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Approximate controllability of impulsive fractional stochastic differential equations with state-dependent delay

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Abstract

This work is concerned with the approximate controllability of nonlinear fractional impulsive stochastic differential system under the assumption that the corresponding linear system is approximately controllable. Using fractional calculus, stochastic analysis, and the technique of stochastic control theory, a new set of sufficient conditions for the approximate controllability of a fractional impulsive stochastic differential system is obtained. The results in this paper are generalizations and continuations of the recent results on this issue. An example is given to illustrate the efficiency of the main results.

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Keywords: approximate controllability; stochastic differential system; state-dependent delay; nonlocal conditions

1 Introduction

In the last few decades, fractional differential systems have provided an excellent tool in electrochemistry, physics, porous media, control theory, engineering *etc.*, due to the descriptions of memory and hereditary properties of various materials and processes. The research of fractional systems has received more and more attention very recently, see the monographs of Kilbas *et al.* [1], Miller and Ross [2], Podlubny [3], the recent papers [4–7] and so forth. Controllability is one of the important concepts both in mathematics and in control theory. Controllability of deterministic control systems has been well developed by using different kinds of methods, which can be found in [8–12]. The approximate controllability, the weaker concept of controllability, has received much attention recently. In this case it is possible to steer the system to an arbitrary small neighborhood of the final state (see, for example, [13–17]). However, the extension of the deterministic controllability concepts to stochastic control system has rarely been reported. As a matter of fact, the accurate analysis or assessment subjected to a realistic environment has to take into account the potential randomness in the system properties, such as fluctuations in the stock market or noise in a communication network. All these problems in mathematics are modeled and described by stochastic differential equations or stochastic integro-differential equations with delay and impulse. Stochastic control theory is a stochastic generalization of classical control theory [12, 18]. The biggest difficulty is the analysis of a stochastic control system and stochastic calculations induced by the stochastic process.

In the deterministic case, Mahmudov and Zorlu [15] discussed the approximate controllability of fractional evolution equations with a compact analytic semigroup. Ganesh *et al.* [13] derived a set of sufficient conditions for the approximate controllability of a class of fractional integro-differential evolution equations. For more work, see [14, 16, 17] and the references therein. In the stochastic case, by using the stochastic analysis technique and the methods directly from deterministic control problems, Sakthivel *et al.* [12] considered the approximate controllability of fractional stochastic evolution equations. Ahmed [19] considered the approximate controllability of impulsive neutral stochastic differential equations with finite delay and fractional Brownian motion in a Hilbert space, and a new set of sufficient conditions for approximate controllability was formulated and proved.

On the other hand, the stochastic differential equation with delay is a special type of stochastic functional differential equations. Delay differential equations arise in many biological and physical applications, and it often forces us to consider variable or state-dependent delays. The stochastic functional differential equations with state-dependent delay have many important applications in mathematical models of real phenomena, and the study of this type of equations has received much attention in recent years. Guendouzi and Benzatout [20] studied the existence of mild solutions for a class of impulsive stochastic differential inclusions with state-dependent delay. Sakthivel and Ren [21] studied the approximate controllability of fractional differential equations with state-dependent delay.

However, approximate controllability of stochastic differential system with state-dependent delay and impulse has rarely been considered in the literature. Therefore, this work aims to study the fractional impulsive stochastic control system with state-dependent delay:

$$\begin{cases} {}^cD_t^\alpha x(t) = Ax(t) + f(t, x_{\rho(t,x_t)}) + Bu(t) + g(t, x_{\rho(t,x_t)})\frac{dw(t)}{dt}, & t \in J := [0,T], t \neq t_k; \\ \Delta x(t_k) = I_k(x(t_k^-)), \quad k = 1,2,\ldots,n; \\ x(0) + h(x) = x_0 = \phi \in \mathfrak{B}, \end{cases} \tag{1.1}$$

where ${}^cD_t^q$ is the Caputo fractional derivative of order $0 < \alpha < 1$, the state variable x takes values in a Hilbert space H, $\Delta x(t_k) := x(t_k^+) - x(t_k^-)$ represents the jump in the state x at time t_k, $0 < t_1 < t_2 < \cdots < t_n < T$. The history x_s represents the function defined by $x_s : (-\infty, 0] \to H$, $x_s(\theta) = x(s+\theta)$ belongs to some abstract phase space $\mathfrak{B}$ described axiomatically and $\rho : J \times \mathfrak{B} \to (-\infty, T]$ is a continuous function. A is the infinitesimal generator of a compact semigroup $\{S(t), t \geq 0\}$ on the Hilbert space H, the control function u is given in $L^2([0,T], U)$, U is a Hilbert space, B is a bounded linear operator from U to H. Let K be another Hilbert space, suppose $\{W(t)\}_{t\geq 0}$ is a given K-valued Brownian motion or Wiener process with a finite trace nuclear covariance operator $Q \geq 0$ defined on a complete probability space $(\Omega, \mathscr{F}, P)$. Denote $PC(J, L^2(\Omega, \mathscr{F}, P; H)) = \{x(t)$ is continuous everywhere except for some t_k at which $x(t_k^-)$ and $x(t_k^+)$ exist and $x(t_k^-) = x(t_k)\}$ be the Banach space with the norm $\|x\|_{PC} = \sup_{t\in J}|x(t)| < \infty$. $PC(J, L^2)$ is the closed subspace of $PC(J, L^2(\Omega, \mathscr{F}, P; H))$ consisting of a measurable and $\mathscr{F}_t$-adapted H-valued process $x(\cdot) \in PC(J, L^2(\Omega, \mathscr{F}, P; H))$ with the norm $\|x\|^2 = \sup\{E\|x(t)\|^2, t \in J\}$. The functions f, g, I_k, h are appropriate functions to be specified later.

The outline of this paper is as follows. In Section 2, we recall some preliminary notations and results which will be needed in the sequel. Section 3 is devoted to studying the approximate controllability of the system (1.1) provided that the corresponding linear system is approximately controllable. Finally, an example is given to illustrate the effectiveness of the main results.

2 Preliminaries

In this section, we will introduce some preliminary definitions, notations, and results, to establish our main results.

Let $(\Omega, \mathscr{F}, P)$ be a complete probability space with a filtration $\{\mathscr{F}_t\}_{t\geq 0}$. An H-valued random variable is an $\mathscr{F}$ measurable function $x(t): \Omega \to H$, and the collection of random variable $C = \{x(t,\omega): \Omega \to H|_{t\in J}\}$ is a stochastic process. Here we suppress the dependence on $\omega \in \Omega$ and write $x(t)$ instead of $x(t,\omega)$. Let $\{\beta_n\}_{n\geq 1}$ be a sequence of real valued independent Brownian motions. Set $W(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n}\beta_n e_n$, $t \geq 0$, where $\{e_n\}_{n\geq 1}$ is complete orthonormal system in K. Assume $Q \in L(K,K)$ be an operator satisfying $Qe_n = \lambda_n e_n$ with $\operatorname{tr}(Q) < \infty$. Then the above K-valued stochastic process $W(t)$ is a Q-Wiener process. Denote $\mathscr{F}_t = \sigma(W(s): 0 \leq s \leq t)$ be the σ-algebra generated by W and $\mathscr{F}_T = \mathscr{F}_t$. Take $\phi \in L(K,H)$ and define $\|\phi\|_Q^2 = \operatorname{tr}(\phi Q\phi^*) = \sum_{n=1}^{\infty} \|\sqrt{\lambda_n}\phi e_n\|^2 < \infty$, then ϕ is called a Q-Hilbert Schmidt operator and we denote its set as $L_Q(K,H)$ with the norm $\|\phi\|_Q^2 = \langle\phi,\phi\rangle$.

Throughout this paper, we assume that the abstract phase space $(\mathfrak{B}, \|\cdot\|_{\mathfrak{B}})$ is a seminorm linear space of $\mathscr{F}_0$-measurable functions mapping $J_0 = (-\infty, 0]$ to H and satisfying the following fundamental axioms [22]:

- (i) If $x: (-\infty, T) \to H$ is continuous on $[0,T)$ and $x_0 \in \mathfrak{B}$, then for each $t \in [0,T)$ the following conditions hold:
  - (a) $x_t \in \mathfrak{B}$;
  - (b) $\|x(t)\| \leq K_1\|x_t\|_{\mathfrak{B}}$;
  - (c) $\|x_t\|_{\mathfrak{B}} \leq K_2(t)\|x_0\|_{\mathfrak{B}} + K_3(t)\sup\{\|x(s)\|; 0 \leq s \leq T\}$,

  where $K_1 > 0$ is a constant, $K_2, K_3: [0,\infty) \to [0,\infty)$, K_2 is locally bounded, K_3 is continuous. Moreover, K_i $(i = 1,2,3)$ are independent of x.
- (ii) For the function $x(\cdot)$ in (i), x_t is a $\mathfrak{B}$-valued continuous functions on $[0,T)$.
- (iii) The space $\mathfrak{B}$ is complete.

Let $x: (-\infty, T] \to H$ be an $\mathscr{F}_t$-adapted measurable process such that we have the $\mathscr{F}_0$-adapted process $x_0 = \phi(t) \in L^2(\Omega, \mathfrak{B})$, then

$$E\|x_t\|_{\mathfrak{B}}^2 \leq \overline{K}_2 E\|\phi\|_{\mathfrak{B}}^2 + \overline{K}_3 \sup_{0\leq s\leq T}\bigl\{E\bigl\|x(s)\bigr\|^2\bigr\},$$

where $\overline{K}_2 = \sup_{t\in J} K_2(t)$, $\overline{K}_3 = \sup_{t\in J} K_3(t)$.

The next lemma is proved using the phase spaces axioms.

Lemma 2.1 ([23]) *Let $\phi \in \mathfrak{B}$ and $I = (-\infty, 0]$ be such that $\phi_t \in \mathfrak{B}$ for each $t \in I$. Assume that there exists a locally bounded function $H^{\phi}: I \to [0,\infty)$ such that $E\|\phi_t\|_{\mathfrak{B}}^2 \leq H^{\phi}(t)E\|\phi\|_{\mathfrak{B}}^2$ for $t \in I$. Let $x: (-\infty, T] \to H$ be functions such that $x_0 = \phi$ and $x \in PC(J, L^2)$, then*

$$E\|x_s\|_{\mathfrak{B}}^2 \leq (\overline{H}_2 + \eta)E\|\phi\|_{\mathfrak{B}}^2 + \overline{H}_3 \sup\bigl\{E\bigl\|x(\theta)\bigr\|^2; \theta \in \bigl[0, \max\{0,s\}\bigr]\bigr\}, \quad s \in (-\infty, T],$$

where $\eta = \sup_{t\in I} H^{\phi}(t)$, $\overline{H}_2 = \sup_{t\in J} \overline{K}_2(t)$, $\overline{H}_3 = \sup_{t\in J} \overline{K}_3(t)$.

Definition 2.1 A stochastic process $J \times \Omega \to H$ is called a mild solution of the system (1.1) if

(i) $x(t)$ is measurable and $\mathscr{F}_t$-adapted for each $t \in J$;

(ii) $x(t) \in H$ satisfies the following integral equation:

$$\begin{aligned} x(t) &= \mathscr{T}\left[\phi(0) - h(x)\right] + \int_0^t (t-s)^{\alpha-1} \mathscr{S}(t-s) f(s, x_{\rho(s,x_s)})\, ds \\ &\quad + \int_0^t (t-s)^{\alpha-1} \mathscr{S}(t-s) Bu(s)\, ds \\ &\quad + \int_0^t (t-s)^{\alpha-1} \mathscr{S}(t-s) g(s, x_{\rho(s,x_s)})\, dw(s) + \sum_{0<t_k<t} \mathscr{S}(t-t_k) I_k\left(x\left(t_k^-\right)\right); \end{aligned}$$

(iii) $x_0(\cdot) = \phi \in \mathfrak{B}$ on $(-\infty, 0]$ satisfying $\|\phi\|_{\mathfrak{B}} < \infty$, where

$$\begin{aligned} &\mathscr{T}(t) = \int_0^\infty \xi_\alpha(\theta) S\left(t^\alpha \theta\right) d\theta, \qquad \mathscr{S}(t) = \alpha \int_0^\infty \theta \xi_\alpha(\theta) S\left(t^\alpha \theta\right) d\theta, \\ &\xi_\alpha(\theta) = \frac{1}{\alpha} \theta^{-1-\frac{1}{\alpha}} \overline{\omega}_\alpha\left(\theta^{-\frac{1}{\alpha}}\right), \qquad \overline{\omega}_\alpha(\theta) = \frac{1}{\pi} \sum_{n=1}^\infty (-1)^{n-1} \theta^{-\alpha n-1} \frac{\Gamma(n\alpha+1)}{n!} \sin(n\pi\alpha), \\ &\quad \theta \in (0, \infty). \end{aligned}$$

Here ξ_α is a probability density function on $(0, \infty)$, that is, $\xi_\alpha(\theta) \geq 0$ and $\int_0^\infty \xi_\alpha(\theta) = 1$.

Lemma 2.2 ([24]) *The operators $\mathscr{T}$ and $\mathscr{S}$ have the following properties*:

(i) *For any fixed $t \geq 0$, the operators $\mathscr{T}(t)$ and $\mathscr{S}(t)$ are linear and bounded operators, i.e., for any $x \in H$,*

$$\left\| \mathscr{T}(t)x \right\| \leq M\|x\| \quad \text{and} \quad \left\| \mathscr{S}(t)x \right\| \leq \frac{\alpha M}{\Gamma(1+\alpha)} \|x\|.$$

(ii) *The operators $\mathscr{T}(t)$ and $\mathscr{S}(t)$ are strongly continuous for all $t \geq 0$, it means that for every $x \in H$ and $0 \leq t' < t'' \leq T$, we have*

$$\left\| \mathscr{T}\left(t''\right)x - \mathscr{T}\left(t'\right)x \right\| \to 0 \quad \text{and} \quad \left\| \mathscr{S}\left(t''\right)x - \mathscr{S}\left(t'\right)x \right\| \to 0 \quad \text{as } t' \to t''.$$

(iii) *For every $t > 0$, $\mathscr{T}(t)$ and $\mathscr{S}(t)$ are also compact operators if $S(t)$ is compact for all $t > 0$.*

Assume that the linear fractional differential system

$$\begin{cases} {}^cD_t^\alpha x(t) = Ax(t) + Bu(t), & t \in J; \\ x(0) = x_0 \end{cases} \tag{2.1}$$

is approximately controllable. It is convenient at this position to introduce the controllability operator associated with (2.1), thus

$$\Gamma_0^T = \int_0^T (T-s)^{\alpha-1} \mathscr{S}(T-s) BB^* \mathscr{S}^*(T-s)\, ds,$$

where B^* and $\mathscr{S}^*$ are the adjoint of B and $\mathscr{S}$, respectively. It is straightforward that the operator Γ_0^T is a linear bounded operator.

Let $x(T;x_0,u)$ be the state value of (1.1) at terminal time T corresponding to the control u and the initial value x_0. Introduce the set $\mathfrak{R}(T,x_0) = \{x(T;x_0,u) : u \in L^2([0,T],U)\}$, which is called the reachable set of system (1.1) at terminal time T, its closure in H is denoted by $\overline{\mathfrak{R}(T,x_0)}$.

Definition 2.2 ([15]) The system (1.1) is said to be approximately controllable on $[0,T]$ if $\overline{\mathfrak{R}(T,x_0)} = L^2(\Omega,H)$, that is, given an arbitrary $\epsilon > 0$, it is possible to steer from the point x_0 to within a distance ϵ from all points in the state space H at time T.

Lemma 2.3 ([14]) *The linear fractional control system* (2.1) *is approximately controllable on* $[0,T]$ *if and only if* $\lambda(\lambda I + \Gamma_0^T) \to 0$ *as* $\lambda \to 0^+$ *in the strong operator topology.*

Lemma 2.4 ([25]) *For any* $\tilde{x}_T \in L^2(\Omega,H)$, *there exists* $\tilde{\phi} \in L^2(\Omega, L^2(J, L_Q(K,H)))$ *such that* $\tilde{x}_T = E\tilde{x}_T + \int_0^T \tilde{\phi}(s)\,dw(s)$.

Lemma 2.5 (Krasnoselskii fixed point theorem [26, 27]) *Let M be a closed, convex, and nonempty subset of a Banach space X. Let A, B be the operators such that*

(i) $Ax + By \in M$, *wherever* $x, y \in M$;

(ii) A *is compact and continuous*;

(iii) B *is a contraction mapping.*

Then there exists $z \in M$ *such that* $z = Az + Bz$.

3 Main results

In this section, we aim to establish the approximate controllability of the system (1.1). To do this, we need the following assumptions.

(H_0) The function $t \to \phi_t$ is well defined from $\mathcal{Z}(\rho^-) = \{\rho(s,\psi); (s,\psi) \in J \times \mathfrak{B}, \rho(s,\psi) \leq 0\}$ into $\mathfrak{B}$ and there exists a continuous and bounded function $H^\phi : \mathcal{Z}(\rho^-) \to (0,\infty)$ such that $E\|x_t\|_{\mathfrak{B}}^2 \leq H^\phi(t)E\|\phi\|_{\mathfrak{B}}^2$ for every $t \in \mathcal{Z}(\rho^-)$.

(H_f) The function $f : J \times \mathfrak{B} \to H$ is continuous and there exist two constants M_f and L_f such that

$$E\|f(t,x)\|^2 \leq M_f(1 + \|x\|_{\mathfrak{B}}^2), \qquad E\|f(t,x) - f(t,y)\| \leq L_f\|x - y\|_{\mathfrak{B}}^2.$$

(H_g) The function $g : J \times \mathfrak{B} \to H$ is continuous and there exist two constants M_g and L_g such that

$$E\|g(t,x)\|^2 \leq M_g(1 + \|x\|_{\mathfrak{B}}^2), \qquad E\|g(t,x) - g(t,y)\| \leq L_g\|x - y\|_{\mathfrak{B}}^2.$$

(H_I) The function $I_k : H \to H$ are continuous and there exist nondecreasing continuous functions $M_{I_K} : R^+ \to R^+$ such that for each $x \in H$,

$$E\|I_k(x)\|^2 \leq M_{I_K}(E\|x\|^2) \quad \text{and} \quad \liminf_{r\to\infty} \frac{M_{I_K}(r)}{r} = \gamma_k < \infty.$$

(H_h) h is continuous and there exists some constant M_h satisfying

$$E\|h(t,x)\|^2 \leq M_h(1 + \|x\|_{\mathfrak{B}}^2).$$

Now, for any $\lambda > 0$ and $\tilde{x}_T \in L^2(\mathscr{F}, H)$, we define the control function

$$\begin{aligned} u^\lambda(t,x) &= B^*\mathscr{S}^*(T-t)R(\lambda,\Gamma_0^T)\left[E\tilde{x}_T + \int_0^t \tilde{\phi}(s)\,dw(s) - \mathscr{T}(T)(\phi(0) - h(x))\right] \\ &\quad - B^*\mathscr{S}^*(T-t)\int_0^t (T-s)^{\alpha-1}R(\lambda,\Gamma_s^T)\mathscr{S}(T-s)f(s,x_{\rho(s,x_s)})\,ds \\ &\quad - B^*\mathscr{S}^*(T-t)\int_0^t (T-s)^{\alpha-1}R(\lambda,\Gamma_s^T)\mathscr{S}(T-s)g(s,x_{\rho(s,x_s)})\,dw(s) \\ &\quad - B^*\mathscr{S}^*(T-t)R(\lambda,\Gamma_0^T)\sum_{0<t_k<T}\mathscr{T}(T-t_k)I_k(x(t_k^-)), \end{aligned} \tag{3.1}$$

where $R(\lambda,\Gamma_0^T) = (\lambda I + \Gamma_0^T)^{-1}$.

Theorem 3.1 *If the hypotheses* (H_0), (H_f), (H_g), (H_h), (H_I) *are satisfied, then the fractional Cauchy problem* (1.1) *with* $u = u^\lambda(t,x)$ *has at least one mild solution provided that*

$$\begin{aligned} &L_{r^*}\overline{H}_3 + L_r < 1, \\ &\frac{2M^2}{\Gamma^2(\alpha)}\left(\frac{T^{2\alpha}}{\alpha^2}L_f + \frac{T^{2\alpha-1}}{2\alpha-1}L_g\right)\overline{H}_3^2 < 1, \end{aligned} \tag{3.2}$$

where

$$\begin{aligned} L_r &= 6M^2 n\sum_{k=1}^n \gamma_k\left(1 + \frac{7M^4M_B^2T^{2\alpha}}{\lambda^2\Gamma^4(\alpha)\alpha^2}\right), \\ L_{r^*} &= 6\left(M^2M_h + \frac{T^{2\alpha}M^2}{\alpha^2\Gamma^2(\alpha)}M_f + \frac{T^{2\alpha-1}M^2}{(2\alpha-1)\Gamma^2(\alpha)}M_g\right)\left(1 + \frac{7M^4M_B^4T^{2\alpha}}{\lambda^2\Gamma^4(\alpha)\alpha^2}\right). \end{aligned}$$

Proof It is convenient to divide the proof into several steps.

Step 1: We can claim that $\Phi\mathfrak{B}_r \subset \mathfrak{B}_r$. If this is not true, then for any $r > 0$, there exist $x^r \in \mathfrak{B}_r$ and $t^r \in J$ such that $r < \|\Phi x^r(t^r)\|^2$, from Lemma 2.1, it follows that $E\|x^r_{\rho(t,x^r_{t^r})}\| \le (H_2 + \eta)\|\phi\|^2_{\mathfrak{B}} + \overline{H}_3 r := r^*$, then one can see that

$$\begin{aligned} &E\|u^\lambda(t^r,x^r)\|^2 \\ &\quad\le \frac{7M_B^2M^2}{\lambda^2\Gamma^2(\alpha)}\left[E\|x_T\|^2 + \int_0^t E\|\tilde{\phi}(s)\|^2\,ds + M^2E\|\phi(0)\|^2 + M^2E\|h(x^r)\|^2\right] \\ &\qquad + \frac{7M_B^2M^2}{\lambda^2\Gamma^2(\alpha)}\int_0^{t^r}(T-s)^{\alpha-1}\,ds\int_0^{t^r}(T-s)^{\alpha-1}E\|\mathscr{S}(T-s)f(s,x^r_{\rho(s,x^r_s)})\|^2\,ds \\ &\qquad + \frac{7M_B^2M^2}{\lambda^2\Gamma^2(\alpha)}\int_0^{t^r}(T-s)^{2\alpha-2}E\|\mathscr{S}(T-s)g(s,x^r_{\rho(s,x^r_s)})\|^2\,ds \\ &\qquad + \frac{7M_B^2M^4n}{\lambda^2\Gamma^2(\alpha)}\sum_{k=1}^n E\|I_k(x(t_k^-))\|^2\,ds \\ &\quad\le \frac{7M_B^2M^2}{\lambda^2\Gamma^2(\alpha)}\left[E\|x_T\|^2 + \int_0^T E\|\tilde{\phi}(s)\|^2\,ds + M^2E\|\phi(0)\|^2 + M^2M_h(1+r^*)\right] \\ &\qquad + \frac{7M_B^2M^4T^{2\alpha}}{\lambda^2\alpha^2\Gamma^4(\alpha)}M_f(1+r^*) + \frac{7M_B^2M^4T^{2\alpha-1}}{\lambda^2\Gamma^4(\alpha)(2\alpha-1)}M_g(1+r^*) + \frac{7M_B^2M^4n}{\lambda^2\Gamma^2(\alpha)}\sum_{k=1}^n M_{I_k}r \end{aligned}$$

and

$$\begin{aligned}
r &< E\left\|\Phi x^r(t^r)\right\|^2 \\
&\le 6M^2\left(E\|\phi(0)\|^2 + E\left\|h(x^r)\right\|^2\right) + 6M^2 n\sum_{k=1}^{n} E\left\|I_k\left(x^r\left(t_k^{r-}\right)\right)\right\|^2 \\
&\quad + 6E\left\|\int_0^{t^r}\left(t^r-s\right)^{\alpha-1}\mathscr{S}\left(t^r-s\right)Bu(s)\,ds\right\|^2 \\
&\quad + 6E\left\|\int_0^{t^r}\left(t^r-s\right)^{\alpha-1}\mathscr{S}\left(t^r-s\right)f\left(s,x^r_{\rho(s,x^r_s)}\right)ds\right\|^2 \\
&\quad + 6E\left\|\int_0^{t^r}\left(t^r-s\right)^{\alpha-1}\mathscr{S}\left(t^r-s\right)g\left(s,x^r_{\rho(s,x^r_s)}\right)dw(s)\right\|^2 \\
&\le 6M^2\left[E\|\phi(0)\|^2 + M_h\left(1+r^*\right)\right] + 6M^2 n\sum_{k=1}^{n} M_{I_k}(r) \\
&\quad + \frac{6T^\alpha M^2 M_B^2}{\alpha\Gamma^2(\alpha)}\int_0^{t^r}\left(t^r-s\right)^{\alpha-1}E\left\|u^\lambda(s)\right\|^2 ds \\
&\quad + \frac{6T^\alpha M^2}{\alpha\Gamma^2(\alpha)}\int_0^{t^r}\left(t^r-s\right)^{\alpha-1}E\left\|f\left(s,x^r_{\rho(s,x^r_s)}\right)\right\|^2 ds \\
&\quad + \frac{6M^2}{\Gamma^2(\alpha)}\int_0^{t^r}\left(t^r-s\right)^{2\alpha-2}E\left\|g\left(s,x^r_{\rho(s,x^r_s)}\right)\right\|^2 ds \\
&\le 6M^2\left[E\|\phi(0)\|^2 + M_h\left(1+r^*\right)\right] + 6M^2 n\sum_{k=1}^{n} M_{I_k}(r) + \frac{6T^{2\alpha}M^2M_B^2}{\alpha^2\Gamma^2(\alpha)}E\left\|u^\lambda(s)\right\|^2 \\
&\quad + \frac{6T^{2\alpha}M^2}{\alpha^2\Gamma^2(\alpha)}M_f\left(1+r^*\right) + \frac{6T^{2\alpha-1}M^2}{(2\alpha-1)\Gamma^2(\alpha)}M_g\left(1+r^*\right).
\end{aligned}$$

Dividing both sides by r and taking the limit as $r \to \infty$, we obtain

$$\begin{aligned}
1 &\le 6\left[\left(M^2M_h + \frac{T^{2\alpha}M^2}{\alpha^2\Gamma^2(\alpha)}M_f + \frac{T^{2\alpha-1}M^2}{(2\alpha-1)\Gamma^2(\alpha)}M_g\right)H_3 + M^2 n\sum_{k=1}^{n}\gamma_k\right] \\
&\quad \times\left(1 + \frac{7M^4M_B^4T^{2\alpha}}{\lambda^2\Gamma^4(\alpha)\alpha^2}\right),
\end{aligned}$$

which is a contradiction to our assumption. Thus, for each $r > 0$, there exists some positive number r such that $\Phi\mathfrak{B}_r \subset \mathfrak{B}_r$.

Next, we denote $\Phi(x) = \Phi_1(x) + \Phi_2(x)$, where

$$\begin{aligned}
\Phi_1(x)(t) &= \int_0^t (t-s)^{\alpha-1}\mathscr{S}(t-s)f\left(s,x_{\rho(s,x_s)}\right)ds + \int_0^t (t-s)^{\alpha-1}\mathscr{S}(t-s)g\left(s,x_{\rho(s,x_s)}\right)dw(s), \\
\Phi_2(x)(t) &= \mathscr{T}(t)\left[\phi(0) - h(x)\right] + \int_0^t (t-s)^{\alpha-1}\mathscr{S}(t-s)Bu^\lambda(s)\,ds \\
&\quad + \sum_{0<t_k<t}\mathscr{T}(t-t_k)I_k\left(x\left(t_k^-\right)\right).
\end{aligned}$$

Step 2: $\Phi_1(x)$ is contractive. Let $x, y \in \mathfrak{B}_r$, then

$$\begin{aligned}
&E\big\|\Phi_1(x)(t) - \Phi_1(y)(t)\big\|^2 \\
&\quad\le 2E\bigg\|\int_0^t (t-s)^{\alpha-1}\mathscr{S}(t-s)\big[f(s,x_{\rho(s,x_s)}) - f(s,y_{\rho(s,y_s)})\big]\,ds\bigg\|^2 \\
&\qquad + 2E\bigg\|\int_0^t (t-s)^{\alpha-1}\mathscr{S}(t-s)\big[g(s,x_{\rho(s,x_s)}) - g(s,y_{\rho(s,y_s)})\big]\,dw(s)\bigg\|^2 \\
&\quad\le \frac{2M^2T^\alpha}{\Gamma^2(\alpha)\alpha}\int_0^t (t-s)^{\alpha-1}E\big\|f(s,x_{\rho(s,x_s)}) - f(s,y_{\rho(s,y_s)})\big\|^2\,ds \\
&\qquad + \frac{2M^2}{\Gamma^2(\alpha)}\int_0^t (t-s)^{2\alpha-2}E\big\|g(s,x_{\rho(s,x_s)}) - g(s,y_{\rho(s,y_s)})\big\|^2\,ds \\
&\quad\le \frac{2M^2T^{2\alpha}}{\Gamma^2(\alpha)\alpha^2}L_f\big\|x_{\rho(s,x_s)} - y_{\rho(s,y_s)}\big\|^2 + \frac{2M^2T^{2\alpha-1}}{\Gamma^2(\alpha)(2\alpha-1)}L_g\big\|x_{\rho(s,x_s)} - y_{\rho(s,y_s)}\big\|^2 \\
&\quad\le \frac{2M^2}{\Gamma^2(\alpha)}\bigg(\frac{T^{2\alpha}}{\alpha^2}L_f + \frac{T^{2\alpha-1}}{2\alpha-1}L_g\bigg)\overline{H}_3^2\sup_{0\le s\le T}E\big\|x(s) - y(s)\big\|^2 \\
&\quad\le \frac{2M^2}{\Gamma^2(\alpha)}\bigg(\frac{T^{2\alpha}}{\alpha^2}L_f + \frac{T^{2\alpha-1}}{2\alpha-1}L_g\bigg)\overline{H}_3^2\big\|x(s) - y(s)\big\|^2 := L_0\big\|x(s) - y(s)\big\|^2,
\end{aligned}$$

where $L_0 < 1$, hence $\Phi_1 x$ is contractive.

Step 3: Φ_2 maps bounded sets to bounded sets in $\mathfrak{B}_r$. We have

$$\begin{aligned}
&E\big\|\Phi_2(x)(t)\big\|^2 \\
&\quad\le 3E\big\|\mathscr{T}(t)\big(\phi(0) - h(x)\big)\big\|^2 + 3E\big\|\mathscr{T}(t-t_k)I_k\big(x\big(t_k^-\big)\big)\big\|^2 \\
&\qquad + 3E\bigg\|\int_0^t (t-s)^{\alpha-1}\mathscr{S}(t-s)Bu^\lambda(s)\,ds\bigg\|^2 \\
&\quad\le 6M^2\big[E\big\|\phi(0)\big\|^2 + M_h\big(1+r^*\big)\big] + 3M^2n\sum_{k=1}^n M_{I_k}E\big\|x\big(t_k^-\big)\big\|^2 \\
&\qquad + \frac{3T^\alpha M^2M_B^2}{\alpha\Gamma^2(\alpha)}\int_0^t (t-s)^{\alpha-1}E\big\|u^\lambda(s)\big\|^2\,ds \\
&\quad\le 6M^2\big[E\big\|\phi(0)\big\|^2 + M_h\big(1+r^*\big)\big] + 3M^2n\sum_{k=1}^n M_{I_k}r + \frac{3T^{2\alpha}M^2M_B^2}{\alpha^2\Gamma^2(\alpha)}E\big\|u^\lambda\big\|^2.
\end{aligned}$$

Therefore, for each $x \in \mathfrak{B}_r$, $E\|\Phi_2(x)(t)\|^2$ is bounded.

Step 4: Φ_2 is equicontinuous. Let $0 < t < t+h \le T$, where t, $t+h$ and $|h|$ is sufficiently small. By means of Hölder's inequality and the compactness of $\mathscr{T}$ and $\mathscr{S}$, we have

$$\begin{aligned}
&E\big\|\Phi_2x(t+h) - \Phi_2x(t)\big\|^2 \\
&\quad\le 5E\big\|\big[\mathscr{T}(t+h) - \mathscr{T}(t)\big]\big[\phi(0) - h(x)\big]\big\|^2 \\
&\qquad + 5E\bigg\|\sum_{k=1}^n\big[\mathscr{T}(t+h-t_k) - \mathscr{T}(t-t_k)\big]I_kx\big(t_k^-\big)\bigg\|^2 \\
&\qquad + 5E\bigg\|\int_0^t\big[(t+h-s)^{\alpha-1} - (t-s)^{\alpha-1}\big]\mathscr{S}(t+h-s)Bu^\lambda(s)\,ds\bigg\|^2
\end{aligned}$$

$$+ 5E\left\| \int_t^{t+h} (t+h-s)^{\alpha-1} \mathscr{S}(t+h-s)Bu^\lambda(s)\,ds \right\|^2$$
$$+ 5E\left\| \int_0^t (t-s)^{\alpha-1} \big[\mathscr{S}(t+h-s) - \mathscr{S}(t-s)\big]Bu^\lambda(s)\,ds \right\|^2$$
$$\leq 10\epsilon^2\big[E\|\phi(0)\|^2 + M_h(1+r^*)\big] + 5n\epsilon^2\sum_{k=1}^n M_{I_k} r + \frac{5M^2M_B^2}{\alpha^2\Gamma^2(\alpha)}E\|u^\lambda(s)\|^2 h^{2\alpha}$$
$$+ \frac{5M^2M_B^2}{\alpha^2\Gamma^2(\alpha)}E\|u^\lambda(s)\|^2\big[(t+h)^\alpha - t^\alpha - h^\alpha\big]^2 + \frac{5\epsilon^2M_B^2T^{2\alpha}}{\alpha^2}E\|u^\lambda(s)\|^2.$$

Consequently, $E\|\Phi_2 x(t+h) - \Phi_2 x(t)\|^2 \to 0$ as $h \to 0$.

Step 5: $V(t) = \{\Phi_2 x(t), x \in \mathfrak{B}_r\}$ is relatively compact in $\mathscr{B}_r$. For any $\epsilon \in (0,t)$ and $\delta > 0$, we define

$$\begin{aligned}
&\Phi_2^{\epsilon,\delta}x(t)\\
&\quad= \int_\delta^\infty \xi_\alpha(\theta)S(t^\alpha\theta)\big[\phi(0) - h(x)\big]\,d\theta + \sum_{0<t_k<t}\int_\delta^\infty \xi_\alpha(\theta)S\big((t-t_k)^\alpha\theta\big)I_k\big(x(t_k^-)\big)\,d\theta\\
&\qquad+ \alpha\int_0^{t-\epsilon}\int_\delta^\infty \theta(t-s)^{\alpha-1}\xi_\alpha(\theta)S\big((t-s)^\alpha\theta\big)Bu^\lambda(s)\,d\theta\,ds\\
&\quad= S(\epsilon^\alpha\delta)\int_\delta^\infty \xi_\alpha(\theta)S\big(t^\alpha\theta - \epsilon^\alpha\delta\big)\big[\phi(0) - h(x)\big]\,d\theta\\
&\qquad+ S(\epsilon^\alpha\delta)\sum_{0<t_k<t}\int_\delta^\infty \xi_\alpha(\theta)S\big((t-t_k)^\alpha\theta - \epsilon^\alpha\delta\big)I_k\big(x(t_k^-)\big)\,d\theta\\
&\qquad+ \alpha S(\epsilon^\alpha\delta)\int_0^{t-\epsilon}\int_\delta^\infty \theta(t-s)^{\alpha-1}\xi_\alpha(\theta)S\big((t-s)^\alpha\theta - \epsilon^\alpha\delta\big)Bu^\lambda(s)\,d\theta\,ds.
\end{aligned}$$

Because of the compactness of $S(\epsilon^\alpha\delta)$, the set $V^{\epsilon,\delta}(t) = \{\Phi_2^{\epsilon,\delta}x(t), x \in \mathfrak{B}_r\}$ is relatively compact for every $\epsilon \in (0,t)$. Furthermore, $\Phi_2^{\epsilon,\delta}x(t)$ is convergent to $\Phi_2 x(t)$ as $\epsilon \to 0$ and $\delta \to 0$, then $V(t) = \{\Phi_2 x(t), x \in \mathscr{B}_r\}$ is also relatively compact. By the Arzela-Ascoli theorem, Φ_2 is completely continuous. So, by means of the Krasnoselskii fixed point theorem, the operator Φ has a fixed point, which is a mild solution of (1.1). □

Theorem 3.2 *Assume the linear stochastic system* (2.1) *is of approximate controllability, and the conditions of Theorem* 3.1 *hold. Further, assume that the functions f and g are bounded uniformly, then the fractional nonlinear stochastic system* (1.1) *is approximately controllable*.

Proof Let x^λ be a solution of (1.1), then it is not difficult to see that

$$\begin{aligned}
x^\lambda(T) &= \tilde{x}_T - \lambda\big(\lambda I + \Gamma_0^T\big)^{-1}\left[E\tilde{x}_T + \int_0^T \tilde{\phi}(s)\,dw(s) - \mathscr{T}(T)\big(\phi(0) - h\big(x^\lambda(T)\big)\big)\right]\\
&\quad+ \lambda\int_0^T (T-s)^{\alpha-1}\big(\lambda I + \Gamma_s^T\big)^{-1}\mathscr{S}(T-s)f\big(s, x^\lambda_{\rho(s,x_s^\lambda)}\big)\,ds\\
&\quad+ \lambda\int_0^T (T-s)^{\alpha-1}\big(\lambda I + \Gamma_s^T\big)^{-1}\mathscr{S}(T-s)g\big(s, x^\lambda_{\rho(s,x_s^\lambda)}\big)\,dw(s)\\
&\quad+ \lambda\big(\lambda I + \Gamma_0^T\big)^{-1}\sum_{0<t_k<T}\mathscr{T}(T-t_k)I_k\big(x(t_k^-)\big).
\end{aligned}$$

In view of the uniformly boundedness of f and g, there are subsequences still denoted by $f(s,x^\lambda(s))$ and $g(s,x^\lambda(s))$, which converge weakly to $f(s)$ and $g(s)$. On the other hand, the operator $\lambda(\lambda I+\Gamma_s^T)^{-1}\to 0$ strongly from the assumption of Theorem 3.2 as $\lambda\to 0^+$ for all $s\in[0,T]$, moreover, $\|\lambda(\lambda I+\Gamma_s^T)^{-1}\|<1$. Thus, the Lebesgue dominated convergence theorem and the compactness of $\mathscr{S}$ and $\mathscr{T}$ yield

$$\begin{aligned}
&E\big\|x^\lambda(T)-\tilde{x}_T\big\|^2\\
&\quad\le 6E\left\|\lambda\big(\lambda I+\Gamma_0^T\big)^{-1}\left[E\tilde{x}_T+\int_0^T\tilde{\phi}(s)\,dw(s)-\mathscr{T}(T)\big(\phi(0)-h\big(x^\lambda(T)\big)\big)\right]\right\|^2\\
&\qquad+6E\left(\int_0^T(T-s)^{\alpha-1}\big\|\lambda\big(\lambda I+\Gamma_s^T\big)^{-1}\big\|\cdot\big\|\mathscr{S}(T-s)\big[f\big(s,x^\lambda_{\rho(s,x_s^\lambda)}\big)-f(s)\big]\big\|\,ds\right)^2\\
&\qquad+6E\left(\int_0^T(T-s)^{\alpha-1}\big\|\lambda\big(\lambda I+\Gamma_s^T\big)^{-1}\mathscr{S}(T-s)f(s)\big\|\,ds\right)^2\\
&\qquad+6E\left(\int_0^T(T-s)^{\alpha-1}\big\|\lambda\big(\lambda I+\Gamma_s^T\big)^{-1}\big\|\cdot\big\|\mathscr{S}(T-s)\big[g\big(s,x^\lambda_{\rho(s,x_s^\lambda)}\big)-g(s)\big]\big\|\,dw(s)\right)^2\\
&\qquad+6E\left(\int_0^T(T-s)^{\alpha-1}\big\|\lambda\big(\lambda I+\Gamma_s^T\big)^{-1}\mathscr{S}(T-s)g(s)\big\|\,dw(s)\right)^2\\
&\qquad+6E\big\|\lambda\big(\lambda I+\Gamma_0^T\big)^{-1}\mathscr{T}(T-t_k)I_k\big(x^\lambda\big(t_k^-\big)\big)\big\|^2\to 0,\quad\text{as }\lambda\to 0.
\end{aligned}$$

This gives the approximate controllability of (1.1), which completes the proof. □

4 Example

Example 4.1 As a simple application, we consider a control system governed by the fractional stochastic partial differential equation with state-dependent delay that is impulsive and of the form

$$\begin{cases}
{}^cD_t^\alpha x(t,z)=\frac{\partial^2}{\partial z^2}x(t,z)+\mu(t,z)+\int_{-\infty}^t a(s-t)x(s-\rho_1(t)\rho_2(\|x(t)\|),z)\,ds\\
\qquad\qquad+\big[\int_{-\infty}^t b(s-t)x(s-\rho_1(t)\rho_2(\|x(t)\|),z)\,ds\big]\frac{d\beta(t)}{dt},\\
\qquad\qquad t\in[0,T]\backslash\{t_1,t_2,\ldots,t_n\},\\
x(t,0)=x(t,\pi)=0,\\
x(\tau,z)=\phi(\tau,z),\quad \tau\le 0,z\in[0,\pi],\\
\triangle x(t_k,z)=\int_{-\infty}^{t_k}p(t_k-s)x(s,z)\,dz,\quad k=1,2,\ldots,n,
\end{cases}\tag{4.1}$$

where $0<t_1<t_2<\cdots<t_n<T$ are prefixed numbers, $\rho_i:[0,\infty)\to[0,\infty)$, $i=1,2$, $a,b:R\to R$ are continuous. Let $\beta(t)$ stands for a standard one-dimensional Wiener process in $H=L^2[0,\pi]$ defined on a stochastic space $(\Omega,\mathscr{F},P)$ and $\phi\in\mathfrak{B}=PC\times L^2(g,H)$ ($g:(-\infty,-r]\to R$ is a positive function) be the phase space used in Hino *et al.* [22].

Let $H=U=L^2[0,\pi]$ and define the operator A by $A\omega=\omega''$ with the domain $D(A)=\{\omega(\cdot)\in L^2[0,\pi],\omega,\omega'\text{ are absolutely continuous},\omega''\in L^2[0,\pi],\omega(0)=\omega(\pi)\}$. Then $A\omega=\sum_{n=1}^\infty -n^2\langle\omega,e_n\rangle e_n$, $\omega\in D(A)$, where $e_n(z)=\sqrt{\frac{2}{\pi}}\sin nz$ is the orthogonal basis of eigenvectors of A. Clearly, A generates a compact analytic semigroup $\{S(t)\}_{t>0}$ in H and it can be written by $S(t)\omega=\sum_{n=1}^\infty e^{-n^2t}\langle\omega,e_n\rangle e_n$, $\omega\in X$.

Defining the maps $\rho, f, g : J \times \mathfrak{B} \to H$ by $\rho(t,\varphi)(z) = t - \rho_1(t)\rho_2(\|\varphi(0,z)\|)$, $f(t,\varphi)(z) = \int_{-\infty}^{0} a(s)\varphi(s,z)\,ds$, $g(t,\varphi)(z) = \int_{-\infty}^{0} b(s)\varphi(s,z)\,ds$, $I_k(x)(z) = \int_0^{t_k} p(t_k - s)x(s,z)\,dz$. Taking $B : U \to H$ by $Bu(t)(z) = \mu(t,z)$, $0 \le z \le \pi$, $u \in U$, where $\mu : [0,T] \times [0,\pi] \to H$ is continuous.

Now, under the above preparations, we can represent the partial stochastic differential equation (4.1) in the abstract form (1.1). Moreover, f and g are bounded linear operators, $\|f\| \le M_f$, $\|g\| \le M_g$, and $\|I_k\| \le M_{I_k}$. All the conditions of Theorem 3.2 are fulfilled, so we can claim that the system (4.1) is approximately controllable.

Remark 4.1 This paper has investigated the approximate controllability of fractional stochastic impulsive functional differential system with state-dependent delay in Hilbert spaces. Firstly we have constructed the control function by using the stochastic analysis technique, it is very important for us to study the approximate controllability. On basis of this control function, a set of sufficient conditions for the approximate controllability of the control system has been obtained with the help of the strong continuous semigroup, and the Krasnoselskii fixed point theorem.

The differential control systems with delay often arise in applications, the control system with state-dependent delay considered in this paper can be come bake to the usual control system with delay as $\rho(t, x_t) = t$. Therefore, our main results are the generalizations and continuations of the recent results on the differential control systems with delay. Moreover, one can consider the neutral stochastic functional differential equations according to the method in this paper.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Each of the authors, XZ, CZ, and CG, contributed to each part of this work equally. All authors read and approved the final manuscript.

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