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Some Tauberian theorems for four-dimensional Euler and Borel summability

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Abstract

The four-dimensional summability methods of Euler and Borel are studied as mappings from absolutely convergent double sequences into themselves. Also the following Tauberian results are proved: if $x = (x_{m,n})$ is a double sequence that is mapped into ℓ_2 by the four-dimensional Borel method and the double sequence x satisfies $\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}|\Delta_{10}x_{m,n}|\sqrt{mn} < \infty$ and $\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}|\Delta_{01}x_{m,n}|\sqrt{mn} < \infty$, then x itself is in ℓ_2.

MSC: Primary 40B05; secondary 40C05

Keywords: Tauberian condition; Euler-Knopp means; Borel method; four-dimensional summability method; double sequences; Pringsheim limit

1 Introduction

The best-known notion of convergence for double sequences is convergence in the sense of Pringsheim. Recall that a double sequence $x = \{x_{k,l}\}$ of complex (or real) numbers is called convergent to a scalar L in the sense of Pringsheim (denoted by P-$\lim x = L$) if for every $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that $|x_{k,l} - L| < \epsilon$ whenever $k, l > N$. Such an x is described more briefly as 'P-convergent'. It is easy to verify that $x = \{x_{k,l}\}$ converges in the sense of Pringsheim if and only if for every $\epsilon > 0$ there exists an integer $N = N(\epsilon)$ such that $|x_{i,j} - x_{k,l}| < \epsilon$ whenever $\min\{i, j, k, l\} \geq N$. A double sequence $x = \{x_{k,l}\}$ is bounded if there exists a positive number M such that $|x_{m,n}| \leq M$ for all m and n, that is, if $\sup_{m,n} |x_{m,n}| < \infty$.

A double sequence $x = \{x_{k,l}\}$ is said to convergence regularly if it converges in the sense of Pringsheim and, in addition, the following finite limits exist:

$$\lim_{m\to\infty} x_{m,n} = \ell_n \quad (n = 1, 2, \dots),$$

$$\lim_{n\to\infty} x_{m,n} = \mathcal{L}_m \quad (m = 1, 2, \dots).$$

Let $A = (a_{m,n,k,l})$ denote a four-dimensional summability method that maps the complex double sequence x into the double sequence Ax where the mnth term of Ax is as follows:

$$(Ax)_{m,n} = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} a_{m,n,k,l}x_{k,l}.$$

In [1] Robison presented the following notion of regularity for four-dimensional matrix transformation and a Silverman-Toeplitz type characterization of such a notion.

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Definition 1.1 The four-dimensional matrix A is said to be RH-regular if it maps every bounded P-convergent sequence into a P-convergent sequence with the same P-limit.

The assumption of boundedness was added because a double sequence which is P-convergent is not necessarily bounded. Along these same lines, Robison and Hamilton presented a Silverman-Toeplitz type multidimensional characterization of regularity in [2] and [1].

Theorem 1.1 (Hamilton [2], Robison [1]) *The four-dimensional matrix A is RH-regular if and only if*

RH_1: $P\text{-}\lim_{m,n} a_{m,n,k,l} = 0$ *for each k and l*;
RH_2: $P\text{-}\lim_{m,n} \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} = 1$;
RH_3: $P\text{-}\lim_{m,n} \sum_{k=0}^{\infty} |a_{m,n,k,l}| = 0$ *for each l*;
RH_4: $P\text{-}\lim_{m,n} \sum_{l=0}^{\infty} |a_{m,n,k,l}| = 0$ *for each k*;
RH_5: $\sum_{k,l=0,0}^{\infty,\infty} |a_{m,n,k,l}|$ *is P-convergent*;
RH_6: *there exist finite positive integers Δ and Γ such that* $\sum_{k,l>\Gamma} |a_{m,n,k,l}| < \Delta$.

The set of all absolutely convergent double sequences will be denoted ℓ_2, that is,

$$\ell_2 = \left\{ x = \{x_{k,l}\} : \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} |x_{k,l}| < \infty \right\}.$$

For single sequences, in [3] Fridy and Roberts proved the following Tauberian theorem.

Theorem 1.2 *If B is a Borel matrix and $x = (x_k)$ is a sequence such that Bx in $\ell = \{x = (x_k) : \sum_{k=1}^{\infty} |x_k| < \infty\}$ and*

$$\sum_{r=1}^{\infty} |\Delta x_r| \sqrt{r} < \infty,$$

then x is in ℓ.

Our aim is to extend the results in [3] from single absolutely convergent sequences to double absolutely convergent sequences. In [4], Patterson proved that the matrix $A = (a_{m,n,k,l})$ determines an ℓ_2-ℓ_2 method if and only if

$$\sup_{k,l} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} |a_{m,n,k,l}| < \infty. \tag{1.1}$$

2 Euler-Knopp and Borel ℓ_2-ℓ_2 methods

The four-dimensional Euler-Knopp method, for any complex numbers r_1 and r_2, is defined by

$$\begin{aligned} &E_{r_1,r_2}[m,n,k,l] \\ &\quad = \begin{cases} \binom{m}{k}\binom{n}{l} r_1^{k+1}(1-r_1)^{m-k} r_2^{l+1}(1-r_2)^{n-l}, & \text{if } k \le m, l \le n, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

An application of the Maclaurin series expansion of $(1-z_1)^{k+1}(1-z_2)^{l+1}$ shows that each column sum of E_{r_1,r_2} converges absolutely to $\frac{1}{r_1 r_2}$ provided that $0 < r_1 \leq 1$ and $0 < r_2 \leq 1$. If $0 < r_1 < 1$ and $0 < r_2 < 1$, then $P\text{-}\lim_{m,n} E_{r_1,r_2}[m,n,m,n] = 0$, so $E^{-1}_{r_1,r_2}$ is not an ℓ_2-ℓ_2 matrix. We summarize this as follows.

Theorem 2.1 *The four-dimensional Euler-Knopp method E_{r_1,r_2} is a sum-preserving ℓ_2-ℓ_2 matrix for which $\ell_{2_{E_{r_1,r_2}}} \neq \ell_2$ if and only if $0 < r_1 < 1$ and $0 < r_2 < 1$, where $\ell_{2_{E_{r_1,r_2}}}$ is the summability field of E_{r_1,r_2}.*

The four-dimensional Borel method B is given by the matrix

$$b_{m,n,k,l} = \frac{e^{-(m+n)} m^k n^l}{k!l!}, \quad m,n,k,l = 0,1,2,3,\ldots.$$

By a direct application of Theorem 3.1 in [4], one can show that B is an ℓ_2-ℓ_2 matrix.

Theorem 2.2 *If $r_1 > 0$ and $r_2 > 0$ and $x = (x_{k,l})$ is a double sequence such that $E_{r_1,r_2}x$ is in ℓ_2, then Bx is in ℓ_2.*

Proof We use the familiar technique of showing that BE_{r_1,r_2} is an ℓ_2-ℓ_2 matrix. Since $Bx = BE^{-1}_{r_1,r_2}E_{r_1,r_2}x$, this will ensure that Bx is in ℓ_2 whenever $E_{r_1,r_2}x$ in ℓ_2. Since $E^{-1}_{r_1,r_2} = E_{\frac{1}{r_1},\frac{1}{r_2}}$ we replace $s_1 = \frac{1}{r_1}$ and $s_2 = \frac{1}{r_2}$ and show that BE_{s_1,s_2} is an ℓ_2-ℓ_2 matrix for all positive s_1 and s_2. The *mnkl*th term of BE_{s_1,s_2} is given by

$$\begin{aligned}
&BE_{s_1,s_2}[m,n,k,l] \\
&\quad= \sum_{i=k}^{\infty}\sum_{j=l}^{\infty} \frac{e^{-(m+n)} m^i n^j}{i!j!} \binom{i}{k}\binom{j}{l} (1-s_1)^{i-k} s_1^k (1-s_2)^{j-l} s_2^l \\
&\quad= \frac{e^{-(m+n)} m^k n^l s_1^k s_2^l}{k!l!} \sum_{i=k}^{\infty}\sum_{j=l}^{\infty} \frac{m^{i-k} n^{j-k}}{(i-k)!(j-l)!} (1-s_1)^{i-k}(1-s_2)^{j-l} \\
&\quad= \frac{(ms_1)^k (ns_2)^l e^{-(ms_1+ns_2)}}{k!l!}.
\end{aligned}$$

Summing the (k,l)th column of BE_{s_1,s_2}, we get

$$\begin{aligned}
\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \bigl|BE_{s_1,s_2}[m,n,k,l]\bigr| &= \frac{1}{k!l!}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} (ms_1)^k (ns_2)^l e^{-(ms_1+ns_2)} \\
&= O\biggl(\frac{1}{k!l!}\int_0^{\infty}\int_0^{\infty} (t_1 s_1)^k (t_2 s_2)^l e^{-(t_1 s_1 + t_2 s_2)}\,dt_1\,dt_2\biggr) \\
&= O\biggl(\frac{1}{s_1 s_2}\biggr).
\end{aligned}$$

Hence,

$$\sup_{k,l} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \bigl|BE_{s_1,s_2}[m,n,k,l]\bigr| < \infty,$$

so BE_{s_1,s_2} is an ℓ_2-ℓ_2 matrix. $\square$

Theorem 2.2 and the ℓ_2-ℓ_2 property of E_{r_1,r_2} lead to the following result.

Theorem 2.3 *The four-dimensional Borel matrix determines an ℓ_2-ℓ_2 method.*

In addition to the inclusion relation given in Theorem 2.2, we can also show that the ℓ_2-ℓ_2 method B is strictly stronger than all E_{r_1,r_2} methods by the following example.

Example 2.1 Suppose $r_1 > 0$ and $r_2 > 0$ and $x_{k,l} = (-s_1)^k(-s_2)^l$ where $s_1 \geq -1 + \frac{2}{r_1}$ and $s_2 \geq -1 + \frac{2}{r_2}$; then Bx is in ℓ_2 but E_{r_1,r_2} is not in ℓ_2. Let us consider the following methods:

$$\begin{aligned}(Bx)_{m,n} &= \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} e^{-(m+n)}\frac{m^k}{k!}\frac{n^l}{l!}(-s_1)^k(-s_2)^l \\ &= e^{-(m+n)}e^{-(s_1m+s_2n)} = e^{-[(s_1+1)m+(s_2+1)n]}\end{aligned}$$

and

$$\begin{aligned}(E_{r_1,r_2}x)_{m,n} &= \sum_{k=0}^{m}\sum_{l=0}^{n}\binom{m}{k}\binom{n}{l}(1-r_1)^{m-k}(-r_1s_1)^k(1-r_2)^{n-l}(-r_2s_2)^l \\ &= (1-r_1-r_1s_1)^m(1-r_2-r_2s_2)^n.\end{aligned}$$

By solving $-1 < 1 - r_1 - r_1s_1 < 1$ and $-1 < 1 - r_2 - r_2s_2 < 1$, we see that $E_{r_1,r_2}x$ is in ℓ_2 if and only if $-1 < s_1 < -1 + \frac{2}{r_1}$ and $-1 < s_2 < -1 + \frac{2}{r_2}$.

3 Tauberian theorems

To prove Theorem 3.1 we need the following lemma.

Lemma 3.1 *If*

$$b_{m,n,k,l} = \frac{e^{-(m+n)}m^k n^l}{k!l!}$$

and r and s are positive integers, then

(i)

$$\sum_{m=r+1}^{\infty}\sum_{n=s+1}^{\infty}\sum_{k=0}^{r}\sum_{l=0}^{s} b_{m,n,k,l} = O(\sqrt{rs})$$

and

(ii)

$$\sum_{m=0}^{r}\sum_{n=0}^{s}\sum_{k=r+1}^{\infty}\sum_{l=s+1}^{\infty} b_{m,n,k,l} = O(\sqrt{rs}).$$

Proof Let $p = [\sqrt{r}]$ and $q = [\sqrt{s}]$, and let us write the sum in (i) as

$$\sum_{m=r+1}^{\infty}\sum_{n=s+1}^{\infty}\sum_{k=0}^{r-p}\sum_{l=0}^{s-q} b_{m,n,k,l} + \sum_{m=r+1}^{\infty}\sum_{n=s+1}^{\infty}\sum_{k=r-p+1}^{r}\sum_{l=s-q+1}^{s} b_{m,n,k,l} = \phi_{r,s} + \varphi_{r,s}.$$

If $s_1 < m$ and $s_2 < n$, then

$$\begin{aligned}\sum_{k=0}^{s_1}\sum_{l=0}^{s_2}\frac{m^m n^l}{k!l!} &= \frac{m^{s_1}n^{s_2}}{s_1!s_2!}\left(1+\frac{s_1}{m}+\frac{s_1}{m}\frac{s_1-1}{m}+\cdots\right)\left(1+\frac{s_2}{n}+\frac{s_2}{n}\frac{s_2-1}{n}+\cdots\right)\\ &\le \frac{m^{s_1}n^{s_2}}{s_1!s_2!}\left(1+\frac{s_1}{m}+\left(\frac{s_1}{m}\right)^2+\cdots\right)\left(1+\frac{s_2}{m}+\left(\frac{s_2}{m}\right)^2+\cdots\right)\\ &= \frac{m^{s_1}n^{s_2}}{s_1!s_2!}\frac{m}{m-s_1}\frac{n}{n-s_2}.\end{aligned}$$

In $\phi_{r,s}$, let $s_1 = r-p$, $s_2 = s-q$, and

$$\max_{m\ge r+1;n\ge s+1}\frac{mn}{(m-r+p)(n-s+q)} = \frac{(r+1)(s+1)}{(p+1)(q+1)} \le (\sqrt{r}+1)(\sqrt{s}+1),$$

thus

$$\phi_{r,s} < (\sqrt{r}+1)(\sqrt{s}+1)\frac{1}{(r-p)!(s-q)!}\sum_{m=r+1}^{\infty}\sum_{n=s+1}^{\infty}e^{-(m+n)}m^{r-p}n^{s-q} \le (\sqrt{r}+1)(\sqrt{s}+1).$$

In $\varphi_{r,s}$, observe that

$$\sum_{k=r-p+1}^{r}\sum_{l=s-q+1}^{s}b_{m,n,k,l} \le \sqrt{rs}\max_{k\ge r;l\ge s}b_{m,n,k,l} = \sqrt{rs}e^{-(m+n)}\frac{m^r n^s}{r!s!},$$

thus

$$\varphi_{r,s} \le \sqrt{rs}\frac{1}{r!s!}\sum_{m=r+1}^{\infty}\sum_{n=s+1}^{\infty}e^{-(m+n)}m^r n^s \le \sqrt{rs}.$$

Hence, (i) is proved. Next write the sum in (ii) as

$$\sum_{m=0}^{r}\sum_{n=0}^{s}\sum_{k=r+1}^{r+p-1}\sum_{l=s+1}^{s+q-1}b_{m,n,k,l} + \sum_{m=0}^{r}\sum_{n=0}^{s}\sum_{k=r+p}^{\infty}\sum_{l=s+q}^{\infty}b_{m,n,k,l} = \lambda_{r,s}+\mu_{r,s}.$$

Assume that $\lambda_{r,s} = 0$ if $p = 1$, $q = 1$. Then

$$\begin{aligned}\lambda_{r,s} &\le (p-1)(q-1)\sum_{m=0}^{r}\sum_{n=0}^{s}e^{-(m+n)}\max_{k>r;l>s}\frac{m^k n^l}{k!l!}\\ &\le (\sqrt{r}-1)(\sqrt{s}-1)\frac{1}{(r+1)!(s+1)!}\sum_{m=0}^{r}\sum_{n=0}^{s}e^{-(m+n)}m^{r+1}n^{s+1}\\ &\le (\sqrt{r}-1)(\sqrt{s}-1).\end{aligned}$$

If $s_1 \ge m$ and $s_2 \ge n$, then

$$\begin{aligned}&\sum_{k=s_1}^{\infty}\sum_{l=s_2}^{\infty}\frac{m^k n^l}{k!l!}\\ &\quad= \frac{m^{s_1}n^{s_2}}{s_1!s_2!}\left(1+\frac{m}{s_1+1}+\frac{m}{s_1+1}\frac{m}{s_1+2}+\cdots\right)\left(1+\frac{n}{s_2+1}+\frac{n}{s_2+1}\frac{n}{s_2+2}+\cdots\right)\end{aligned}$$

$$\leq \frac{m^{s_1}n^{s_2}}{s_1!s_2!}\left(1+\frac{m}{s_1+1}+\left(\frac{m}{s_1+1}\right)^2+\cdots\right)\left(1+\frac{n}{s_2+1}+\left(\frac{n}{s_2+1}\right)^2+\cdots\right)$$
$$= \frac{m^{s_1}n^{s_2}}{s_1!s_2!}\frac{(s_1+1)(s_2+1)}{(s_1+1-m)(s_2+1-n)}.$$

Let $s_1 = r + p$ and $s_2 = s + q$, we have

$$\mu_{r,s} \leq \frac{1}{(r+p)!(s+q)!}\sum_{m=0}^{r}\sum_{n=0}^{s} e^{-(m+n)}m^{r+p}n^{s+q}\left(\frac{r+p+1}{r+p+1-m}\right)\left(\frac{s+q+1}{s+q+1-n}\right)$$
$$\leq \frac{r+p+1}{p+1}\frac{s+q+1}{q+1}\frac{1}{(r+p)!(s+q)!}\sum_{m=0}^{r}\sum_{n=0}^{s} e^{-(m+n)}m^{r+p}n^{s+q}$$
$$\leq (\sqrt{r}+1)(\sqrt{s}+1).$$

Thus the lemma is proved. □

We are now ready to prove the following result.

Theorem 3.1 *If x is a double sequence such that Bx is in ℓ_2,*

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}|\Delta_{10}x_{s,r}|\sqrt{rs} < \infty \tag{3.1}$$

and

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}|\Delta_{01}x_{s,r}|\sqrt{rs} < \infty, \tag{3.2}$$

then x in ℓ_2 where $\Delta_{10}x_{r,s} = x_{r,s} - x_{r+1,s}$ and $\Delta_{01}x_{r,s} = x_{r,s} - x_{r,s+1}$.

Proof It is suffices to show that $Bx - x$ is in ℓ_2; that is,

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left|\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} b_{m,n,k,l}x_{k,l} - x_{m,n}\right| < \infty.$$

Since

$$\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} b_{m,n,k,l} = 1$$

for each m, n, the above sum can be written as

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left|\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} b_{m,n,k,l}(x_{k,l} - x_{m,n})\right|$$

and we need only show the following:

$$S = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} b_{m,n,k,l}|x_{k,l} - x_{m,n}| < \infty.$$

Let $S = S_1 + S_2$, where

$$S_1 = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{m}\sum_{l=0}^{n} b_{m,n,k,l}|x_{k,l} - x_{m,n}|$$

and

$$S_2 = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=m+1}^{\infty}\sum_{l=n+1}^{\infty} b_{m,n,k,l}|x_{k,l} - x_{m,n}|.$$

Since

$$|x_{k,l} - x_{m,n}| = |x_{m,n} - x_{k,l}| = \left|\sum_{s=m}^{k-1}\Delta_{10}x_{s,n} + \sum_{r=n}^{l-1}\Delta_{01}x_{k,r}\right|,$$

$$\begin{aligned}
S_1 &\leq \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{m-1}\sum_{l=0}^{n-1} b_{m,n,k,l}\left(\sum_{s=m}^{k-1}|\Delta_{10}x_{s,n}| + \sum_{r=n}^{l-1}|\Delta_{01}x_{k,r}|\right) \\
&\leq \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}|\Delta_{10}x_{r,s}| \sum_{m=r+1}^{\infty}\sum_{n=s+1}^{\infty}\sum_{k=0}^{r}\sum_{l=0}^{s} b_{m,n,k,l} \\
&\quad + \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}|\Delta_{01}x_{r,s}| \sum_{m=r+1}^{\infty}\sum_{n=s+1}^{\infty}\sum_{k=0}^{r}\sum_{l=0}^{s} b_{m,n,k,l} \\
&= \left(\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}|\Delta_{10}x_{r,s}| + \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}|\Delta_{01}x_{r,s}|\right)\zeta_{r,s}, \quad \text{say.}
\end{aligned}$$

Also,

$$\begin{aligned}
S_2 &\leq \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=m+1}^{\infty}\sum_{l=n+1}^{\infty} b_{m,n,k,l}\left(\sum_{s=m}^{k-1}|\Delta_{10}x_{s,n}| + \sum_{r=n}^{l-1}|\Delta_{01}x_{k,r}|\right) \\
&\leq \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}|\Delta_{10}x_{r,s}| \sum_{m=0}^{r}\sum_{n=0}^{s}\sum_{k=r+1}^{\infty}\sum_{l=s+1}^{\infty} b_{m,n,k,l} \\
&\quad + \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}|\Delta_{01}x_{r,s}| \sum_{m=0}^{r}\sum_{n=0}^{s}\sum_{k=r+1}^{\infty}\sum_{l=s+1}^{\infty} b_{m,n,k,l} \\
&= \left(\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}|\Delta_{10}x_{r,s}| + \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}|\Delta_{01}x_{r,s}|\right)\varsigma_{r,s}, \quad \text{say.}
\end{aligned}$$

By Lemma 3.1, $\zeta_{r,s} = O(\sqrt{rs})$ and $\varsigma_{r,s} = O(\sqrt{rs})$, we have

$$S_1 + S_2 \leq \lambda\left(\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}|\Delta_{10}x_{s,r}|\sqrt{rs} + \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}|\Delta_{01}x_{s,r}|\sqrt{rs}\right) < \infty,$$

which proves the theorem. □

Combining Theorem 3.1 with Theorem 2.2, we are lead to the following ℓ_2-ℓ_2 Tauberian theorem for the four-dimensional Euler-Knopp means.

Theorem 3.2 *If $r_1 > 0$, $r_2 > 0$, and x is a double sequence satisfying* (3.1) *such that E_{r_1,r_2} is in ℓ_2, then x is in ℓ_2.*

Example 3.1 The following double sequence is not mapped into ℓ_2 by B or by E_{r_1,r_2}, with $r_1 > 0$, $r_2 > 0$. Define $x = \{x_{k,l}\}$ by

$$x_{0,0} = \frac{\pi^2}{3} \quad \text{and} \quad \Delta_{01}x_{k,j} = \frac{1}{(j+1)^2}, \qquad \Delta_{10}x_{i,0} = \frac{1}{(i+1)^2}, \quad i,j = 1,2,3,\ldots.$$

Then x satisfies (3.1) and (3.2), but x is not in ℓ_2 because if $k \geq 1$ and $l \geq 1$,

$$\begin{aligned} x_{k,l} &= x_{0,0} - \sum_{i=0}^{k-1} \Delta_{10}x_{i,0} - \sum_{j=0}^{l-1} \Delta_{01}x_{k,j} \\ &= \frac{\pi^2}{3} - \sum_{i=0}^{k-1} \frac{1}{(i+1)^2} - \sum_{j=0}^{l-1} \frac{1}{(j+1)^2} \\ &= \frac{\pi^2}{6} - \sum_{i=0}^{k-1} \frac{1}{(i+1)^2} + \frac{\pi^2}{6} - \sum_{j=0}^{l-1} \frac{1}{(j+1)^2} \\ &= \sum_{i=k}^{\infty} \frac{1}{(i+1)^2} + \sum_{j=l}^{\infty} \frac{1}{(j+1)^2} \sim \frac{1}{k} - \frac{1}{l}. \end{aligned}$$

Hence, by Theorem 3.1, Bx is not in ℓ_2.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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