuation theorem to study the existence of  $2kT$ -periodic solution of (1.6), we rewrite (1.6) as

$$\begin{cases} x'(t) = \frac{y(t)}{\sqrt{1-y^2(t)}}, \\ y'(t) = -f\left(\frac{y(t)}{\sqrt{1-y^2(t)}}\right) - g(x(t)) + e_k(t). \end{cases} \tag{2.2}$$

Obviously, if  $z(t) = (x(t), y(t))^T$  is a  $2kT$ -periodic solution of (2.2), then  $x(t)$  must be a  $2kT$ -periodic solution of (1.6). Hence, the problem of finding a  $2kT$ -periodic solution of (1.6) reduces to finding one of (2.2).

Now, we set

$$X_k = Y_k = \{z : z(t) = (x(t), y(t))^T \in C(\mathbb{R}^1, \mathbb{R}^2), z(t) = z(t + 2kT)\},$$

with the norm  $\|z\| = \max\{\|x\|_\infty, \|y\|_\infty\}$ , where

$$\|x\|_\infty = \max_{t \in [0, 2kT]} |x(t)|, \quad \|y\|_\infty = \max_{t \in [0, 2kT]} |y(t)|.$$

Clearly,  $X_k$  and  $Y_k$  are Banach spaces. Meanwhile, let

$$L : X_k \supset \text{Dom } L \rightarrow Y_k, \quad Lz = z' = (x'(t), y'(t))^T,$$

where

$$\text{Dom } L = \{z : z = (x(t), y(t))^T \in C^1(\mathbb{R}, \mathbb{R}^2), z(t) = z(t + 2kT)\}.$$

Define a nonlinear operator  $N : X_k \rightarrow Y_k$  by

$$Nz = \begin{pmatrix} \frac{y(t)}{\sqrt{1-y^2(t)}} \\ -f\left(\frac{y(t)}{\sqrt{1-y^2(t)}}\right) - g(x(t)) + e(t) \end{pmatrix}.$$

Then the system (2.2) can be written to  $Lz = Nz$ .

It is easy to see that  $\ker L = \mathbb{R}^2$  and  $\text{Im } L = \{u \in Y_k : \int_{-kT}^{kT} u(s) ds = 0\}$ . So  $L$  is a Fredholm operator with index zero.

Let  $P : X_k \rightarrow \ker L$  and  $Q : Y_k \rightarrow \text{Im } Q$  be defined by

$$Pz = \frac{1}{2kT} \int_{-kT}^{kT} z(s) ds, \quad Qu = \frac{1}{2kT} \int_{-kT}^{kT} u(s) ds,$$

and denote by  $K$  the inverse of  $L|_{\ker P \cap \text{Dom } L}$ . Then,  $\ker L = \text{Im } Q = \mathbb{R}^2$  and

$$Ku(t) = \int_{-kT}^{kT} G(t, s)u(s) ds, \tag{2.3}$$

where

$$G(t, s) = \begin{cases} \frac{s}{2kT}, & -kT \leq s < t \leq kT, \\ \frac{s-kT}{2kT}, & -kT \leq t \leq s \leq kT. \end{cases}$$

It follows from (2.3) that  $N$  is  $L$ -compact on  $\overline{\Omega}$ , where  $\Omega$  is an open, bounded subset of  $X_k$ .

### 3 Main result

For the sake of convenience, we give the following fundamental assumptions.

(H<sub>1</sub>) There exist  $\alpha$  and  $\beta$  with  $\beta > \alpha > 0$  such that  $\forall x \in \mathbb{R}$ ,

$$\alpha x^2 \leq xf(x) \leq \beta x^2.$$

(H<sub>2</sub>) There exists  $\gamma > 0$  such that

$$xg(x) \leq -\gamma x^2.$$

(H<sub>3</sub>)  $e(t)$  is bounded on  $\mathbb{R}$  and

$$0 < d := \max \left\{ \left( \int_{-\infty}^{+\infty} e^2(t) dt \right)^{\frac{1}{2}}, \sup_{t \in \mathbb{R}} |e(t)| \right\} < +\infty.$$

Obviously, the conditions (H<sub>1</sub>), (H<sub>2</sub>), and (H<sub>3</sub>) are simplistic and reasonable.

**Theorem 3.1** *Let (H<sub>1</sub>), (H<sub>2</sub>), and (H<sub>3</sub>) hold. Then for each  $k \in \mathbb{N}$ , (1.6) has at least one  $2kT$ -periodic solution.*

*Proof* We consider the auxiliary system of the system (2.2),

$$\begin{cases} x'(t) = \lambda \frac{y(t)}{\sqrt{1-y^2(t)}} = \lambda \psi(y(t)), \\ y'(t) = -\lambda f(\psi(y(t))) - \lambda g(x(t)) + \lambda e_k(t), \end{cases} \tag{3.1}$$

where  $\lambda \in (0, 1]$  is a parameter. Firstly, we will prove that the set of all possible  $2kT$ -periodic solutions of the system (3.1) is bounded.

Obviously, the system (3.1) is equivalent to the following equation:

$$\left( \frac{\frac{1}{\lambda} x'(t)}{\sqrt{1 + \frac{1}{\lambda^2} x'^2(t)}} \right)' + \lambda f\left(\frac{1}{\lambda} x'(t)\right) + \lambda g(x(t)) = \lambda e_k(t). \tag{3.2}$$

Multiplying (3.2) by  $x'$  and integrating from  $-kT$  to  $kT$ , we have

$$\lambda \int_{-kT}^{kT} f\left(\frac{1}{\lambda} x'(t)\right) x'(t) dt + \lambda \int_{-kT}^{kT} g(x(t)) x'(t) dt = \lambda \int_{-kT}^{kT} e_k(t) x'(t) dt. \tag{3.3}$$

By (H<sub>1</sub>), we obtain

$$\begin{aligned} \left| \lambda \int_{-kT}^{kT} f\left(\frac{1}{\lambda} x'(t)\right) x'(t) dt \right| &\geq \lambda^2 \int_{-kT}^{kT} f\left(\frac{1}{\lambda} x'(t)\right) \frac{x'(t)}{\lambda} dt \\ &\geq \alpha \int_{-kT}^{kT} |x'(t)|^2 dt. \end{aligned} \tag{3.4}$$

By (1.7), we get

$$\begin{aligned} \int_{-kT}^{kT} |e_k(t)|^2 dt &= \int_{-kT}^{kT-\varepsilon_0} |e(t)|^2 dt + \int_{kT-\varepsilon_0}^{kT} |e(kT - \varepsilon_0) \\ &\quad + \frac{e(-kT) - e(kT - \varepsilon_0)}{\varepsilon_0} \cdot (t - kT + \varepsilon_0)|^2 dt \\ &\leq \int_{-kT}^{kT-\varepsilon_0} |e(t)|^2 dt + \int_{kT-\varepsilon_0}^{kT} \max\{|e(kT - \varepsilon_0)|^2, |e(-kT)|^2\} dt \end{aligned}$$

$$\begin{aligned} &\leq \int_{-\infty}^{+\infty} |e(t)|^2 dt + \int_{kT-\varepsilon_0}^{kT} d^2 dt \\ &\leq d^2(1 + \varepsilon_0). \end{aligned} \tag{3.5}$$

Noticing that  $\int_{-kT}^{kT} g(x(t))x'(t) dt = 0$ , we have from (3.3), (3.4), and (3.5)

$$\begin{aligned} \alpha \int_{-kT}^{kT} |x'(t)|^2 dt &\leq \left| \lambda \int_{-kT}^{kT} f\left(\frac{1}{\lambda}x'(t)\right)x'(t) dt \right| \\ &\leq \lambda \left( \left| \int_{-kT}^{kT} g(x(t))x'(t) dt \right| + \left| \int_{-kT}^{kT} e_k(t)x'(t) dt \right| \right) \\ &\leq \lambda \left( \int_{-kT}^{kT} e_k^2(t) dt \right)^{\frac{1}{2}} \cdot \left( \int_{-kT}^{kT} |x'(t)|^2 dt \right)^{\frac{1}{2}} \\ &\leq \lambda d \sqrt{1 + \varepsilon_0} \cdot \left( \int_{-kT}^{kT} |x'(t)|^2 dt \right)^{\frac{1}{2}}, \end{aligned}$$

i.e.,

$$\left( \int_{-kT}^{kT} |x'(t)|^2 dt \right)^{\frac{1}{2}} \leq \frac{\lambda d}{\alpha} \sqrt{1 + \varepsilon_0}. \tag{3.6}$$

Hence, there exists a positive constant  $D_1$  independent of  $k$  and  $\lambda$  such that

$$\|x'\|_2 \leq D_1. \tag{3.7}$$

Multiplying (3.2) by  $x$  and integrating from  $-kT$  to  $kT$ , we obtain

$$\begin{aligned} &\int_{-kT}^{kT} \left( \frac{\frac{1}{\lambda}x'(t)}{\sqrt{1 + \frac{1}{\lambda^2}x'^2(t)}} \right)' x(t) dt + \lambda \int_{-kT}^{kT} f\left(\frac{1}{\lambda}x'(t)\right)x(t) dt + \lambda \int_{-kT}^{kT} g(x(t))x(t) dt \\ &= \lambda \int_{-kT}^{kT} e_k(t)x(t) dt. \end{aligned} \tag{3.8}$$

Since  $x'(t) = \lambda \frac{y(t)}{\sqrt{1-y^2(t)}}$ , we get  $y(t) = \frac{\frac{1}{\lambda}x'(t)}{\sqrt{1 + \frac{1}{\lambda^2}x'^2(t)}}$ . Then

$$\begin{aligned} \int_{-kT}^{kT} \left( \frac{\frac{1}{\lambda}x'(t)}{\sqrt{1 + \frac{1}{\lambda^2}x'^2(t)}} \right)' x(t) dt &= \int_{-kT}^{kT} y'(t)x(t) dt \\ &= - \int_{-kT}^{kT} y(t)x'(t) dt \\ &= -\lambda \int_{-kT}^{kT} \frac{y^2(t)}{\sqrt{1 - y^2(t)}} dt. \end{aligned} \tag{3.9}$$

It follows from (3.8) and (3.9) that

$$- \int_{-kT}^{kT} g(x(t))x(t) dt \leq - \int_{-kT}^{kT} f\left(\frac{1}{\lambda}x'(t)\right)x(t) dt + \lambda \int_{-kT}^{kT} e_k(t)x(t) dt. \tag{3.10}$$

By (H<sub>2</sub>), we have

$$\int_{-kT}^{kT} g(x(t))x(t) dt \leq - \int_{-kT}^{kT} \gamma x^2(t) dt = -\gamma \|x\|_2^2. \tag{3.11}$$

By (H<sub>1</sub>) and (3.6), we obtain

$$\begin{aligned} - \int_{-kT}^{kT} f\left(\frac{1}{\lambda}x'(t)\right)x(t) dt &\leq \beta \int_{-kT}^{kT} \left|\frac{1}{\lambda}x'(t)\right| |x(t)| dt \\ &\leq \beta \left(\int_{-kT}^{kT} \left|\frac{1}{\lambda}x'(t)\right|^2 dt\right)^{\frac{1}{2}} \cdot \left(\int_{-kT}^{kT} |x(t)|^2 dt\right)^{\frac{1}{2}} \\ &\leq \frac{\beta}{\lambda} \cdot \|x'\|_2 \cdot \|x\|_2 \\ &\leq \frac{\beta d}{\alpha} \sqrt{1 + \varepsilon_0} \|x\|_2. \end{aligned} \tag{3.12}$$

By (3.5), we get

$$\begin{aligned} \int_{-kT}^{kT} e_k(t)x(t) dt &\leq \left(\int_{-kT}^{kT} e_k^2(t) dt\right)^{\frac{1}{2}} \cdot \left(\int_{-kT}^{kT} |x(t)|^2 dt\right)^{\frac{1}{2}} \\ &\leq d\sqrt{1 + \varepsilon_0} \|x\|_2. \end{aligned} \tag{3.13}$$

Then we get from (3.10), (3.11), (3.12), and (3.13)

$$\gamma \|x\|_2^2 \leq \frac{\beta d}{\alpha} \sqrt{1 + \varepsilon_0} \|x\|_2 + d\sqrt{1 + \varepsilon_0} \|x\|_2,$$

*i.e.*,

$$\|x\|_2 \leq \frac{\alpha + \beta}{\alpha \gamma} d\sqrt{1 + \varepsilon_0}.$$

Therefore, there exists a positive constant  $D_2$  independent of  $k$  and  $\lambda$  such that

$$\|x\|_2 \leq D_2. \tag{3.14}$$

Thus, by using Lemma 2.2, we have

$$\begin{aligned} |x(t)| &\leq (2T)^{-\frac{1}{2}} \left(\int_{t-T}^{t+T} |x(s)|^2\right)^{\frac{1}{2}} + T(2T)^{-\frac{1}{2}} \left(\int_{t-T}^{t+T} |x'(s)|^2\right)^{\frac{1}{2}} \\ &\leq (2T)^{-\frac{1}{2}} \left(\int_{t-kT}^{t+kT} |x(s)|^2\right)^{\frac{1}{2}} + T(2T)^{-\frac{1}{2}} \left(\int_{t-kT}^{t+kT} |x'(s)|^2\right)^{\frac{1}{2}} \\ &= (2T)^{-\frac{1}{2}} \left(\int_{-kT}^{kT} |x(s)|^2\right)^{\frac{1}{2}} + T(2T)^{-\frac{1}{2}} \left(\int_{-kT}^{kT} |x'(s)|^2\right)^{\frac{1}{2}} \\ &\leq (2T)^{-\frac{1}{2}} D_2 + T(2T)^{-\frac{1}{2}} D_1. \end{aligned}$$

Hence, there exists a positive constant  $M_1$  independent of  $k$  and  $\lambda$  such that

$$\|x\|_\infty \leq M_1.$$

In what follows, we prove that there exists a constant  $\varepsilon_1$  with  $0 < \varepsilon_1 \ll 1$  such that

$$|y(t)| < 1 - \varepsilon_1, \quad \forall t \in \mathbb{R}. \tag{3.15}$$

Since  $\|x\|_\infty \leq M_1$  and  $g$  is continuous, there exists  $M_2 > 0$  such that

$$-M_2 < -g(x(t)) + e_k(t) < M_2, \quad \forall t \in \mathbb{R}.$$

By  $(H_1)$ , we get

$$f(x) \geq \alpha x, \quad \forall x > 0.$$

Now we prove by contradiction that

$$y(t) \leq \frac{M_2}{\sqrt{M_2^2 + \alpha^2}}, \quad \forall t \in \mathbb{R}.$$

Assume that there exist  $t_2^* > t_1^*$  such that

$$y(t_1^*) = \frac{M_2}{\sqrt{M_2^2 + \alpha^2}}, \quad y(t_2^*) > \frac{M_2}{\sqrt{M_2^2 + \alpha^2}}$$

and

$$y(t) > \frac{M_2}{\sqrt{M_2^2 + \alpha^2}}, \quad \forall t \in (t_1^*, t_2^*).$$

Noticing that  $\lambda \in (0, 1]$ , we have  $\forall t \in (t_1^*, t_2^*)$ ,

$$y'(t) = \lambda(-f(\psi(y(t))) - g(x(t)) + e_k(t)) < 0,$$

which is a contradiction. By  $(H_1)$ , we get

$$f(x) \leq \alpha x, \quad \forall x < 0.$$

By using a similar argument, we can prove that

$$y(t) \geq -\frac{M_2}{\sqrt{M_2^2 + \alpha^2}}, \quad \forall t \in \mathbb{R}.$$

Thus

$$-\frac{M_2}{\sqrt{M_2^2 + \alpha^2}} \leq y(t) \leq \frac{M_2}{\sqrt{M_2^2 + \alpha^2}}, \quad \forall t \in \mathbb{R}.$$

Therefore, (3.15) holds.



Putting

$$\Omega = \{ (x(t), y(t)) \in C(\mathbb{R}^1, \mathbb{R}^2) : \|x\|_\infty < M_1 + 1, \|y\|_\infty < 1 - \varepsilon_1 \},$$

we can give a standard argument on  $\Omega$  by using Lemma 2.1 (also see [13] or [14]) and find that the system (2.2) has at least one  $2kT$ -periodic solution. Equivalently, (1.6) has at least one  $2kT$ -periodic solution. □

**Theorem 3.2** *Let  $(H_1)$ ,  $(H_2)$ , and  $(H_3)$  hold. Then (1.1) has at least one nontrivial homoclinic solution.*

*Proof* If  $(x_k(t), y_k(t))$  is a solution of (1.6), then

$$\left( \frac{x'_k(t)}{\sqrt{1 + (x'_k(t))^2}} \right)' + f(x'_k(t)) + g(x_k(t)) = e_k(t). \tag{3.16}$$

From Theorem 3.1, we have

$$\|x_k\|_\infty < M_1 + 1, \quad \|y_k\|_\infty < 1 - \varepsilon_1, \tag{3.17}$$

where  $y_k(t) = \frac{x'_k(t)}{\sqrt{1 + (x'_k(t))^2}}$ . Equation (3.16) is equivalent to

$$\begin{cases} x'_k(t) = \frac{y_k(t)}{\sqrt{1 - y_k^2(t)}}, \\ y'_k(t) = -f\left(\frac{y_k(t)}{\sqrt{1 - y_k^2(t)}}\right) - g(x_k(t)) + e_k(t). \end{cases}$$

Since  $f, g, e_k$  are continuous, we have  $y_k(t)$  is continuous differentiable on  $\mathbb{R}$ . Moreover, it follows that  $x'_k(t)$  is also continuous differentiable on  $\mathbb{R}$ . Hence,

$$x''_k(t) = \frac{y'_k(t)}{(1 - y_k^2(t))^{\frac{3}{2}}}. \tag{3.18}$$

Since  $f, g, e_k$  are continuous, we find from (3.17) that there exists a positive constant  $M_3$  independent of  $k$  such that

$$\|y'_k\|_\infty < M_3.$$

Hence, by (3.18), there exists a positive constant  $M_4$  independent of  $k$  such that

$$\|x''_k\|_\infty < M_4.$$

By using Lemma 2.3, we find that there is a function  $x_0 \in C^1(\mathbb{R}, \mathbb{R})$  such that, for each interval  $[a, b] \subset \mathbb{R}$ , there is a subsequence  $\{x_{k_j}(t)\}$  of  $x_k(t)$  with  $x'_{k_j}(t) \rightarrow x'_0(t)$  uniformly on  $[a, b]$ . In what follows, we will show that  $x_0(t)$  is just a homoclinic solution of (1.1).

For all  $a, b \in \mathbb{R}$  with  $a < b$ , there exist a positive integer  $j_0$  and a positive real number  $\varepsilon_2$  such that, for  $j > j_0$ ,  $[a - \varepsilon_2, b + \varepsilon_2] \subset [-k_j T, k_j T - \varepsilon_0]$ . Then for  $j > j_0$ ,

$$\left( \frac{x'_{k_j}(t)}{\sqrt{1 + (x'_{k_j}(t))^2}} \right)' + f(x'_{k_j}(t)) + g(x_{k_j}(t)) = e(t). \tag{3.19}$$

Since  $x'_{k_j}(t) \rightarrow x'_0(t)$  uniformly on  $[a, b]$ , we have

$$-f(x'_{k_j}(t)) - g(x_{k_j}(t)) + e(t) \rightarrow -f(x'_0(t)) - g(x_0(t)) + e(t), \quad \text{uniformly on } [a, b].$$

Since  $\frac{x'_{k_j}(t)}{\sqrt{1+(x'_{k_j}(t))^2}} \rightarrow \frac{x'_0(t)}{\sqrt{1+(x'_0(t))^2}}$  uniformly on  $[a, b]$  and  $\frac{x'_{k_j}(t)}{\sqrt{1+(x'_{k_j}(t))^2}} = y_{k_j}(t)$  is continuous differentiable for  $t \in [a, b]$ , we get

$$\left( \frac{x'_{k_j}(t)}{\sqrt{1+(x'_{k_j}(t))^2}} \right)' \rightarrow \left( \frac{x'_0(t)}{\sqrt{1+(x'_0(t))^2}} \right)', \quad \text{uniformly on } [a, b].$$

Hence, we obtain

$$\left( \frac{x'_0(t)}{\sqrt{1+(x'_0(t))^2}} \right)' + f(x'_0(t)) + g(x_0(t)) = e(t), \quad t \in [a, b]. \tag{3.20}$$

Noticing that  $a, b$  are two arbitrary constants with  $a < b$ , we find that  $x_0 : \mathbb{R} \rightarrow \mathbb{R}$  is a solution of (1.1).

In the following, we prove that  $x_0(t) \rightarrow 0$  and  $x'_0(t) \rightarrow 0$  as  $|t| \rightarrow +\infty$ . Firstly, we prove  $x_0(t) \rightarrow 0$  as  $|t| \rightarrow +\infty$ . Let  $i$  be a positive integer with  $i < k_j$ , then

$$\begin{aligned} \int_{-iT}^{iT} (|x_0(t)|^2 + |x'_0(t)|^2) dt &= \lim_{j \rightarrow +\infty} \int_{-iT}^{iT} (|x_{k_j}(t)|^2 + |x'_{k_j}(t)|^2) dt \\ &\leq \lim_{j \rightarrow +\infty} \int_{-k_j T}^{k_j T} (|x_{k_j}(t)|^2 + |x'_{k_j}(t)|^2) dt \\ &\leq D_1^2 + D_2^2. \end{aligned}$$

Let  $i \rightarrow +\infty$ , then

$$\int_{-\infty}^{\infty} (|x_0(t)|^2 + |x'_0(t)|^2) dt = \lim_{j \rightarrow +\infty} \int_{-iT}^{iT} (|x_0(t)|^2 + |x'_0(t)|^2) dt \leq D_1^2 + D_2^2,$$

which implies

$$\lim_{r \rightarrow +\infty} \int_{|t| \geq r} (|x_0(t)|^2 + |x'_0(t)|^2) dt = 0.$$

Using Lemma 2.2, we have

$$\begin{aligned} |x_0(t)| &\leq (2T)^{-\frac{1}{2}} \left( \int_{t-T}^{t+T} |x_0(s)|^2 dt \right)^{\frac{1}{2}} + T(2T)^{-\frac{1}{2}} \left( \int_{t-T}^{t+T} |x'_0(s)|^2 dt \right)^{\frac{1}{2}} \\ &\leq 2((2T)^{-\frac{1}{2}} + T(2T)^{-\frac{1}{2}}) \left( \int_{t-kT}^{t+kT} (|x_0(s)|^2 + |x'_0(s)|^2) dt \right)^{\frac{1}{2}} \\ &\rightarrow 0, \quad \text{as } |t| \rightarrow +\infty. \end{aligned}$$

Finally, we prove that

$$x'_0(t) \rightarrow 0 \quad \text{as } |t| \rightarrow +\infty. \tag{3.21}$$

From (3.17), we get

$$|x_0(t)| \leq M_1 + 1, \quad |x'_0(t)| \leq \frac{1 - \varepsilon_1}{\sqrt{2\varepsilon_1 - \varepsilon_1^2}}, \quad \text{for } t \in \mathbb{R}.$$

Then we have

$$\left| \left( \frac{x'_0(t)}{\sqrt{1 + (x'_0(t))^2}} \right)' \right| \leq |f(x'_0(t))| + |g(x_0(t))| + |e(t)| \leq M_5,$$

where  $M_5$  is a positive constant.

If (3.21) does not hold, then there exists a sequence  $t_k$  satisfying

$$|t_1| < |t_2| < |t_3| < \dots \quad \text{with } |t_{k+1}| - |t_k| > 1, k = 1, 2, \dots$$

and  $\varepsilon_2 \in (0, \frac{1}{4})$  such that

$$|x'_0(t_k)| \geq \frac{2\varepsilon_2}{\sqrt{1 - (2\varepsilon_2)^2}}, \quad k = 1, 2, \dots$$

Hence we have, for  $t \in [t_k, t_k + \frac{\varepsilon_2}{1+M_5}]$ ,

$$\begin{aligned} |x'_0(t)| &\geq \left| \frac{x'_0(t)}{\sqrt{1 + (x'_0(t))^2}} \right| \\ &= \left| \frac{x'_0(t_k)}{\sqrt{1 + (x'_0(t_k))^2}} + \int_{t_k}^t \left( \frac{x'_0(s)}{\sqrt{1 + (x'_0(s))^2}} \right)' ds \right| \\ &\geq \left| \frac{x'_0(t_k)}{\sqrt{1 + (x'_0(t_k))^2}} \right| - \int_{t_k}^t \left| \left( \frac{x'_0(s)}{\sqrt{1 + (x'_0(s))^2}} \right)' \right| ds \\ &\geq 2\varepsilon_2 - \frac{M_5}{1 + M_5} \cdot \varepsilon_2 \\ &> \varepsilon_2. \end{aligned}$$

Therefore,

$$\int_{-\infty}^{+\infty} |x'_0(t)|^2 dt \geq \sum_{k=1}^{\infty} \int_{t_k}^{t_k + \frac{\varepsilon_2}{1+M_5}} |x'_0(t)|^2 dt = \infty,$$

which is a contradiction. Then (3.21) holds.

By  $(H_3)$ , we have  $e \neq 0$ . Thus it follows from  $f(0) = 0$  and  $g(0) = 0$  that  $x_0$  is nontrivial.

□

#### 4 An example

In this section, we shall construct an example to show the applications of Theorem 3.2.

**Example 4.1** Let  $f(x) = \frac{x^3+x}{\sqrt{1+x^4}}$ ,  $g(x) = -\frac{x^5+x}{\sqrt{1+x^8}}$ . The prescribed mean curvature Rayleigh equation

$$\left(\frac{x'}{\sqrt{1+x^2}}\right)' + f(x'(t)) + g(x(t)) = \exp\{-t^2\} \tag{4.1}$$

has at least one nontrivial homoclinic solution.

*Proof* Obviously, one has

$$\frac{1}{\sqrt{2}} \cdot x^2 \leq xf(x) \leq 2x^2$$

and

$$xg(x) \leq -\frac{1}{\sqrt{2}} \cdot x^2.$$

Then  $(H_1)$  and  $(H_2)$  hold. Since

$$d = \max \left\{ \left( \int_{-\infty}^{+\infty} (\exp\{-t^2\})^2 dt \right)^{\frac{1}{2}}, \sup_{t \in \mathbb{R}} \exp\{-t^2\} \right\} = \sqrt[4]{\frac{\pi}{2}},$$

$(H_3)$  holds. Therefore, (4.1) has at least one nontrivial homoclinic solution. □

#### Competing interests

The authors declare that they have no competing interests.

#### Authors' contributions

All authors have equally contributed in obtaining new results in this article and also read and approved the final manuscript.

#### Author details

<sup>1</sup>Department of Scientific Research Management, Jiujiang University, Jiujiang, 332005, China. <sup>2</sup>School of Science, Jiujiang University, Jiujiang, 332005, China.

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