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Stochastic delay evolution equations driven by sub-fractional Brownian motion

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Abstract
In this paper, we investigate the existence, uniqueness and exponential asymptotic behavior of mild solutions to stochastic delay evolution equations perturbed by a sub-fractional Brownian motion $S_Q^H(t)$: $dX(t) = (AX(t) + f(t, X_t))\,dt + g(t)\,dS_Q^H(t)$ with index $H \in (1/2, 1)$.

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1 Introduction

The fractional Brownian motion (fBm for short) has become an object of intense study, due to its interesting properties and its applications in various scientific areas including telecommunications, turbulence and finance. The fBm with Hurst index $H \in (0,1)$ is a continuous centered Gaussian process $B^H = \{B^H(t), t \geq 0\}$, starting from zero, with covariance

$$R_H(t,s) = \mathbb{E}\big[B^H(t)B^H(s)\big] = \frac{1}{2}\big[t^{2H} + s^{2H} + |t-s|^{2H}\big] \tag{1.1}$$

for all $s, t \geq 0$. For $H = \frac{1}{2}$, B^H coincides with the standard Brownian motion B. B^H is neither a semimartingale nor a Markov process when $H \neq \frac{1}{2}$. The fBm is a suitable generalization of the standard Brownian motion, but exhibits long-range dependence, self-similarity and which has stationary increments. On the other hand, based on the sufficient study of fBm, many authors have proposed to use more general self-similar Gaussian processes and random fields as stochastic models. Such applications have raised many interesting theoretical questions about self-similar Gaussian processes and fields in general. Therefore, some other generalizations of the fBm were introduced. However, contrast to the extensive studies on fBm, there has been little systematic investigation on other self-similar Gaussian processes. The main reason for this is the complexity of dependence structures for self-similar Gaussian processes which do not have stationary increments.

As an extension of Brownian motion, recently, Bojdecki *et al.* [1] introduced and studied a rather special class of self-similar Gaussian process. This process arises from occupation time fluctuations of branching particle systems with Poisson initial condition. This process is called the sub-fractional Brownian motion (sub-fBm in short). The sub-fBm with index

$H \in (0,1)$ is mean zero Gaussian $S^H = \{S^H(t), t \geq 0\}$ starting from zero, with covariance

$$C_H(t,s) = \mathbb{E}\big[S^H(t)S^H(s)\big] = s^{2H} + t^{2H} - \frac{1}{2}\big[(s+t)^{2H} + |t-s|^{2H}\big] \tag{1.2}$$

for all $s,t \geq 0$. For $H = \frac{1}{2}$, S^H coincides with the standard Brownian motion B. S^H is neither a semimartingale nor a Markov process when $H \neq \frac{1}{2}$, so many of the power techniques from stochastic analysis are not available when dealing with S^H. The sub-fBm has properties analogous to those of fBm (self-similarity, long-range dependence, Hölder paths), but it does not have stationary increments. More works for sub-fractional Brownian motion can be found in Bojdecki *et al.* [2, 3], Tudor [4–9] and Yan and Shen [10].

The sub-fractional Brownian motion satisfies the following estimates:

$$\big[\big(2-2^{2H-1}\big)\wedge 1\big]|t-s|^{2H}\mathbb{E}\big|S^H(t)-S^H(s)\big|^2 \leq \big[\big(2-2^{2H-1}\big)\wedge 1\big]|t-s|^{2H}. \tag{1.3}$$

Thus, Kolmogorov's continuity criterion implies that sub-fBm is Hölder continuous of order γ for any $\gamma < H$ on any finite interval. Therefore, if u is a stochastic process with Hölder continuous trajectories of order $\beta > 1-H$, then the path-wise Riemann-Stieltjes integral $\int_0^T u_t(\omega)\,dS^H(t)(\omega)$ exists for all $T \geq 0$ (see Young [11]). Yan *et al.* [12] have used the divergence operator to define the stochastic integrals with respect to sub-fBm with $H > 1/2$ and showed that if $f \in C^2(\mathbb{R})$ and $1/2 < H < 1$, then

$$f\big(S^H(t)\big) = f(0) + \int_0^t f'\big(S^H(s)\big)\,dS^H(s) + H\big(2-2^{2H-1}\big)\int_0^t f''\big(S^H(s)\big)s^{2H-1}\,ds.$$

In 2012, Shen and Chen [13] defined a stochastic integral with respect to sub-fractional Brownian motion S^H with index $H \in (0,1/2)$ that extends the divergence integral from Malliavin calculus, and established versions of the formulas of Itô and Tanaka that hold for all $H \in (0,1/2)$. Tudor [6, 8] characterized the domain of the Wiener integral with respect to a sub-fractional Brownian motion $S^H(s)$ with index $H \in (0,1)$ and stochastic differential equations driven by sub-fractional Brownian motion has also been consider by Mendy [14].

In this paper, we desire to define a stochastic integral with respect to Q-sub-fractional Brownian motion S_Q^H in infinite dimensional space and consider the following stochastic semilinear delay evolution equation:

$$\begin{cases} dX(t) = (AX(t) + f(t,X_t))\,dt + g(t)\,dS_Q^H(t), \\ X(s) = \varphi(s), \quad -r \leq s \leq 0, r \geq 0, \end{cases} \tag{1.4}$$

under suitable conditions on the operator A, the coefficient functions f, g, and the initial value φ. Here $S_Q^H(t)$ denotes an Q-sub-fBm with $H \in (1/2,1)$. The purpose of this paper is to investigate the existence and uniqueness of mild solutions to the stochastic delay differential equation (1.4) and to study its longtime behavior as well.

The contents of the paper are as follows. In Section 2, some necessary preliminaries on the stochastic integration with respect to sub-fBm are established. Also a technical lemma which is crucial in our stability analysis is proved. In Section 3, the existence and uniqueness of mild solutions are proved. In Section 4, we prove that a mild solution, when it exists, is also a weak solution. In Section 5, we establish some sufficient conditions ensuring the exponential decay to zero in mean square of the mild solution of our delay model. In Section 6, an example is given to illustrate the effectiveness of our results.

2 Preliminaries

In this section we introduce the sub-fractional Brownian motion as well as the Wiener integral with respect to it. We also establish some important results which will be needed throughout this paper.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. Now we aim at introducing the Wiener integral with respect to one-dimensional sub-fBm S^H. Fix a time interval $[0, T]$. We denote by $\mathcal{E}$ the linear space of $\mathbb{R}$-valued step functions on $[0, T]$, that is, $\varphi \in \mathcal{E}$ if

$$\varphi(t) = \sum_{i=1}^{n-1} x_i \mathbf{I}_{[t_i, t_{i+1}]}(t),$$

where $t \in [0, T]$, $x_i \in \mathbb{R}$, and $0 = t_1 < t_2 < \cdots < t_n = T$. For $\varphi \in \mathcal{E}$ we define its Wiener integral with respect to S^H as

$$\int_0^T \varphi(s)\, dS^H(s) = \sum_{i=1}^{n} x_i \bigl(S^H_{t_{i+1}} - S^H_{t_i}\bigr).$$

Let $\mathcal{H}_{S^H}$ be the canonical Hilbert space associated to the sub-fBm S^H. That is, $\mathcal{H}_{S^H}$ is the closure of the linear span $\mathcal{E}$ with respect to the scalar product

$$\langle \mathbf{I}_{[0,t]}, \mathbf{I}_{[0,s]} \rangle_{\mathcal{H}_{S^H}} = C_H(t, s).$$

We know that the covariance of sub-fractional Brownian motion can be written as

$$\mathbb{E}\bigl[S^H(t) S^H(s)\bigr] = \int_0^t \int_0^s \phi_H(u, v)\, du\, dv = C_H(s, t), \tag{2.1}$$

where $\phi_H(u, v) = H(2H - 1)[|u - v|^{2H-2} - (u + v)^{2H-2}]$.

Equation (2.1) implies that

$$\langle \varphi, \psi \rangle_{\mathcal{H}_{S^H}} = \int_0^t \int_0^t \varphi_u \psi_v \phi_H(u, v)\, du\, dv \tag{2.2}$$

for any pair step functions φ and ψ on $[0, T]$. Consider the kernel

$$n_H(t, s) = \frac{2^{1-H}\sqrt{\pi}}{\Gamma(H - \frac{1}{2})} s^{3/2-H} \left(\int_s^t \bigl(x^2 - s^2\bigr)^{H-3/2}\, dx \right) \mathbf{I}_{[0,t]}(s). \tag{2.3}$$

By Dzhaparidze and Van Zanten [15], we have

$$C_H(t, s) = c_H^2 \int_0^{s \wedge t} n_H(t, u) n_H(s, u)\, du, \tag{2.4}$$

where

$$c_H^2 = \frac{\Gamma(1 + 2H) \sin(\pi H)}{\pi}.$$

Property (2.4) implies that $C_H(s,t)$ is non-negative definite. Consider the linear operator n_H^* from $\mathcal{E}$ to $L^2([0,T])$ defined by

$$n_H^*(\varphi)(s) := c_H \int_s^r \varphi_r \frac{\partial n_H}{\partial r}(r,s)\,dr.$$

Using (2.2) and (2.4) we have

$$\begin{aligned}
\langle n_H^*\varphi, n_H^*\psi\rangle_{L^2([0,T])} &= c_H^2 \int_0^T \left(\int_s^T \varphi_r \frac{\partial n_H}{\partial r}(r,s)\,dr\right)\left(\int_s^T \psi_u \frac{\partial n_H}{\partial u}(u,s)\,du\right) ds \\
&= c_H^2 \int_0^T \int_0^T \left(\int_0^{r\wedge u} \frac{\partial n_H}{\partial r}(r,s)\frac{\partial n_H}{\partial u}(u,s)\,ds\right)\varphi_r\psi_u\,dr\,du \\
&= c_H^2 \int_0^T \int_0^T \frac{\partial^2 n_H}{\partial r\,\partial u}(r,u)\varphi_r\psi_u\,dr\,du \\
&= H(2H-1)\int_0^T \int_0^T \left[|u-r|^{2H-2} - (u+r)^{2H-2}\right]\varphi_r\psi_u\,dr\,du \\
&= \langle \varphi, \psi\rangle_{\mathcal{H}_{S^H}}. \qquad (2.5)
\end{aligned}$$

As a consequence, the operator n_H^* provides an isometry between the Hilbert space $\mathcal{H}_{S^H}$ and $L^2([0,T])$. Hence, the process W defined by

$$W(t) := S^H\left(\left(n_H^*\right)^{-1}(\mathbf{I}_{[0,t]})\right)$$

is a Wiener process, and S^H has the following Wiener integral representation:

$$S^H(t) = c_H \int_0^t n_H(t,s)\,dW(s)$$

because $(n_H^*)(\mathbf{I}_{[0,t]})(s) = c_H n_H(t,s)$. By Dzhaparidze and Van Zanten [15], we have

$$W(t) = \int_0^t \psi_H(t,s)\,dS^H(s),$$

where

$$\psi_H(t,s) = \frac{s^{H-1/2}}{\Gamma(3/2-H)}\left[t^{H-3/2}\left(t^2-s^2\right)^{1/2-H} - (H-3/2)\int_s^t \left(x^2-s^2\right)^{1/2-H} x^{H-3/2}\,dx\right] \times \mathbf{I}_{[0,t]}(s).$$

In addition, for any $\varphi \in \mathcal{H}_{S^H}$,

$$\int_0^T \varphi(s)\,dS^H(s) = \int_0^T \left(n_H^*\varphi\right)(t)\,dW(t)$$

if and only if $n_H^*\varphi \in L^2([0,T])$.

Also denoting $L^2_{\mathcal{H}_{S^H}}([0,T]) = \{\varphi \in \mathcal{H}_{S^H}, n_H^*\varphi \in L^2([0,T])\}$. Since $H > 1/2$, we have by

(2.5) and Lemma 2.1 of [16],

$$L^2([0,T]) \subset L^{\frac{1}{H}}([0,T]) \subset L^2_{\mathcal{H}_{S^H}}([0,T]). \tag{2.6}$$

Moreover, the following useful result holds:

Lemma 2.1 (Nualart [17]) *For* $\varphi \in L^{1/H}([0,T])$,

$$H(2H-1)\int_0^T\int_0^T |\varphi_r||\varphi_u||u-r|^{2H-2}\,dr\,du \le C_H\|\varphi\|_{L^{\frac{1}{H}}([0,T])},$$

where $C_H = \big(\frac{H(2H-1)}{B(2-2H,H-\frac{1}{2})}\big)^{1/2}$, *with B denoting the beta function*.

Next we are interested in considering a sub-fBm with values in Hilbert space and giving the definition of the corresponding stochastic integral.

Let $(U,\|\cdot\|_U,\langle\cdot\rangle_U)$ and $(K,\|\cdot\|_K,\langle\cdot\rangle_K)$ be two separable Hilbert spaces. Let $L(K,U)$ denote the space of all bounded linear operators from K to U. Let $Q \in L(K,K)$ be a non-negative self-adjoint operator. Denote by $L^0_Q(K,U)$ the space of all $\xi \in L(K,U)$ such that $\xi Q^{\frac{1}{2}}$ is a Hilbert-Schmidt operator. The norm is given by

$$\|\xi\|^2_{L^0_Q(K,U)} = \big\|\xi Q^{\frac{1}{2}}\big\|^2_{HS} = \operatorname{tr}\big(\xi Q\xi^*\big).$$

Then ξ is called a Q-Hilbert-Schmidt operator from K to U.

Let $\{S^H_n(t)\}_{n\in\mathbb{N}}$ be a sequence of one-dimensional standard sub-fractional Brownian motions mutually independent on $(\Omega,\mathcal{F},\mathbb{P})$. When one considers the following series:

$$\sum_{n=1}^{\infty} S^H_n(t)e_n, \quad t \ge 0,$$

where $\{e_n\}_{n\in\mathbb{N}}$ is a complete orthonormal basis in K, this series does not necessarily converge in the space K. Thus we consider a K-valued stochastic process $S^H_Q(t)$ given formally by the following series:

$$S^H_Q(t) = \sum_{n=1}^{\infty} S^H_n(t)Q^{\frac{1}{2}}e_n, \quad t \ge 0.$$

If Q is a non-negative self-adjoint trace class operator, then this series converges in the space K, that is, we have $S^H_Q(t) \in L^2(\Omega,K)$. Then we say that the above $S^H_Q(t)$ is a K-valued Q-cylindrical sub-fractional Brownian motion with covariance operator Q. For example, if $\{\sigma_n\}_{n\in\mathbb{N}}$ is a bounded sequence of non-negative real numbers such that $Qe_n = \sigma_n e_n$, assuming that Q is a nuclear operator in K (that is, $\sum_{n=1}^{\infty}\sigma_n < \infty$), then the stochastic process

$$S^H_Q(t) = \sum_{n=1}^{\infty} S^H_n(t)Q^{\frac{1}{2}}e_n = \sum_{n=1}^{\infty}\sqrt{\sigma_n}S^H_n(t)e_n, \quad t \ge 0,$$

is well defined as a K-valued Q-cylindrical sub-fractional Brownian motion.

Let $\varphi : [0, T] \to L_Q^0(K, U)$ such that

$$\sum_{n=1}^{\infty} \left\| n_H^*\left(\varphi Q^{1/2} e_n\right) \right\|_{L^2([0,T];U)} < \infty. \tag{2.7}$$

Definition 2.1 Let $\varphi : [0, T] \to L_Q^0(K, U)$ satisfy (2.7). Then its stochastic integral with respect to the sub-fBm S_Q^H is defined, for $t \geq 0$, as follows:

$$\begin{aligned} \int_0^t \varphi(s)\, dS_Q^H(s) &:= \sum_{n=1}^{\infty} \int_0^t \varphi(s) Q^{1/2} e_n \, dS_n^H(s) \\ &= \sum_{n=1}^{\infty} \int_0^t \left(n_H^*\left(\varphi Q^{1/2} e_n\right)\right)(s)\, dW(s). \end{aligned}$$

Notice that if

$$\sum_{n=1}^{\infty} \left\| \varphi Q^{1/2} e_n \right\|_{L^{1/H}([0,T];U)} < \infty, \tag{2.8}$$

then in particular (2.7) holds, which follows immediately from (2.6).

The following lemma is obtained as a simple application of Lemma 2.1.

Lemma 2.2 *For any $\varphi : [0, T] \to L_Q^0(K, U)$ such that (2.8) holds, and for any $\alpha, \beta \in [0, T]$ with $\alpha > \beta$,*

$$\mathbb{E}\left\| \int_\alpha^\beta \varphi(s)\, dS_Q^H(s) \right\|_U^2 \leq C_H(\alpha - \beta)^{2H-1} \sum_{n=1}^{\infty} \int_\alpha^\beta \left\| \varphi(s) Q^{1/2} e_n \right\|_U^2 ds.$$

If, in addition,

$$\sum_{n=1}^{\infty} \left\| \varphi(s) Q^{1/2} e_n \right\|_U^2 \textit{ is uniformly convergent for } t \in [0, T], \tag{2.9}$$

then

$$\mathbb{E}\left\| \int_\alpha^\beta \varphi(s)\, dS_Q^H(s) \right\|_U^2 \leq C_H(\alpha - \beta)^{2H-1} \int_\alpha^\beta \left\| \varphi(s) \right\|_{L_Q^0(K,U)}^2 ds. \tag{2.10}$$

Proof Let $\{e_n\}_{n\in\mathbb{N}}$ be the complete orthonormal basis of K introduced above. Applying Lemma 2.1 we can obtain

$$\begin{aligned} \mathbb{E}\left\| \int_\alpha^\beta \varphi(s)\, dS_Q^H(s) \right\|_U^2 &= \mathbb{E}\left\| \sum_{n=1}^{\infty} \int_\alpha^\beta \varphi(s) Q^{1/2} e_n \, dS^H(s) \right\|_U^2 \\ &= \sum_{n=1}^{\infty} \mathbb{E}\left\| \int_\alpha^\beta \varphi(s) Q^{1/2} e_n \, dS^H(s) \right\|_U^2 \\ &= \sum_{n=1}^{\infty} H(2H-1) \int_\alpha^\beta \int_\alpha^\beta \left\| \varphi(t) Q^{1/2} e_n \right\|_U \left\| \varphi(s) Q^{1/2} e_n \right\|_U \end{aligned}$$

$$\begin{aligned}
&\times \left[|t-s|^{2H-2}-(t+s)^{2H-2}\right] dt\, ds \\
&\leq \sum_{n=1}^{\infty} H(2H-1) \int_{\alpha}^{\beta} \int_{\alpha}^{\beta} \left\| \varphi(t) Q^{1/2} e_n \right\|_U \left\| \varphi(s) Q^{1/2} e_n \right\|_U \\
&\quad \times |t-s|^{2H-2}\, dt\, ds \\
&\leq C_H \sum_{n=1}^{\infty} \left(\int_{\alpha}^{\beta} \left\| \varphi(s) Q^{1/2} e_n \right\|_U^{1/H} ds \right)^{2H} \\
&\leq C_H (\alpha-\beta)^{2H-1} \sum_{n=1}^{\infty} \int_{\alpha}^{\beta} \left\| \varphi(s) Q^{1/2} e_n \right\|_U^2 ds.
\end{aligned}$$

The second assertion is an immediate consequence of the Weierstrass M-test. □

Remark 2.1 If $\{\sigma_n\}_{n\in\mathbb{N}}$ is a bounded sequence of non-negative real numbers such that the nuclear operator Q satisfies $Qe_n = \sigma_n e_n$, assuming that there exists a positive constant k_φ such that

$$\left\| \varphi(t) \right\|_{L_Q^2(K,U)} \leq k_\varphi, \quad \text{uniformly in } \in [0,T],$$

then (2.9) holds automatically.

3 Existence and uniqueness of mild solution

We denote by $C(a,b;L^2(\Omega;U)) = C(a,b;L^2(\Omega,\mathcal{F},\mathbb{P};U))$ the Banach space of all continuous functions from $[a,b]$ into $L^2(\Omega;U)$ equipped with sup norm. Let us consider two fixed real numbers $r \geq 0$ and $T > 0$. If $x \in C(-r,T;L^2(\Omega;U))$ for each $t \in [0,T]$ we denote $x_t \in C(-r,0;L^2(\Omega;U))$ the function defined by $x_t(\theta) = x(t+\theta)$, for $\theta \in [-r,0]$.

In this section we consider the existence and uniqueness of mild solutions to the following stochastic evolution equation with delays:

$$\begin{cases} dX(t) = (AX(t) + f(t,X_t))\, dt + g(t)\, dS_Q^H(t), & t \in [0,T], \\ X(t) = \varphi(t), & t \in [-r,0], \end{cases} \tag{3.1}$$

where $S_Q^H(t)$ is the sub-fractional Brownian motion which was introduced in the previous section, the initial data $\varphi \in C(-r,0;L^2(\Omega;U))$, and $A : \mathrm{Dom}(A) \subset U \to U$ is the infinitesimal generator of a strongly continuous semigroup $S(\cdot)$ on U, that is, for $t \geq 0$, we have

$$\left\| S(t) \right\|_U \leq M e^{\rho t}, \quad M \geq 1, \rho \in \mathbb{R}.$$

$f : [0,T] \times C(-r,0;U) \to U$ is a family of nonlinear operators defined for almost every t which satisfy:

(f.1) The mapping $t \in [0,T] \to f(t,\xi) \in U$ is Lebesgue measurable for all $\xi \in C(-r,0;L^2(\Omega;U))$.

(f.2) There exists a constant $C > 0$ such that for any $x, y \in C(-r,T;U)$ and $t \in [0,T]$,

$$\int_0^t \left\| f(s,x_s) - f(s,y_s) \right\|_U^2 ds \leq C \int_{-r}^t \left\| x(s) - y(s) \right\|_U^2 ds.$$

(f.3) $\int_0^T \|f(s,0)\|_U^2\,ds < \infty$.

Moreover, for $g : [0,T] \to L_Q^0(K,U)$ we assume the following conditions: for the complete orthonormal basis $\{e_n\}_{n\in\mathbb{N}}$ in K, we have:

(g.1) $\sum_{n=1}^{\infty} \|gQ^{1/2}e_n\|_{L^2([0,T];U)} < \infty$.

(g.2) $\sum_{n=1}^{\infty} \|g(t)Q^{1/2}e_n\|_U$ is uniformly convergent for $t \in [0,T]$.

Definition 3.1 A U-valued process $X(t)$ is called a mild solution of (3.1) if $X \in C(-r,T;L^2(\Omega;U))$, $X(t) = \varphi(t)$, for $t \in [-r,0]$ and for $t \in [0,T]$, satisfies

$$X(t) = S(t)\varphi(0) + \int_0^t S(t-s)f(s,X_s)\,ds + \int_0^t S(t-s)g(s)\,dS_Q^H(t) \quad \mathbb{P}\text{-a.s.} \tag{3.2}$$

Notice that, thanks to (g.1) and the fact that $H \in (1/2,1)$, (2.8) holds, which implies that the stochastic integral in (3.2) is well defined since $S(\cdot)$ is a strongly continuous semigroup. Moreover, (g.1) together with (g.2) immediately imply that, for every $t \in [0,T]$,

$$\int_0^t \|g(s)\|_{L_Q^0(K,U)}^2\,ds < \infty.$$

Theorem 3.1 *Under the assumptions on A and conditions* (f.1)-(f.3) *and* (g.1)-(g.2), *for every $\varphi \in C(-r,0;L^2(\Omega,U))$ there exists a unique mild solution X to* (3.1).

Proof We can assume that $\rho > 0$, otherwise we can take $\rho_0 > 0$ such that, for $t \geq 0$, $\|S(t)\|_U \leq Me^{\rho_0 t}$.

We start the proof by checking the uniqueness of solutions. Assume that X, Y are two mild solutions of (3.1). Then

$$\begin{aligned}\mathbb{E}\|X(t)-Y(t)\|_U^2 &\leq t\mathbb{E}\int_0^t \|S(t-s)(f(s,X_s)-f(s,Y_s))\|_U^2\,ds \\ &\leq tM^2e^{2\rho t}\mathbb{E}\int_0^t \|f(s,X_s)-f(s,Y_s)\|_U^2\,ds \\ &\leq tM^2e^{2\rho t}C\int_0^t \mathbb{E}\|X(s)-Y(s)\|_U^2\,ds \\ &\leq tM^2e^{2\rho t}C\int_0^t \sup_{0\leq\tau\leq s}\mathbb{E}\|X(\tau)-Y(\tau)\|_U^2\,ds,\end{aligned}$$

and therefore, since $X = Y$ over the interval $[-r,0]$, by taking the supremum in the above inequality,

$$\sup_{0\leq\theta\leq t}\mathbb{E}\|X(\theta)-Y(\theta)\|_U^2 \leq TM^2e^{2\rho T}C\int_0^t \sup_{0\leq\tau\leq s}\mathbb{E}\|X(\tau)-Y(\tau)\|_U^2\,ds.$$

The Gronwall's lemma implies now the uniqueness result.

Now we prove the existence of solutions to problem (3.1).

First of all, we check that the well-defined stochastic integral possesses the repaired regularity. To this end, let us consider $\sigma > 0$ small enough. We have

$$\begin{aligned}
&\mathbb{E}\left\| \int_0^{t+\sigma} S(t+\sigma-s)g(s)\,dS_Q^H(s) - \int_0^t S(t-s)g(s)\,dS_Q^H(s)\right\|_U^2 \\
&\quad\le 2\mathbb{E}\left\| \int_0^{t+\sigma} \bigl(S(t+\sigma-s)-S(t-s)\bigr)g(s)\,dS_Q^H(s)\right\|_U^2 \\
&\qquad + 2\mathbb{E}\left\| \int_t^{t+\sigma} S(t-s)g(s)\,dS_Q^H(s)\right\|_U^2 \\
&\quad := J_1 + J_2.
\end{aligned}$$

Applying inequality (2.10) to J_1, we obtain

$$\begin{aligned}
J_1 &\le 2C_H t^{2H-1} \int_0^t \bigl\| S(t-s)\bigl(S(\sigma)-Id\bigr)g(s)\bigr\|_{L_Q^0(K,U)}^2\,ds \\
&\le C_H t^{2H-1} M^2 e^{2\rho t} \int_0^t \bigl\| \bigl(S(\sigma)-Id\bigr)g(s)\bigr\|_{L_Q^0(K,U)}^2\,ds \to 0
\end{aligned}$$

when $\sigma \to 0$ thanks to the Lebesgue majorant theorem, since, for every s fixed,

$$S(\sigma)g(s) \to g(s), \qquad \bigl\|S(\sigma)g(s)\bigr\|_{L_Q^0(K,U)} \le Me^{\rho\sigma}\bigl\|g(s)\bigr\|_{L_Q^0(K,U)}^2.$$

Applying now (2.10) to J_2, we have

$$J_2 \le 2C_H \sigma^{2H-1} M^2 e^{2\rho\sigma} \int_t^{t+\sigma} \bigl\|g(s)\bigr\|_{L_Q^0(K,U)}^2\,ds \to 0$$

when $\sigma \to 0$. Therefore the stochastic integral belongs to the space $C(-r,T;L^2(\Omega;U))$.

We denote $X_0 = 0$ and define by recurrence a sequence $\{X^n\}_{n\in\mathbb{N}}$ of processes as

$$\begin{cases} X^n(t) = S(t)\varphi(0) + \int_0^t S(t-s)f(s,X_s^{n-1})\,ds + \int_0^t S(t-s)g(s)\,dS_Q^H(s), & t\in[0,T], \\ X^n(t) = \varphi(t), \quad t\in[-r,0]. \end{cases} \tag{3.3}$$

The sequence (3.3) is well defined, since $X^0 = 0 \in C(-r,T;L^2(\Omega;U))$ and given $X^{n-1} \in C(-r,T;L^2(\Omega;U))$, let us check that $X^n \in C(-r,T;L^2(\Omega;U))$ as well. To this end, let us consider $\sigma > 0$ sufficiently small. Then

$$\begin{aligned}
\bigl\|X^n(t+\sigma)-X^n(t)\bigr\|_U^2 &\le 2\left\| \int_0^t \bigl(S(t+\sigma-s)-S(t-s)\bigr)f\bigl(s,X_s^{n-1}\bigr)\,ds\right\|_U^2 \\
&\quad + 2\left\| \int_t^{t+\sigma} S(t+\sigma-s)f\bigl(s,X_s^{n-1}\bigr)\,ds\right\|_U^2 \\
&:= I_1 + I_2.
\end{aligned}$$

On the one hand,

$$\mathbb{E}I_1 \le 2M^2 te^{2\rho t}\mathbb{E}\int_0^t \bigl\|\bigl(S(\sigma)-Id\bigr)f\bigl(s,X_s^{n-1}\bigr)\bigr\|_U^2\,ds \to 0$$

when $\sigma \to 0$ thanks to the Lebesgue majorant theorem, since, for each s fixed,

$$S(\sigma)f\big(s,X_s^{n-1}\big) \to f\big(s,X_s^{n-1}\big), \qquad \big\|S(\sigma)f\big(s,X_s^{n-1}\big)\big\|_U \le Me^{\rho\sigma}\big\|f\big(s,X_s^{n-1}\big)\big\|_U$$

and

$$\mathbb{E}\int_0^t \big\|f\big(s,X_s^{n-1}\big)\big\|_U^2\,ds \le C\mathbb{E}\int_{-r}^t \big\|X^{n-1}(s)\big\|_U^2\,ds + \mathbb{E}\int_0^t \big\|f(s,0)\big\|_U^2\,ds$$

due to conditions (f.2) and (f.3) and the fact that $X^{n-1} \in C(-r,T;L^2(\Omega;U))$.

On the other hand,

$$\begin{aligned} I_2 &\le 2\sigma M^2e^{2\rho\sigma}\int_t^{t+\sigma} \big\|f\big(s,X_s^{n-1}\big)-f(s,0)\big\|_U^2\,ds + 2\sigma M^2e^{2\rho\sigma}\int_t^{t+\sigma}\big\|f(s,0)\big\|_U^2\,ds \\ &\le 2\sigma M^2e^{2\rho\sigma}C\int_{-r}^{t+\sigma}\big\|X^{n-1}(s)\big\|_U^2\,ds + 2\sigma M^2e^{2\rho\sigma}\int_t^{t+\sigma}\big\|f(s,0)\big\|_U^2\,ds, \end{aligned}$$

so that

$$\mathbb{E}I_2 \le 2\sigma M^2e^{2\rho\sigma}C\int_{-r}^{t+\sigma}\mathbb{E}\big\|X^{n-1}(s)\big\|_U^2\,ds + 2\sigma M^2e^{2\rho\sigma}\int_t^{t+\sigma}\big\|f(s,0)\big\|_U^2\,ds \to 0$$

when $\sigma \to 0$.

Next, we want to show that $\{X^n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $C(-r,T;L^2(\Omega;U))$.

Firstly, for $t \in [0,T]$ and $n \in \mathbb{N}$, since $X^n = X^{n-1}$ on $[-r,0]$, we have

$$\big\|X^{n+1}(t)-X^n(t)\big\|_U^2 \le tM^2e^{2\rho t}C\int_0^t \big\|X^n(s)-X^{n-1}(s)\big\|_U^2\,ds$$

and this implies

$$\mathbb{E}\big\|X^{n+1}(t)-X^n(t)\big\|_U^2 \le tM^2e^{2\rho t}C\int_0^t \sup_{0\le\tau\le s}\big\|X^n(\tau)-X^{n-1}(\tau)\big\|_U^2\,ds.$$

Defining

$$G^n(t) = \sup_{0\le\theta\le t}\mathbb{E}\big\|X^{n+1}(\theta)-X^n(\theta)\big\|_U^2,$$

we obtain

$$G^n(t) \le k\int_0^t G^{n-1}(s)\,ds, \quad \forall n \ge 2$$

for $k = TM^2e^{2\rho T}C$. Consequently, by iteration we can obtain for $\forall t \in [0,T]$,

$$G^n(t) \le \frac{k^{n-1}T^{n-1}}{(n-1)!}G^1(T), \quad \forall n \ge 2.$$

Since $X^{n+1}(t) = X^n(t)$, $\forall t \in [-r,0]$, the last estimate implies that $\{X^n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $C(-r,T;L^2(\Omega;U))$.

Finally, we check that the limit X of the sequence $\{X^n\}_{n\in\mathbb{N}}$ is a solution of (3.1). But this is straightforward, taking into account that X^n is defined by (3.3) and that f satisfies (f.2), so that, in particular, when $n \to \infty$,

$$\mathbb{E}\left\|\int_0^t S(t-s)\bigl(f\bigl(s,X_s^{n-1}\bigr)-f(s,X_s)\bigr)\,ds\right\|_U^2 \le tM^2e^{2\rho t}C\int_0^t \mathbb{E}\bigl\|X^{n-1}(s)-X(s)\bigr\|_U^2\,ds \to 0,$$

and therefore X is the unique (mild) solution of (3.1). □

4 Existence of weak solutions

In this section we prove that the mild solution to system (3.1) is also a weak solution. First of all we recall the definition of weak solution according to Da Prato and Zabczyk [18].

Definition 4.1 An U-valued process $X(t)$, $t \in [-r, T]$ is called a weak solution of (3.1) if $X(t) = \varphi(t)$, for $t \in [-r, 0]$, and for all $\xi \in D(A^*)$ and all $t \in [0, T]$,

$$\begin{aligned}\langle X(t),\xi\rangle_U &= \langle\varphi(0),\xi\rangle_U + \int_0^t \bigl(\langle X(s),A^*\xi\rangle_U + \langle f(s,X_s),\xi\rangle_U\bigr)\,ds \\ &\quad + \int_0^t \langle g(s),\xi\rangle_U\,dS_Q^H(s) \quad \mathbb{P}\text{-a.s.}\end{aligned}$$

Theorem 4.1 *Under the assumptions of Theorem* 3.1, *the mild solution of* (3.1) *is also a weak solution.*

Proof For each $\xi \in D(A^*)$ it follows that

$$\begin{aligned}&\mathbb{E}\biggl[\biggl|\int_0^t \langle X(s),A^*\xi\rangle_U\,ds - \int_0^t \langle S(s)\varphi(0),A^*\xi\rangle_U\,ds - \int_0^t\int_0^s \langle S(s-\tau)f(\tau,X_\tau),A^*\xi\rangle_U\,d\tau\,ds \\ &\qquad - \int_0^t\int_0^s \langle S(s-\tau)g(\tau),A^*\xi\rangle_U\,dS_Q^H(\tau)\,ds\biggr|\biggr] \\ &\quad\le \int_0^t \mathbb{E}\biggl[\biggl|\langle X(s),A^*\rangle_U - \langle S(s)\varphi(0),A^*\xi\rangle_U - \int_0^s \langle S(s-\tau)f(\tau,X_\tau),A^*\xi\rangle_U\,d\tau \\ &\qquad - \int_0^s \langle S(s-\tau)g(\tau),A^*\xi\rangle_U\,dS_Q^H(\tau)\biggr|\biggr]\,ds \\ &\quad= \int_0^t \mathbb{E}\biggl[\biggl|\biggl\langle X(s)-S(s)\varphi(0)-\int_0^s S(s-\tau)f(\tau,X_\tau)\,d\tau \\ &\qquad - \int_0^s S(s-\tau)g(\tau)\,dS_Q^H(\tau),A^*\xi\biggr\rangle_U\biggr|\biggr]\,ds \\ &\quad= 0.\end{aligned}$$

Thus, for a.e. $\omega \in \Omega$, we have

$$\begin{aligned}\int_0^t \langle X(s),A^*\xi\rangle_U\,ds &= \int_0^t \langle S(s)\varphi(0),A^*\xi\rangle_U\,ds + \int_0^t\int_0^s \langle S(s-\tau)f(\tau,X_\tau),A^*\xi\rangle_U\,d\tau\,ds \\ &\quad + \int_0^t\int_0^s \langle S(s-\tau)g(\tau),A^*\xi\rangle_U\,dS_Q^H(\tau)\,ds. \qquad (4.1)\end{aligned}$$

Now we use the fact that, for $\in D(A^*)$, $\frac{d}{dt}S^*(t)\xi = S^*(t)A^*\xi$. We can obtain

$$\int_0^t \langle S(s)\varphi(0), A^*\xi\rangle_U \, ds = \int_0^t \langle \varphi(0), S^*(s)A^*\xi\rangle_U \, ds = \langle S(t)\varphi(0) - \varphi(0), \xi\rangle_U.$$

On the other hand, using Fubini's theorem we have

$$\begin{aligned}
&\int_0^t \int_0^s \langle S(s-\tau)f(\tau, X_\tau), A^*\xi\rangle_U \, d\tau \, ds \\
&\quad = \int_0^t \int_\tau^t \langle \mathbf{I}_{(0,s]}(\tau)f(\tau, X_\tau), S^*(s-\tau)A^*\xi\rangle_U \, ds \, d\tau \\
&\quad = \int_0^t \langle S(t-\tau)f(\tau, X_\tau) - f(\tau, X_\tau), \xi\rangle_U \, d\tau.
\end{aligned}$$

Finally,

$$\begin{aligned}
&\int_0^t \int_0^s \langle S(s-\tau)g(\tau), A^*\xi\rangle_U \, dS_Q^H(\tau) \, ds \\
&\quad = \int_0^t \int_\tau^t \langle \mathbf{I}_{(0,s]}(\tau)g(\tau), S^*(s-\tau)A^*\xi\rangle_U \, ds \, dS_Q^H(\tau) \\
&\quad = \int_0^t \langle g(\tau), S^*(t-\tau)\xi - \xi\rangle_U \, dS_Q^H(\tau) \\
&\quad = \int_0^t \langle S(t-\tau)g(\tau), \xi\rangle_U \, dS_Q^H(\tau) - \int_0^t \langle g(\tau), \xi\rangle_U \, dS_Q^H(\tau).
\end{aligned}$$

Therefore by (4.1) for a.e. $\omega \in \Omega$, it follows that

$$\begin{aligned}
\int_0^t \langle AX(s), \xi\rangle_U \, ds &= \int_0^t \langle X(s), A^*\xi\rangle_U \, ds \\
&= \langle S(t)\varphi(0) - \varphi(0), \xi\rangle_U + \int_0^t \langle S(t-\tau)f(\tau, X_\tau) - f(\tau, X_\tau), \xi\rangle_U \, d\tau \\
&\quad + \int_0^t \langle S(t-\tau)g(\tau), \xi\rangle_U \, dS_Q^H(\tau) - \int_0^t \langle g(\tau), \xi\rangle_U \, dS_Q^H(\tau) \\
&= \langle X(t) - \varphi(0), \xi\rangle_U + \int_0^t \big(\langle X(\tau), A^*\xi\rangle_U + \langle f(\tau, X_\tau), \xi\rangle_U\big) \, d\tau \\
&\quad + \int_0^t \langle g(\tau), \xi\rangle_U \, dS_Q^H(\tau).
\end{aligned}$$

Consequently, it follows that almost surely

$$\langle X(t), \xi\rangle_U = \langle \varphi(0), \xi\rangle_U + \int_0^t \big(\langle X(s), A^*\xi\rangle_U + \langle f(s, X_s), \xi\rangle_U\big) \, ds + \int_0^t \langle g(s), \xi\rangle_U \, dS_Q^H(s),$$

which means that $X(t)$ is the weak solution to (3.1). □

5 Exponential decay of solutions in mean square

In this section we investigate the exponential stability of the mild solutions to (3.1). We impose the following conditions:

Condition 1 The operator A is a closed linear operator generating a strongly continuous semigroup $S(t)$, $t \geq 0$, on the separable Hilbert space U and satisfies

$$\|S(t)\|_U \leq Me^{-\lambda t}, \quad \forall t \geq 0, \text{where } M \geq 1, \lambda > 0.$$

Condition 2 There exists a constant $C \geq 0$ such that for any $x, y \in C(-r, T; U)$ and for all $t \geq 0$,

$$\int_0^t e^{ms} \|f(s, x_s) - f(s, y_s)\|_U^2 \leq C \int_{-r}^t e^{ms} \|x(s) - y(s)\|_U^2 \, ds \quad \text{for all } 0 \leq m \leq \lambda$$

and

$$\int_0^\infty e^{\lambda s} \|f(s, 0)\|_U^2 \, ds < \infty.$$

Condition 3 In addition to assumptions (g.1) and (g.2), assume

$$\int_0^\infty e^{\lambda s} \|g(s)\|_{L_Q^0(K,U)}^2 \, ds < \infty.$$

The following theorem shows the exponential decay to zero in mean square, with an explicit exponential decay rate γ.

Theorem 5.1 *In addition to Conditions* 1-3, *assume that the mild solution $X(t)$ of system* (3.1) *corresponding to initial function $\varphi \in C(-r, 0; L^2(\Omega; U))$, exists for all $t \geq -r$, and that*

$$\lambda^2 > 6CM^2. \tag{5.1}$$

Then there exists a constant $\gamma > 0$ such that

$$\limsup_{t\to\infty} \left(\frac{1}{t}\right) \log \mathbb{E}\|X(t)\|_U^2 \leq -\gamma.$$

In other words, every mild solution exponentially decays to zero in mean square.

Proof By (5.1) we can choose an $\theta > 0$ such that $\gamma = \lambda - \theta - 6CM^2\lambda^{-1} > 0$. Then for this γ we have

$$\begin{aligned}\mathbb{E}\|X(t)\|_U^2 &\leq 3\mathbb{E}\|S(t)\varphi(0)\|_U^2 + 3\mathbb{E}\left\|\int_0^t S(t-s)f(s, X_s)\, ds\right\|_U^2 \\ &\quad + 3\mathbb{E}\left\|\int_0^t S(t-s)g(s)\, dS_Q^H(s)\right\|_U^2.\end{aligned}$$

Therefore, by Condition 1 and Lemma 2.2 we have

$$\begin{aligned}\mathbb{E}\|X(t)\|_U^2 &\leq 3\mathbb{E}\|S(t)\varphi(0)\|_U^2 + 3M^2 \int_0^t e^{-\lambda(t-s)}\, ds \int_0^t e^{-\lambda(t-s)} \mathbb{E}\|f(s, X_s)\|_U^2 \, ds \\ &\quad + 3C_H M^2 t^{2H-1} \int_0^t e^{-2\lambda(t-s)} \|g(s)\|_{L_Q^0(K,U)}^2 \, ds\end{aligned}$$

$$\le 3M^2e^{-2\lambda t}\mathbb{E}\|\varphi(0)\|_U^2 + \frac{3}{\lambda}M^2\int_0^t e^{-\lambda(t-s)}\mathbb{E}\|f(s,X_s)\|_U^2\,ds + 3C_HM^2t^{2H-1}\int_0^t e^{-2\lambda(t-s)}\|g(s)\|^2_{L_Q^0(K,U)}\,ds$$

and

$$e^{\lambda t}\mathbb{E}\|X(t)\|_U^2 \le 3M^2\mathbb{E}\|\varphi(0)\|_U^2 + \frac{3}{\lambda}M^2\int_0^t e^{\lambda s}\mathbb{E}\|f(s,X_s)\|_U^2\,ds + 3C_HM^2t^{2H-1}\int_0^t e^{\lambda s}\|g(s)\|^2_{L_Q^0(K,U)}\,ds.$$

Then, for the chosen parameter θ, we obtain

$$e^{(\lambda-\theta)t}\mathbb{E}\|X(t)\|_U^2 \le 3M^2e^{-\theta t}\mathbb{E}\|\varphi(0)\|_U^2 + \frac{3}{\lambda}M^2\int_0^t e^{(\lambda-\theta)s}\mathbb{E}\|f(s,X_s)\|_U^2\,ds + 3C_HM^2t^{2H-1}e^{-\theta t}\int_0^t e^{\lambda s}\|g(s)\|^2_{L_Q^0(K,U)}\,ds.$$

On the one hand, Condition 3 ensures the existence of a positive constant C_1 such that

$$3C_HM^2t^{2H-1}e^{-\theta t}\int_0^t e^{\lambda s}\|g(s)\|^2_{L_Q^0(K,U)}\,ds \le C_1 \quad \text{for all } t \ge 0,$$

hence,

$$e^{(\lambda-\theta)t}\mathbb{E}\|X(t)\|_U^2 \le C_1 + 3M^2\mathbb{E}\|\varphi(0)\|_U^2 + \frac{3}{\lambda}M^2\int_0^t e^{(\lambda-\theta)s}\mathbb{E}\|f(s,X_s)\|_U^2\,ds.$$

On the other hand, in view of Condition 2, there exists another positive constant C_2 such that

$$\begin{aligned}\int_0^t e^{(\lambda-\theta)s}\mathbb{E}\|f(s,X_s)\|_U^2\,ds &\le 2C\int_{-r}^0 e^{(\lambda-\theta)s}\mathbb{E}\|\varphi(s)\|_U^2\,ds + 2C\int_0^t e^{(\lambda-\theta)s}\mathbb{E}\|X(s)\|_U^2\,ds \\ &\quad + 2\int_0^t e^{(\lambda-\theta)s}\|f(s,0)\|_U^2\,ds \\ &\le C_2 + 2C\int_{-r}^0 \mathbb{E}\|\varphi(s)\|_U^2\,ds + 2C\int_0^t e^{(\lambda-\theta)s}\mathbb{E}\|X(s)\|_U^2\,ds.\end{aligned}$$

Thus, we obtain

$$\begin{aligned}e^{(\lambda-\theta)t}\mathbb{E}\|X(t)\|_U^2 &\le C_1 + 3M^2\lambda^{-1}C_2 + 6CM^2\lambda^{-1}\int_{-r}^0 \mathbb{E}\|\varphi(s)\|_U^2\,ds \\ &\quad + 6CM^2\lambda^{-1}\int_0^t e^{(\lambda-\theta)s}\mathbb{E}\|X(s)\|_U^2\,ds \\ &= C_3 + 6CM^2\lambda^{-1}\int_0^t e^{(\lambda-\theta)s}\mathbb{E}\|X(s)\|_U^2\,ds,\end{aligned}$$

where C_3 is a suitable positive constant. Therefore, Gronwall's lemma implies that

$$e^{(\lambda-\theta)t}\mathbb{E}\|X(t)\|_U^2 \le C_3e^{6CM^2\lambda^{-1}t}.$$

Consequently,

$$\mathbb{E}\|X(t)\|_U^2 \le C_3 e^{(6CM^2\lambda^{-1}-\lambda+\theta)t} = C_3 e^{-\gamma t}.$$

The proof is complete. □

Remark 5.1 A remarkable fact is that the decay rate γ is independent of H. Indeed, in the case of considering a Q-Brownian motion, *i.e.*, the case $H = 1/2$, instead of our sub-fBm S_Q^H, the condition on λ would be exactly (5.1) (to check this assertion, it is enough to take into account the isometry for classical Wiener integrals). In other words, whenever the stochastic integral is well defined, and under Conditions 1-3, the rate of the exponential decay to zero in mean square is insensitive to the parameter H.

Remark 5.2 Theorem 5.1 remains true if we replace the first part of Condition 2 by Condition 4 below.

Condition 4 For any $x \in C(-r, T; U)$,

$$\int_0^t e^{ms}\|f(s,X_s)\|_U^2 \le C_2 + 2C\int_{-r}^t e^{ms}\|X(s)\|_U^2\,ds \quad \text{for all } 0 \le m \le \lambda.$$

6 An example

In this section, an example is provided to illustrate the obtained theory.

Let $K = L^2(0,\pi)$ and $e_n = \sqrt{\frac{2}{\pi}}\sin(nx)$, $n \in \mathbb{N}$. Then $\{e_n\}_{n\in\mathbb{N}}$ is a complete orthonormal basis in K. Let $U = L^2(0,\pi)$ and $A = \frac{\partial^2}{\partial x^2}$ with domain $D(A) = L_0^1(0,\pi) \cap L^2(0,\pi)$. Then it is well known that $Au = -\sum_{n=1}^\infty n^2\langle u, e_n\rangle_U e_n$ for any $u \in U$, and A is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators $S(t): U \to U$, where $S(t)u = \sum_{n=1}^\infty e^{-n^2 t}\langle u, e_n\rangle_U e_n$ and for $\forall t \ge 0$, $\|S(t)\|_U \le e^{-t}$. In order to define the operator $Q: K \to K$, we choose a sequence $\{\sigma_n\}_{n\ge 1} \subset \mathbb{R}^+$ and set $Qe_n = \sigma_n e_n$, and assume that $\operatorname{tr}(Q) = \sum_{n=1}^\infty \sqrt{\sigma_n} < \infty$. Define the process $S_Q^H(s)$ by

$$S_Q^H(t) = \sum_{n=1}^\infty \sqrt{\sigma_n} S_n^H(t) e_n,$$

where $H \in (1/2, 1)$ and $\{S_n^H\}_{n\in\mathbb{N}}$ is a sequence of two-sided one-dimensional sub-fractional Brownian motions mutually independent.

Then we consider the following stochastic evolution equation:

$$\begin{cases} du(t,x) = [\frac{\partial^2}{\partial x^2}u(t,x) + b(t)\cdot u(t,x(t-r))]\,dt \\ \qquad\qquad + g(t)\,dS_Q^H(t), \quad t \in [0,T], x \in [0,\pi], \\ u(t,0) = u(t,\pi) = 0, \quad t \in [0,T], \\ u(t,x) = \varphi(t,x), \quad t \in [-\tau, 0], x \in [0,\pi], \end{cases} \tag{6.1}$$

where $r > 0$ and $b, g: \mathbb{R}^+ \to \mathbb{R}$ are continuous functions such that g satisfies Condition 3 above and b satisfies

$$\int_0^\infty e^{\lambda s}|b(s)|^2\,ds < \infty.$$

Observe that the fact $\int_0^\infty e^{\lambda_s}|b(s)|^2\,ds < \infty$ implies that $b(t)$ is bounded for all $t \geq 0$. Denote by b_0 the smallest upper bound of the b. Taking

$$f(t,\varphi_t)(\eta) = \sin(t)\cdot\varphi(\eta_t).$$

Thus, for any $x, y \in C(-r,T;U)$, and for all $t \geq 0$, one has

$$\int_0^t \|f(s,x_s) - f(s,y_s)\|_U^2\,ds \leq b_0^2 \int_{-r}^t \|x(s) - y(s)\|^2\,ds$$

and

$$\int_0^t e^{ms}\|f(s,x_s) - f(s,y_s)\|_U^2\,ds \leq b_0^2 \int_{-r}^t e^{ms}\|x(s) - y(s)\|_U^2\,ds.$$

Then we can check that there exists a unique mild solution to (6.1) according to Theorem 3.1.

If we assume, in addition, that

$$b_0^2 < \frac{1}{6},$$

then any mild solution to (6.1) decays exponentially to zero in mean square.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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