Jin *Boundary Value Problems* (2023) 2023:111
https://doi.org/10.1186/s13661-023-01801-7

Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

Existence of exponential attractors for the coupled system of suspension bridge equations

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Abstract
In this paper, we investigate the asymptotic behavior of the coupled system of suspension bridge equations. Under suitable assumptions, we obtain the existence of exponential attractors by using the decomposing technique of operator semigroup.

Keywords: Coupled suspension bridge equations; Decomposition of operator; Exponential attractors

1 Introduction

In the paper, we consider the following system, which describes the vibrating beam equation coupled with a vibrating string equation:

$$\begin{cases} u_{tt} + \alpha u_{xxxx} + \delta_1 u_t + k(u-v)^+ + f_B(u) = h_B, & \text{in } (0,L) \times \mathbb{R}^+, \\ v_{tt} - \beta v_{xx} + \delta_2 v_t - k(u-v)^+ + f_S(v) = h_S, & \text{in } (0,L) \times \mathbb{R}^+ \end{cases} \tag{1}$$

with the simply supported boundary conditions at both ends

$$\begin{aligned} &u(0,t) = u(L,t) = u_{xx}(0,t) = u_{xx}(L,t) = 0, \quad t \geq 0, \\ &v(0,t) = v(L,t) = 0 \quad t \geq 0, \end{aligned} \tag{2}$$

and the initial-value conditions

$$\begin{aligned} &u(x,0) = u_0, \qquad u_t(x,0) = u_1, \quad x \in (0,L), \\ &v(x,0) = v_0, \qquad v_t(x,0) = v_1, \quad x \in (0,L), \end{aligned} \tag{3}$$

where the first equation of (1) represents the vibration of the road bed in the vertical direction and the second equation describes that of the main cable from which the road bed is suspended by the tie cables (see [1]). $k > 0$ denotes the spring constants of the ties, $\alpha > 0$ and $\beta > 0$ are the flexural rigidity of the structure and the coefficient of tensile strength of the cable, respectively. $\delta_1, \delta_2 > 0$ are constants, the force term $h_B, h_S \in L^2(0,L)$.

We assume that the nonlinear functions $f_B \in C^3(\mathbb{R})$ and $f_S \in C^2(\mathbb{R})$ satisfy the following conditions:

$$(F_1) \quad \liminf_{|s|\to+\infty} \frac{|f_B(s)|}{s} \geq \delta, \qquad \liminf_{|s|\to+\infty} \frac{|f_S(s)|}{s} \geq \delta;$$

$$(F_2) \quad \left|f_B(s)\right|, \left|f_S(s)\right| \leq C_0\left(1 + |s|^p\right), \quad \forall p \geq 1,$$

for any $s \in \mathbb{R}$, where C_0, δ are positive constants.

As is well known, the suspension bridge equations were presented by Lazer and Mckenna as new problems in the field of nonlinear analysis [2]. In [3], the authors obtained the existence and uniqueness of a weak solution for $k > -1$ and showed decay estimates of the solution for the suspension problem. Similar models have been studied by many authors [4–14]. In [4], Ma and Zhong obtained the existence of weak solutions for suspension bridge equations, and the existence of strong solutions and strong global attractors was also achieved in [5]. Park and Kang [6] showed the existence of pullback $\mathcal{D}$-attractors for nonautonomous suspension bridge equations. In [7], Kang obtained the existence of global attractors for suspension bridge equations with memory, and Park and Kang [8] investigated the existence of global attractors for suspension bridge equations with nonlinear damping. In [14], Jia and Ma obtained the existence of exponential attractors for strong damped Kirchhoff type suspension bridge equations by using the decomposing technique of operator semigroup.

For the coupled suspension bridge equations, Ahmed and Harbi discussed this problem in [1], pointed out that the system is conservative and asymptotically stable with respect to the rest state for $k > 0, f_B(u) \equiv 0 \equiv f_S(v)$, and showed that the Cauchy problem of system (1) has at least one weak solution. Holubová and Matas considered the initial-boundary value problem for the more general nonlinear string-beam system in [15] and obtained the existence and uniqueness of the weak solution by the Faedo–Galerkin method. In [16], Litcanu investigated the existence of weak T-periodic solutions of (1) and obtained a regularity result when $k(u-v)^+ = \phi(u,v), f_B(u) \equiv 0 \equiv f_S(v)$. About the long time behavior of solutions for suspension bridge model, Ma and Zhong [17] achieved the existence of global attractor of a weak solution for autonomous coupled suspension bridge equations. In the sequel, they [18] obtained the existence of strong solutions and compact global attractors for autonomous coupled suspension bridge equations. In [19], Ma and Wang obtained pullback attractors for coupled suspension bridge equations. To our knowledge, although Jia and Ma investigated the existence of exponential attractors for single suspension bridge equations, the existence of exponential attractors of (1) has no any results, while it is just our concern.

The remaining paper is organized as follows. In Sect. 2, we introduce some notations and recall several abstract results. In Sect. 3, we prove the existence of exponential attractors for the coupled system of suspension bridge equations by using the decomposing technique of operator semigroup introduced in [20, 21].

2 Preliminaries

We consider the Hilbert spaces that will be used in our paper. Let

$$Y_0 = L^2(0,L), \qquad Y_1 = H_0^1(0,L), \qquad Y_2 = D(A) = H^2(0,L) \cap H_0^1(0,L),$$
$$Y_3 = D\left(A^2\right) = \left\{u \in H^2(0,L) \mid A^2 u \in L^2(0,L)\right\},$$

where $A = -\frac{\partial^2}{\partial x^2}$, $A^2 = \frac{\partial^4}{\partial x^4}$, and, $(\cdot,\cdot)$, $\|\cdot\|$ denote the scalar product and the norm of $L^2(0,L)$.

Moreover, we introduce spaces E_0 and E_1 as follows:

$$E_0 = Y_2 \times Y_1 \times Y_0 \times Y_0, \qquad E_1 = Y_3 \times Y_2 \times Y_2 \times Y_1,$$

and endow norms (denoted by $\|\cdot\|_s$)

$$\|(u,v)\|_0^2 = \alpha\|\Delta u\|^2 + \beta\|\nabla v\|^2 + \|u\|^2 + \|v\|^2$$

and

$$\|(u,v)\|_1^2 = \alpha\|\Delta^2 u\|^2 + \beta\|\Delta v\|^2 + \|\Delta u\|^2 + \|\nabla v\|^2.$$

By the Poincaré inequality, there exist constants $\lambda_1, \lambda_2 > 0$ such that

$$\|\nabla u\| \geq \lambda_1\|u\|, \qquad \|\Delta u\| \geq \lambda_2\|u\|, \quad \forall u \in Y_2,$$

let $\lambda = \min\{\lambda_1, \lambda_2\}$, we have

$$\|\nabla u\| \geq \lambda\|u\|, \qquad \|\Delta u\| \geq \lambda\|u\|, \quad \forall u \in Y_2. \tag{4}$$

In the following, we recall some abstract results, see [18, 20–22] for more details.

Definition 1 ([22]) A compact set $\mathcal{E} \subset E_0$ is called an exponential attractor or an inertial set for the semigroup $S(t)$ if the following conditions hold:

(i) $\mathcal{E}$ is invariant of $S(t)$, that is, $S(t)\mathcal{E} \subset \mathcal{E}$ for every $t \geq 0$;

(ii) $\dim_F \mathcal{E} < \infty$, that is, $\mathcal{E}$ has finite fractal dimension;

(iii) There exist an increasing function $J : \mathbb{R}^+ \to \mathbb{R}^+$ and $\nu > 0$ such that, for any set $\mathcal{B} \subset E_0$ with $\sup_{z_0 \in \mathcal{B}} \|z_0\|_0 \leq R$, there holds

$$\operatorname{dist}_{E_0}(S(t)\mathcal{B}, \mathcal{E}) \leq J(R)e^{-\nu t}.$$

Theorem 2 ([20, 21]) *Let $\mathcal{X} \subset E_0$ be a compact invariant subset. Assume that there exists a time $t_* > 0$ such that the following hold*:

(i) *the map*

$$(t, z_0) \mapsto S(t)z_0 : [0, t_*] \times \mathcal{X} \to \mathcal{X}$$

is Lipschitz continuous;

(ii) *the map $S(t_*) : \mathcal{X} \to \mathcal{X}$ admits a decomposition of the form*

$$S(t_*) = S_0 + S_1, \qquad S_0 : \mathcal{X} \to E_0, \quad S_1 : \mathcal{X} \to E_1,$$

where S_0 and S_1 satisfy the conditions

$$\|S_0(z_1) - S_0(z_2)\|_0 \leq \frac{1}{8}\|z_1 - z_2\|_0, \quad \forall z_1, z_2 \in \mathcal{X},$$

and

$$\left\|S_1(z_1) - S_1(z_2)\right\|_1 \le C_* \|z_1 - z_2\|_0, \quad \forall z_1, z_2 \in \mathcal{X},$$

for some $C_* > 0$.

Then there exists an invariant compact set $\mathcal{E} \subset \mathcal{X}$ such that

$$\mathrm{dist}_{E_0}\big(S(t)\mathcal{X}, \mathcal{E}\big) \le J_0 e^{-\frac{\log 2}{t_*}t}, \tag{5}$$

where

$$J_0 = 2L_* \sup_{z_0 \in \mathcal{X}} \|z_0\|_0 e^{\frac{\log 2}{t_*}}, \tag{6}$$

and L^* is the Lipschitz constant of the map $S(t_*) : \mathcal{X} \to \mathcal{X}$. Moreover,

$$\dim_F \mathcal{E} \le 1 + \frac{\log N_*}{\log 2}, \tag{7}$$

where N_* is the minimum number of $\frac{1}{8C_*}$-balls of E_0 necessary to cover the unit ball of E_1.

Lemma 3 ([18]) *Under assumptions* $(F1)$–$(F2)$, *the semigroup* $\{S(t)\}_{t\ge 0}$ *corresponding to problem* (1) *has a bounded absorbing set* $\mathcal{B}_0$ *in* E_0.

Lemma 4 ([18]) *Assume that conditions* $(F1)$–$(F2)$ *hold, the semigroup* $\{S(t)\}_{t\ge 0}$ *corresponding to problem* (1) *has a bounded absorbing set* $\mathcal{B}_1$ *in* E_1.

3 Existence of exponential attractors

In this section, we first state the result about the well-posedness of problem (1). Under assumptions, we can derive an existence result by the standard Faedo–Galerkin method (see [15, 18]).

Theorem 5 *Suppose that* $k > 0$, $\alpha, \beta, \delta_1, \delta_2 > 0$ *and* $(F1)$–$(F2)$ *hold. If* $h_B, h_S \in L^2(0, L)$, $(u_0, v_0, u_1, v_1) \in E_1$, *then for any given* $T > 0$, *there exists a unique solution* (u, v) *of* (1)–(3) *such that*

$$u \in C\big([0,T], Y_2\big), \qquad u_t \in C\big([0,T], Y_0\big),$$
$$v \in C\big([0,T], Y_1\big), \qquad v_t \in C\big([0,T], Y_0\big).$$

Furthermore, $(u_0, v_0, u_1, v_1) \to (u(t), v(t), u_t(t), v_t(t))$ *is continuous in* E_1.

Consequently, it admits to define a C^0 semigroup

$$S(t) : (u_0, v_0, u_1, v_1) \to \big(u(t), v(t), u_t(t), v_t(t)\big), \quad t \in \mathbb{R}^+,$$

and it maps E_1 into itself.

To obtain the existence of exponential attractors, we need to prove some lemmas as follows.

Lemma 6 *Given any $R > 0$ and any two initial data $z_1 = (u_{11}, v_{11}, u_{12}, v_{12})$, $z_2 = (u_{21}, v_{21}, u_{22}, v_{22}) \in E_0$ such that $\|z_i\|_0 \le R$, there holds*

$$\|S(t)z_1 - S(t)z_2\|_0 \le e^{Kt}\|z_1 - z_2\|_0, \quad \forall t \in \mathbb{R}^+, \tag{8}$$

for some $K = K(R) > 0$.

Proof Given two solutions $z^1 = (u^1, v^1, u^1_t, v^1_t)$ and $z^2 = (u^2, v^2, u^2_t, v^2_t)$, corresponding to different initial data z_1 and z_2, the difference $z^1 - z^2 = (\omega^1, \omega^2, \omega^1_t, \omega^2_t)$ fulfills

$$\begin{aligned}
&\frac{d}{dt}\left(\alpha\|\Delta\omega^1\|^2 + \beta\|\nabla\omega^2\|^2 + \|\omega^1_t\|^2 + \|\omega^2_t\|^2\right) + 2\delta_1\|\omega^1_t\|^2 + 2\delta_2\|\omega^2_t\|^2 \\
&\quad + 2\left(k(u^1 - v^1)^+ - k(u^2 - v^2)^+, \omega^1_t\right) + 2\left(f_B(u^1) - f_B(u^2), \omega^1_t\right) \\
&\quad - 2\left(k(u^1 - v^1)^+ - k(u^2 - v^2)^+, \omega^2_t\right) + 2\left(f_S(v^1) - f_S(v^2), \omega^2_t\right) = 0.
\end{aligned} \tag{9}$$

Using (4) and Hölder's inequality, we have

$$\begin{aligned}
-2\left(k(u^1 - v^1)^+ - k(u^2 - v^2)^+, \omega^1_t\right) &\le 2k\left\|(u^1 - v^1)^+ - (u^2 - v^2)^+\right\|\|\omega^1_t\| \\
&\le 2k\|\omega^1 - \omega^2\|\|\omega^1_t\| \\
&\le \frac{2k}{\lambda^2}\|\Delta\omega^1\|^2 + \frac{2k}{\lambda^2}\|\nabla\omega^2\|^2 + k\|\omega^1_t\|^2,
\end{aligned} \tag{10}$$

$$\begin{aligned}
2\left(k(u^1 - v^1)^+ - k(u^2 - v^2)^+, \omega^2_t\right) &\le 2k\left\|(u^1 - v^1)^+ - (u^2 - v^2)^+\right\|\|\omega^2_t\| \\
&\le 2k\|\omega^1 - \omega^2\|\|\omega^2_t\| \\
&\le \frac{2k}{\lambda^2}\|\Delta\omega^1\|^2 + \frac{2k}{\lambda^2}\|\nabla\omega^2\|^2 + k\|\omega^2_t\|^2.
\end{aligned} \tag{11}$$

By $(F2)$ and Lemma 3, as well as the Sobolev embedding theorems, we know that $f_B(u)$, $f'_B(u), f''_B(u), f_S(u)$, $f'_S(u), f''_S(u)$ are uniformly bounded in L^∞. That is, there exists a constant $M > 0$ such that

$$|f_B(u)|_{L^\infty}, |f'_B(u)|_{L^\infty}, |f''_B(u)|_{L^\infty}, |f_S(u)|_{L^\infty}, |f'_S(u)|_{L^\infty}, |f''_S(u)|_{L^\infty} \le M. \tag{12}$$

Therefore

$$\begin{aligned}
-2\left(f_B(u^1) - f_B(u^2), \omega^1_t\right) &= -2\left(f'_B(\theta u^1 + (1-\theta)u^2)\omega^1, \omega^1_t\right) \\
&\le 2\|f'_B(\theta u^1 + (1-\theta)u^2\|_\infty\|\omega^1\|\|\omega^1_t\| \le 2M\|\omega^1\|\|\omega^1_t\| \\
&\le \frac{M}{\lambda^2}\|\Delta\omega^1\|^2 + M\|\omega^1_t\|^2,
\end{aligned} \tag{13}$$

$$\begin{aligned}
-2\left(f_S(v^1) - f_S(v^2), \omega^2_t\right) &= -2\left(f'_S(\theta v^1 + (1-\theta)v^2)\omega^2, \omega^2_t\right) \\
&\le 2\|f'_S(\theta v^1 + (1-\theta)v^2\|_\infty\|\omega^2\|\|\omega^2_t\| \le 2M\|\omega^2\|\|\omega^2_t\| \\
&\le \frac{M}{\lambda^2}\|\Delta\omega^2\|^2 + M\|\omega^2_t\|^2.
\end{aligned} \tag{14}$$

Combining with the above estimates, we have

$$\begin{aligned}&\frac{d}{dt}\left(\alpha\left\|\Delta\omega^1\right\|^2+\beta\left\|\nabla\omega^2\right\|^2+\left\|\omega_t^1\right\|^2+\left\|\omega_t^2\right\|^2\right)\\&\quad\le\frac{4k+M}{\lambda^2}\left\|\Delta\omega^1\right\|^2+\frac{4k+M}{\lambda^2}\left\|\nabla\omega^2\right\|^2+(k+M)\left\|\omega_t^1\right\|^2+(k+M)\left\|\omega_t^2\right\|^2.\end{aligned}\tag{15}$$

Thus, we can find a positive constant $K=\max\{\frac{4k+M}{\alpha\lambda^2},\frac{4k+M}{\beta\lambda^2},k+M\}$ such that

$$\begin{aligned}&\frac{d}{dt}\left(\alpha\left\|\Delta\omega^1\right\|^2+\beta\left\|\nabla\omega^2\right\|^2+\left\|\omega_t^1\right\|^2+\left\|\omega_t^2\right\|^2\right)\\&\quad\le K\left(\alpha\left\|\Delta\omega^1\right\|^2+\beta\left\|\nabla\omega^2\right\|^2+\left\|\omega_t^1\right\|^2+\left\|\omega_t^2\right\|^2\right).\end{aligned}\tag{16}$$

The assertion follows from the Gronwall lemma. □

Lemma 7 *There exists $C\ge 0$ such that*

$$\sup_{z_0\in\mathcal{B}_1}\left\|z_t(t)\right\|_0\le C.$$

Proof From (1) we have

$$u_{tt}=-\alpha\Delta^2u-\delta_1u_t-k(u-v)^+-f_B(u)+h_B$$

and

$$v_{tt}=\beta\Delta v-\delta_2v_t+k(u-v)^+-f_S(u)+h_S.$$

By exploiting Lemma 3, Lemma 4, and (12), we get

$$\|u_{tt}\|\le\alpha\left\|\Delta^2u\right\|+\delta_1\|u_t\|+k\|u-v\|+\left\|f_B(u)\right\|+\|h_B\|\le C\tag{17}$$

and

$$\|v_{tt}\|\le\beta\|\Delta v\|+\delta_2\|v_t\|+k\|u-v\|+\left\|f_S(u)\right\|+\|h_S\|\le C.\tag{18}$$

Further, by virtue of Lemma 4, we can get $\|\Delta u_t\|\le C$, $\|\nabla v_t\|\le C$, thus

$$\left\|z_t(t)\right\|_0^2=\alpha\|\Delta u_t\|^2+\beta\|\nabla v_t\|^2+\|u_{tt}\|^2+\|v_{tt}\|^2\le C.\tag{19}$$

Namely,

$$\sup_{z_0\in\mathcal{B}_1}\left\|z_t(t)\right\|_0\le C.$$

□

Like the method in [21], we define

$$\mathcal{X}=\overline{\bigcup_{\tau\ge t_1}S(\tau)\mathcal{B}_1}^{E_0}.\tag{20}$$

Lemma 8 *For every $T > 0$, the mapping $(t, z_0) \mapsto S(t)z_0$ is Lipschitz continuous on $[0, T] \times \mathcal{X}$.*

Proof For $z_1, z_2 \in \mathcal{X}$ and $t_1, t_2 \in [0, T]$, we have

$$\left\|S(t_1)z_1 - S(t_2)z_2\right\|_0 \leq \left\|S(t_1)z_1 - S(t_1)z_2\right\|_0 + \left\|S(t_1)z_2 - S(t_2)z_2\right\|_0. \tag{21}$$

The first term of the above inequality is handled by estimate (8). Concerning the second one,

$$\left\|S(t_1)z_2 - S(t_2)z_2\right\|_0 = \left\|z(t_1) - z(t_2)\right\|_0 \leq \left|\int_{t_1}^{t_2} \left\|z_t(\tau)\right\|_0 \, d\tau\right| \leq C|t_1 - t_2|. \tag{22}$$

Hence

$$\left\|S(t_1)z_1 - S(t_2)z_2\right\|_0 \leq L\left[|t_1 - t_2| + \|z_1 - z_2\|_0\right] \tag{23}$$

for some $L = L(T) \geq 0$. □

Lemma 9 *Let $\mathcal{X} \subset E_0$ be a compact invariant subset. Assume that there exists a time $t_* > 0$ such that the map $S(t_*) : \mathcal{X} \to \mathcal{X}$ admits a decomposition of the form*

$$S(t_*) = S_0 + S_1, \quad S_0 : \mathcal{X} \to E_0, \quad S_1 : \mathcal{X} \to E_1,$$

where S_0 satisfies

$$\left\|S_0(z_1) - S_0(z_2)\right\|_0 \leq \frac{1}{8}\|z_1 - z_2\|_0, \quad \forall z_1, z_2 \in \mathcal{X},$$

and S_1 satisfies

$$\left\|S_1(z_1) - S_1(z_2)\right\|_1 \leq C_*\|z_1 - z_2\|_0, \quad \forall z_1, z_2 \in \mathcal{X},$$

for some $C_* > 0$.

Proof For $z_0 \in \mathcal{X}$, we denote by $S_0(t)z_0$ the solution at time t of the linear homogeneous problem associated with (1)–(3), and let $S_1(t)z_0 = S(t)z_0 - S_0(t)z_0$.

Given two solutions

$$z^1(t) = \left(u^1, v^1; u^1_t, v^1_t\right) \quad \text{and} \quad z^2(t) = \left(u^2, v^2; u^2_t, v^2_t\right)$$

originating from $z_1, z_2 \in \mathcal{X}$, respectively.

Set $\bar{z} = z^1 - z^2 = (\bar{u}, \bar{v}; \bar{u}_t, \bar{v}_t)$ and decompose $\bar{z}$ into the sum

$$\bar{z} = \bar{z}_d + \bar{z}_c = (\omega_1, \omega_3; \omega_{1t}, \omega_{3t}) + (\omega_2, \omega_4; \omega_{2t}, \omega_{4t}),$$

where $\bar{z}_d$ satisfies

$$\begin{cases} \omega_{1tt} + \alpha\Delta^2\omega_1 + \delta_1\omega_{1t} = 0, \\ \omega_{3tt} - \beta\Delta\omega_3 + \delta_2\omega_{3t} = 0, \\ \bar{z}_d(0) = z_1 - z_2, \end{cases} \tag{24}$$

and $\bar{z}_c$ satisfies

$$\begin{cases} \omega_{2tt} + \alpha\Delta^2\omega_2 + \delta_1\omega_{2t} + k(u^1 - v^1)^+ - k(u^2 - v^2)^+ + f_B(u^1) - f_B(u^2) = 0, \\ \omega_{4tt} - \beta\Delta\omega_4 + \delta_2\omega_{4t} - k(u^1 - v^1)^+ + k(u^2 - v^2)^+ + f_S(v^1) - f_S(v^2) = 0, \\ \bar{z}_c(0) = 0. \end{cases} \tag{25}$$

It is apparent that $\bar{z}_d(t) = S_0(t)z_1 - S_0(t)z_2$ and $\bar{z}_c(t) = S_1(t)z_1 - S_1(t)z_2$.

For (24), taking the scalar product of the first and second equations of (24) with $2\omega_{1t} + \delta_1\omega_1$ and $2\omega_{3t} + \delta_2\omega_3$ in $L^2(0,L)$, respectively, we infer that

$$\begin{aligned} &\frac{d}{dt}\Big(\|\omega_{1t}\|^2 + \alpha\|\Delta\omega_1\|^2 + \delta_1(\omega_{1t},\omega_1) + \frac{1}{2}\delta_1^2\|\omega_1\|^2 + \|\omega_{3t}\|^2 + \beta\|\nabla\omega_3\|^2 + \delta_2(\omega_{3t},\omega_3) \\ &\quad + \frac{1}{2}\delta_2^2\|\omega_3\|^2\Big) + \delta_1\big(\|\omega_{1t}\|^2 + \alpha\|\Delta\omega_1\|^2\big) + \delta_2\big(\|\omega_{3t}\|^2 + \beta\|\nabla\omega_3\|^2\big) = 0. \end{aligned} \tag{26}$$

Denote

$$\begin{aligned} E(t) &= \|\omega_{1t}\|^2 + \alpha\|\Delta\omega_1\|^2 + \delta_1(\omega_{1t},\omega_1) + \frac{1}{2}\delta_1^2\|\omega_1\|^2 \\ &\quad + \|\omega_{3t}\|^2 + \beta\|\nabla\omega_3\|^2 + \delta_2(\omega_{3t},\omega_3) + \frac{1}{2}\delta_2^2\|\omega_3\|^2. \end{aligned}$$

Due to the inequalities $\delta_1(\omega_{1t},\omega_1) \le \frac{1}{2}\|\omega_{1t}\|^2 + \frac{1}{2}\delta_1^2\|\omega_1\|^2$, $\delta_2(\omega_{3t},\omega_3) \le \frac{1}{2}\|\omega_{3t}\|^2 + \frac{1}{2}\delta_2^2\|\omega_3\|^2$, we have

$$\begin{aligned} E(t) &\le \frac{3}{2}\|\omega_{1t}\|^2 + \alpha\|\Delta\omega_1\|^2 + \delta_1^2\|\omega_1\|^2 + \frac{3}{2}\|\omega_{3t}\|^2 + \beta\|\nabla\omega_3\|^2 + \delta_2^2\|\omega_3\|^2 \\ &\le \frac{3}{2}\|\omega_{1t}\|^2 + \left(\alpha + \frac{\delta_1^2}{\lambda^2}\right)\|\Delta\omega_1\|^2 + \frac{3}{2}\|\omega_{3t}\|^2 + \left(\beta + \frac{\delta_2^2}{\lambda^2}\right)\|\nabla\omega_3\|^2. \end{aligned} \tag{27}$$

Let $\kappa = \max\{\frac{3}{2}, 1 + \frac{\delta_1^2}{\lambda^2\alpha}, 1 + \frac{\delta_2^2}{\lambda^2\beta}\} > 0$, we get

$$E(t) \le \kappa\|\bar{z}_d(t)\|_0^2. \tag{28}$$

Meanwhile

$$E(t) \ge \frac{1}{2}\|\omega_{1t}\|^2 + \alpha\|\Delta\omega_1\|^2 + \frac{1}{2}\|\omega_{3t}\|^2 + \beta\|\nabla\omega_3\|^2 \ge \frac{1}{2}\|\bar{z}_d(t)\|_0^2. \tag{29}$$

Therefore, $E(t)$ satisfies

$$\frac{1}{2}\|\bar{z}_d(t)\|_0^2 \le E(t) \le \kappa\|\bar{z}_d(t)\|_0^2. \tag{30}$$

Let $\delta = \min\{\delta_1, \delta_2\} > 0$, by (26), we deduce that

$$\frac{d}{dt}E(t) \le -\delta \left\| \overline{z}_d(t) \right\|_0^2. \tag{31}$$

Combining (30) with (31), we get

$$\frac{d}{dt}E(t) \le -\frac{\delta}{\kappa}E(t). \tag{32}$$

Using (30) and the Gronwall lemma, we end up with

$$\left\| \overline{z}_d(t) \right\|_0^2 \le 2\kappa e^{-\frac{\delta}{\kappa}t} \left\| \overline{z}_d(0) \right\|_0^2, \tag{33}$$

namely,

$$\left\| S_0(t)z_1 - S_0(t)z_2 \right\|_0 \le \sqrt{2\kappa} e^{-\frac{\delta}{2\kappa}t} \|z_1 - z_2\|_0. \tag{34}$$

Choose $t_* = \frac{2\kappa}{\delta} \ln 8\sqrt{2\kappa}$, we get

$$\left\| S_0(t_*)z_1 - S_0(t_*)z_2 \right\|_0 \le \frac{1}{8} \|z_1 - z_2\|_0. \tag{35}$$

For system (25), choose $0 < \varepsilon < 1$. Taking the scalar product of the first and second equations of (25) with $\Delta^2\phi = \Delta^2\omega_{2t} + \varepsilon\Delta^2\omega_2$ and $-\Delta\psi = -\Delta\omega_{4t} - \varepsilon\Delta\omega_4$ in $L^2(0, L)$, respectively, we find

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\Delta\phi\|^2 + \alpha\left\|\Delta^2\omega_2\right\|^2 + \|\nabla\psi\|^2 + \beta\|\Delta\omega_4\|^2\right) \\
&\quad + \alpha\varepsilon\left\|\Delta^2\omega_2\right\|^2 + (\delta_1 - \varepsilon)\|\Delta\phi\|^2 - \varepsilon(\delta_1 - \varepsilon)\left(\omega_2, \Delta^2\phi\right) \\
&\quad + \beta\varepsilon\|\Delta\omega_4\|^2 + (\delta_2 - \varepsilon)\|\nabla\psi\|^2 - \varepsilon(\delta_2 - \varepsilon)(\omega_4, -\Delta\psi) \\
&\quad + \left(k\left(u^1 - v^1\right)^+ - k\left(u^2 - v^2\right)^+, \Delta^2\phi\right) + \left(-k\left(u^1 - v^1\right)^+ + k\left(u^2 - v^2\right)^+, -\Delta\psi\right) \\
&\quad + \left(f_B\left(u^1\right) - f_B\left(u^2\right), \Delta^2\phi\right) + \left(f_S\left(v^1\right) - f_S\left(v^2\right), -\Delta\psi\right) = 0.
\end{aligned} \tag{36}$$

Thanks to Young's inequality and Hölder's inequality, we have

$$\begin{aligned}
&\left(k\left(u^1 - v^1\right)^+ - k\left(u^2 - v^2\right)^+, \Delta^2\phi\right) \\
&\quad = \frac{d}{dt}\left(k\left(u^1 - v^1\right)^+ - k\left(u^2 - v^2\right)^+, \Delta^2\omega_2\right) + \varepsilon\left(k\left(u^1 - v^1\right)^+ - k\left(u^2 - v^2\right)^+, \Delta^2\omega_2\right) \\
&\qquad - \left(k\left(u^1 - v^1\right)_t^+ - k\left(u^2 - v^2\right)_t^+, \Delta^2\omega_2\right) \\
&\quad \ge \frac{d}{dt}\left(k\left(u^1 - v^1\right)^+ - k\left(u^2 - v^2\right)^+, \Delta^2\omega_2\right) + \varepsilon\left(k\left(u^1 - v^1\right)^+ - k\left(u^2 - v^2\right)^+, \Delta^2\omega_2\right) \\
&\qquad - k\left\|\left(u^1 - v^1\right)_t^+ - \left(u^2 - v^2\right)_t^+\right\|\left\|\Delta^2\omega_2\right\| \\
&\quad \ge \frac{d}{dt}\left(k\left(u^1 - v^1\right)^+ - k\left(u^2 - v^2\right)^+, \Delta^2\omega_2\right) + \varepsilon\left(k\left(u^1 - v^1\right)^+ - k\left(u^2 - v^2\right)^+, \Delta^2\omega_2\right) \\
&\qquad - k\|\bar{u}_t - \bar{v}_t\|\left\|\Delta^2\omega_2\right\|
\end{aligned}$$

$$\begin{aligned}&\geq \frac{d}{dt}\left(k\left(u^1-v^1\right)^+-k\left(u^2-v^2\right)^+,\Delta^2\omega_2\right)+\varepsilon\left(k\left(u^1-v^1\right)^+-k\left(u^2-v^2\right)^+,\Delta^2\omega_2\right)\\&\quad-k\|\bar{u}_t\|\left\|\Delta^2\omega_2\right\|-\|\bar{v}_t\|\left\|\Delta^2\omega_2\right\|\\&\geq \frac{d}{dt}\left(k\left(u^1-v^1\right)^+-k\left(u^2-v^2\right)^+,\Delta^2\omega_2\right)+\varepsilon\left(k\left(u^1-v^1\right)^+-k\left(u^2-v^2\right)^+,\Delta^2\omega_2\right)\\&\quad-\frac{\varepsilon\alpha}{4}\left\|\Delta^2\omega_2\right\|^2-\frac{8k^2}{\varepsilon\alpha}\|\bar{u}_t\|^2-\frac{8k^2}{\varepsilon\alpha}\|\bar{v}_t\|^2\end{aligned}\tag{37}$$

and

$$\begin{aligned}&\left(-k\left(u^1-v^1\right)^++k\left(u^2-v^2\right)^+,-\Delta\psi\right)\\&\quad=-k\left(\left(u^1-v^1\right)_x^+-\left(u^2-v^2\right)_x^+,\nabla\psi\right)\\&\quad\geq -k\left\|\left(u^1-v^1\right)_x^+-\left(u^2-v^2\right)_x^+\right\|\|\nabla\psi\|\geq -k\|\nabla\bar{u}-\nabla\bar{v}\|\|\nabla\psi\|\\&\quad\geq -k\|\nabla\bar{u}\|\|\nabla\psi\|-k\|\nabla\bar{v}\|\|\nabla\psi\|\geq -\frac{\delta_2}{4}\|\nabla\psi\|^2-\frac{8k^2}{\delta_2}\|\nabla\bar{u}\|^2-\frac{8k^2}{\delta_2}\|\nabla\bar{v}\|^2.\end{aligned}\tag{38}$$

Denote $\varphi(t)=\theta u^1(t)+(1-\theta)u^2(t)$, $\sigma(t)=\theta v^1(t)+(1-\theta)v^2(t)$, applying Lemma 3, we have

$$\begin{aligned}&\left\|\varphi_t(t)\right\|\leq\theta\left\|u_t^1(t)\right\|+(1-\theta)\left\|u_t^2(t)\right\|\leq R_0,\\&\left\|\nabla\sigma(t)\right\|\leq\theta\left\|\nabla v^1(t)\right\|+(1-\theta)\left\|\nabla v^2(t)\right\|\leq R_0.\end{aligned}\tag{39}$$

By (12) and (39), we achieve

$$\begin{aligned}\left(f_B\left(u^1\right)-f_B\left(u^2\right),\Delta^2\phi\right)&=\left(f_B'\left(\varphi(t)\right)\bar{u},\Delta^2\omega_{2t}\right)+\varepsilon\left(f_B'\left(\varphi(t)\right)\bar{u},\Delta^2\omega_2\right)\\&=\frac{d}{dt}\left(f_B'\left(\varphi(t)\right)\bar{u},\Delta^2\omega_2\right)+\varepsilon\left(f_B'\left(\varphi(t)\right)\bar{u},\Delta^2\omega_2\right)\\&\quad-\left(f_B''\left(\varphi(t)\right)\varphi_t(t)\bar{u},\Delta^2\omega_2\right)-\left(f_B'\left(\varphi(t)\right)\bar{u}_t,\Delta^2\omega_2\right)\\&\geq\frac{d}{dt}\left(f_B'\left(\varphi(t)\right)\bar{u},\Delta^2\omega_2\right)+\varepsilon\left(f_B'\left(\varphi(t)\right)\bar{u},\Delta^2\omega_2\right)\\&\quad-MR_0\|\bar{u}\|\left\|\Delta^2\omega_2\right\|-M\|\bar{u}_t\|\left\|\Delta^2\omega_2\right\|\\&\geq\frac{d}{dt}\left(f_B'\left(\varphi(t)\right)\bar{u},\Delta^2\omega_2\right)+\varepsilon\left(f_B'\left(\varphi(t)\right)\bar{u},\Delta^2\omega_2\right)\\&\quad-\frac{\varepsilon\alpha}{4}\left\|\Delta^2\omega_2\right\|^2-\frac{8M^2R_0^2}{\varepsilon\alpha}\|\bar{u}\|^2-\frac{8M^2}{\varepsilon\alpha}\|\bar{u}_t\|^2\end{aligned}\tag{40}$$

and

$$\begin{aligned}\left(f_S\left(v^1\right)-f_S\left(v^2\right),-\Delta\psi\right)&=\left(f_S'\left(\sigma(t)\right)\bar{v},-\Delta\psi\right)\\&=\left(f_S''\left(\sigma(t)\right)\nabla\sigma(t)\bar{v},\nabla\psi\right)+\left(f_S'\left(\sigma(t)\right)\nabla\bar{v},\nabla\psi\right)\\&\geq -MR_0\|\bar{v}\|\|\nabla\psi\|-M\|\nabla\bar{v}\|\|\nabla\psi\|\\&\geq -\frac{\delta_2}{4}\|\nabla\psi\|^2-\frac{8M^2R_0^2}{\delta_2}\|\bar{v}\|^2-\frac{8M^2}{\delta_2}\|\nabla\bar{v}\|^2.\end{aligned}\tag{41}$$

Therefore, together with (36)–(41), it leads to

$$\begin{aligned}&\frac{d}{dt}\big(\|\Delta\phi\|^2+\alpha\big\|\Delta^2\omega_2\big\|^2+\|\nabla\psi\|^2+\beta\|\Delta\omega_4\|^2+2\big(k\big(u^1-v^1\big)^+-k\big(u^2-v^2\big)^+,\Delta^2\omega_2\big)\\&\quad+2\big(f_B'\big(\varphi(t)\big)\bar{u},\Delta^2\omega_2\big)\big)+\varepsilon\alpha\big\|\Delta^2\omega_2\big\|^2+2(\delta_1-\varepsilon)\|\Delta\phi\|^2-2\varepsilon(\delta_1-\varepsilon)\big(\Delta^2\omega_2,\phi\big)\\&\quad+2\varepsilon\beta\|\Delta\omega_4\|^2+2\bigg(\frac{\delta_2}{2}-\varepsilon\bigg)\|\nabla\psi\|^2-2\varepsilon(\delta_2-\varepsilon)(-\Delta\omega_4,\psi)\\&\quad+2\varepsilon\big(k\big(u^1-v^1\big)^+-k\big(u^2-v^2\big)^+,\Delta^2\omega_2\big)+2\varepsilon\big(f_B'\big(\varphi(t)\big)\bar{u},\Delta^2\omega_2\big)\\&\le2\bigg(\frac{8k^2}{\varepsilon\alpha}\|\bar{u}_t\|^2+\frac{8k^2}{\varepsilon\alpha}\|\bar{v}_t\|^2+\frac{8k^2}{\delta_2}\|\nabla\bar{u}\|^2+\frac{8k^2}{\delta_2}\|\nabla\bar{v}\|^2\\&\quad+\frac{8M^2R_0^2}{\varepsilon\alpha}\|\bar{u}\|^2+\frac{8M^2}{\varepsilon\alpha}\|\bar{u}_t\|^2+\frac{8M^2R_0^2}{\delta_2}\|\bar{v}\|^2+\frac{8M^2}{\delta_2}\|\nabla\bar{v}\|^2\bigg).\end{aligned}\tag{42}$$

Furthermore, by Young's inequality and Hölder's inequality, we have

$$\begin{aligned}&\varepsilon\alpha\big\|\Delta^2\omega_2\big\|^2+2(\delta_1-\varepsilon)\|\Delta\phi\|^2-2\varepsilon(\delta_1-\varepsilon)\big(\Delta^2\omega_2,\phi\big)\\&\ge\varepsilon\alpha\big\|\Delta^2\omega_2\big\|^2+2(\delta_1-\varepsilon)\|\Delta\phi\|^2-\frac{2\varepsilon\delta_1}{\lambda}\big\|\Delta^2\omega_2\big\|\|\Delta\phi\|\\&\ge\frac{\varepsilon\alpha}{2}\big\|\Delta^2\omega_2\big\|^2+2\bigg(\delta_1-\varepsilon-\frac{4\varepsilon\delta_1^2}{\lambda^2\alpha}\bigg)\|\Delta\phi\|^2\end{aligned}\tag{43}$$

and

$$\begin{aligned}&2\varepsilon\beta\|\Delta\omega_4\|^2+2\bigg(\frac{\delta_2}{2}-\varepsilon\bigg)\|\nabla\psi\|^2-2\varepsilon(\delta_2-\varepsilon)(-\Delta\omega_4,\psi)\\&\ge2\varepsilon\beta\|\Delta\omega_4\|^2+(\delta_2-2\varepsilon)\|\nabla\psi\|^2-\frac{2\varepsilon\delta_2}{\lambda}\|\Delta\omega_4\|\|\nabla\psi\|\\&\ge\varepsilon\beta\|\Delta\omega_4\|^2+\bigg(\delta_2-2\varepsilon-\frac{4\varepsilon\delta_2^2}{\lambda^2\beta}\bigg)\|\nabla\psi\|^2.\end{aligned}\tag{44}$$

Thus, we can choose ε small enough such that

$$\delta_1-\varepsilon-\frac{4\varepsilon\delta_1^2}{\lambda^2\alpha}\ge\frac{\delta_1}{2},\qquad\delta_2-2\varepsilon-\frac{4\varepsilon\delta_2^2}{\lambda^2\beta}\ge\frac{\delta_2}{2}.$$

And let $\varepsilon_0=\min\{\frac{\varepsilon}{2},\delta_1,\frac{\delta_2}{2}\}$, we conclude from (42) that

$$\begin{aligned}&\frac{d}{dt}\big(\|\Delta\phi\|^2+\alpha\big\|\Delta^2\omega_2\big\|^2+\|\nabla\psi\|^2+\beta\|\Delta\omega_4\|^2+2\big(k\big(u^1-v^1\big)^+-k\big(u^2-v^2\big)^+,\Delta^2\omega_2\big)\\&\quad+\big(f_B'\big(\varphi(t)\big)\bar{u},\Delta^2\omega_2\big)\big)+\varepsilon_0\big(\|\Delta\phi\|^2+\alpha\big\|\Delta^2\omega_2\big\|^2+\|\nabla\psi\|^2+\beta\|\Delta\omega_4\|^2\\&\quad+2\big(k\big(u^1-v^1\big)^+-k\big(u^2-v^2\big)^+,\Delta^2\omega_2\big)+\big(f_B'\big(\varphi(t)\big)\bar{u},\Delta^2\omega_2\big)\big)\\&\le2\bigg(\frac{8k^2}{\varepsilon\alpha}\|\bar{u}_t\|^2+\frac{8k^2}{\varepsilon\alpha}\|\bar{v}_t\|^2+\frac{8k^2}{\delta_2}\|\nabla\bar{u}\|^2+\frac{8k^2}{\delta_2}\|\nabla\bar{v}\|^2\\&\quad+\frac{8M^2R_0^2}{\varepsilon\alpha}\|\bar{u}\|^2+\frac{8M^2}{\varepsilon\alpha}\|\bar{u}_t\|^2+\frac{8M^2R_0^2}{\delta_2}\|\bar{v}\|^2+\frac{8M^2}{\delta_2}\|\nabla\bar{v}\|^2\bigg).\end{aligned}\tag{45}$$

Therefore, we arrive at

$$\begin{aligned}
&\frac{d}{dt}\Bigg(\bigg\|\sqrt{\frac{\alpha}{2}}\Delta^2\omega_2+\frac{\sqrt{2}k}{\sqrt{\alpha}}\big((u^1-v^1)^+-(u^2-v^2)^+\big)\bigg\|^2+\bigg\|\sqrt{\frac{\alpha}{2}}\Delta^2\omega_2+\frac{\sqrt{2}}{\sqrt{\alpha}}f_B'\big(\varphi(t)\big)\bar{u}\bigg\|^2\\
&\quad+\|\Delta\phi\|^2+\beta\|\Delta\omega_4\|^2+\|\nabla\psi\|^2\Bigg)\\
&\quad+\varepsilon_0\Bigg(\bigg\|\sqrt{\frac{\alpha}{2}}\Delta^2\omega_2+\frac{\sqrt{2}k}{\sqrt{\alpha}}\big((u^1-v^1)^+-(u^2-v^2)^+\big)\bigg\|^2\\
&\quad+\bigg\|\sqrt{\frac{\alpha}{2}}\Delta^2\omega_2+\frac{\sqrt{2}}{\sqrt{\alpha}}f_B'\big(\varphi(t)\big)\bar{u}\bigg\|^2+\|\Delta\phi\|^2+\beta\|\Delta\omega_4\|^2+\|\nabla\psi\|^2\Bigg)\\
&\le\bigg(\frac{16k^2}{\varepsilon\alpha}+\frac{16M^2}{\varepsilon\alpha}\bigg)\|\bar{u}_t\|^2+\frac{16k^2}{\varepsilon\alpha}\|\bar{v}_t\|^2+\frac{16k^2}{\delta_2}\|\nabla\bar{u}\|^2+\bigg(\frac{16k^2}{\delta_2}+\frac{16M^2}{\delta_2}\bigg)\|\nabla\bar{v}\|^2\\
&\quad+\frac{16M^2R_0^2}{\varepsilon\alpha}\|\bar{u}\|^2+\frac{16M^2R_0^2}{\delta_2}\|\bar{v}\|^2\\
&\quad+\frac{4k^2}{\alpha}\int_\Omega\big((u^1-v^1)^+-(u^2-v^2)^+\big)\big((u^1-v^1)_t^+-(u^2-v^2)_t^+\big)\,dx\\
&\quad+\frac{4}{\alpha}\big(f_B''(\varphi)\varphi_t(t)\bar{u},f_B'(\varphi)\bar{u}\big)+\frac{4}{\alpha}\big(f_B'(\varphi)\bar{u}_t,f_B'(\varphi)\bar{u}\big)\\
&\quad+\frac{2\varepsilon_0k^2}{\alpha}\big\|(u^1-v^1)^+-(u^2-v^2)^+\big\|^2+\frac{2\varepsilon_0}{\alpha}\big\|f_B'(\varphi)\bar{u}\big\|^2.
\end{aligned}\tag{46}$$

Moreover, by exploiting conditions (12), (39) and Young's inequality, as well as Hölder's inequality, we have

$$\begin{aligned}
&\frac{4k^2}{\alpha}\int_\Omega\big((u^1-v^1)^+-(u^2-v^2)^+\big)\big((u^1-v^1)_t^+-(u^2-v^2)_t^+\big)\,dx\\
&\le\frac{4k^2}{\alpha}\big\|(u^1-v^1)^+-(u^2-v^2)^+\big\|\big\|(u^1-v^1)_t^+-(u^2-v^2)_t^+\big\|\\
&\le\frac{4k^2}{\alpha}\|\bar{u}-\bar{v}\|\|\bar{u}_t-\bar{v}_t\|\\
&\le\frac{2k^2}{\alpha}\|\bar{u}-\bar{v}\|^2+\frac{2k^2}{\alpha}\|\bar{u}_t-\bar{v}_t\|^2\\
&\le\frac{4k^2}{\alpha}\|\bar{u}\|^2+\frac{4k^2}{\alpha}\|\bar{v}\|^2+\frac{4k^2}{\alpha}\|\bar{u}_t\|^2+\frac{4k^2}{\alpha}\|\bar{v}_t\|^2
\end{aligned}\tag{47}$$

and

$$\frac{4}{\alpha}\big(f_B''(\varphi)\varphi_t(t)\bar{u},f_B'(\varphi)\bar{u}\big)\le\frac{4}{\alpha}M^2R_0\|\bar{u}\|^2,\tag{48}$$

and

$$\frac{4}{\alpha}\big(f_B'(\varphi)\bar{u}_t,f_B'(\varphi)\bar{u}\big)\le\frac{4}{\alpha}M^2\|\bar{u}_t\|\|\bar{u}\|\le\frac{2M^2}{\alpha}\|\bar{u}_t\|^2+\frac{2M^2}{\alpha}\|\bar{u}\|^2,\tag{49}$$

and

$$\frac{2\varepsilon_0k^2}{\alpha}\big\|(u^1-v^1)^+-(u^2-v^2)^+\big\|^2\le\frac{2\varepsilon_0k^2}{\alpha}\|\bar{u}-\bar{v}\|^2\le\frac{4\varepsilon_0k^2}{\alpha}\|\bar{u}\|^2+\frac{4\varepsilon_0k^2}{\alpha}\|\bar{v}\|^2,\tag{50}$$

$$\frac{2\varepsilon_0}{\alpha}\left\|f_B'(\varphi)\bar{u}\right\|^2 \le \frac{2\varepsilon_0 M^2}{\alpha}\|\bar{u}\|^2. \tag{51}$$

Combining with (46)–(51), it leads to

$$\begin{aligned}
&\frac{d}{dt}\Bigg(\left\|\sqrt{\frac{\alpha}{2}}\Delta^2\omega_2 + \frac{\sqrt{2}k}{\sqrt{\alpha}}\big((u^1-v^1)^+ - (u^2-v^2)^+\big)\right\|^2 + \left\|\sqrt{\frac{\alpha}{2}}\Delta^2\omega_2 + \frac{\sqrt{2}}{\sqrt{\alpha}}f_B'\big(\varphi(t)\big)\bar{u}\right\|^2 \\
&\quad + \|\Delta\phi\|^2 + \beta\|\Delta\omega_4\|^2 + \|\nabla\psi\|^2\Bigg) \\
&\quad + \varepsilon_0\Bigg(\left\|\sqrt{\frac{\alpha}{2}}\Delta^2\omega_2 + \frac{\sqrt{2}k}{\sqrt{\alpha}}\big((u^1-v^1)^+ - (u^2-v^2)^+\big)\right\|^2 \\
&\quad + \left\|\sqrt{\frac{\alpha}{2}}\Delta^2\omega_2 + \frac{\sqrt{2}}{\sqrt{\alpha}}f_B'\big(\varphi(t)\big)\bar{u}\right\|^2 + \|\Delta\phi\|^2 + \beta\|\Delta\omega_4\|^2 + \|\nabla\psi\|^2\Bigg) \\
&\le \left(\frac{16k^2}{\varepsilon\alpha} + \frac{16M^2}{\varepsilon\alpha} + \frac{4k^2}{\alpha} + \frac{2M^2}{\alpha}\right)\|\bar{u}_t\|^2 + \left(\frac{16k^2}{\varepsilon\alpha} + \frac{4k^2}{\alpha}\right)\|\bar{v}_t\|^2 \\
&\quad + \left(\frac{16k^2}{\delta_2} + \frac{16M^2}{\delta_2}\right)\|\nabla\bar{v}\|^2 \\
&\quad + \frac{16k^2}{\delta_2}\|\nabla\bar{u}\|^2 + \left(\frac{16M^2R_0^2}{\varepsilon\alpha} + \frac{4k^2}{\alpha} + \frac{4M^2R_0}{\alpha} + \frac{2M^2}{\alpha} + \frac{4\varepsilon_0 k^2}{\alpha} + \frac{4\varepsilon_0 M^2}{\alpha}\right)\|\bar{u}\|^2 \\
&\quad + \left(\frac{16M^2R_0^2}{\delta_2} + \frac{4k^2}{\alpha} + \frac{4\varepsilon_0 k^2}{\alpha}\right)\|\bar{v}\|^2 \\
&\le \left(\frac{16k^2}{\varepsilon\alpha} + \frac{16M^2}{\varepsilon\alpha} + \frac{4k^2}{\alpha} + \frac{2M^2}{\alpha}\right)\|\bar{u}_t\|^2 + \left(\frac{16k^2}{\varepsilon\alpha} + \frac{4k^2}{\alpha}\right)\|\bar{v}_t\|^2 \\
&\quad + \left(\frac{16k^2}{\delta_2} + \frac{16M^2R_0^2}{\varepsilon\alpha} + \frac{4k^2}{\alpha} + \frac{4M^2R_0}{\alpha} + \frac{2M^2}{\alpha} + \frac{4\varepsilon_0 k^2}{\alpha} + \frac{4\varepsilon_0 M^2}{\alpha}\right)/\lambda^2\|\Delta\bar{u}\|^2 \\
&\quad + \left(\frac{16k^2}{\delta_2} + \frac{16M^2}{\delta_2} + \left(\frac{16M^2R_0^2}{\delta_2} + \frac{4k^2}{\alpha} + \frac{4\varepsilon_0 k^2}{\alpha}\right)/\lambda^2\right)\|\nabla\bar{v}\|^2.
\end{aligned} \tag{52}$$

Let

$$\begin{aligned}
\Lambda = \max\Bigg\{&\frac{16k^2}{\varepsilon\alpha} + \frac{16M^2}{\varepsilon\alpha} + \frac{4k^2}{\alpha} + \frac{2M^2}{\alpha}, \Bigg(\frac{16k^2}{\delta_2} + \frac{16M^2R_0^2}{\varepsilon\alpha} + \frac{4k^2}{\alpha} + \frac{4M^2R_0}{\alpha} \\
&+ \frac{2M^2}{\alpha} + \frac{4\varepsilon_0 k^2}{\alpha} + \frac{4\varepsilon_0 M^2}{\alpha}\Bigg) \Big/ \lambda^2\alpha, \left(\frac{16k^2}{\delta_2} + \frac{16M^2}{\delta_2}\right) \Big/ \beta \\
&+ \left(\frac{16M^2R_0^2}{\delta_2} + \frac{4k^2}{\alpha} + \frac{4\varepsilon_0 k^2}{\alpha}\right) \Big/ \lambda^2\beta)\Bigg\}.
\end{aligned}$$

We can deduce from (52) that

$$\begin{aligned}
&\frac{d}{dt}\Bigg(\left\|\sqrt{\frac{\alpha}{2}}\Delta^2\omega_2 + \frac{\sqrt{2}k}{\sqrt{\alpha}}\big((u^1-v^1)^+ - (u^2-v^2)^+\big)\right\|^2 + \left\|\sqrt{\frac{\alpha}{2}}\Delta^2\omega_2 + \frac{\sqrt{2}}{\sqrt{\alpha}}f_B'\big(\varphi(t)\big)\bar{u}\right\|^2 \\
&\quad + \|\Delta\phi\|^2 + \beta\|\Delta\omega_4\|^2 + \|\nabla\psi\|^2\Bigg) \\
&\le \Lambda\big(\alpha\|\Delta\bar{u}\|^2 + \beta\|\nabla\bar{v}\|^2 + \|\bar{u}_t\|^2 + \|\bar{v}_t\|^2\big) = \Lambda\left\|\bar{z}(t)\right\|_0^2 \le \Lambda e^{Kt}\|z_1 - z_2\|_0^2.
\end{aligned} \tag{53}$$

Integrating (53) over $(0,t_*)$, we have that

$$\begin{aligned}
&\left\|\sqrt{\frac{\alpha}{2}}\Delta^2\omega_2(t_*)+\frac{\sqrt{2}k}{\sqrt{\alpha}}\big(\big(u^1(t_*)-v^1(t_*)\big)^+-\big(u^2(t_*)-v^2(t_*)\big)^+\big)\right\|^2\\
&\quad+\left\|\sqrt{\frac{\alpha}{2}}\Delta^2\omega_2(t_*)+\frac{\sqrt{2}}{\sqrt{\alpha}}f_B'\big(\varphi(t_*)\big)\bar{u}(t_*)\right\|^2\\
&\quad+\|\Delta\phi(t_*)\|^2+\beta\|\Delta\omega_4(t_*)\|^2+\|\nabla\psi(t_*)\|^2\\
&\le\left\|\sqrt{\frac{\alpha}{2}}\Delta^2\omega_2(0)+\frac{\sqrt{2}k}{\sqrt{\alpha}}\big(\big(u^1(0)-v^1(0)\big)^+-\big(u^2(0)-v^2(0)\big)^+\big)\right\|^2\\
&\quad+\left\|\sqrt{\frac{\alpha}{2}}\Delta^2\omega_2(0)+\frac{\sqrt{2}}{\sqrt{\alpha}}f_B'\big(\varphi(0)\big)\bar{u}(0)\right\|^2+\|\Delta\phi(0)\|^2+\beta\|\Delta\omega_4(0)\|^2+\|\nabla\psi(0)\|^2\\
&\quad+\int_0^{t_*}\Lambda e^{Kt}\|z_1-z_2\|_0^2\,dt \qquad (54)\\
&\le\frac{\Lambda}{K}\big(e^{Kt_*}-1\big)\|z_1-z_2\|_0^2+\alpha\|\Delta^2\omega_2(0)\|^2\\
&\quad+\frac{4k^2}{\alpha}\big\|\big(u^1(0)-v^1(0)\big)^+-\big(u^2(0)-v^2(0)\big)^+\big\|^2\\
&\quad+\alpha\|\Delta^2\omega_2(0)\|^2+\frac{4}{\alpha}\big\|f_B'\big(\varphi(0)\big)\bar{u}(0)\big\|^2+\|\Delta\phi(0)\|^2+\beta\|\Delta\omega_4(0)\|^2+\|\nabla\psi(0)\|^2\\
&\le\frac{\Lambda}{K}\big(e^{Kt_*}-1\big)\|z_1-z_2\|_0^2+\frac{4k^2}{\alpha}\|\bar{u}(0)-\bar{v}(0)\|^2+\frac{4}{\alpha}\big\|f_B'\big(\varphi(0)\big)\bar{u}(0)\big\|^2\\
&\le\frac{\Lambda}{K}\big(e^{Kt_*}-1\big)\|z_1-z_2\|_0^2+\frac{8k^2}{\alpha}\big(\|\bar{u}(0)\|^2+\|\bar{v}(0)\|^2\big)+\frac{4M^2}{\alpha}\|\bar{u}(0)\|^2\\
&\le\frac{\Lambda}{K}\big(e^{Kt_*}-1\big)+\left(\frac{8k^2}{\alpha}+\frac{4M^2}{\alpha}\right)/\lambda^2\|\Delta\bar{u}(0)\|^2+\frac{8k^2}{\alpha\lambda^2}\|\nabla\bar{v}(0)\|^2\\
&\le\frac{\Lambda}{K}\big(e^{Kt_*}-1\big)+\left(\frac{8k^2}{\alpha}+\frac{4M^2}{\alpha}\right)/\alpha\lambda^2\|z_1-z_2\|_0^2+\frac{8k^2}{\alpha\beta\lambda^2}\|z_1-z_2\|_0^2\\
&\le C_*\|z_1-z_2\|_0^2, \qquad (55)
\end{aligned}$$

where $C_*=\frac{\Lambda}{K}(e^{Kt_*}-1)+\frac{8k^2+4M^2}{\alpha^2\lambda^2}+\frac{8k^2}{\alpha\beta\lambda^2}$. Applying (12), Hölder's inequality, and Cauchy's inequality as well as Lemma 3, we conclude from (54) that

$$\alpha\|\Delta^2\omega_2(t_*)\|^2+\beta\|\Delta\omega_4(t_*)\|^2+\|\Delta\phi(t_*)\|^2+\|\nabla\psi(t_*)\|^2\le C_*\|z_1-z_2\|_0^2,$$

namely,

$$\|\bar{z}_*\|_1^2\le C_*\|z_1-z_2\|_0^2.$$

This completes the proof of Lemma 9. □

Our main result reads as follows.

Theorem 10 *Under conditions (F1)–(F2), the semigroup $S(t)$ acting on E_0 possesses an exponential attractor $\mathcal{E}$.*

Proof Lemma 8, Lemma 9, and Theorem 2 imply the existence of an exponential attractor. □

Funding
Not applicable.

Availability of data and materials
Not applicable.

Declarations

Ethics approval and consent to participate
Not applicable.

Competing interests
The authors declare no competing interests.

Author contributions
This paper is completed by myself.

Received: 27 June 2023 Accepted: 13 November 2023 Published online: 22 November 2023

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