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# Existence of positive solutions for Caputo fractional difference equation

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**Abstract**
In this paper, we studied the following Caputo fractional difference boundary value problem (FBVP): $\Delta_C^\nu y(t) = -f(t+\nu-1, y(t+\nu-1))$, $y(\nu-3) = \Delta y(b+\nu) = \Delta^2 y(\nu-3) = 0$, where $2 < \nu \leq 3$ is a real number, $\Delta_C^\nu y(t)$ is the standard Caputo difference. By means of cone theoretic fixed point theorems, some results on the existence of one or more positive solutions for the above Caputo fractional boundary value problems are obtained.

**MSC:** 26A33; 39A05; 39A12

**Keywords:** Caputo fractional difference; boundary value problem; Green's function; existence of positive solution

## 1 Introduction

In this paper, we discuss the following Caputo fractional difference boundary value problem (FBVP):

$$\begin{cases} \Delta_C^\nu y(t) = -f(t+\nu-1, y(t+\nu-1)), \\ y(\nu-3) = \Delta y(b+\nu) = \Delta^2 y(\nu-3) = 0, \end{cases} \tag{1}$$

where $t \in [0, b+1]_{\mathbb{N}_0}$, $b \geq 5$ is an integer. $f : [\nu-2, \nu-1, \ldots, b+\nu]_{\mathbb{N}_{\nu-2}} \times [0, +\infty) \to [0, +\infty)$ is continuous and $f$ is not identically zero, $2 < \nu \leq 3$, and $\Delta_C^\nu y(t)$ is the standard Caputo difference.

Fractional difference equations appear in useful biological models [1]. Influenced by the results of [1–3], the existence of one or two positive solutions for fractional difference equations has been studied (see [4–14]). Among them, in [4, 5], the authors introduced the fractional sum and difference operators, studied their behavior and developed a complete theory governing their compositions. In [6], Atici and Eloe studied the following two-point boundary value problem for fractional difference equation:

$$\begin{cases} -\Delta^\nu y(t) = f(t+\nu-1, y(t+\nu-1)), \\ y(\nu-2) = 0, \qquad y(b+\nu+1) = 0, \end{cases} \tag{2}$$

where $1 < \nu \leq 2$. They obtained the existence of positive solutions by means of the Krasnosel'skiĭ fixed point theorem. Goodrich gave some new existence results for (2) in [7].

Goodrich [8] also considered the following fractional boundary value problem with nonlocal conditions:

$$\begin{cases} -\Delta^{\nu} y(t) = f(t+\nu-1, y(t+\nu-1)), \\ y(\nu-2) = g(y), \qquad y(b+\nu+1) = 0, \end{cases} \tag{3}$$

where $1 < \nu \leq 2$. He showed a uniqueness theorem by means of the contraction and an existence theorem by means of the Brouwer theorem for (3). He also presented the existence of at least one positive solution to (3) by using the Krasnosel'skiĭ theorem. In [9–12], the authors deduced the existence of one or more positive solutions for fractional difference equations. Recently, Wu and Baleanu introduced some applications of the Caputo fractional difference to discrete chaotic maps in [13, 14]. However, to the best of our knowledge, few papers can be found in the literature dealing with the Caputo fractional difference equation boundary value problems. Motivated by [15], we make some attempt to fill this gap in the existing literature in this paper. We consider the existence of one or more positive solutions for the boundary value problem of the Caputo fractional difference equation (1).

The article is structured as follows. In Section 2, we deduce the Green's function of the FBVP (1). Then we prove that Green's function satisfies the useful properties. In Section 3, we present and prove our main results. In Section 4, we give some examples to illustrate our results.

## 2 The Green's function

We first review some basic results about fractional sums and differences. For any $t \in [0, b+1]_{\mathbb{N}_0}$ and $\nu > 0$, we define

$$t^{\underline{\nu}} = \frac{\Gamma(t+1)}{\Gamma(t+1-\nu)},$$

for which the right-hand side is defined. We appeal to the convention that if $t+1-\nu$ is a pole of the Gamma function and $t+1$ is not a pole, then $t^{\underline{\nu}} = 0$.

The $\nu$th fractional sum of a function $f$ is defined by

$$\Delta^{-\nu} f(t) = \frac{1}{\Gamma(\nu)} \sum_{s=a}^{t-\nu} (t-s-1)^{\underline{\nu-1}} f(s),$$

for $\nu > 0$ and $t \in \{a+\nu, a+\nu+1, \ldots\} = \mathbb{N}_{a+\nu}$. We also define the $\nu$th Caputo fractional difference for $\nu > 0$ by

$$\Delta_C^{\nu} f(t) = \Delta^{-(n-\nu)} \Delta^n f(t) = \frac{1}{\Gamma(n-\nu)} \sum_{s=a}^{t-(n-\nu)} (t-s-1)^{\underline{n-\nu-1}} \Delta_a^n f(s),$$

where $n-1 < \nu \leq n$.

First, we give the following two lemmas, they can be found in the recent papers (see, *e.g.*, [4, 5]).

**Lemma 2.1** [4] *Assume that $\nu > 0$ and $f$ is defined on domains $\mathbb{N}_a$, then*

$$\triangle_{a+(n-\nu)}^{-\nu}\triangle_C^{\nu} f(t) = f(t) - \sum_{k=0}^{n-1} c_k (t-a)^{\underline{k}},$$

*where $c_k \in \mathbb{R}$, $k = 0, 1, \ldots n-1$, and $n-1 < \nu \le n$.*

**Lemma 2.2** [5] *Let $f : \mathbb{N}_{a+\nu} \times \mathbb{N}_a \to \mathbb{R}$ be given. Then*

$$\triangle\left(\sum_{s=a}^{t-\nu} f(t,s)\right) = \sum_{s=a}^{t-\nu} \triangle_t f(t,s) + f(t+1, t+1-\nu), \quad \textit{for } t \in \mathbb{N}_{a+\nu}.$$

In order to get our main results, we first give an important lemmas. This lemma will give a representation for the solution of (1), provided that the solution exists.

**Lemma 2.3** *Let $2 < \nu \le 3$ and $g : [\nu-2, \nu-1, \ldots, b+\nu]_{\mathbb{N}_{\nu-2}} \to \mathbb{R}$ be given. Then the solution of the FBVP*

$$\begin{cases} \triangle_C^{\nu} y(t) = -g(t+\nu-1), \\ y(\nu-3) = \triangle y(b+\nu) = \triangle^2 y(\nu-3) = 0 \end{cases} \tag{4}$$

*is given by*

$$y(t) = \sum_{s=0}^{b+1} G(t,s) g(s+\nu-1),$$

*where the Green's function $G : [\nu-2, \nu-1, \ldots, b+\nu]_{\mathbb{N}_{\nu-2}} \times [0, b+1]_{\mathbb{N}_0} \to \mathbb{R}$ is defined by*

$$G(t,s) = \frac{1}{\Gamma(\nu)} \begin{cases} (\nu-1)(t-\nu+3)(b+\nu-s-1)^{\underline{\nu-2}} - (t-s-1)^{\underline{\nu-1}}, \\ \quad 0 \le s < t-\nu+1 \le b+1, \\ (\nu-1)(t-\nu+3)(b+\nu-s-1)^{\underline{\nu-2}}, \quad 0 \le t-\nu+1 \le s \le b+1. \end{cases}$$

*Proof* We apply Lemma 2.1 to reduce (4) to an equivalent summation equation

$$y(t) = -\frac{1}{\Gamma(\nu)} \sum_{s=0}^{t-\nu} (t-s-1)^{\underline{\nu-1}} g(s+\nu-1) + c_1 + c_2 t + c_3 t^{\underline{2}}$$

for some $c_i \in \mathbb{R}$, $i = 1, 2, 3$. By Lemma 2.2, we have

$$\triangle y(t) = -\frac{1}{\Gamma(\nu)} \sum_{s=0}^{t+1-\nu} (\nu-1)(t-s-1)^{\underline{\nu-2}} g(s+\nu-1) + c_2 + 2c_3 t,$$

$$\triangle^2 y(t) = -\frac{1}{\Gamma(\nu)} \sum_{s=0}^{t+2-\nu} (\nu-1)(\nu-2)(t-s-1)^{\underline{\nu-3}} g(s+\nu-1) + 2c_3.$$

From (4), we get

$$c_1 = -\frac{\nu-3}{\Gamma(\nu-1)} \sum_{s=0}^{b+1} (b+\nu-s-1)^{\underline{\nu-2}} g(s+\nu-1),$$

$$c_2 = \frac{1}{\Gamma(\nu-1)} \sum_{s=0}^{b+1} (b+\nu-s-1)^{\underline{\nu-2}} g(s+\nu-1), \qquad c_3 = 0.$$

Therefore, the solution of the FBVP (4) is

$$\begin{aligned}
y(t) &= -\frac{1}{\Gamma(\nu)} \sum_{s=0}^{t-\nu} (t-s-1)^{\underline{\nu-1}} g(s+\nu-1) - \frac{\nu-3}{\Gamma(\nu-1)} \sum_{s=0}^{b+1} (b+\nu-s-1)^{\underline{\nu-2}} g(s+\nu-1) \\
&\quad + \frac{t}{\Gamma(\nu-1)} \sum_{s=0}^{b+1} (b+\nu-s-1)^{\underline{\nu-2}} g(s+\nu-1) \\
&= \frac{1}{\Gamma(\nu)} \sum_{s=0}^{t-\nu} \bigl[(\nu-1)(t-\nu+3)(b+\nu-s-1)^{\underline{\nu-2}} - (t-s-1)^{\underline{\nu-1}}\bigr] g(s+\nu-1) \\
&\quad + \frac{1}{\Gamma(\nu-1)} \sum_{s=t-\nu+1}^{b+1} (t-\nu+3)(b+\nu-s-1)^{\underline{\nu-2}} g(s+\nu-1) \\
&= \sum_{s=0}^{b+1} G(t,s) g(s+\nu-1).
\end{aligned}$$

□

**Remark** Notice that $G(\nu-3,s)=0$, $G(t,b+2)=0$. $G$ could be extended to $[\nu-3, b+\nu]_{\mathbb{N}_{\nu-3}} \times [0,b+2]_{\mathbb{N}_0}$, so we only discuss $(t,s) \in [\nu-2,b+\nu]_{\mathbb{N}_{\nu-2}} \times [0,b+1]_{\mathbb{N}_0}$.

**Lemma 2.4** *The Green's function G has the following properties*:

(i) $G(t,s) > 0$, $(t,s) \in [\nu-2,b+\nu]_{\mathbb{N}_{\nu-2}} \times [0,b+1]_{\mathbb{N}_0}$.

(ii) $\max_{t\in[\nu-2,b+\nu]_{\mathbb{N}_{\nu-2}}} G(t,s) = G(b+\nu,s)$, $s \in [0,b+1]_{\mathbb{N}_0}$.

(iii) $\min_{\frac{b+\nu}{4} \le t \le \frac{3(b+\nu)}{4}} G(t,s) \ge \frac{1}{4} \max_{t\in[\nu-2,b+\nu]_{\mathbb{N}_{\nu-2}}} G(t,s) = \frac{1}{4} G(b+\nu,s)$, $s \in [0,b+1]_{\mathbb{N}_0}$.

*Proof* (i) From a representation of $G(t,s)$, it is clear that $G(t,s) > 0$.

(ii) By a representation of $G(t,s)$, we have $\triangle_t G(t,s) > 0$ for $0 \le t-\nu+1 \le s \le b+1$ or $0 \le s < t-\nu+1 \le b+1$. Hence $G(t,s)$ is increasing to $t$ for $s \in [0,b+1]_{\mathbb{N}_0}$. From this we may conclude that

$$\max_{t\in[\nu-2,\nu+b]_{\mathbb{N}_{\nu-2}}} G(t,s) = G(b+\nu,s), \quad s \in [0,b+1]_{\mathbb{N}_0}.$$

(iii) Since

$$\frac{G(t,s)}{G(b+\nu,s)} = \begin{cases} \frac{(\nu-1)(t-\nu+3)(b+\nu-s-1)^{\underline{\nu-2}} - (t-s-1)^{\underline{\nu-1}}}{(\nu-1)(b+3)(b+\nu-s-1)^{\underline{\nu-2}} - (b+\nu-s-1)^{\underline{\nu-1}}}, & 0 \le s < t-\nu+1 \le b+1, \\ \frac{(\nu-1)(t-\nu+3)(b+\nu-s-1)^{\underline{\nu-2}}}{(\nu-1)(b+3)(b+\nu-s-1)^{\underline{\nu-2}} - (b+\nu-s-1)^{\underline{\nu-1}}}, & 0 \le t-\nu+1 \le s \le b+1, \end{cases}$$

for $t-\nu+1 \le s$ and $\frac{b+\nu}{4} \le t \le \frac{3(b+\nu)}{4}$, we get

$$\frac{G(t,s)}{G(b+\nu,s)} \ge \frac{(\nu-1)(t-\nu+3)(b+\nu-s-1)^{\underline{\nu-2}}}{(\nu-1)(b+3)(b+\nu-s-1)^{\underline{\nu-2}}} \ge \frac{1}{4}.$$

For $s < t-\nu+1$ and $\frac{b+\nu}{4} \le t \le \frac{3(b+\nu)}{4}$, we have

$$\min_{\frac{b+\nu}{4} \le t \le \frac{3(b+\nu)}{4}} \frac{G(t,s)}{G(b+\nu,s)} = \frac{(\nu-1)(\frac{b+\nu}{4}-\nu+3)(b+\nu-s-1)^{\underline{\nu-2}} - (\frac{b+\nu}{4}-s-1)^{\underline{\nu-1}}}{(\nu-1)(b+3)(b+\nu-s-1)^{\underline{\nu-2}} - (b+\nu-s-1)^{\underline{\nu-1}}}.$$

Now, we want to get

$$\frac{(\nu-1)(\frac{b+\nu}{4}-\nu+3)(b+\nu-s-1)^{\underline{\nu-2}}-(\frac{b+\nu}{4}-s-1)^{\underline{\nu-1}}}{(\nu-1)(b+3)(b+\nu-s-1)^{\underline{\nu-2}}-(b+\nu-s-1)^{\underline{\nu-1}}}\geq\frac{1}{4}, \tag{5}$$

that is,

$$\begin{aligned}&(\nu-1)\biggl(\frac{b+\nu}{4}-\nu+3\biggr)(b+\nu-s-1)^{\underline{\nu-2}}-\biggl(\frac{b+\nu}{4}-s-1\biggr)^{\underline{\nu-1}}\\&\quad\geq\frac{1}{4}(\nu-1)(b+3)(b+\nu-s-1)^{\underline{\nu-2}}-\frac{1}{4}(b+\nu-s-1)^{\underline{\nu-1}}.\end{aligned}$$

In fact

$$\frac{b+\nu}{4}-\nu+3\geq\frac{b+3}{4},\qquad\frac{\frac{1}{4}(b+\nu-s-1)^{\underline{\nu-1}}}{(\frac{b+\nu}{4}-s-1)^{\underline{\nu-1}}}\geq 1.$$

So (5) holds, which completes the proof. □

**Lemma 2.5** *If* $f:[\nu-2,\nu-1,\dots,b+\nu]_{\mathbb{N}_{\nu-2}}\times[0,+\infty)\to[0,\infty)$ *is a continuous function. Then the solutions y of the FBVP* (1) *satisfy*

$$\min_{\frac{b+\nu}{4}\leq t\leq\frac{3(b+\nu)}{4}}y(t)\geq\frac{1}{4}\max_{[\nu-2,b+\nu]_{\mathbb{N}_{\nu-2}}}\bigl|y(t)\bigr|.$$

*Proof* Taking into account Lemma 2.4, we have

$$\begin{aligned}\min_{\frac{b+\nu}{4}\leq t\leq\frac{3(b+\nu)}{4}}\bigl|y(t)\bigr|&\geq\frac{1}{4}\sum_{s=0}^{b+1}G(b+\nu,s)f\bigl(s+\nu-1,y(s+\nu-1)\bigr)\\&\geq\frac{1}{4}\max_{t\in[\nu-2,b+\nu]_{\mathbb{N}_{\nu-2}}}\sum_{s=0}^{b+1}G(t,s)f\bigl(s+\nu-1,y(s+\nu-1)\bigr)\\&=\frac{1}{4}\max_{[\nu-2,b+\nu]_{\mathbb{N}_{\nu-2}}}\bigl|y(t)\bigr|.\end{aligned}$$

□

The following fixed point theorem will play a major role in our main results.

**Lemma 2.6** [16] *Let* $\mathcal{B}$ *a Banach space and let* $\mathcal{K}\subseteq\mathcal{B}$ *be a cone. Assume that* $\Omega_1$ *and* $\Omega_2$ *are bounded open sets contained in* $\mathcal{B}$ *such that* $0\in\Omega_1$ *and* $\overline{\Omega}_1\subseteq\Omega_2$. *Further, assume that* $T:\mathcal{K}\cap(\overline{\Omega}_2\backslash\Omega_1)\to\mathcal{K}$ *is a completely continuous operator. If either*

(i) $\|Ty\|\leq\|y\|$ *for* $y\in\mathcal{K}\cap\partial\Omega_1$ *and* $\|Ty\|\geq\|y\|$ *for* $y\in\mathcal{K}\cap\partial\Omega_2$; *or*

(ii) $\|Ty\|\geq\|y\|$ *for* $y\in\mathcal{K}\cap\partial\Omega_1$ *and* $\|Ty\|\leq\|y\|$ *for* $y\in\mathcal{K}\cap\partial\Omega_2$.

*Then the operator T has at least one fixed point in* $\mathcal{K}\cap(\overline{\Omega}_2\backslash\Omega_1)$.

## 3 Main results

For the sake of convenience, we denote

$$f_0=\liminf_{y\to 0}\min_{t\in[\nu-2,b+\nu]_{\mathbb{N}_{\nu-2}}}\frac{f(t,y)}{y},\qquad f^0=\limsup_{y\to 0}\max_{t\in[\nu-2,b+\nu]_{\mathbb{N}_{\nu-2}}}\frac{f(t,y)}{y},$$

$$f_\infty = \liminf_{y\to+\infty} \min_{t\in[\nu-2,b+\nu]_{\mathbb{N}_{\nu-2}}} \frac{f(t,y)}{y}, \qquad f^\infty = \limsup_{y\to+\infty} \max_{t\in[\nu-2,b+\nu]_{\mathbb{N}_{\nu-2}}} \frac{f(t,y)}{y},$$

$$\frac{1}{A} = \sum_{s=0}^{b+1} G(b+\nu,s), \qquad \frac{1}{B} = \frac{1}{4} \sum_{s=[\frac{b+\nu}{4}-\nu+1]}^{[\frac{3(b+\nu)}{4}-\nu+1]} G\left(\left[\frac{b-\nu}{2}\right]+\nu,s\right).$$

We use Lemma 2.6 to establish the existence of positive solutions to the FBVP (1). To this end, one or several of the following conditions will be needed.

(H) $f:[\nu-2,\nu-1,\dots,b+\nu]_{\mathbb{N}_{\nu-2}} \times [0,+\infty) \to [0,+\infty)$ is continuous.
($H_1$) There is a number $p>0$ such that $f(t,y) < Ap$ for $0 \le y \le p$ and $\nu-2 \le t \le b+\nu$.
($H_2$) There is a number $p>0$ such that $f(t,y) > Bp$ for $\frac{1}{4}p \le y \le p$ and $\frac{b+\nu}{4} \le t \le \frac{3(b+\nu)}{4}$.
($H_3$) $f_0 > B, f_\infty > B$.
($H_4$) $f^0 < A, f^\infty < A$.
($H_5$) $f_0 > B, f^\infty < A$.
($H_6$) $f^0 < A, f_\infty > B$.
($H_3^*$) $f_0 = +\infty, f_\infty = +\infty$.
($H_4^*$) $f^0 = 0, f^\infty = 0$.

Let

$$\mathcal{B} = \bigl\{y:[\nu-3,b+\nu]_{\mathbb{N}_{\nu-3}} \to \mathbb{R}, y(\nu-3) = \triangle y(b+\nu) = \triangle^2 y(\nu-3) = 0\bigr\}.$$

Then $\mathcal{B}$ is a Banach space with respect to the norm $\|y\| = \max_{t\in[\nu-3,b+\nu]_{\mathbb{N}_{\nu-3}}} |y(t)|$. We define a cone in $\mathcal{B}$ by

$$\mathcal{K} = \left\{y \in \mathcal{B} : y(t) \ge 0, \min_{\frac{b+\nu}{4} \le t \le \frac{3(b+\nu)}{4}} y(t) \ge \frac{1}{4}\|y\|\right\}.$$

Now consider the operator $T$ defined by

$$(Ty)(t) = \sum_{s=0}^{b+1} G(t,s) f\bigl(s+\nu-1, y(s+\nu-1)\bigr). \tag{6}$$

It is easy to see that $y = y(t)$ is a solution of the FBVP (1) if and only if $y = y(t)$ is a fixed point of $T$. We shall obtain sufficient conditions for the existence of fixed points of $T$. First, we notice that $T$ is a summation operator on a discrete finite set. Hence, $T$ is trivially completely continuous. From (6), then

$$\begin{aligned} \min_{\frac{b+\nu}{4} \le t \le \frac{3(b+\nu)}{4}} (Ty)(t) &\ge \frac{1}{4} \sum_{s=0}^{b+1} G(b+\nu,s) f\bigl(s+\nu-1, y(s+\nu-1)\bigr) \\ &\ge \frac{1}{4} \max_{t\in[\nu-2,b+\nu]_{\mathbb{N}_{\nu-2}}} \sum_{s=0}^{b+1} G(t,s) f\bigl(s+\nu-1, y(s+\nu-1)\bigr) \\ &= \frac{1}{4}\|Ty\|, \end{aligned}$$

hence $T\mathcal{K} \subset \mathcal{K}$.

In the sequel, let $\Omega_\lambda = \{y \in \mathcal{K} : \|y\| < \lambda\}$, for $\lambda > 0$, $\partial\Omega_\lambda = \{y \in \mathcal{K} : \|y\| = \lambda\}$.

**Theorem 3.1** *Assume that there exist two different positive numbers $r$ and $R$ such that $f$ satisfies condition* $(\mathrm{H}_1)$ *at $r$ and condition* $(\mathrm{H}_2)$ *at $R$. Then the FBVP* (1) *has at least one positive solution $y_0 \in \mathcal{K}$ satisfying* $\min\{r,R\} \le \|y_0\| \le \max\{r,R\}$.

*Proof* We know that $T : \mathcal{K} \to \mathcal{K}$, and $T$ is completely continuous. Without loss of generality suppose that $r < R$. Note that for $y \in \partial\Omega_r$, we have $\|y\| = r$, so that condition $(\mathrm{H}_1)$ holds for all $y \in \partial\Omega_r$. Then

$$\begin{aligned}(Ty)(t) &\le \sum_{s=0}^{b+1} G(b+\nu,s) f\bigl(s+\nu-1, y(s+\nu-1)\bigr) \\ &\le Ar \sum_{s=0}^{b+1} G(b+\nu,s) \\ &= r,\end{aligned}$$

*i.e.*, we have $\|Ty\| \le \|y\|$ for $y \in \mathcal{K} \cap \partial\Omega_r$.

Note that for $y \in \partial\Omega_R$, we have $\|y\| = R$, so condition $(\mathrm{H}_2)$ holds for all $y \in \partial\Omega_R$. Since $[\frac{b-\nu}{2}] + \nu \in [\frac{b+\nu}{4}, \frac{3(b+\nu)}{4}]$,

$$\begin{aligned}(Ty)\left(\left[\frac{b-\nu}{2}\right]+\nu\right) &= \sum_{s=0}^{b+1} G\left(\left[\frac{b-\nu}{2}\right]+\nu, s\right) f\bigl(s+\nu-1, y(s+\nu-1)\bigr) \\ &\ge \frac{1}{4} \sum_{s=[\frac{b+\nu}{4}-\nu+1]}^{[\frac{3(b+\nu)}{4}-\nu+1]} G\left(\left[\frac{b-\nu}{2}\right]+\nu, s\right) f\bigl(s+\nu-1, y(s+\nu-1)\bigr) \\ &\ge \frac{BR}{4} \sum_{s=[\frac{b+\nu}{4}-\nu+1]}^{[\frac{3(b+\nu)}{4}-\nu+1]} G\left(\left[\frac{b-\nu}{2}\right]+\nu, s\right) \\ &= R,\end{aligned}$$

*i.e.*, we have $\|Ty\| \ge \|y\|$ for $y \in \mathcal{K} \cap \partial\Omega_R$.

By Lemma 2.6 that $T$ has at least one fixed point, say, $y_0 \in \overline{\Omega}_R \backslash \Omega_r$. This function $y_0(t)$ is a positive solution of (1) and satisfies $r \le \|y_0\| \le R$. The proof is complete. □

**Theorem 3.2** *Assume that conditions* (H), $(\mathrm{H}_1)$, *and* $(\mathrm{H}_3)$ *hold. Then the FBVP* (1) *has at least two positive solutions $y_1$ and $y_2$ with* $0 < \|y_1\| < p \le \|y_2\|$.

*Proof* Suppose that $(\mathrm{H}_3)$ holds. Since $f_0 > B$, there exist $\varepsilon > 0$ and $0 < r_0 < p$ such that

$$f(t,y) \ge (B+\varepsilon)y, \quad 0 \le y \le r_0, t \in [\nu-2, b+\nu]_{\mathbb{N}_{\nu-2}}.$$

Let $r_1 \in (0, r_0)$. Thus for $y \in \partial\Omega_{r_1}$, we have

$$\begin{aligned}(Ty)\left(\left[\frac{b-\nu}{2}\right]+\nu\right) &\ge \sum_{s=0}^{b+1} G\left(\left[\frac{b-\nu}{2}\right]+\nu, s\right)(B+\varepsilon)y \\ &\ge (B+\varepsilon)\cdot\frac{1}{4}\|y\| \sum_{s=[\frac{b+\nu}{4}-\nu+1]}^{[\frac{3(b+\nu)}{4}-\nu+1]} G\left(\left[\frac{b-\nu}{2}\right]+\nu, s\right)\end{aligned}$$

$$\begin{aligned} &> B \cdot \frac{1}{4}\|y\| \sum_{s=[\frac{b+\nu}{4}-\nu+1]}^{[\frac{3(b+\nu)}{4}-\nu+1]} G\left(\left[\frac{b-\nu}{2}\right]+\nu, s\right) \\ &= r_1, \end{aligned}$$

from which we see that $\|Ty\| > \|y\|$ for $y \in \mathcal{K} \cap \partial\Omega_{r_1}$.

On the other hand, since $f_\infty > B$, there exist $\eta > 0$ and $R_0 > 0$ such that

$$f(t,y) \geq (B+\eta)y, \quad y \geq R_0, t \in [\nu-2, b+\nu]_{\mathbb{N}_{\nu-2}}.$$

Choose $R_1 > \max\{4R_0, p\}$. For $y \in \partial\Omega_{R_1}$, since $y(t) \geq \frac{1}{4}\|y\| > R_0$ for $\frac{b+\nu}{4} \leq t \leq \frac{3(b+\nu)}{4}$, we have

$$\begin{aligned} (Ty)\left(\left[\frac{b-\nu}{2}\right]+\nu\right) &\geq \sum_{s=0}^{b+1} G\left(\left[\frac{b-\nu}{2}\right]+\nu, s\right)(B+\eta)y \\ &\geq (B+\eta) \cdot \frac{1}{4}\|y\| \sum_{s=[\frac{b+\nu}{4}-\nu+1]}^{[\frac{3(b+\nu)}{4}-\nu+1]} G\left(\left[\frac{b-\nu}{2}\right]+\nu, s\right) \\ &> B \cdot \frac{1}{4}\|y\| \sum_{s=[\frac{b+\nu}{4}-\nu+1]}^{[\frac{3(b+\nu)}{4}-\nu+1]} G\left(\left[\frac{b-\nu}{2}\right]+\nu, s\right) \\ &= R_1, \end{aligned}$$

from which we see that $\|Ty\| > \|y\|$ for $y \in \mathcal{K} \cap \partial\Omega_{R_1}$.

For any $y \in \partial\Omega_p$, from $(H_1)$, we have $f(t,y) < Ap$, $t \in [\nu-2, b+\nu]_{\mathbb{N}_{\nu-2}}$, then

$$\begin{aligned} (Ty)(t) &= \sum_{s=0}^{b+1} G(t,s) f\bigl(s+\nu-1, y(s+\nu-1)\bigr) \\ &\leq \sum_{s=0}^{b+1} G(b+\nu, s)Ap \\ &= p, \end{aligned}$$

from which we see that $\|Ty\| \leq \|y\|$ for $y \in \mathcal{K} \cap \partial\Omega_p$.

Therefore, by Lemma 2.6, we complete the proof. □

By the proof of Theorem 3.2, we obtain the following corollary.

**Corollary 3.1** *Assume that* (H) *and* $(H_1)$ *hold,* $(H_3)$ *is replaced by* $(H_3^*)$. *Then the FBVP* (1) *has at least two positive solutions* $y_1$ *and* $y_2$ *with* $0 < \|y_1\| < p \leq \|y_2\|$.

**Theorem 3.3** *Suppose that conditions* (H), $(H_2)$, *and* $(H_4)$ *hold,* $f > 0$ *for* $t \in [\nu-2, b+\nu]_{\mathbb{N}_{\nu-2}}$. *Then the FBVP* (1) *has at least two positive solutions* $y_1$ *and* $y_2$ *with* $0 < \|y_1\| < p < \|y_2\|$.

*Proof* Suppose that $(H_4)$ holds. Since $f^0 < A$, one can find $\varepsilon > 0$ $(\varepsilon < A)$ and $0 < r_0 < p$ such that

$$f(t,y) \leq (A-\varepsilon)y, \quad 0 \leq y \leq r_0, t \in [\nu-2, b+\nu]_{\mathbb{N}_{\nu-2}}.$$

Let $r_2 \in (0, r_0)$, for $y \in \partial\Omega_{r_2}$, we get

$$\begin{aligned}(Ty)(t) &= \sum_{s=0}^{b+1} G(t,s) f\bigl(s+\nu-1, y(s+\nu-1)\bigr)\\ &\le \sum_{s=0}^{b+1} G(b+\nu,s)(A-\varepsilon)r_2\\ &< Ar_2 \sum_{s=0}^{b+1} G(b+\nu,s)\\ &= r_2,\end{aligned}$$

from which we see that $\|Ty\| < \|y\|$ for $y \in \partial\Omega_{r_2}$.

On the other hand, since $f^\infty < A$, there exist $0 < \sigma < A$ and $R_0 > 0$ such that

$$f(t,y) \le \sigma y, \quad y \ge R_0, t \in [\nu-2, b+\nu]_{\mathbb{N}_{\nu-2}}.$$

Let $M = \max_{(t,y)\in[\nu-2,b+\nu]\times[0,R_0]} f(t,y)$, then $0 \le f(t,y) \le \sigma y + M$, $0 \le y < +\infty$. Let $R_2 > \max\{p, \frac{M}{A-\sigma}\}$, for $y \in \partial\Omega_{R_2}$, we have

$$\begin{aligned}\|Ty\| &\le \sum_{s=0}^{b+1} G(b+\nu,s) f\bigl(s+\nu-1, y(s+\nu-1)\bigr)\\ &\le \bigl(\sigma\|y\| + M\bigr) \sum_{s=0}^{b+1} G(b+\nu,s)\\ &= (\sigma R_2 + M)\cdot\frac{1}{A}\\ &< R_2.\end{aligned}$$

Therefore, we have $\|Ty\| \le \|y\|$ for $y \in \partial\Omega_{R_2}$.

Finally, for any $y \in \partial\Omega_p$, since $\frac{1}{4}p \le y(t) \le p$ for $\frac{b+\nu}{4} \le t \le \frac{3(b+\nu)}{4}$, we have

$$\begin{aligned}(Ty)\left(\left[\frac{b-\nu}{2}\right]+\nu\right) &= \sum_{s=0}^{b+1} G\left(\left[\frac{b-\nu}{2}\right]+\nu, s\right) f\bigl(s+\nu-1, y(s+\nu-1)\bigr)\\ &> B\cdot\frac{1}{4}p \sum_{s=[\frac{b+\nu}{4}-\nu+1]}^{[\frac{3(b+\nu)}{4}-\nu+1]} G\left(\left[\frac{b-\nu}{2}\right]+\nu, s\right)\\ &= p = \|y\|,\end{aligned}$$

from which we see that $\|Ty\| > \|y\|$ for $y \in \mathcal{K}\cap\partial\Omega_p$.

By Lemma 2.6, the proof is complete. □

From the proof of Theorem 3.3, we get the following corollary.

**Corollary 3.2** *Assume that* (H) *and* ($H_2$) *hold*, ($H_4$) *is replaced by* ($H_4^*$). *Then the FBVP* (1) *has at least two positive solutions.*

From the proof of Theorem 3.2 and 3.3, we get some theorems and corollaries.

**Theorem 3.4** *Suppose that condition* (H) *and* ($H_5$) *are satisfied. Then the FBVP* (1) *has at least one positive solution.*

**Corollary 3.3** *Suppose that* (H) *holds. Also assume that* $f_0 = +\infty, f^\infty = 0$. *Then the FBVP* (1) *has at least one positive solution.*

**Theorem 3.5** *Suppose that* (H) *and* ($H_6$) *hold. Then the FBVP* (1) *has at least one positive solution.*

**Corollary 3.4** *Suppose that* (H) *holds. Also assume that* $f^0 = 0, f_\infty = +\infty$. *Then the FBVP* (1) *has at least one positive solution.*

## 4 Some examples

In this section, we present some examples to validate our main conclusions. In the following examples, we take $\nu = \frac{17}{8}$ and $b = 19$. A computation shows that $A \approx 0.00273$, $B \approx 0.0255$.

**Example 4.1** Consider the following Caputo fractional difference boundary value problem:

$$\begin{cases} \triangle_C^{\frac{17}{8}} y(t) = -\frac{1}{51} e^{t-20} (\frac{1}{5} y^{\frac{1}{2}}(t + \frac{9}{8}) + \frac{1}{64} y^{\frac{3}{2}}(t + \frac{9}{8})), \\ y(-\frac{7}{8}) = \triangle y(\frac{169}{8}) = \triangle^2 y(-\frac{7}{8}) = 0, \end{cases} \tag{7}$$

where $f(t,y) = \frac{1}{51} e^{t-\frac{169}{8}} (\frac{1}{5} y^{\frac{1}{2}} + \frac{1}{64} y^{\frac{3}{2}})$, thus $f_0 = f_\infty = +\infty$. Taking $p = 4$, we get

$$f(t,y) \le \frac{1}{51} \left( \frac{1}{5} \cdot 4^{\frac{1}{2}} + \frac{1}{64} \cdot 4^{\frac{3}{2}} \right) = 0.010294 < Ap$$

for $0 \le y \le p$ and $\nu - 2 \le t \le b + \nu$. All conditions in Corollary 3.1 are satisfied. Applying Corollary 3.1, the FBVP (7) has at least two positive solutions $y_1$ and $y_2$ such that $0 < \|y_1\| < 4 < \|y_2\|$.

**Example 4.2** Consider the following boundary value problem:

$$\begin{cases} \triangle_C^{\frac{17}{8}} y(t) = -20(2 + \cos(t + \frac{9}{8})) y^2(t + \frac{9}{8}) e^{-2y(t+\frac{9}{8})}, \\ y(-\frac{7}{8}) = \triangle y(\frac{169}{8}) = \triangle^2 y(-\frac{7}{8}) = 0. \end{cases} \tag{8}$$

Let $f(t,y) = 20(2 + \cos t) y^2 e^{-2y}$, then $f^0 = f^\infty = 0$. We choose $p = 4$, when $\frac{1}{4} p \le y \le p$ and $\frac{b+\nu}{4} \le t \le \frac{3(b+\nu)}{4}$, we get

$$f(t,y) \ge 20(2-1) p^2 e^{-2p} \approx 0.026837 p \ge Bp.$$

All conditions in Corollary 3.2 are satisfied. Thus, the FBVP (8) has at least two positive solutions $y_1$ and $y_2$ such that $0 < \|y_1\| < 4 < \|y_2\|$.

**Example 4.3** Consider the following boundary value problem:

$$\begin{cases} \triangle_C^{\frac{17}{8}} y(t) = -\frac{1}{400} y(t+\frac{9}{8})(1+\frac{10}{1+y^2(t+\frac{9}{8})}), \\ y(-\frac{7}{8}) = \triangle y(\frac{169}{8}) = \triangle^2 y(-\frac{7}{8}) = 0, \end{cases} \tag{9}$$

where $f(t,y) = \frac{1}{400} y(1+\frac{10}{1+y^2})$, it is easy to compute that $f^\infty = 0.0025 < A$ and $f_0 = 0.0275 > B$, which yields the condition ($H_5$). By Theorem 3.4, the FBVP (9) has at least one positive solution.

**Example 4.4** Consider the following boundary value problem:

$$\begin{cases} \triangle_C^{\frac{17}{8}} y(t) = -f(t+\nu-1, y(t+\nu-1)), \\ y(-\frac{7}{8}) = \triangle y(\frac{169}{8}) = \triangle^2 y(-\frac{7}{8}) = 0, \end{cases} \tag{10}$$

where $f(t,y) = \frac{1}{367+t} \cdot \frac{10y}{10e^{-2y}+e^{-y}+1}$, it is easy to compute that $f^0 \approx 0.002269 < A$ and $f_\infty \approx 0.025765 > B$. Thus, the conditions (H) and ($H_6$) hold. Applying Theorem 3.5, we can find that the FBVP (10) has at least one positive solution.

**Remark** If $f(t,y) = h(t)g(y)$, our conclusions hold also.

**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**
HQ conceived of the study and participated in its design and coordination. SK drafted the manuscript. ZJ participated in the design of the study and the sequence correction. All authors read and approved the final manuscript.

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