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Oscillation of second-order nonlinear delay differential equations with nonpositive neutral coefficients

Qi Li, Rui Wang, Feng Chen and Tongxing Li*

*Correspondence:
litongx2007@163.com
Qingdao Technological University,
Feixian, Shandong 273400,
P.R. China

Abstract
We present several oscillation criteria for a second-order nonlinear delay differential equation with a nonpositive neutral coefficient. Two examples are given to illustrate the main results.

MSC: 34K11

Keywords: oscillation; second-order neutral equation; nonlinear delay differential equation

1 Introduction

In this work, we study the oscillation of a nonlinear second-order neutral delay differential equation

$$\big(r(t)\big(z'(t)\big)^{\alpha}\big)' + q(t)f\big(x\big(\sigma(t)\big)\big) = 0, \quad t \geq t_0 > 0, \tag{1.1}$$

where $z(t) = x(t) - p(t)x(\tau(t))$ and $\alpha > 0$ is the ratio of two odd integers. Throughout, we assume that the following hypotheses are satisfied:

(H$_1$) $r, p, q \in \mathrm{C}([t_0, \infty), \mathbb{R})$, $r(t) > 0$, $0 \leq p(t) \leq p_0 < 1$, $q(t) \geq 0$, and q is not identically zero for large t;

(H$_2$) $\tau \in \mathrm{C}([t_0, \infty), \mathbb{R})$, $\tau(t) \leq t$, and $\lim_{t\to\infty} \tau(t) = \infty$;

(H$_3$) $\sigma \in \mathrm{C}^1([t_0, \infty), \mathbb{R})$, $\sigma'(t) > 0$, $\sigma(t) \leq t$, and $\lim_{t\to\infty} \sigma(t) = \infty$;

(H$_4$) $f \in \mathrm{C}(\mathbb{R}, \mathbb{R})$, $uf(u) > 0$ for all $u \neq 0$, and there exists a positive constant k such that

$$\frac{f(u)}{u^{\alpha}} \geq k \quad \text{for all } u \neq 0.$$

By a solution to (1.1), we mean a function $x \in \mathrm{C}([T_x, \infty), \mathbb{R})$, $T_x \geq t_0$ which has the property $r(z')^{\alpha} \in \mathrm{C}^1([T_x, \infty), \mathbb{R})$ and satisfies (1.1) on the interval $[T_x, \infty)$. We consider only those solutions of (1.1) which satisfy condition $\sup\{|x(t)| : t \geq T\} > 0$ for all $T \geq T_x$ and assume that (1.1) possesses such solutions. As usual, a solution of (1.1) is called oscillatory if it has arbitrarily large zeros on $[T_x, \infty)$; otherwise, it is said to be nonoscillatory. Equation (1.1) is termed oscillatory if all its solutions oscillate.

In recent years, there has been increasing interest in studying oscillation of solutions to different classes of differential equations due to the fact that they have numerous applications in natural sciences and engineering; see, *e.g.*, Hale [1] and Wong [2]. In particular, many papers deal with oscillatory behavior of second-order and third-order delay differential equations; see, for instance, [2–15] and the references cited therein.

In what follows, we provide some background details regarding the study of oscillation of second-order differential equations which motivated our study. Oscillation criteria for (1.1) and its particular cases have been reported in [2–5, 9–12, 14, 15]. A commonly used assumption is

$$-1 < p(t) \leq 0,$$

although several authors studied the oscillation of (1.1) in the case where

$$-\infty < -p_0 \leq p(t) \leq 0.$$

In particular, Wong [2] and Yang *et al.* [15] obtained several oscillation theorems for (1.1) under the assumptions that

$$0 \leq p(t) \leq p_0 < 1 \tag{1.2}$$

and

$$\tau(t) = t - \tau_0 \leq t \quad \text{and} \quad \sigma(t) = t - \sigma_0 \leq t; \tag{1.3}$$

see also the paper by Qin *et al.* [14] where inaccuracies in [15] were pointed out.

Baculíková and Džurina [6] investigated the asymptotic properties of the couple of third-order neutral differential equations

$$\left(r(t)\left(\left[x(t) \pm p(t)x\left(\delta(t)\right)\right]''\right)^{\gamma}\right)' + q(t)x^{\gamma}\left(\tau(t)\right) = 0$$

assuming that

$$\int_{t_0}^{\infty} r^{-1/\alpha}(t)\,\mathrm{d}t = \infty. \tag{1.4}$$

On the basis of the ideas exploited by [6], we derive some new oscillation results for (1.1). In the sequel, all functional inequalities are assumed to hold for all t large enough. Without loss of generality, we can deal only with positive solutions of (1.1).

2 Lemmas

In this section, we give two lemmas that will be useful for establishing oscillation criteria for (1.1).

Lemma 2.1 *Let conditions* (H_1)-(H_4) *and* (1.4) *be satisfied and assume that* x *is a positive solution of* (1.1). *Then* z *satisfies the following two possible cases*:

(C_1) $z(t) > 0$, $z'(t) > 0$, $(r(t)(z'(t))^{\alpha})' \leq 0$;

(C_2) $z(t) < 0$, $z'(t) > 0$, $(r(t)(z'(t))^\alpha)' \le 0$,

for $t \ge t_1$, *where* $t_1 \ge t_0$ *is sufficiently large.*

Proof Suppose that there exists a $t_1 \ge t_0$ such that $x(t) > 0$, $x(\tau(t)) > 0$, and $x(\sigma(t)) > 0$ for $t \ge t_1$. It follows from (1.1) that

$$\bigl(r(t)\bigl(z'(t)\bigr)^\alpha\bigr)' \le -kq(t)x^\alpha\bigl(\sigma(t)\bigr) \le 0. \tag{2.1}$$

Hence, $r(z')^\alpha$ is nonincreasing and of one sign. That is, there exists a $t_2 \ge t_1$ such that $z'(t) > 0$ or $z'(t) < 0$ for $t \ge t_2$.

If $z'(t) > 0$ for $t \ge t_2$, then we have (C_1) or (C_2). We prove now that $z'(t) < 0$ cannot occur. If $z'(t) < 0$ for $t \ge t_2$, then

$$r(t)\bigl(z'(t)\bigr)^\alpha \le -c < 0$$

for $t \ge t_2$, where $c = -r(t_2)(z'(t_2))^\alpha > 0$. Thus, we conclude that

$$z(t) \le z(t_2) - c^{1/\alpha}\int_{t_2}^{t} r^{-1/\alpha}(s)\,\mathrm{d}s.$$

By virtue of condition (1.4), $\lim_{t\to\infty} z(t) = -\infty$. We consider now the following two cases separately.

Case 1. If x is unbounded, then there exists a sequence $\{t_k\}$ such that $\lim_{k\to\infty} t_k = \infty$ and $\lim_{k\to\infty} x(t_k) = \infty$, where $x(t_k) = \max\{x(s); t_0 \le s \le t_k\}$. Since $\lim_{t\to\infty} \tau(t) = \infty$, $\tau(t_k) > t_0$ for all sufficiently large k. By $\tau(t) \le t$,

$$x\bigl(\tau(t_k)\bigr) = \max\bigl\{x(s); t_0 \le s \le \tau(t_k)\bigr\} \le \max\bigl\{x(s); t_0 \le s \le t_k\bigr\} = x(t_k).$$

Therefore, for all large k,

$$z(t_k) = x(t_k) - p(t_k)x\bigl(\tau(t_k)\bigr) \ge \bigl(1 - p(t_k)\bigr)x(t_k) > 0,$$

which contradicts the fact that $\lim_{t\to\infty} z(t) = -\infty$.

Case 2. If x is bounded, then z is also bounded, which contradicts $\lim_{t\to\infty} z(t) = -\infty$. Hence, z satisfies one of the cases (C_1) and (C_2). This completes the proof. □

Lemma 2.2 *Assume that x is a positive solution of* (1.1) *and z satisfies case* (C_2). *Then*

$$\lim_{t\to\infty} x(t) = 0.$$

Proof By $z < 0$ and $z' > 0$, we deduce that

$$\lim_{t\to\infty} z(t) = l \le 0,$$

where l is a finite constant. That is, z is bounded. As in the proof of Case 1 in Lemma 2.1, x is also bounded. Using the fact that x is bounded, we obtain

$$\limsup_{t\to\infty} x(t) = a, \quad 0 \le a < \infty.$$

We claim that $a = 0$. If $a > 0$, then there exists a sequence $\{t_m\}$ such that $\lim_{m\to\infty} t_m = \infty$ and $\lim_{m\to\infty} x(t_m) = a$. Let $\varepsilon = a(1 - p_0)/2p_0$. Then for all large m, $x(\tau(t_m)) < a + \varepsilon$, and so

$$0 \geq \lim_{m\to\infty} z(t_m) \geq \lim_{m\to\infty} x(t_m) - p_0(a + \varepsilon) = \frac{a(1 - p_0)}{2} > 0,$$

which is a contradiction. Thus, $a = 0$ and $\lim_{t\to\infty} x(t) = 0$. The proof is complete. □

3 Oscillation results

In what follows, we denote

$$\xi(t) = r^{1/\alpha}(t) \int_{t_1}^{t} r^{-1/\alpha}(s)\,\mathrm{d}s,$$

where $t_1 \geq t_0$ is sufficiently large.

Theorem 3.1 *Let conditions* (H_1)-(H_4) *and* (1.4) *be satisfied. If there exists a positive function* $\rho \in \mathrm{C}^1([t_0,\infty),\mathbb{R})$ *such that, for all sufficiently large* $t_1 \geq t_0$,

$$\int^{\infty} \left[k\rho(t)q(t) - \frac{\rho'_+(t)r(\sigma(t))}{\xi^{\alpha}(\sigma(t))} \right] \mathrm{d}t = \infty, \tag{3.1}$$

where $\rho'_+(t) = \max\{0, \rho'(t)\}$, *then every solution* x *of* (1.1) *is either oscillatory or satisfies* $\lim_{t\to\infty} x(t) = 0$.

Proof Without loss of generality, we may assume that there exists a $t_1 \geq t_0$ such that $x(t) > 0$, $x(\tau(t)) > 0$, and $x(\sigma(t)) > 0$ for $t \geq t_1$. Then we have (2.1). From Lemma 2.1, z satisfies one of the cases (C_1) and (C_2). We consider each of two cases separately.

Suppose first that case (C_1) holds. By the definition of z,

$$x(t) = z(t) + p(t)x\bigl(\tau(t)\bigr) \geq z(t). \tag{3.2}$$

Using (2.1) and condition $\sigma(t) \leq t$, we conclude that

$$r(t)\bigl(z'(t)\bigr)^{\alpha} \leq r\bigl(\sigma(t)\bigr)\bigl(z'\bigl(\sigma(t)\bigr)\bigr)^{\alpha}. \tag{3.3}$$

It follows now from (2.1) that

$$z(t) = z(t_1) + \int_{t_1}^{t} \frac{(r(s)(z'(s))^{\alpha})^{1/\alpha}}{r^{1/\alpha}(s)}\,\mathrm{d}s \geq z'(t)r^{1/\alpha}(t) \int_{t_1}^{t} r^{-1/\alpha}(s)\,\mathrm{d}s = \xi(t)z'(t). \tag{3.4}$$

We define a function ω by

$$\omega(t) = \rho(t)\frac{r(t)(z'(t))^{\alpha}}{z^{\alpha}(\sigma(t))}, \quad t \geq t_1.$$

Then $\omega(t) > 0$ for $t \geq t_1$ and

$$\omega'(t) = \rho'(t)\frac{r(t)(z'(t))^{\alpha}}{z^{\alpha}(\sigma(t))} + \rho(t)\frac{(r(t)(z'(t))^{\alpha})'}{z^{\alpha}(\sigma(t))} - \rho(t)\frac{\alpha r(t)(z'(t))^{\alpha}z'(\sigma(t))\sigma'(t)}{z^{\alpha+1}(\sigma(t))}. \tag{3.5}$$

Using (2.1) and (3.2)-(3.5), we get

$$\omega'(t) \le \rho'(t)\frac{r(t)(z'(t))^\alpha}{z^\alpha(\sigma(t))} - k\rho(t)q(t) \le \rho'_+(t)\frac{r(\sigma(t))}{\xi^\alpha(\sigma(t))} - k\rho(t)q(t). \tag{3.6}$$

Integrating (3.6) from t_2 ($t_2 > t_1$) to t, we obtain

$$\int_{t_2}^{t}\left[k\rho(s)q(s) - \frac{\rho'_+(s)r(\sigma(s))}{\xi^\alpha(\sigma(s))}\right]\mathrm{d}s \le \omega(t_2),$$

which contradicts (3.1).

If z satisfies (C_2), then $\lim_{t\to\infty} x(t) = 0$ due to Lemma 2.2. The proof is complete. □

Let $\rho(t) = 1$. We can obtain the following criterion for (1.1) using Theorem 3.1.

Corollary 3.1 *Let conditions* (H_1)-(H_4) *and* (1.4) *be satisfied. If*

$$\int^\infty q(t)\,\mathrm{d}t = \infty,$$

then the conclusion of Theorem 3.1 *remains intact.*

Theorem 3.2 *Let* $\alpha \ge 1$ *hold and conditions* (H_1)-(H_4) *and* (1.4) *be satisfied. If there exists a positive function* $\rho \in \mathrm{C}^1([t_0,\infty),\mathbb{R})$ *such that, for all sufficiently large* $t_1 \ge t_0$,

$$\int^\infty \left[k\rho(t)q(t) - \frac{(\rho'_+(t))^2 r(\sigma(t))}{4\alpha\rho(t)\sigma'(t)\xi^{\alpha-1}(\sigma(t))}\right]\mathrm{d}t = \infty, \tag{3.7}$$

where $\rho'_+(t) = \max\{0,\rho'(t)\}$, *then the conclusion of Theorem* 3.1 *remains intact.*

Proof As above, suppose that x is a positive solution of (1.1). By virtue of Lemma 2.1, z satisfies one of (C_1) and (C_2). We discuss each of the two cases separately.

Assume first that z has property (C_1). We obtain (3.3) and (3.4). Define now ω as in the proof of Theorem 3.1. Then $\omega > 0$ and

$$\omega'(t) \le -k\rho(t)q(t) + \frac{\rho'_+(t)}{\rho(t)}\omega(t) - \alpha\sigma'(t)\rho(t)\frac{r(t)(z'(t))^\alpha}{z^\alpha(\sigma(t))}\frac{z'(\sigma(t))}{z(\sigma(t))}. \tag{3.8}$$

On the other hand, by (3.3) and (3.4),

$$\frac{z'(\sigma(t))}{z(\sigma(t))} = \frac{1}{r(\sigma(t))}\frac{r(\sigma(t))(z'(\sigma(t)))^\alpha}{z^\alpha(\sigma(t))}\left(\frac{z(\sigma(t))}{z'(\sigma(t))}\right)^{\alpha-1} \ge \frac{\xi^{\alpha-1}(\sigma(t))}{r(\sigma(t))}\frac{r(t)(z'(t))^\alpha}{z^\alpha(\sigma(t))}. \tag{3.9}$$

Substituting (3.9) into (3.8), we obtain

$$\begin{aligned}\omega'(t) &\le -k\rho(t)q(t) + \frac{\rho'_+(t)}{\rho(t)}\omega(t) - \alpha\sigma'(t)\frac{\xi^{\alpha-1}(\sigma(t))}{r(\sigma(t))\rho(t)}\omega^2(t)\\ &\le -k\rho(t)q(t) + \frac{(\rho'_+(t))^2 r(\sigma(t))}{4\alpha\rho(t)\sigma'(t)\xi^{\alpha-1}(\sigma(t))}.\end{aligned} \tag{3.10}$$

Integrating (3.10) from t_2 ($t_2 > t_1$) to t, we have

$$\int_{t_2}^{t}\left[k\rho(s)q(s) - \frac{(\rho'_+(s))^2 r(\sigma(s))}{4\alpha\rho(s)\sigma'(s)\xi^{\alpha-1}(\sigma(s))}\right]\mathrm{d}s \le \omega(t_2),$$

which contradicts (3.7).

If z satisfies (C_2), then $\lim_{t\to\infty} x(t) = 0$ when using Lemma 2.2. This completes the proof. □

4 Examples and discussion

Example 4.1 For $t \ge 1$, consider a second-order neutral differential equation

$$\left[x(t) - \frac{1}{2}x\left(t - \frac{\pi}{2}\right)\right]'' + 8x\left(t - \frac{\pi}{2}\right) = 0. \tag{4.1}$$

It follows from Corollary 3.1 that every solution x of (4.1) is either oscillatory or satisfies property $\lim_{t\to\infty} x(t) = 0$. For instance, $x(t) = \sin 4t$ is an oscillatory solution of this equation.

Example 4.2 For $t \ge 1$, consider a second-order nonlinear neutral differential equation

$$\left(t^2\left(z'(t)\right)^3\right)' + \gamma t^{-2}x^3\left(\frac{t}{2}\right) = 0, \tag{4.2}$$

where $z(t) = x(t) - x(t/3)/2$ and $\gamma > 0$ is a constant. Let $\alpha = 3$, $r(t) = t^2$, $q(t) = t^{-2}$, $\sigma(t) = t/2$, $k = \gamma$, and $\rho(t) = t$, and note that $\xi(t) \ge 3k_0 t$ for every $k_0 \in (0,1)$. By virtue of Theorem 3.2, every solution x of (4.2) is either oscillatory or satisfies $\lim_{t\to\infty} x(t) = 0$ if $\gamma > 1/(54k_0^2)$ for some $k_0 \in (0,1)$. However, it follows from Theorem 3.1 that every solution x of (4.2) is either oscillatory or satisfies $\lim_{t\to\infty} x(t) = 0$ if $\gamma > 2/(27k_0^3)$ for some $k_0 \in (0,1)$. Note that

$$\frac{2}{27k_0^3} > \frac{1}{54k_0^2}$$

for every $k_0 \in (0,1)$. Therefore, Theorem 3.2 improves Theorem 3.1 in some cases. But Theorem 3.1 can be applied to (1.1) in the case when $0 < \alpha < 1$. Observe that results reported in [2, 14, 15] cannot be applied to (4.2) since (1.3) fails to hold for this equation.

Remark 4.1 We establish two classes of oscillation criteria for (1.1) without requiring the restrictive conditions (1.3). Note that these results are based on the assumption (1.2) and, as fairly noticed by one of the referees, these results cannot be applied to the case where $p(t) = 1$.

Remark 4.2 Note that Theorems 3.1 and 3.2 and Corollary 3.1 guarantee that every solution x of (1.1) is either oscillatory or satisfies $\lim_{t\to\infty} x(t) = 0$ and, unfortunately, these results cannot distinguish solutions with different behaviors. Since the sign of z is not known, it is not easy to establish sufficient conditions which ensure that all solutions of (1.1) are just oscillatory and do not satisfy $\lim_{t\to\infty} x(t) = 0$.

Remark 4.3 On the basis of Remarks 4.1 and 4.2, two interesting problems for future research can be formulated as follows:

(P1) Is it possible to establish oscillation criteria for (1.1) without requiring condition (1.2)?

(P2) Is it possible to suggest a different method to study (1.1) and obtain some sufficient conditions which ensure that all solutions of (1.1) are oscillatory?

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All four authors contributed equally to this work. They all read and approved the final version of the manuscript.

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