Lv et al. *Boundary Value Problems* (2024) 2024:50
https://doi.org/10.1186/s13661-024-01857-z

Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

Longtime dynamics of solutions for higher-order (m_1, m_2)-coupled Kirchhoff models with higher-order rotational inertia and nonlocal damping

Penghui Lv[1*], Yuan Yuan[2] and Guoguang Lin[3]

*Correspondence:
18487279097@163.com
[1] Applied Technology College of Soochow University, Suzhou, Jiangsu 215325, China
Full list of author information is available at the end of the article

Abstract

The Kirchhoff model is derived from the vibration problem of stretchable strings. This paper focuses on the longtime dynamics of a higher-order (m_1, m_2)-coupled Kirchhoff system with higher-order rotational inertia and nonlocal damping. We first obtain the state of the model's solutions in different spaces through prior estimation. After that, we immediately prove the existence and uniqueness of their solutions in different spaces through the Faedo-Galerkin method. Subsequently, we prove their family of global attractors using the compactness theorem. Finally, we reflect on the subsequent research of the model and point out relevant directions for further research on the model. In this way, we systematically study the longtime dynamics of the higher-order (m_1, m_2)-coupled Kirchhoff model with higher-order rotational inertia, thus enriching the relevant findings of higher-order coupled Kirchhoff models and laying a theoretical foundation for future practical applications.

Mathematics Subject Classification: 35B41; 35G31

Keywords: Higher-order (m_1, m_2)-coupled Kirchhoff system; Higher-order rotational inertia; Family of global attractors

1 Introduction

This study considers the longtime dynamics of the following higher-order (m_1, m_2)-coupled Kirchhoff model in a bounded smooth domain $\Omega \subset \mathbb{R}^n$:

$$\begin{cases} (1+\alpha(-\Delta)^{m_1})u_{tt} + N_1(\|\nabla^{m_1}u\|^2)(-\Delta)^{m_1}u_t + M(\|\nabla^{m_1}u\|^2 + \|\nabla^{m_2}v\|^2)(-\Delta)^{m_1}u \\ \quad + g_1(u,v) = f_1(x), \\ (1+\beta(-\Delta)^{m_2})v_{tt} + N_2(\|\nabla^{m_2}v\|^2)(-\Delta)^{m_2}v_t + M(\|\nabla^{m_1}u\|^2 + \|\nabla^{m_2}v\|^2)(-\Delta)^{m_2}v \\ \quad + g_2(u,v) = f_2(x), \end{cases} \tag{1}$$

under the following boundary conditions:

$$
\begin{aligned}
&u(x) = 0, \qquad \frac{\partial^i u}{\partial \mathbf{n}^i} = 0, \quad i = 1,\dots,m_1 - 1, m_1 > 1, \\
&v(x) = 0, \qquad \frac{\partial^j v}{\partial \mathbf{n}^j} = 0, \quad j = 1,\dots,m_2 - 1, m_2 > 1,
\end{aligned} \tag{2}
$$

and the following initial conditions:

$$
\begin{aligned}
&u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), \qquad v(x,0) = v_0(x), \\
&v_t(x,0) = v_1(x), \quad x \in \Omega,
\end{aligned} \tag{3}
$$

where Δ is the Laplace operator, $\alpha \in (0,1]$ and $\beta \in (0,1]$ are rotational coefficients, N_1, N_2, M_1, and M_2 are scalar functions specified later, g_1 and g_2 are the given source terms, and f_1 and f_2 are the given functions.

Equation (1) is a set of generalized higher-order quasilinear wave equations. The proposed equation in this paper originated from the stretchable string vibration problem established by Kirchhoff in 1883:

$$
\rho h \frac{\partial^2 u}{\partial t^2} = \left\{ p_0 + \frac{Eh}{2L} \int_0^L \left(\frac{\partial u}{\partial x} \right)^2 \mathrm{d}x \right\} \frac{\partial^2 u}{\partial t^2}, \tag{4}
$$

where $u = u(x,t)$ is the lateral displacement at space coordinate x and time coordinate t, $0 < x < L$, $t \geq 0$, E is the Young's modulus, ρ is the mass density, h is the cross-sectional area, L is the length, and p_0 is the initial axial tension. In recent decades, the long-term behaviors of Kirchhoff equations in various forms have attracted much academic attention, and for abundant research results for some related system, we can refer to [1–12].

For instance, Chueshov [1] studied the well-posedness and long-term dynamic behaviors of the following Kirchhoff equation with a nonlinear strong damping term:

$$
u_{tt} + \sigma\left(\|\nabla u\|^2\right)\Delta u_t - \phi\left(\|\nabla u\|^2\right)\Delta u + f(u) = h(x). \tag{5}
$$

Moreover, Lin, Lv, and Lou [2] studied the global dynamics of the following generalized nonlinear Kirchhoff–Boussinesq equation with strong damping:

$$
u_{tt} + \alpha u_t - \beta \Delta u_t + \Delta^2 u = \operatorname{div}\left(g\left(|\nabla u|^2\right)\nabla u\right) + \Delta h(u) + f(x). \tag{6}
$$

This paper proved that the semigroup conformed to the squeezing property of the system, while demonstrating the existence of an exponential attractor. Then, the spectral interval theory verified that the system had an inertial manifold.

Nakao [3] investigated the initial-boundary value problem of a quasilinear Kirchhoff-type wave equation with standard dissipation u_t:

$$
u_{tt} - \left(1 + \left\|\nabla u(t)\right\|_2^2\right)\Delta u + u_t + g(x,u) = f(x). \tag{7}
$$

Under an external force, the stretchable string undergoes elastic deformation. Over time, elastic mechanics methods may not fully reflect the actual long-term characteristics of the string, and increasing attention is directed to the long-term properties of the strings with the rotational inertia effect. Wave equations with rotational inertia have become a research hotspot in mathematics and physics.

Chueshov and Lasiecka [13] proposed a plate model with rotational inertia,

$$(1-\alpha\Delta)u_{tt} + \Delta^2 u - \beta\Delta u_t + \left(Q - \|\nabla u\|^2\right)\Delta u = P(u, u_t), \tag{8}$$

where $\alpha \geq 0$ represents the rotational inertia parameter, $\beta > 0$ is the damping coefficient, Q is a parameter describing the internal stress acting on the plate, and P is a function representing the lateral load that may depend on u and u_t. When $\alpha > 0$, this model of transverse inertia becomes the Rayleigh plate equation, and (8) is a pure hyperbolic problem. When $\alpha = 0$, (8) becomes a Berger plate model with structural damping. Chueshov and Lasiecka [13] studied the well-posedness and longtime dynamic behavior of (8).

Niimura [14] studied the long-term dynamic behavior of autonomous beam equations with nonlocal structural damping and rotational inertia under initial boundary value conditions:

$$(1-\alpha\Delta)u_{tt} + \Delta^2 u - N\left(\|\nabla u\|^2\right)\Delta u_t - M\left(\|\nabla u\|^2\right)\Delta u + f(u) = h(x). \tag{9}$$

The well-posedness of the global solution was established, and the existence of a global attractor was proved for the autonomous infinite dynamical system corresponding to $\alpha \in [0,1]$, while the existence of an exponential attractor was demonstrated.

With the advance of research, scholars have shifted their focus on the dynamics of higher-order Kirchhoff equations. Ye and Tao [15] studied the initial-boundary value problem of the following higher-order Kirchhoff-type equation with a nonlinear dissipation term:

$$u_{tt} + \Phi\left(\left\|D^m u\right\|^2\right)(-\Delta)^m u + a|u_t|^{q-2}u_t = b|u|^{r-2}u. \tag{10}$$

Lin and Zhu [16] studied the initial-boundary value problem of the following nonlinear nonlocal higher-order Kirchhoff-type equations:

$$u_{tt} + M\left(\left\|D^m u\right\|^2\right)(-\Delta)^m u + \beta(-\Delta)^m u_t + g(x, u_t) = f(x). \tag{11}$$

The existence and uniqueness of the solutions were demonstrated, and the existence of a global attractor family was confirmed using the compactness method, thus obtaining the finite Hausdorff and fractal dimensions.

Ding and Yang [17] investigated the well-posedness, regularity, and longtime behavior of solutions for an extensible beam equation with fractional rotational inertia and structural nonlinear damping: $(1 + (-\Delta)^\theta)u_{tt} + \Delta^2 u - M(\|\nabla u\|^2)\Delta u + N(\|\nabla u\|^2)(-\Delta)^\omega u_t + f(u) = g$, where the dissipative index is $\omega \in (0,1]$, and the rotational index is $\theta \in [0,\omega)$. To the best of our knowledge, a comprehensive study on the long-term dynamics of coupled Kirchhoff models incorporating rotational inertia is yet to be reported.

Originating from physics, a system coupling measures the dependence of two entities on each other. With suitable conditions or parameters, a connected system can be coupled, and its potential energy can enable the generation of new functions by combining the structural functions of different systems. As mathematical equations derived from physics, the Kirchhoff model is naturally considered a coupled system, and scholars gradually considered the dynamics of coupled Kirchhoff equations. For example, Wang and Zhang [18] studied the long-term dynamics of coupled beam equations with strong damping under nonlinear boundary conditions. Lin and Zhang [19] studied the initial boundary value problem of the following Kirchhoff coupling group with a source term and strong damping:

$$\begin{cases} u_{tt} - \beta\Delta u_t - M(\|\nabla u\|^2 + \|\nabla v\|^2)\Delta u + g_1(u,v) = f_1(x), \\ v_{tt} - \beta\Delta v_t - M(\|\nabla u\|^2 + \|\nabla v\|^2)\Delta v + g_2(u,v) = f_2(x). \end{cases} \tag{12}$$

The finite Hausdorff dimension of the global attractor was obtained in a previous work [19].

In recent years, Lin et al. [20–23] explored the dynamics of higher-order coupled Kirchhoff equations and obtained a series of excellent results.

Few existing studies have focused on higher-order coupled Kirchhoff problems, and higher-order (m_1, m_2)-coupled Kirchhoff models with a nonlinear strong damping have not been studied. The main difficulties lie in the estimation and processing of the harmonic term and the nonlinear damping term. In addition, the nonlinear damping also brings challenges when proving the uniqueness. Therefore, we propose a higher-order coupled Kirchhoff model with higher-order rotational inertia. Under reasonable assumptions, this paper overcame these difficulties by using Hölder, Young, Poincaré, and Gagliardo–Nirenberg inequalities, thus obtaining the global solution and the global attractor family. The conclusions could fill the gap of the global attractor family for higher-order coupled models with higher-order rotational inertia (regardless of whether m_1 equals m_2) and lay the foundation for subsequent engineering applications.

The rest of this paper is organized as follows. Section 2 provides the fundamentals for this work and states the main results. Section 3 proves the main results. Finally, the summary and prospects are presented in Sect. 4.

2 Preparatory knowledge and statement of main results

This section introduces the assumptions for this work and presents the main results.

In this paper, $\|\cdot\|$ and $(\cdot,\cdot)$ denote the norm and inner product in $H = L^2(\Omega)$. Let $V_k = D((-\Delta)^{\frac{k}{2}})$ be the scale of the Hilbert space generated by the Laplacian with Dirichlet boundary condition on H and endowed with standard inner product and norm, respectively, $(\cdot,\cdot)_{V_k} = ((-\Delta)^{\frac{k}{2}}\cdot, (-\Delta)^{\frac{k}{2}}\cdot)$, and $\|\cdot\|_{V_k} = \|(-\Delta)^{\frac{k}{2}}\cdot\|$. The main goal here is to study the well-posedness and long-term dynamics of problem (1)–(3) under the following set of assumptions:

(A1) Function $M(s)$ is continuous on the interval $[0, +\infty)$, $M(s) \in C^1(\mathbb{R}^+)$, and
 1) $M'(s) \geq 0$,
 2) $M(0) \equiv M_0 > 0$.

(A2) For any $u, v \in H$, if $J(u,v) = \int_\Omega [G_1(u,v) + G_2(u,v)]\,\mathrm{d}x$, where $G_1(u,v) = \int_0^u g_1(s,v)\,\mathrm{d}s$, $G_2(u,v) = \int_0^v g_2(u,s)\,\mathrm{d}s$, then for any $\mu \geq 0$, there exist $C_1 \geq 0$, $C_\mu \geq 0$, $C'_\mu \geq 0$ such that

$$G_1(u,v) + G_2(u,v) - C_1 J(u,v) + \mu\left(\left\|\nabla^{m_1} u\right\|^2 + \left\|\nabla^{m_2} v\right\|^2\right) \geq -C_\mu,$$

$$J(u,v) + \mu\left(\left\|\nabla^{m_1}u\right\|^2 + \left\|\nabla^{m_2}v\right\|^2\right) \geq -C'_\mu.$$

(A3) Function $g_j(u,v) \in C^1(\mathbb{R})$ $(j = 1,2)$, and

$$\begin{aligned} &\left|g_j(u,v)\right| \leq C_2\left(1 + |u|^{p_j} + |v|^{q_j}\right); \\ &\left|g_{ju}(u,v)\right| \leq C_3\left(1 + |u|^{p_j-1} + |v|^{q_j}\right); \\ &\left|g_{jv}(u,v)\right| \leq C_4\left(1 + |u|^{p_j} + |v|^{q_j-1}\right). \end{aligned}$$

Specifically, when $n = 1,2$, $1 \leq p_j(q_j)$; when $3 \leq n \leq 2m$, $1 \leq p_j(q_j) \leq \frac{n}{n-2}$; when $2m < n$, $1 \leq p_j(q_j) \leq \frac{n}{n-2m}$, where $m = \min\{m_1, m_2\}$.

(A4) One has $N_j(s_j) \geq N_{j0}$, where N_{j0} $(j = 1,2)$ are positive constants and $\rho_1, \rho_2 > 0$. Thus, $M(s_1 + s_2) - \rho_1 N_1(s_1) - \rho_2 N_2(s_2) > 0$.

Then, the research phase space of this study is obtained as follows:

$$\begin{aligned} &V_0 = H, \qquad V_1 = H_0^1(\Omega), \qquad V_k = H^k(\Omega) \cap H_0^1(\Omega), \\ &X_{\alpha 0 \times \beta 0} = V_{m_1} \times V_{m_1} \times V_{m_2} \times V_{m_2}, \\ &X_{\alpha k_1 \times \beta k_2} = V_{m_1+k_1} \times V_{m_1+k_1} \times V_{m_2+k_2} \times V_{m_2+k_2}, \\ &k_1 = 0,1,2,\dots,m_1, \qquad k_2 = 0,1,2,\dots,m_2, \end{aligned}$$

and the norms of the corresponding spaces are as follows:

$$\begin{aligned} \left\|(u,y_1,v,y_2)\right\|^2_{X_{\alpha k_1 \times \beta k_2}} = {}& \left\|\nabla^{m_1+k_1}u\right\|^2 + \left\|\nabla^{k_1}y_1\right\|^2 + \alpha\left\|\nabla^{m_1+k_1}y_1\right\|^2 + \left\|\nabla^{m_2+k_2}v\right\|^2 \\ &+ \left\|\nabla^{k_2}y_2\right\|^2 + \beta\left\|\nabla^{m_2+k_2}y_2\right\|^2. \end{aligned}$$

Meanwhile, the general form of the Poincaré inequality is $\lambda_1\|\nabla^r u\|^2 \leq \|\nabla^{r+1}u\|^2$, where λ_1 is the first eigenvalue of $-\Delta$ with a homogeneous Dirichlet boundary on Ω. In this paper, C_i is a constant, and $C(\cdot)$ is a constant depending on the parameters in the parentheses.

The main results of this paper are as follows.

Theorem 1 *Suppose that assumptions* (A1)–(A4) *hold. If* $f_1 \in V_{k_1}$, $f_2 \in V_{k_2}$ *and initial data* $(u_0,u_1,v_0,v_1) \in X_{\alpha k_1 \times \beta k_2}$, $k_1 = 0,1,2,\dots,m_1$, $k_2 = 0,1,2,\dots,m_2$, *then for* $\forall \alpha, \beta \in (0,1]$, *problem* (1)–(3) *admits a unique solution* (u,v) *satisfying*

$$\begin{aligned} &u \in L^\infty(0,\infty; V_{m_1+k_1}); \\ &u_t \in L^\infty(0,\infty; V_{m_1+k_1}) \cap L^2(0,T; V_{m_1+k_1}); \\ &v \in L^\infty(0,\infty; V_{m_2+k_2}); \\ &v_t \in L^\infty(0,\infty; V_{m_2+k_2}) \cap L^2(0,T; V_{m_2+k_2}). \end{aligned}$$

Theorem 2 *Suppose that assumptions* (A1)–(A4) *hold. If* $f_1 \in V_{m_1}$, $f_2 \in V_{m_2}$ *and initial data* $(u_0,u_1,v_0,v_1) \in X_{\alpha m_1 \times \beta m_2}$, *then for* $\forall \alpha, \beta \in (0,1]$, *problem* (1)–(3) *has a global attractor family* $\mathcal{A}$ *in* $X_{\alpha 0 \times \beta 0}$:

$$\mathcal{A} = \{A_{\alpha k_1 \times \beta k_2}\}, \qquad A_{\alpha k_1 \times \beta k_2} = \omega(B_{\alpha k_1 \times \beta k_2,0}) = \bigcap_{\tau \geq 0} \overline{\bigcup_{t \geq \tau} S(t) B_{\alpha k_1 \times \beta k_2,0}},$$

$$k_1 = 1, 2, \ldots, m_1, \qquad k_2 = 1, 2, \ldots, m_2,$$

where $B_{\alpha k_1\times\beta k_2,0} = \{(u,u_t,v,v_t)\in X_{\alpha k_1\times\beta k_2} : \|(u,u_t,v,v_t)\|^2_{X_{\alpha k_1\times\beta k_2}} = \|\nabla^{m_1+k_1}u\|^2 + \|\nabla^{k_1}u_t\|^2 + \alpha\|\nabla^{m_1+k_1}u_t\|^2 + \|\nabla^{m_2+k_2}v\|^2 + \|\nabla^{k_2}v_t\|^2 + \beta\|\nabla^{m_2+k_2}v_t\|^2 \le C(R_{\alpha 0\times\beta 0}) + C(R_{\alpha k_1\times\beta k_2})\}$ *are bounded absorbing sets in* $X_{\alpha 0\times\beta 0}$, $B_{\alpha k_1\times\beta k_2,0}$ *are compact in* $X_{\alpha 0\times\beta 0}$, $A_{\alpha k_1\times\beta k_2}\subset X_{\alpha 0\times\beta 0}$. *Moreover,*

1) $S(t)A_{\alpha k_1\times\beta k_2} = A_{\alpha k_1\times\beta k_2}$, *for all* $t\ge 0$,
2) *Sets* $A_{\alpha k_1\times\beta k_2}$ *attract all bounded sets in* $X_{\alpha 0\times\beta 0}$, *i.e., for all* $B_{\alpha k_1\times\beta k_2}\subset X_{\alpha 0\times\beta 0}$, *they are mapped to bounded sets in* $X_{\alpha 0\times\beta 0}$, *and*

$$\begin{aligned}&\operatorname{dist}\bigl(S(t)B_{\alpha k_1\times\beta k_2}, A_{\alpha k_1\times\beta k_2}\bigr)\\&\quad=\sup_{x\in B_{\alpha k_1\times\beta k_2}}\inf_{y\in A_{\alpha k_1\times\beta k_2}}\bigl\|S(t)x-y\bigr\|_{X_{\alpha 0\times\beta 0}}\to 0\quad (t\to\infty),\end{aligned}$$

where $\{S(t)\}_{t\ge 0}$ *is the solution semigroup generated by problem* (1)–(3).

3 Proof of the main results

This section presents the proof of the existence and uniqueness of the solutions and the family of global attractors for problem (1)–(3).

Let $\varepsilon > 0$ be small enough, and $\frac{\lambda_1^{m_1}}{2}N_{10} - 2 - 4\varepsilon - 2\varepsilon^2 \ge 0$, $\frac{N_{10}}{4\alpha} - 2\varepsilon - \varepsilon^2 \ge 0$, $\frac{\lambda_1^{m_2}}{2}N_{20} - 2 - 4\varepsilon - 2\varepsilon^2 \ge 0$, and $\frac{N_{20}}{4\beta} - 2\varepsilon - \varepsilon^2 \ge 0$.

Lemma 1 ([24]) *Let* $y:\mathbb{R}^+\to\mathbb{R}^+$ *be an absolutely continuous positive function, which satisfies the following differential inequality for some* $\delta > 0$:

$$\frac{\mathrm{d}}{\mathrm{d}t}y(t) + 2\delta y(t) \le g(t)y(t) + K,\quad t > 0,$$

where $K\ge 0$, *and* $a\ge 0$ *if* $t\ge s\ge 0$ *so that* $\int_s^t g(\tau)\,\mathrm{d}\tau \le \delta(t-s)+a$. *Then,*

$$y(t)\le \mathrm{e}^a y(0)\mathrm{e}^{-\delta t} + \frac{K\mathrm{e}^a}{\delta},\quad t\ge 0.$$

Lemma 2 ([16]) *Let* X *be a Banach space, then the continuous operator semigroup* $\{S(t)\}_{t\ge 0}$ *satisfies the following*:

(1) *Semigroup* $\{S(t)\}_{t\ge 0}$ *is uniformly bounded in* X, *i.e., for all* $R_0 > 0$, *there exists a positive constant* $C_0(R_0)$ *that when* $\|u\|_X\le R_0$,

$$\bigl\|S(t)u\bigr\|_X \le C_0(R_0),\quad \textit{for all } t\in[0,+\infty);$$

(2) *There exists a bounded absorbing set* B_0 *in* X, *and for any bounded set* $B\subset X$, *there exists a moment* t_0 *such that*

$$S(t)B\subset B_0,\quad t\ge t_0;$$

(3) *If* $t > 0$, *and* $S(t)$ *is a fully continuous operator,*

then the semigroup $\{S(t)\}_{t\geq 0}$ *has a global attractor* A *in* X, *and*

$$A = \omega(B_0) = \bigcap_{\tau\geq 0}\overline{\bigcup_{t\geq\tau} S(t)B_0}.$$

Lemma 3 *Suppose that assumptions* (A1)–(A4) *hold. If* $f_j \in H$ $(j = 1,2)$ *and initial data* $(u_0,u_1,v_0,v_1) \in X_{\alpha 0\times\beta 0}$, *then for* $R_{\alpha 0\times\beta 0} > 0$, *there exist positive constants* $C(R_{\alpha 0\times\beta 0})$ *and* $t_{\alpha 0\times\beta 0}$ *so that when* $t \geq t_{\alpha 0\times\beta 0}$, (u,y_1,v,y_2) *determined by problem* (1)–(3) *satisfies*

$$\begin{aligned}\left\|(u,y_1,v,y_2)\right\|^2_{X_{\alpha 0\times\beta 0}} &= \left\|\nabla^{m_1}u\right\|^2 + \|y_1\|^2 + \alpha\left\|\nabla^{m_1}y_1\right\|^2 + \left\|\nabla^{m_2}v\right\|^2 \\ &\quad + \|y_2\|^2 + \beta\left\|\nabla^{m_2}y_2\right\|^2 \leq C(R_{\alpha 0\times\beta 0}), \quad \text{for } \forall\alpha,\beta\in(0,1],\end{aligned} \tag{13}$$

where $y_1 = u_t + \varepsilon u$, $y_2 = v_t + \varepsilon v$.

Proof Multiplying the first equation of (1) by y_1 in H and the second by y_2 in H, we have

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\bigg[\|y_1\|^2 + \alpha\left\|\nabla^{m_1}y_1\right\|^2 + \|y_2\|^2 + \beta\left\|\nabla^{m_2}y_2\right\|^2 \\ &\qquad + \int_0^{\|\nabla^{m_1}u\|^2+\|\nabla^{m_2}v\|^2} M(\tau)\,\mathrm{d}\tau + 2J(u,v)\bigg] \\ &\qquad + \varepsilon M\left(\left\|\nabla^{m_1}u\right\|^2 + \left\|\nabla^{m_2}v\right\|^2\right)\cdot\left(\left\|\nabla^{m_1}u\right\|^2 + \left\|\nabla^{m_2}v\right\|^2\right) \\ &\qquad - \varepsilon\left(\|y_1\|^2 + \alpha\left\|\nabla^{m_1}y_1\right\|^2\right) - \varepsilon\left(\|y_2\|^2 + \beta\left\|\nabla^{m_2}y_2\right\|^2\right) \\ &\qquad + \varepsilon^2\left((u,y_1) + \alpha\left((-\Delta)^{m_1}u,y_1\right) + (v,y_2) + \beta\left((-\Delta)^{m_2}v,y_2\right)\right) \\ &\qquad + N_1\left(\left\|\nabla^{m_1}u\right\|^2\right)\left\|\nabla^{m_1}y_1\right\|^2 + N_2\left(\left\|\nabla^{m_2}v\right\|^2\right)\left\|\nabla^{m_2}y_2\right\|^2 \\ &\qquad - \varepsilon N_1\left(\left\|\nabla^{m_1}u\right\|^2\right)\left(\nabla^{m_1}y_1,\nabla^{m_1}u\right) - \varepsilon N_2\left(\left\|\nabla^{m_2}v\right\|^2\right)\left(\nabla^{m_2}y_2,\nabla^{m_2}v\right) \\ &\qquad + \varepsilon\left(g_1(u,v),u\right) + \varepsilon\left(g_2(u,v),v\right) = (f_1,y_1) + (f_2,y_2).\end{aligned} \tag{14}$$

Using Hölder, Young, and Poincaré inequalities, we have

$$\begin{aligned}&-\varepsilon\left(\|y_1\|^2 + \alpha\left\|\nabla^{m_1}y_1\right\|^2 + \|y_2\|^2 + \beta\left\|\nabla^{m_2}y_2\right\|^2\right) \\ &\qquad + \varepsilon^2\left((u,y_1) + \alpha\left((-\Delta)^{m_1}u,y_1\right)\right) + \varepsilon^2\left((v,y_2) + \beta\left((-\Delta)^{m_2}v,y_2\right)\right) \\ &\geq\left(-\varepsilon - \frac{\varepsilon^2}{2}\right)\left(\|y_1\|^2 + \alpha\left\|\nabla^{m_1}y_1\right\|^2 + \|y_2\|^2 + \beta\left\|\nabla^{m_2}y_2\right\|^2\right) \\ &\qquad - \frac{\varepsilon^2}{2}\left(\|u\|^2 + \alpha\left\|\nabla^{m_1}u\right\|^2 + \|v\|^2 + \beta\left\|\nabla^{m_2}v\right\|^2\right) \\ &\geq\left(-\varepsilon - \frac{\varepsilon^2}{2}\right)\left(\|y_1\|^2 + \alpha\left\|\nabla^{m_1}y_1\right\|^2\right) + \left(-\varepsilon - \frac{\varepsilon^2}{2}\right)\left(\|y_2\|^2 + \beta\left\|\nabla^{m_2}y_2\right\|^2\right) \\ &\qquad - \frac{\varepsilon^2}{2}\left(\lambda_1^{-m_1} + \alpha\right)\left\|\nabla^{m_1}u\right\|^2 - \frac{\varepsilon^2}{2}\left(\lambda_1^{-m_2} + \beta\right)\left\|\nabla^{m_2}v\right\|^2,\end{aligned} \tag{15}$$

$$\begin{aligned}&N_1\left(\left\|\nabla^{m_1}u\right\|^2\right)\left\|\nabla^{m_1}y_1\right\|^2 + N_2\left(\left\|\nabla^{m_2}v\right\|^2\right)\left\|\nabla^{m_2}y_2\right\|^2 \\ &\qquad - \varepsilon N_1\left(\left\|\nabla^{m_1}u\right\|^2\right)\left(\nabla^{m_1}y_1,\nabla^{m_1}u\right) - \varepsilon N_2\left(\left\|\nabla^{m_2}v\right\|^2\right)\left(\nabla^{m_2}y_2,\nabla^{m_2}v\right)\end{aligned}$$

$$\begin{aligned}&\geq \frac{1}{2}N_1\big(\|\nabla^{m_1}u\|^2\big)\|\nabla^{m_1}y_1\|^2+\frac{1}{2}N_2\big(\|\nabla^{m_2}v\|^2\big)\|\nabla^{m_2}y_2\|^2\\&\quad-\frac{\varepsilon^2}{2}N_1\big(\|\nabla^{m_1}u\|^2\big)\|\nabla^{m_1}u\|^2-\frac{\varepsilon^2}{2}N_2\big(\|\nabla^{m_2}v\|^2\big)\|\nabla^{m_2}v\|^2,\end{aligned}\tag{16}$$

$$(f_1,y_1)+(f_2,y_2)\leq\|f_1\|\|y_1\|+\|f_2\|\|y_2\|\leq\frac{1}{2}\|y_1\|^2+\frac{1}{2}\|y_2\|^2+\frac{1}{2}\|f_1\|^2+\frac{1}{2}\|f_2\|^2.\tag{17}$$

Inserting the above estimates into (14), we have

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\bigg[\|y_1\|^2+\alpha\|\nabla^{m_1}y_1\|^2+\|y_2\|^2+\beta\|\nabla^{m_2}y_2\|^2\\&\qquad+\int_0^{\|\nabla^{m_1}u\|^2+\|\nabla^{m_2}v\|^2}M(\tau)\,\mathrm{d}\tau+2J(u,v)\bigg]\\&\qquad+\frac{1}{2}N_1\big(\|\nabla^{m_1}u\|^2\big)\|\nabla^{m_1}y_1\|^2-\bigg(\frac{1}{2}+\varepsilon+\frac{\varepsilon^2}{2}\bigg)\|y_1\|^2\\&\qquad-\bigg(\varepsilon+\frac{\varepsilon^2}{2}\bigg)\alpha\|\nabla^{m_1}y_1\|^2+\frac{1}{2}N_2\big(\|\nabla^{m_2}v\|^2\big)\|\nabla^{m_2}y_2\|^2\\&\qquad-\bigg(\frac{1}{2}+\varepsilon+\frac{\varepsilon^2}{2}\bigg)\|y_2\|^2-\bigg(\varepsilon+\frac{\varepsilon^2}{2}\bigg)\beta\|\nabla^{m_2}y_2\|^2\\&\qquad+\varepsilon M\big(\|\nabla^{m_1}u\|^2+\|\nabla^{m_2}v\|^2\big)\big(\|\nabla^{m_1}u\|^2+\|\nabla^{m_2}v\|^2\big)\\&\qquad-\frac{\varepsilon^2}{2}\big(N_1\big(\|\nabla^{m_1}u\|^2\big)+\lambda_1^{-m_1}+\alpha\big)\|\nabla^{m_1}u\|^2\\&\qquad-\frac{\varepsilon^2}{2}\big(N_2\big(\|\nabla^{m_2}v\|^2\big)+\lambda_1^{-m_2}+\beta\big)\|\nabla^{m_2}v\|^2\\&\quad\leq-\varepsilon\big(g_1(u,v),u\big)-\varepsilon\big(g_2(u,v),v\big)+\frac{1}{2}\|f_1\|^2+\frac{1}{2}\|f_2\|^2.\end{aligned}\tag{18}$$

According to (A1),

$$\begin{aligned}&\varepsilon M\big(\|\nabla^{m_1}u\|^2+\|\nabla^{m_2}v\|^2\big)\big(\|\nabla^{m_1}u\|^2+\|\nabla^{m_2}v\|^2\big)\\&\quad\geq\frac{\varepsilon}{4}\int_0^{\|\nabla^{m_1}u\|^2+\|\nabla^{m_2}v\|^2}M(\tau)\,\mathrm{d}\tau\\&\qquad+\frac{3\varepsilon}{4}M\big(\|\nabla^{m_1}u\|^2+\|\nabla^{m_2}v\|^2\big)\cdot\big(\|\nabla^{m_1}u\|^2+\|\nabla^{m_2}v\|^2\big),\end{aligned}\tag{19}$$

and according to (A2),

$$-\varepsilon\big(g_1(u,v),u\big)-\varepsilon\big(g_2(u,v),v\big)\leq-\varepsilon C_1J(u,v)+\varepsilon\mu\big(\|\nabla^{m_1}u\|^2+\|\nabla^{m_2}v\|^2\big)+\varepsilon C_\mu.\tag{20}$$

Inserting (18) and (19) into (20), we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\bigg[\|y_1\|^2+\alpha\|\nabla^{m_1}y_1\|^2+\|y_2\|^2+\beta\|\nabla^{m_2}y_2\|^2\\&\qquad+\int_0^{\|\nabla^{m_1}u\|^2+\|\nabla^{m_2}v\|^2}M(\tau)\,\mathrm{d}\tau+2J(u,v)\bigg]\\&\qquad+\bigg(\frac{\lambda_1^{m_1}}{2}N_1\big(\|\nabla^{m_1}u\|^2\big)-1-2\varepsilon-\varepsilon^2\bigg)\|y_1\|^2\end{aligned}$$

$$
\begin{aligned}
&+\left(\frac{1}{2\alpha}N_1\left(\left\|\nabla^{m_1}u\right\|^2\right)-2\varepsilon-\varepsilon^2\right)\alpha\left\|\nabla^{m_1}y_1\right\|^2\\
&+\left(\frac{\lambda_1^{m_2}}{2}N_2\left(\left\|\nabla^{m_2}v\right\|^2\right)-1-2\varepsilon-\varepsilon^2\right)\|y_2\|^2\\
&+\left(\frac{1}{2\beta}N_2\left(\left\|\nabla^{m_2}v\right\|^2\right)-2\varepsilon-\varepsilon^2\right)\beta\left\|\nabla^{m_2}y_2\right\|^2\\
&+\frac{\varepsilon}{2}\int_0^{\|\nabla^{m_1}u\|^2+\|\nabla^{m_2}v\|^2}M(\tau)\,\mathrm{d}\tau+2\varepsilon C_1J(u,v)\\
&+\left(\frac{3\varepsilon}{2}M\left(\left\|\nabla^{m_1}u\right\|^2+\left\|\nabla^{m_2}v\right\|^2\right)-2\varepsilon\mu-\varepsilon^2N_1\left(\left\|\nabla^{m_1}u\right\|^2\right)-\varepsilon^2\lambda_1^{-m_1}\right)\left\|\nabla^{m_1}u\right\|^2\\
&+\left(\frac{3\varepsilon}{2}M\left(\left\|\nabla^{m_1}u\right\|^2+\left\|\nabla^{m_2}v\right\|^2\right)-2\varepsilon\mu-\varepsilon^2N_2\left(\left\|\nabla^{m_2}v\right\|^2\right)-\varepsilon^2\lambda_1^{-m_2}\right)\left\|\nabla^{m_2}v\right\|^2\\
\le{}& 2\varepsilon C_\mu+\|f_1\|^2+\|f_2\|^2.
\end{aligned}
\tag{21}
$$

Let $H_1(t)=\|y_1\|^2+\alpha\|\nabla^{m_1}y_1\|^2+\|y_2\|^2+\beta\|\nabla^{m_2}y_2\|^2+\int_0^{\|\nabla^{m_1}u\|^2+\|\nabla^{m_2}v\|^2}M(\tau)\,\mathrm{d}\tau+2J(u,v)$ and $\sigma_1=\min\{\frac{\lambda_1^{m_1}}{2}N_{10}-2-4\varepsilon-2\varepsilon^2,\frac{1}{2\alpha}N_{10}-2\varepsilon-\varepsilon^2,\frac{\lambda_1^{m_2}}{2}N_{20}-2-4\varepsilon-2\varepsilon^2,\frac{1}{2\beta}N_{10}-2\varepsilon-\varepsilon^2,\frac{\varepsilon}{2},\varepsilon C_1\}$. Then we can infer from (21) that

$$
\frac{\mathrm{d}}{\mathrm{d}t}H_1(t)+\sigma_1H_1(t)\le 2\varepsilon C_\mu+\|f_1\|^2+\|f_2\|^2. \tag{22}
$$

According to Gronwall's inequality, we have

$$
H_1(t)\le H_1(0)\mathrm{e}^{-\sigma_1t}+\frac{2\varepsilon C_\mu+\|f_1\|^2+\|f_2\|^2}{\sigma_1}, \tag{23}
$$

and

$$
\begin{aligned}
H_1(t)\ge{}&\|y_1\|^2+\alpha\left\|\nabla^{m_1}y_1\right\|^2+\|y_2\|^2+\beta\left\|\nabla^{m_2}y_2\right\|^2\\
&+M_0\left(\left\|\nabla^{m_1}u\right\|^2+\left\|\nabla^{m_2}v\right\|^2\right)+2J(u,v)\\
\ge{}&\|y_1\|^2+\alpha\left\|\nabla^{m_1}y_1\right\|^2+\|y_2\|^2+\beta\left\|\nabla^{m_2}y_2\right\|^2\\
&+\frac{M_0}{2}\left(\left\|\nabla^{m_1}u\right\|^2+\left\|\nabla^{m_2}v\right\|^2\right)-2C'_\mu\\
\ge{}& C_5\left(\|y_1\|^2+\|y_2\|^2+\left\|\nabla^{m_1}u\right\|^2+\left\|\nabla^{m_2}v\right\|^2\right)-2C'_\mu,
\end{aligned}
\tag{24}
$$

according to (A1) and (A2), where $\mu=\frac{M_0}{4}$ and $C_5=\min\{1,\frac{M_0}{2}\}$. Thus,

$$
\begin{aligned}
\left\|(u,y_1,v,y_2)\right\|^2_{X_{\alpha0\times\beta0}}={}&\left\|\nabla^{m_1}u\right\|^2+\|y_1\|^2+\alpha\left\|\nabla^{m_1}y_1\right\|^2\\
&+\left\|\nabla^{m_2}v\right\|^2+\|y_2\|^2+\beta\left\|\nabla^{m_2}y_2\right\|^2\\
\le{}&\frac{(H_1(t)+2C'_\mu)}{C_5}\\
\le{}&\frac{H_1(0)\mathrm{e}^{-\sigma_1t}+2C'_\mu}{C_5}+\frac{2\varepsilon C_\mu+\|f_1\|^2+\|f_2\|^2}{\sigma_1C_5},
\end{aligned}
\tag{25}
$$

i.e.,

$$\overline{\lim_{t\to\infty}}\left\|(u,y_1,v,y_2)\right\|^2_{X_{\alpha0\times\beta0}} \le \frac{2C'_\mu}{C_5} + \frac{2\varepsilon C_\mu + \|f_1\|^2 + \|f_2\|^2}{\sigma_1 C_5} = R_{\alpha0\times\beta0}. \tag{26}$$

Therefore, there exist positive constants $C(R_{\alpha0\times\beta0})$ and $t_{\alpha0\times\beta0}$ such that, whenever $t \ge t_{\alpha0\times\beta0}$,

$$\begin{aligned}\left\|(u,y_1,v,y_2)\right\|^2_{X_{\alpha0\times\beta0}} &= \left\|\nabla^{m_1}u\right\|^2 + \|y_1\|^2 + \alpha\left\|\nabla^{m_1}y_1\right\|^2 \\ &\quad + \left\|\nabla^{m_2}v\right\|^2 + \|y_2\|^2 + \beta\left\|\nabla^{m_2}y_2\right\|^2 \\ &\le C(R_{\alpha0\times\beta0}).\end{aligned} \tag{27}$$

Thus, Lemma 3 is proved. □

Lemma 4 *Suppose that assumptions* (A1)–(A4) *hold. If* $f_1 \in V_{k_1}, f_2 \in V_{k_2}$, $k_1 = 1,2,\ldots,m_1$, $k_2 = 1,2,\ldots,m_2$, *and initial data* $(u_0,u_1,v_0,v_1) \in X_{\alpha k_1\times\beta k_2}$, *then, for* $R_{\alpha k_1\times\beta k_2} > 0$, *there exist positive constants* $C(R_{\alpha k_1\times\beta k_2})$ *and* $t_{\alpha k_1\times\beta k_2}$ *such that, whenever* $t \ge t_{\alpha k_1\times\beta k_2}$, (u,y_1,v,y_2) *determined by problem* (1)–(3) *satisfies*

$$\begin{aligned}\left\|(u,y_1,v,y_2)\right\|^2_{X_{\alpha k_1\times\beta k_2}} &= \left\|\nabla^{m_1+k_1}u\right\|^2 + \left\|\nabla^{k_1}y_1\right\|^2 + \alpha\left\|\nabla^{m_1+k_1}y_1\right\|^2 \\ &\quad + \left\|\nabla^{m_2+k_2}v\right\|^2 + \left\|\nabla^{k_2}y_2\right\|^2 + \beta\left\|\nabla^{m_2+k_2}y_2\right\|^2 \\ &\le C(R_{\alpha k_1\times\beta k_2}), \quad \text{for } \forall\alpha,\beta\in(0,1],\end{aligned} \tag{28}$$

where $y_1 = u_t + \varepsilon u$, $y_2 = v_t + \varepsilon v$.

Proof Multiplying the first equation of (1) by $(-\Delta)^{k_1}y_1$, $k_1 = 1,2,\ldots,m_1$ in H and the second by $(-\Delta)^{k_2}y_2$, $k_2 = 1,2,\ldots,m_2$ in H and then integrating over Ω, we have

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big[\left\|\nabla^{k_1}y_1\right\|^2 + \alpha\left\|\nabla^{m_1+k_1}y_1\right\|^2 + \left\|\nabla^{k_2}y_2\right\|^2 + \beta\left\|\nabla^{m_2+k_2}y_2\right\|^2 \\ &\quad + M\big(\left\|\nabla^{m_1}u\right\|^2 + \left\|\nabla^{m_2}v\right\|^2\big)\big(\left\|\nabla^{m_1+k_1}u\right\|^2 + \left\|\nabla^{m_2+k_2}v\right\|^2\big)\Big] \\ &\quad + \varepsilon M\big(\left\|\nabla^{m_1}u\right\|^2 + \left\|\nabla^{m_2}v\right\|^2\big)\cdot\big(\left\|\nabla^{m_1+k_1}u\right\|^2 + \left\|\nabla^{m_2+k_2}v\right\|^2\big) \\ &\quad - \varepsilon\big(\left\|\nabla^{k_1}y_1\right\|^2 + \alpha\left\|\nabla^{m_1+k_1}y_1\right\|^2\big) - \varepsilon\big(\left\|\nabla^{k_2}y_2\right\|^2 + \beta\left\|\nabla^{m_2+k_2}y_2\right\|^2\big) \\ &\quad + \varepsilon^2\big(\big(\nabla^{k_1}u,\nabla^{k_1}y_1\big) + \alpha\big(\nabla^{m_1+k_1}u,\nabla^{m_1+k_1}y_1\big)\big) \\ &\quad + \varepsilon^2\big(\big(\nabla^{k_2}v,\nabla^{k_2}y_2\big) + \beta\big(\nabla^{m_2+k_2}v,\nabla^{m_2+k_2}y_2\big)\big) \\ &\quad + N_1\big(\left\|\nabla^{m_1}u\right\|^2\big)\left\|\nabla^{m_1+k_1}y_1\right\|^2 + N_2\big(\left\|\nabla^{m_2}v\right\|^2\big)\left\|\nabla^{m_2+k_2}y_2\right\|^2 \\ &\quad - \varepsilon N_1\big(\left\|\nabla^{m_1}u\right\|^2\big)\big(\nabla^{m_1+k_1}y_1,\nabla^{m_1+k_1}u\big) \\ &\quad - \varepsilon N_2\big(\left\|\nabla^{m_2}v\right\|^2\big)\big(\nabla^{m_2+k_2}y_2,\nabla^{m_2+k_2}v\big) \\ &\quad + \big(g_1(u,v),(-\Delta)^{k_1}y_1\big) + \big(g_2(u,v),(-\Delta)^{k_2}y_2\big) \\ &= \frac{\|\nabla^{m_1+k_1}u\|^2 + \|\nabla^{m_2+k_2}v\|^2}{2}\frac{\mathrm{d}}{\mathrm{d}t}M\big(\left\|\nabla^{m_1}u\right\|^2 + \left\|\nabla^{m_2}v\right\|^2\big) \\ &\quad + \big(f_1,(-\Delta)^{k_1}y_1\big) + \big(f_2,(-\Delta)^{k_2}y_2\big).\end{aligned} \tag{29}$$

Using Hölder, Young, and Poincaré inequalities, we then have

$$\begin{aligned}
&-\varepsilon\big(\|\nabla^{k_1}y_1\|^2+\alpha\|\nabla^{m_1+k_1}y_1\|^2\big)-\varepsilon\big(\|\nabla^{k_2}y_2\|^2+\beta\|\nabla^{m_2+k_2}y_2\|^2\big)\\
&\quad+\varepsilon^2\big((\nabla^{k_1}u,\nabla^{k_1}y_1)+\alpha(\nabla^{m_1+k_1}u,\nabla^{m_1+k_1}y_1)\big)\\
&\quad+\varepsilon^2\big((\nabla^{k_2}v,\nabla^{k_2}y_2)+\beta(\nabla^{m_2+k_2}v,\nabla^{m_2+k_2}y_2)\big)\\
&\geq\Big(-\varepsilon-\frac{\varepsilon^2}{2}\Big)\big(\|\nabla^{k_1}y_1\|^2+\alpha\|\nabla^{m_1+k_1}y_1\|^2\big)\\
&\quad+\Big(-\varepsilon-\frac{\varepsilon^2}{2}\Big)\big(\|\nabla^{k_2}y_2\|^2+\beta\|\nabla^{m_2+k_2}y_2\|^2\big)\\
&\quad-\frac{\varepsilon^2}{2}\big(\|\nabla^{k_1}u\|^2+\alpha\|\nabla^{m_1+k_1}u\|^2\big)-\frac{\varepsilon^2}{2}\big(\|\nabla^{k_2}v\|^2+\beta\|\nabla^{m_2+k_2}v\|^2\big)\\
&\geq-\Big(\varepsilon+\frac{\varepsilon^2}{2}\Big)\big(\|\nabla^{k_1}y_1\|^2+\alpha\|\nabla^{m_1+k_1}y_1\|^2\big)\\
&\quad-\Big(\varepsilon+\frac{\varepsilon^2}{2}\Big)\big(\|\nabla^{k_2}y_2\|^2+\beta\|\nabla^{m_2+k_2}y_2\|^2\big)\\
&\quad-\frac{\varepsilon^2}{2}\big(\lambda_1^{-m_1}+\alpha\big)\|\nabla^{m_1+k_1}u\|^2-\frac{\varepsilon^2}{2}\big(\lambda_1^{-m_2}+\beta\big)\|\nabla^{m_2+k_2}v\|^2, \qquad (30)
\end{aligned}$$

$$\begin{aligned}
&N_1\big(\|\nabla^{m_1}u\|^2\big)\|\nabla^{m_1+k_1}y_1\|^2+N_2\big(\|\nabla^{m_2}v\|^2\big)\|\nabla^{m_2+k_2}y_2\|^2\\
&\quad-\varepsilon N_1\big(\|\nabla^{m_1}u\|^2\big)\big(\nabla^{m_1+k_1}y_1,\nabla^{m_1+k_1}u\big)-\varepsilon N_2\big(\|\nabla^{m_2}v\|^2\big)\big(\nabla^{m_2+k_2}y_2,\nabla^{m_2+k_2}v\big)\\
&\geq\frac{1}{2}N_1\big(\|\nabla^{m_1}u\|^2\big)\|\nabla^{m_1+k_1}y_1\|^2+\frac{1}{2}N_2\big(\|\nabla^{m_2}v\|^2\big)\|\nabla^{m_2+k_2}y_2\|^2\\
&\quad-\frac{\varepsilon^2}{2}N_1\big(\|\nabla^{m_1}u\|^2\big)\|\nabla^{m_1+k_1}u\|^2-\frac{\varepsilon^2}{2}N_2\big(\|\nabla^{m_2}v\|^2\big)\|\nabla^{m_2+k_2}v\|^2, \qquad (31)
\end{aligned}$$

$$\begin{aligned}
&\big(g_1(u,v),(-\Delta)^{k_1}y_1\big)+\big(g_2(u,v),(-\Delta)^{k_2}y_2\big)\\
&\leq\|g_1(u,v)\|\|\nabla^{2k_1}y_1\|+\|g_2(u,v)\|\|\nabla^{2k_2}y_2\|\\
&\leq\frac{N_{10}}{8}\|\nabla^{m_1+k_1}y_1\|^2+\frac{2\lambda_1^{k_1-m_1}}{N_{10}}\|g_1(u,v)\|^2\\
&\quad+\frac{N_{20}}{8}\|\nabla^{m_2+k_2}y_2\|^2+\frac{2\lambda_1^{k_2-m_2}}{N_{20}}\|g_2(u,v)\|^2, \qquad (32)
\end{aligned}$$

$$\begin{aligned}
&\big(f_1,(-\Delta)^{k_1}y_1\big)+\big(f_2,(-\Delta)^{k_2}y_2\big)\\
&\leq\|\nabla^{k_1}f_1\|\|\nabla^{k_1}y_1\|+\|\nabla^{k_2}f_2\|\|\nabla^{k_2}y_2\|\\
&\leq\frac{1}{2}\|\nabla^{k_1}y_1\|^2+\frac{1}{2}\|\nabla^{k_2}y_2\|^2+\frac{1}{2}\|\nabla^{k_1}f_1\|^2+\frac{1}{2}\|\nabla^{k_2}f_2\|^2, \qquad (33)
\end{aligned}$$

and

$$\begin{aligned}
\|g_1(u,v)\|^2&=\int_\Omega|g_1(u,v)|^2\,\mathrm{d}x\leq\int_\Omega\big|C_2\big(1+|u|^{p_1}+|v|^{q_1}\big)\big|^2\,\mathrm{d}x\\
&\leq C_6\int_\Omega\big(1+|u|^{2p_1}+|v|^{2q_1}\big)\,\mathrm{d}x\leq C_7\big(1+\|u\|_{2p_1}^{2p_1}+\|v\|_{2q_1}^{2q_1}\big), \qquad (34)
\end{aligned}$$

$$\|g_2(u,v)\|^2\leq C_8\big(1+\|u\|_{2p_2}^{2p_2}+\|v\|_{2q_2}^{2q_2}\big), \qquad (35)$$

according to (A3). Furthermore, based on the Gagliardo–Nirenberg inequality, we can conclude that

$$\begin{cases} \|u\|_{2p_j}^{2p_j} \le C_{9j}\|\nabla^{m_1}u\|^{\frac{n(p_j-1)}{m_1}}\|u\|^{\frac{2m_1p_j-n(p_j-1)}{m_1}}, \\ \|v\|_{2q_j}^{2q_j} \le C_{10j}\|\nabla^{m_2}v\|^{\frac{n(q_j-1)}{m_2}}\|v\|^{\frac{2m_2q_j-n(q_j-1)}{m_2}}. \end{cases}$$

Thus, we have

$$\left\|g_1(u,v)\right\|^2 + \left\|g_2(u,v)\right\|^2 \le C(R_{\alpha 0\times\beta 0}). \tag{36}$$

Inserting (31)–(33) and (36) into (29), we have

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big[\left\|\nabla^{k_1}y_1\right\|^2 + \alpha\left\|\nabla^{m_1+k_1}y_1\right\|^2 + \left\|\nabla^{k_2}y_2\right\|^2 \\
&\quad + \beta\left\|\nabla^{m_2+k_2}y_2\right\|^2 + M\big(\left\|\nabla^{m_1}u\right\|^2 + \left\|\nabla^{m_2}v\right\|^2\big)\big(\left\|\nabla^{m_1+k_1}u\right\|^2 + \left\|\nabla^{m_2+k_2}v\right\|^2\big)\Big] \\
&\quad + \frac{N_1(\|\nabla^{m_1}u\|^2)\lambda_1^{m_1} - 2 - 4\varepsilon - 2\varepsilon^2}{4}\left\|\nabla^{k_1}y_1\right\|^2 \\
&\quad + \left(\frac{2N_1(\|\nabla^{m_1}u\|^2) - N_{10}}{8\alpha} - \varepsilon - \frac{\varepsilon^2}{2}\right)\alpha\left\|\nabla^{m_1+k_1}y_1\right\|^2 \\
&\quad + \frac{N_2(\|\nabla^{m_2}v\|^2)\lambda_1^{m_2} - 2 - 4\varepsilon - 2\varepsilon^2}{4}\left\|\nabla^{k_2}y_2\right\|^2 \\
&\quad + \left(\frac{2N_2(\|\nabla^{m_2}v\|^2) - N_{20}}{8\beta} - \varepsilon - \frac{\varepsilon^2}{2}\right)\beta\left\|\nabla^{m_2+k_2}y_2\right\|^2 \\
&\quad + \frac{\varepsilon}{2}M\big(\left\|\nabla^{m_1}u\right\|^2 + \left\|\nabla^{m_2}v\right\|^2\big)\big(\left\|\nabla^{m_1+k_1}u\right\|^2 + \left\|\nabla^{m_2+k_2}v\right\|^2\big) \\
&\quad + \bigg(\frac{\varepsilon}{2}M\big(\left\|\nabla^{m_1}u\right\|^2 + \left\|\nabla^{m_2}v\right\|^2\big) - \frac{\varepsilon^2}{2}N_1\big(\left\|\nabla^{m_1}u\right\|^2\big) \\
&\quad - \frac{\varepsilon^2}{2}\big(\lambda_1^{-m_1} + \alpha\big)\bigg)\left\|\nabla^{m_1+k_1}u\right\|^2 \\
&\quad + \bigg(\frac{\varepsilon}{2}M\big(\left\|\nabla^{m_1}u\right\|^2 + \left\|\nabla^{m_2}v\right\|^2\big) - \frac{\varepsilon^2}{2}N_2\big(\left\|\nabla^{m_2}v\right\|^2\big) \\
&\quad - \frac{\varepsilon^2}{2}\big(\lambda_1^{-m_2} + \beta\big)\bigg)\left\|\nabla^{m_2+k_2}v\right\|^2 \\
&\le \frac{\|\nabla^{m_1+k_1}u\|^2 + \|\nabla^{m_2+k_2}v\|^2}{2}\frac{\mathrm{d}}{\mathrm{d}t}M\big(\left\|\nabla^{m_1}u\right\|^2 + \left\|\nabla^{m_2}v\right\|^2\big) + \frac{1}{2}\left\|\nabla^{k_1}f_1\right\|^2 \\
&\quad + \frac{1}{2}\left\|\nabla^{k_2}f_2\right\|^2 + C(R_0,\lambda_1) \\
&\le \big(\left\|\nabla^{m_1+k_1}u\right\|^2 + \left\|\nabla^{m_2+k_2}v\right\|^2\big)\cdot M'\big(\left\|\nabla^{m_1}u\right\|^2 + \left\|\nabla^{m_2}v\right\|^2\big) \\
&\quad \times \big(\big(\nabla^{m_1}u, \nabla^{m_1}u_t\big) + \big(\nabla^{m_2}v, \nabla^{m_2}v_t\big)\big) \\
&\quad + \frac{1}{2}\left\|\nabla^{k_1}f_1\right\|^2 + \frac{1}{2}\left\|\nabla^{k_2}f_2\right\|^2 + C(R_0,\lambda_1) \\
&\le C_9\big(\left\|\nabla^{m_1}u_t\right\| + \left\|\nabla^{m_2}v_t\right\|\big)\cdot\big(\left\|\nabla^{m_1+k_1}u\right\|^2 + \left\|\nabla^{m_2+k_2}v\right\|^2\big) + \frac{1}{2}\left\|\nabla^{k_1}f_1\right\|^2 \\
&\quad + \frac{1}{2}\left\|\nabla^{k_2}f_2\right\|^2 + C(R_{\alpha 0\times\beta 0},\lambda_1).
\end{aligned} \tag{37}$$

Letting $H_2(t) = \|\nabla^{k_1}y_1\|^2 + \alpha\|\nabla^{m_1+k_1}y_1\|^2 + \|\nabla^{k_2}y_2\|^2 + \beta\|\nabla^{m_2+k_2}y_2\|^2 + M(\|\nabla^{m_1}u\|^2 + \|\nabla^{m_2}v\|^2)\cdot(\|\nabla^{m_1+k_1}u\|^2 + \|\nabla^{m_2+k_2}v\|^2)$ and $\sigma_2 = \min\{\frac{\lambda_1^{m_1}N_{10}-2-4\varepsilon-2\varepsilon^2}{4}, \frac{\lambda_1^{m_2}N_{20}-2-4\varepsilon-2\varepsilon^2}{4}, \frac{N_{10}}{8\alpha} - \varepsilon - \frac{\varepsilon^2}{2}, \frac{N_{20}}{8\beta} - \varepsilon - \frac{\varepsilon^2}{2}, \frac{\varepsilon}{2}\}$, we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}H_2(t) + \sigma_2 H_2(t)\\&\quad\le C_{11}\big(\|\nabla^{m_1}u_t\| + \|\nabla^{m_2}v_t\|\big)H_2(t) + \|\nabla^{k_1}f_1\|^2 + \|\nabla^{k_2}f_2\|^2 + C(R_{\alpha 0\times\beta 0},\lambda_1).\end{aligned}\tag{38}$$

Taking the scalar product of (1) in H with u_t, v_t, we have

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\bigg[\|u_t\|^2 + \alpha\|\nabla^{m_1}u_t\|^2 + \|v_t\|^2 + \beta\|\nabla^{m_2}v_t\|^2\\&\quad+ \int_0^{\|\nabla^{m_1}u\|^2+\|\nabla^{m_2}v\|^2} M(\tau)\,\mathrm{d}\tau + 2J(u,v) - 2(f_1,u) - 2(f_2,v)\bigg]\\&\quad+ N_1\big(\|\nabla^{m_1}u\|^2\big)\|\nabla^{m_1}u_t\|^2 + N_2\big(\|\nabla^{m_2}v\|^2\big)\|\nabla^{m_2}v_t\|^2 = 0,\end{aligned}\tag{39}$$

and integrating (39) on $(0,t)$ yields

$$\begin{aligned}&\int_0^t\big(\|\nabla^{m_1}u_t(\tau)\|^2 + \|\nabla^{m_2}v_t(\tau)\|^2\big)\,\mathrm{d}\tau\\&\quad\le \frac{1}{\min\{N_{10},N_{20}\}}\int_0^t\big(N_1\big(\|\nabla^{m_1}u(\tau)\|^2\big)\|\nabla^{m_1}u_t(\tau)\|^2\\&\qquad+ N_2\big(\|\nabla^{m_2}v(\tau)\|^2\big)\|\nabla^{m_2}v_t(\tau)\|^2\big)\,\mathrm{d}\tau\\&\quad\le \frac{1}{\min\{N_{10},N_{20}\}}\bigg(\|u_1\|^2 + \alpha\|\nabla^{m_1}u_1\|^2 + \|v_1\|^2 + \beta\|\nabla^{m_2}v_1\|^2\\&\qquad+ \int_0^{\|\nabla^{m_1}u_0\|^2+\|\nabla^{m_2}v_0\|^2} M(\tau)\,\mathrm{d}\tau + 2J(u_0,v_0) - 2(f_1,u_0) - 2(f_2,v_0)\bigg) \le C_{12}.\end{aligned}\tag{40}$$

Then,

$$C_{11}\int_s^t\big(\big(\|\nabla^{m_1}u_t(\tau)\| + \|\nabla^{m_2}v_t(\tau)\|\big)\big)\,\mathrm{d}\tau \le \frac{\sigma_2}{2}(t-s) + a \tag{41}$$

for $t > s \ge 0$ and some $a > 0$. Based on (38), (41), and Lemma 1, we obtain

$$H_2(t) \le C_{13}H_2(0)\mathrm{e}^{-\frac{\sigma_2}{2}t} + C_{14}. \tag{42}$$

According to (A1), we have

$$\begin{aligned}H_2(t) &\ge \|\nabla^{k_1}y_1\|^2 + \alpha\|\nabla^{m_1+k_1}y_1\|^2 + \|\nabla^{k_2}y_2\|^2 + \beta\|\nabla^{m_2+k_2}y_2\|^2\\&\quad+ M\big(\|\nabla^{m_1}u\|^2 + \|\nabla^{m_2}v\|^2\big)\cdot\big(\|\nabla^{m_1+k_1}u\|^2 + \|\nabla^{m_2+k_2}v\|^2\big)\\&\ge C_{15}\big(\|\nabla^{k_1}y_1\|^2 + \alpha\|\nabla^{m_1+k_1}y_1\|^2 + \|\nabla^{k_2}y_2\|^2 + \beta\|\nabla^{m_2+k_2}y_2\|^2\\&\quad+ \|\nabla^{m_1+k_1}u\|^2 + \|\nabla^{m_2+k_2}v\|^2\big),\end{aligned}\tag{43}$$

and then,

$$\begin{aligned}\left\|(u,y_1,v,y_2)\right\|^2_{X_{\alpha k_1\times\beta k_2}} &= \left\|\nabla^{k_1}y_1\right\|^2+\alpha\left\|\nabla^{m_1+k_1}y_1\right\|^2+\left\|\nabla^{k_2}y_2\right\|^2+\beta\left\|\nabla^{m_2+k_2}y_2\right\|^2\\&\quad+\left\|\nabla^{m_1+k_1}u\right\|^2+\left\|\nabla^{m_2+k_2}v\right\|^2\le\frac{C_{13}H_2(0)\mathrm{e}^{-\frac{\sigma_2}{2}t}+C_{14}}{C_{15}},\end{aligned}\tag{44}$$

i.e.,

$$\overline{\lim_{t\to\infty}}\left\|(u,y_1,v,y_2)\right\|^2_{X_{\alpha k_1\times\beta k_2}}\le R_{\alpha k_1\times\beta k_2}.\tag{45}$$

Therefore, there exist positive constants $C(R_{\alpha k_1\times\beta k_2})$ and $t_{\alpha k_1\times\beta k_2}$ such that whenever $t\ge t_{\alpha k_1\times\beta k_2}$, the obtained (u,y_1,v,y_2) satisfies

$$\begin{aligned}\left\|(u,y_1,v,y_2)\right\|^2_{X_{\alpha k_1\times\beta k_2}} &= \left\|\nabla^{k_1}y_1\right\|^2+\alpha\left\|\nabla^{m_1+k_1}y_1\right\|^2+\left\|\nabla^{k_2}y_2\right\|^2+\beta\left\|\nabla^{m_2+k_2}y_2\right\|^2\\&\quad+\left\|\nabla^{m_1+k_1}u\right\|^2+\left\|\nabla^{m_2+k_2}v\right\|^2\\&\le C(R_{\alpha k_1\times\beta k_2}),\quad k_1=1,2,\dots,m_1,k_2=1,2,\dots,m_2.\end{aligned}\tag{46}$$

Thus, Lemma 4 is proved. □

Proof of Theorem 1 According to previous findings [16] and the Faedo–Galerkin method, problem (1)–(3) has global solutions, which follows by combining with Lemmas 3 and 4.

Let (u^1,v^1) and (u^2,v^2) be two solutions of problem (1)–(3) corresponding to the same initial data, respectively, $w=u^1-u^2$, $z=v^1-v^2$. Then, (w,z) solves

$$\begin{cases}(1+\alpha(-\Delta)^{m_1})w_{tt}+\frac{1}{2}\sigma_{12}(t)(-\Delta)^{m_1}w_t+\frac{1}{2}\Phi_{12}(t)(-\Delta)^{m_1}w\\\quad+G_1(u^1,u^2,v^1,v^2;t)=0,\\(1+\beta(-\Delta)^{m_2})z_{tt}+\frac{1}{2}\sigma_{34}(t)(-\Delta)^{m_2}z_t+\frac{1}{2}\Phi_{12}(t)(-\Delta)^{m_2}z\\\quad+G_2(u^1,u^2,v^1,v^2;t)=0,\end{cases}\tag{47}$$

where

$$\begin{aligned}&\sigma_{12}=\sigma_1(t)+\sigma_2(t),\qquad\Phi_{12}(t)=\Phi_1(t)+\Phi_2(t),\\&\sigma_i(t)=N_1\left(\left\|\nabla^{m_1}u^i\right\|^2\right),\qquad\Phi_i(t)=M\left(\left\|\nabla^{m_1}u^i\right\|^2+\left\|\nabla^{m_2}v^i\right\|^2\right),\quad i=1,2,\\&\sigma_{34}=\sigma_3(t)+\sigma_4(t),\qquad\sigma_j(t)=N_2\left(\left\|\nabla^{m_2}v^j\right\|^2\right),\quad j=3,4,\\&G_1\left(u^1,u^2,v^1,v^2;t\right)\\&\quad=\frac{1}{2}\left\{\left[\sigma_1(t)-\sigma_2(t)\right](-\Delta)^{m_1}\left(u_t^1+u_t^2\right)+\left[\Phi_1(t)-\Phi_2(t)\right](-\Delta)^{m_1}\left(u^1+u^2\right)\right\}\\&\qquad+g_1(u_1,v_1)-g_1(u_2,v_2),\\&G_2\left(u^1,u^2,v^1,v^2;t\right)\\&\quad=\frac{1}{2}\left\{\left[\sigma_3(t)-\sigma_4(t)\right](-\Delta)^{m_2}\left(v_t^1+v_t^2\right)+\left[\Phi_1(t)-\Phi_2(t)\right](-\Delta)^{m_2}\left(v^1+v^2\right)\right\}\\&\qquad+g_2(u_1,v_1)-g_2(u_2,v_2).\end{aligned}$$

According to Lemma 3,

$$\sigma_{12}' \le C(R_{\alpha 0\times\beta 0})\big(\big\|\nabla^{m_1}u_t^1\big\| + \big\|\nabla^{m_1}u_t^2\big\|\big), \sigma_{34}' \le C(R_{\alpha 0\times\beta 0})\big(\big\|\nabla^{m_2}v_t^1\big\| + \big\|\nabla^{m_2}v_t^2\big\|\big).$$

Taking the scalar product of (47) in H with w_t, z_t, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\bigg[\|w_t\|^2 + \alpha\big\|\nabla^{m_1}w_t\big\|^2 + \|z_t\|^2 + \beta\big\|\nabla^{m_2}z_t\big\|^2 \\
&\quad + \frac{1}{4}\Phi_0\cdot\big(\big\|\nabla^{m_1}w\big\|^2 + \big\|\nabla^{m_2}z\big\|^2\big)\bigg] \\
&\quad + \frac{1}{2}\sigma_{12}(t)\big\|\nabla^{m_1}w_t\big\|^2 + \frac{1}{2}\sigma_{34}(t)\big\|\nabla^{m_2}z_t\big\|^2 \\
&\quad + \big(G_1\big(u^1,u^2,v^1,v^2;t\big),w_t\big) + \big(G_2\big(u^1,u^2,v^1,v^2;t\big),z_t\big) = 0.
\end{aligned} \tag{48}$$

According to Lemma 3 and (A1), $M_0 \le M \le C(R_0,H_1(0)) \equiv M_1$. When $\frac{\mathrm{d}}{\mathrm{d}t}(\|\nabla^{m_1}w\|^2 + \|\nabla^{m_2}z\|^2) \ge 0$, we have $\Phi_0 = 2M_0$; otherwise $\Phi_0 = 2M_1$.

Letting $(G_1(u^1,u^2,v^1,v^2;t),w_t) = G_{11} + G_{12} + G_{13}$ and $(G_2(u^1,u^2,v^1,v^2;t),z_t) = G_{21} + G_{22} + G_{23}$, we have

$$\begin{aligned}
G_{11} &= \frac{1}{2}\big(\sigma_1(t) - \sigma_2(t)\big)\big(\nabla^{m_1}\big(u_t^1 + u_t^2\big),\nabla^{m_1}w_t\big) \\
&\le C(R_{\alpha 0\times\beta 0})\big(\big\|\nabla^{m_1}u_t^1\big\| + \big\|\nabla^{m_1}u_t^2\big\|\big)\big\|\nabla^{m_1}w\big\|\big\|\nabla^{m_1}w_t\big\| \\
&\le \frac{\sigma_{120}}{8}\big\|\nabla^{m_1}w_t\big\|^2 + \frac{2C(R_{\alpha 0\times\beta 0})}{\sigma_{120}}\big(\big\|\nabla^{m_1}u_t^1\big\|^2 + \big\|\nabla^{m_1}u_t^2\big\|^2\big)\big\|\nabla^{m_1}w\big\|^2,
\end{aligned} \tag{49}$$

$$\begin{aligned}
G_{12} &= \frac{1}{2}\big(\Phi_1(t) - \Phi_2(t)\big)\big(\nabla^{m_1}\big(u^1 + u^2\big),\nabla^{m_1}w_t\big) \\
&\le C(R_{\alpha 0\times\beta 0})\big(\big\|\nabla^{m_1}w\big\| + \big\|\nabla^{m_2}z\big\|\big)\big\|\nabla^{m_1}w_t\big\| \\
&\le \frac{\sigma_{120}}{8}\big\|\nabla^{m_1}w_t\big\|^2 + \frac{2C(R_{\alpha 0\times\beta 0})}{\sigma_{120}}\big(\big\|\nabla^{m_1}w\big\|^2 + \big\|\nabla^{m_2}z\big\|^2\big),
\end{aligned} \tag{50}$$

$$G_{13} = \big(g_1(u_1,v_1) - g_1(u_2,v_2),w_t\big) \le C(R_{\alpha 0\times\beta 0})\big(\|w_t\|^2 + \big\|\nabla^{m_1}w\big\|^2 + \big\|\nabla^{m_2}z\big\|^2\big), \tag{51}$$

$$\begin{aligned}
G_{21} &= \frac{1}{2}\big(\sigma_3(t) - \sigma_4(t)\big)\big(\nabla^{m_2}\big(v_t^1 + v_t^2\big),\nabla^{m_2}z_t\big) \\
&\le C(R_{\alpha 0\times\beta 0})\big(\big\|\nabla^{m_2}v_t^1\big\| + \big\|\nabla^{m_2}v_t^2\big\|\big)\big\|\nabla^{m_2}z\big\|\big\|\nabla^{m_2}z_t\big\| \\
&\le \frac{\sigma_{340}}{8}\big\|\nabla^{m_2}z_t\big\|^2 + \frac{2C(R_{\alpha 0\times\beta 0})}{\sigma_{340}}\big(\big\|\nabla^{m_2}v_t^1\big\|^2 + \big\|\nabla^{m_2}v_t^2\big\|^2\big)\big\|\nabla^{m_2}z\big\|^2,
\end{aligned} \tag{52}$$

$$\begin{aligned}
G_{22} &= \frac{1}{2}\big(\Phi_1(t) - \Phi_2(t)\big)\big(\nabla^{m_2}\big(v^1 + v^2\big),\nabla^{m_2}z_t\big) \\
&\le C(R_{\alpha 0\times\beta 0})\big(\big\|\nabla^{m_1}w\big\| + \big\|\nabla^{m_2}z\big\|\big)\big\|\nabla^{m_2}z_t\big\| \\
&\le \frac{\sigma_{340}}{8}\big\|\nabla^{m_2}z_t\big\|^2 + \frac{2C(R_{\alpha 0\times\beta 0})}{\sigma_{340}}\big(\big\|\nabla^{m_1}w\big\|^2 + \big\|\nabla^{m_2}z\big\|^2\big),
\end{aligned} \tag{53}$$

$$G_{23} = \big(g_2(u_1,v_1) - g_2(u_2,v_2),z_t\big) \le C(R_{\alpha 0\times\beta 0})\big(\|z_t\|^2 + \big\|\nabla^{m_1}w\big\|^2 + \big\|\nabla^{m_2}z\big\|^2\big), \tag{54}$$

where $\sigma_{120} = 2N_{10}$ and $\sigma_{340} = 2N_{20}$.

Inserting (49)–(54) into (48), we have

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\bigg[\|w_t\|^2+\alpha\left\|\nabla^{m_1}w_t\right\|^2+\|z_t\|^2+\beta\left\|\nabla^{m_2}z_t\right\|^2\\
&\quad+\frac{1}{4}\Phi_0\cdot\left(\left\|\nabla^{m_1}w\right\|^2+\left\|\nabla^{m_2}z\right\|^2\right)\bigg]\\
&\le C_{16}\left(1+\left\|\nabla^{m_1}u_t^1\right\|^2+\left\|\nabla^{m_1}u_t^2\right\|^2+\left\|\nabla^{m_2}v_t^1\right\|^2+\left\|\nabla^{m_2}v_t^2\right\|^2\right)\\
&\quad\times\bigg[\|w_t\|^2+\alpha\left\|\nabla^{m_1}w_t\right\|^2+\|z_t\|^2+\beta\left\|\nabla^{m_2}z_t\right\|^2\\
&\quad+\frac{1}{4}\Phi_0\cdot\left(\left\|\nabla^{m_1}w\right\|^2+\left\|\nabla^{m_2}z\right\|^2\right)\bigg].
\end{aligned}\tag{55}$$

Solving this differential inequality yields

$$\begin{aligned}
&\bigg[\|w_t\|^2+\alpha\left\|\nabla^{m_1}w_t\right\|^2+\|z_t\|^2+\beta\left\|\nabla^{m_2}z_t\right\|^2\\
&\quad+\frac{1}{4}\Phi_0\cdot\left(\left\|\nabla^{m_1}w\right\|^2+\left\|\nabla^{m_2}z\right\|^2\right)\bigg]\\
&\le\bigg[\|w_1\|^2+\alpha\left\|\nabla^{m_1}w_1\right\|^2+\|z_1\|^2+\beta\left\|\nabla^{m_2}z_1\right\|^2\\
&\quad+\frac{1}{4}\Phi_0\cdot\left(\left\|\nabla^{m_1}w_0\right\|^2+\left\|\nabla^{m_2}z_0\right\|^2\right)\bigg]\\
&\quad\times\exp\left(\int_0^t C_{17}\left(1+\left\|\nabla^{m_1}u_t^1\right\|^2+\left\|\nabla^{m_1}u_t^2\right\|^2+\left\|\nabla^{m_2}v_t^1\right\|^2+\left\|\nabla^{m_2}v_t^2\right\|^2\right)\mathrm{d}s\right).
\end{aligned}\tag{56}$$

Thus, the uniqueness of the solutions is proved.

Therefore, problem (1)–(3) possess a unique solution (u,v). Theorem 1 is proved. □

According to Theorem 1, we can define a nonlinear operator $\{S(t)\}_{t\ge0}$ the on space $X_{\alpha0\times\beta0}: S(t)(u_0,u_1,v_0,v_1)=(u,u_t,v,v_t)$, for all $t\ge0$. Theorem 1 shows that $\{S(t)\}_{t\ge0}$ forms a continuous semigroup in $X_{\alpha0\times\beta0}$. Before proving the existence of a family of global attractors, we first present their definition.

Definition 1 Let X_0 be a Banach space and $\{S(t)\}_{t\ge0}$ a continuous operator semigroup. If there exists a compact set $A_{k_1\times k_2}$ satisfying the following conditions:

(i) (Invariance) All $A_{k_1\times k_2}$ are invariant sets under the action of semigroup $\{S(t)\}_{t\ge0}$,

$$S(t)A_{k_1\times k_2}=A_{k_1\times k_2},\quad\forall t\ge0;$$

(ii) (Attractiveness) All $A_{k_1\times k_2}$ attract all bounded sets in X_0, i.e., for any bounded $B\subset X_0$,

$$\operatorname{dist}\left(S(t)B,A_{k_1\times k_2}\right)=\sup_{x\in B}\inf_{y\in A_{k_1\times k_2}}\left\|S(t)x-y\right\|_{X_0}\to0,\quad\text{as } t\to\infty.$$

In particular, when $t\to\infty$, all trajectories $S(t)u_0$ from u_0 converge to $A_{k_1\times k_2}$, i.e.,

$$\operatorname{dist}\left(S(t)u_0,A_{k_1\times k_2}\right)\to0,\quad\text{as } t\to\infty.$$

then, a compact set A_k is a global attractor of the semigroup $\{S(t)\}_{t\geq 0}$. Let $\mathcal{A} = \{A_{k_1\times k_2} \subset X_0 : k_1 = 1,2,\dots,m_1, k_2 = 1,2,\dots,m_2\}$ be a family of subsets in X_0. Then $\mathcal{A}$ is called the global attractor family in X_0.

Proof of Theorem 2 By Lemma 3, for all $R_{\alpha 0\times\beta 0} > 0$, we have $\|(u_0,u_1,v_0,v_1)\|_{X_{\alpha 0\times\beta 0}} \leq R_{\alpha 0\times\beta 0}$. Thus,

$$\begin{aligned}\left\|S(t)(u_0,u_1,v_0,v_1)\right\|^2_{X_{\alpha 0\times\beta 0}} &= \left\|\nabla^{m_1}u\right\|^2 + \|u_t\|^2 + \alpha\left\|\nabla^{m_1}u_t\right\|^2 + \left\|\nabla^{m_2}v\right\|^2 \\ &\quad + \|v_t\|^2 + \beta\left\|\nabla^{m_2}v_t\right\|^2 \\ &\leq C(R_{\alpha 0\times\beta 0}),\end{aligned}$$

indicating that $\{S(t)\}_{t\geq 0}$ are uniformly bounded in $X_{\alpha 0\times\beta 0}$.

Further,

$$\begin{aligned}B_{\alpha k_1\times\beta k_2,0} = \Big\{&(u,u_t,v,v_t)\in X_{\alpha k_1\times\beta k_2}: \\ &\left\|(u,u_t,v,v_t)\right\|^2_{X_{\alpha k_1\times\beta k_2}} = \left\|\nabla^{m_1+k_1}u\right\|^2 + \left\|\nabla^{k_1}u_t\right\|^2 + \alpha\left\|\nabla^{m_1+k_1}u_t\right\|^2 \\ &+ \left\|\nabla^{m_2+k_2}v\right\|^2 + \left\|\nabla^{k_2}v_t\right\|^2 + \beta\left\|\nabla^{m_2+k_2}v_t\right\|^2 \leq C(R_{\alpha 0\times\beta 0}) + C(R_{\alpha k_1\times\beta k_2})\Big\}\end{aligned}$$

are bounded absorbing sets of the semigroup $\{S(t)\}_{t\geq 0}$ in $X_{\alpha 0\times\beta 0}$.

Because $X_{\alpha k_1\times\beta k_2} \hookrightarrow\hookrightarrow X_{\alpha 0\times\beta 0}$ are compactly embedded, i.e., bounded sets in $X_{\alpha k_1\times\beta k_2}$ are compact sets in $X_{\alpha 0\times\beta 0}$, the solution semigroup $\{S(t)\}_{t\geq 0}$ is a fully continuous operator.

To sum up, we obtained the global attractor family $\mathcal{A} = \{A_{\alpha k_1\times\beta k_2}\}$ of the solution semigroup $\{S(t)\}_{t\geq 0}$ in $X_{\alpha 0\times\beta 0}$, and

$$A_{\alpha k_1\times\beta k_2} = \omega(B_{\alpha k_1\times\beta k_2,0}) = \bigcap_{\tau\geq 0}\overline{\bigcup_{t\geq\tau} S(t)B_{\alpha k_1\times\beta k_2,0}},$$

$$A_{\alpha k_1\times\beta k_2} \subset X_{\alpha 0\times\beta 0}, \quad k_1 = 1,2,\dots,m_1, k_2 = 1,2,\dots,m_2, \text{ for } \forall\alpha,\beta\in(0,1].$$

Theorem 2 is proved. □

Note 1 Lemma 4 and Theorem 2 show that bounded absorbing sets

$$\begin{aligned}B_{\alpha k_1\times\beta k_2,0} = \Big\{&(u,u_t,v,v_t)\in X_{\alpha k_1\times\beta k_2}: \\ &\left\|(u,u_t,v,v_t)\right\|^2_{X_{\alpha k_1\times\beta k_2}} = \left\|\nabla^{m_1+k_1}u\right\|^2 + \left\|\nabla^{k_1}u_t\right\|^2 + \alpha\left\|\nabla^{m_1+k_1}u_t\right\|^2 \\ &+ \left\|\nabla^{m_2+k_2}v\right\|^2 + \left\|\nabla^{k_2}v_t\right\|^2 + \beta\left\|\nabla^{m_2+k_2}v_t\right\|^2 \leq C(R_{\alpha 0\times\beta 0}) + C(R_{\alpha k_1\times\beta k_2})\Big\}\end{aligned}$$

are compact sets in $X_{\alpha 0\times\beta 0}$. Therefore, based on condition 3 in Lemma 2, the operator semigroup $\{S(t)\}_{t\geq 0}$ only needs to be a continuous operator. According to Theorem 1, the semigroup $\{S(t)\}_{t\geq 0}$ is already continuous. Thus, the global attractor family $\mathcal{A} = \{A_{\alpha k_1\times\beta k_2}\}$ of problem (1)–(3) in $X_{\alpha 0\times\beta 0}$ can also be obtained.

4 Summary and prospects

This paper investigated higher-order (m_1,m_2)-coupled Kirchhoff systems with higher-order rotational inertia and nonlocal damping. For the first time, we systematically defined the family of global attractors of problem (1)–(3) and proved its existence. The findings enriched the relevant findings on higher-order coupled Kirchhoff models and laid a theoretical foundation for future practical applications.

Despite defining and proving the existence of the global attractor family of the higher-order (m_1, m_2)-coupled Kirchhoff system, many questions concerning such models still require further investigation:

1. The higher-order (m_1, m_2)-coupled Kirchhoff system in this paper is autonomous, while the relatively complex nonautonomous higher-order (m_1, m_2)-coupled Kirchhoff systems and higher-order (m_1, m_2)-coupled Kirchhoff systems with delays have not been studied. Thus, it is very meaningful to study the asymptotic behaviors of such systems;
2. This paper focused mainly on the global attractor family of dynamic systems, while many other properties were not explored, such as the dimension estimate, the exponential attractor family, and the inertial manifold family. The scarce relevant theoretical results warrant further research efforts.

Acknowledgements
The authors would like to thank the anonymous referees for their careful reading of the paper.

Author contributions
All authors contributed to the writing of this paper. All authors read and approved the final manuscript.

Funding
This work was partially supported by the basic science (NATURAL SCIENCE) research project of colleges and universities in Jiangsu Province (21KJB110013) and the fundamental research fund of Yunnan Education Department (2020J0908).

Data availability
No datasets were generated or analysed during the current study.

Declarations

Competing interests
The authors declare no competing interests.

Author details
[1]Applied Technology College of Soochow University, Suzhou, Jiangsu 215325, China. [2]Jiangsu Keyida Environmental Protection Technology Co., LTD., Yancheng, Jiangsu 224005, China. [3]School of Mathematics and Statistics, Yunnan University, Kunming, Yunnan 650500, China.

Received: 10 February 2024 Accepted: 27 March 2024 Published online: 04 April 2024

References

1. Chueshov, I.: Long-time dynamics of Kirchhoff wave models with strong nonlinear damping. J. Differ. Equ. **252**, 1229–1262 (2012)
2. Lin, G., Lv, P., Lou, R.: Exponential attractors and inertial manifolds for a class of nonlinear generalized Kirchhoff–Boussinesq model. Far East J. Math. Sci. **101**(9), 1913–1945 (2017)
3. Nakao, M.: An attractor for a nonlinear dissipative wave equation of Kirchhoff type. J. Math. Anal. Appl. **353**(2), 652–659 (2009)
4. Cao, Y., Zhao, Q.: Asymptotic behavior of global solutions to a class of mixed pseudo-parabolic Kirchhoff equations. Appl. Math. Lett. **118**, 107119 (2021)
5. Ma, H., Zhong, C.: Attractors for the Kirchhoff equations with strong nonlinear damping. Appl. Math. Lett. **74**, 127–133 (2017)
6. Ghisi, M.: Global solutions for dissipative Kirchhoff strings with non-Lipschitz nonlinear term. J. Differ. Equ. **230**(1), 128–139 (2006)
7. Qin, D., Tang, X., Zhang, J.: Ground states for planar Hamiltonian elliptic systems with critical exponential growth. J. Differ. Equ. **308**, 130–159 (2022)
8. Papageorgiou, N.S., Zhang, J., Zhang, W.: Solutions with sign information for noncoercive double phase equations. J. Geom. Anal. **34**(1), 14 (2024)
9. Papadopoulos, P.G., Stavrakakis, N.M.: Global existence and blow-up results for an equation of Kirchhoff type on $\mathbb{R}^N$. Topol. Methods Nonlinear Anal. **17**(1), 91–109 (2001)
10. Li, Q., Nie, J., Zhang, W.: Existence and asymptotics of normalized ground states for a Sobolev critical Kirchhoff equation. J. Geom. Anal. **33**(4), 126 (2023)

11. Li, Q., Rădulescu, V.D., Zhang, W.: Normalized ground states for the Sobolev critical Schrödinger equation with at least mass critical growth. Nonlinearity **37**(2), 025018 (2024)
12. Zhang, J., Zhou, H., Mi, H.: Multiplicity of semiclassical solutions for a class of nonlinear Hamiltonian elliptic system. Adv. Nonlinear Anal. **13**, 20230139 (2024)
13. Chueshov, I., Lasiecka, I.: Long-Time Behavior of Second Order Evolution Equations with Nonlinear Damping. Am. Math. Soc., Providence (2008)
14. Niimura, T.: Attractors and their stability with respect to rotational inertia for nonlinear extensible beam equations. Discrete Contin. Dyn. Syst. **40**, 2561–2591 (2020)
15. Ye, Y.J., Tao, X.X.: Initial boundary value problem for higher-order nonlinear Kirchhoff-type equation. Acta Math. Sinica (Chin. Ser.) **62**(6), 923–938 (2019)
16. Lin, G.-G., Zhu, C.-Q.: Asymptotic behavior of solutions for a class of nonlinear higher-order Kirchhoff-type equations. J. Yunnan Univ. Nat. Sci. **41**(05), 7–15 (2019)
17. Ding, P., Yang, Z.: Longtime behavior for an extensible beam equation with rotational inertia and structural nonlinear damping. J. Math. Anal. Appl. **496**(1), 124785 (2020)
18. Wang, Y., Zhang, J.: Long-time dynamics of solutions for a class of coupling beam equations with nonlinear boundary conditions. Math. Appl. **33**(01), 25–35 (2020)
19. Lin, G., Zhang, M.: The estimates of the upper bounds of Hausdorff dimensions for the global attractor for a class of nonlinear coupled Kirchhoff-type equations. Adv. Pure Math. **8**(1), 1–10 (2018)
20. Lin, G., Yang, S.: Hausdorff dimension and fractal dimension of the global attractor for the higher-order coupled Kirchhoff-type equations. J. Appl. Math. Phys. **05**(12), 2411–2424 (2017)
21. Lin, G., Hu, L.: Estimate on the dimension of global attractor for nonlinear higher-order coupled Kirchhoff type equations. Adv. Pure Math. **08**(1), 11–24 (2018)
22. Lin, G., Xia, X.: The exponential attractor for a class of Kirchhoff-type equations with strongly damped terms and source terms. J. Appl. Math. Phys. **06**(7), 1481–1493 (2018)
23. Lv, P., Liu, Y., Yu, S.: Long-term dynamic behavior of a higher-order coupled Kirchhoff model with nonlinear strong damping. J. Math. **2022**, 7044906 (2022)
24. Pata, V., Zelik, S.: A remark on the damped wave equation. Commun. Pure Appl. Anal. **5**(3), 611–616 (2006)

Publisher's Note

Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.