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A new weighted fractional operator with respect to another function via a new modified generalized Mittag–Leffler law

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Abstract

In this paper, new generalized weighted fractional derivatives with respect to another function are derived in the sense of Caputo and Riemann–Liouville involving a new modified version of a generalized Mittag–Leffler function with three parameters, as well as their corresponding fractional integrals. In addition, several new and existing operators of nonsingular kernels are obtained as special cases of our operator. Many important properties related to our new operator are introduced, such as a series version involving Riemann–Liouville fractional integrals, weighted Laplace transforms with respect to another function, etc. Finally, an example is given to illustrate the effectiveness of the new results.

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1 Introduction

Recently, fractional calculus has developed as an exciting and widely appreciated subject of research. There are various research papers referring to fractional differential equations and inclusions with a variety of boundary conditions published in this topic. These researches include the theoretical evolvement, numerical strategies, as well as comprehensive applications in the mathematical modeling of biological, physical, and engineering phenomena, we refer readers to the works [1–12] and references cited therein.

In fractional calculus, there appeared two types of nonlocal operators, the first type is with a singular kernel such as Caputo [3], Riemann–Liouville [3], Katugampola [13], Hadamard [3], and Hilfer [2] operators. Also, these operators were extended to weighted fractional operators with respect to another function [14–17]. In fact, the singularity of kernels caused problems sometimes in numerical analysis. For this reason, the second type of nonlocal operators appeared that created a new fractional operator with nonsingular kernel by Caputo and Fabrizio [18] in 2015. Then, Atangana and Baleanu (AB) [19] introduced a new nonsingular fractional operator via a function of Mittag–Leffler. These two operators attracted the attention of many researchers to study various problems with

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applications [20]. Afterwards, Fernandez and Baleanu [21] generalized the AB operator to differointegration with respect to another function, and recently Abdeljawad et al. [22] extended this to higher fractional orders. In 2018, Abdeljawad and Baleanu [23] derived a new AB operator with the generalized Mittag–Leffler kernels of three parameters. In 2019, Al-Refai and Jarrah [24] defined the weighted Caputo–Fabrizio–Caputo (CFC) fractional operator respect to another function. In 2020, the weighted Atangana–Baleanu–Caputo (ABC) fractional derivative was presented by Al-Refai [25]. Also, in 2020 Hattaf [26], proposed a new fractional operator of a nonsingular kernel in the Caputo and Riemann–Liouville sense that is generalized to some definitions existing in the literature.

Motivated by the above various operators of nonsingular kernels, we aim in this paper to present new generalized weighted fractional derivatives and integrals with respect to another function, which cover all existing forms of nonlocal, nonsingular fractional derivatives and integrals. For this, in Section 2 some definitions and basics are presented. Section 3 is concerned with new generalized weighted fractional derivatives with respect to another function in the sense of Caputo and Riemann–Liouville, as well as their corresponding fractional integrals. Indeed, this section proves many propositions and lemmas related to our fractional derivatives and integrals.

2 Background materials

In this section, we present some background materials, which support us in order to reach the desired results.

Definition 2.1 ([14]) Let $\phi : [\iota, \tau] \to \mathbb{R}$ be an increasing function on $[\iota, \tau]$. For $\mu > 0$, the μth left-sided weighted ϕ-Riemann–Liouville fractional integral for an integrable function $\mathfrak{g} : [\iota, \tau] \to \mathbb{R}$ with respect to another function $\phi(\mathrm{u})$, is given by

$$\left({}^{R}\mathfrak{I}^{\mu}_{\iota,\omega,\phi}\mathfrak{g}\right)(\mathrm{u}) = \frac{1}{\omega(\mathrm{u})\Gamma(\mu)} \int_{\iota}^{\mathrm{u}} \left(\phi(\mathrm{u}) - \phi(\mathrm{v})\right)^{\mu-1} \phi'(\mathrm{v})(\omega\mathfrak{g})(\mathrm{v})d\mathrm{v}, \tag{2.1}$$

where $\Gamma(\mu) = \int_0^{+\infty} e^{-\mathrm{u}}\mathrm{u}^{\mu-1}d\mathrm{u}, \mu > 0$. Further,

$${}^{R}\mathfrak{I}^{\mu}_{\iota,\omega,\phi}\left[\frac{[\phi(\mathrm{u}) - \phi(\iota)]^{\varrho}}{\omega(\mathrm{u})}\right] = \frac{\Gamma(\varrho + 1)}{\Gamma(\mu + \varrho + 1)} \frac{[\phi(\mathrm{u}) - \phi(\iota)]^{\mu+\varrho}}{\omega(\mathrm{u})}. \tag{2.2}$$

Lemma 2.2 ([14]) *Let $\mathbb{L}_{\omega,\phi}$ be the weighted Laplace transform with respect to another function ϕ, and $\mathfrak{g}$ be a piecewise-continuous function on $[\iota, \tau]$. Then,*

(i) $\mathbb{L}_{\omega,\phi}\{({}^{R}\mathfrak{I}^{\mu}_{\iota,\omega,\phi}\mathfrak{g})(\mathrm{u})\}(s) = s^{-\mu}\mathbb{L}_{\omega,\phi}\{\mathfrak{g}(\mathrm{u})\}(s)$;

(ii) $\mathbb{L}_{\omega,\phi}\{D^{1}_{\omega,\phi}\mathfrak{g}(\mathrm{u})\}(s) = s\mathbb{L}_{\omega,\phi}\{\mathfrak{g}(\mathrm{u})\}(s) - (\omega\mathfrak{g})(\iota)$;

(iii) $\mathbb{L}_{\omega,\phi}\{\omega^{-1}(\mathrm{u})(\phi(\mathrm{u}) - \phi(\iota))^{\eta-1}\}(s) = \frac{\Gamma(\eta)}{s^{\eta}}, \eta > 1$.

Definition 2.3 ([23]) The generalized Mittag–Leffler function of three parameters is defined as

$$\mathbb{E}^{\sigma}_{\beta,\mu}(\mathrm{r}) = \sum_{k=0}^{\infty} (\sigma)_k \frac{\mathrm{r}^k}{k!\Gamma(\beta k + \mu)}, \tag{2.3}$$

where $\mathrm{r} \in \mathbb{C}$ and $(\sigma)_k$ is the Pochhammer symbol, such that $(\sigma)_k = \sigma(\sigma + 1)\dots(\sigma + k - 1)$, and $(\sigma)_k = \frac{\Gamma(\sigma+k)}{\Gamma(\sigma)}, (1)_k = k!$.

For simplicity, we define the following modified version of the generalized Mittag–Leffler function as follows:

$$\mathbb{E}^{\gamma,\sigma}_{\beta,\mu}(\lambda,\mathrm{r}) = r^{\mu-1}\mathbb{E}^{\sigma}_{\beta,\mu}\big(\lambda\mathrm{r}^{\gamma}\big) = \sum_{k=0}^{\infty}(\sigma)_k \frac{\lambda^k \mathrm{r}^{k\gamma+\mu-1}}{k!\Gamma(\beta k+\mu)}, \tag{2.4}$$

such that $\mathrm{r}\in\mathbb{C}, 0\neq\lambda\in\mathbb{R}, \gamma,\mu,\sigma>0$.

3 Main results

In this section, we will introduce new weighted generalized fractional derivatives and integrals with respect to another function.

3.1 New fractional derivatives

First, we will define a generalized weighted fractional derivative in the Riemann–Liouville sense with respect to another function.

Definition 3.1 Consider $\phi:[\iota,\tau]\to\mathbb{R}$ to be an increasing function with $\phi'(\mathrm{u})\neq 0$, $\forall\mathrm{u}\in[\iota,\tau]$. Let $\alpha\in(0,1], \beta,\gamma>0, \mathrm{Re}(\mu)>0, \sigma\in\mathbb{R}$, and $\mathfrak{g}(\mathrm{u})$ is continuous on $[\iota,\tau]$. The new αth left-sided generalized weighted fractional derivative in the Riemann–Liouville sense of a function $\mathfrak{g}$ with respect to another function $\phi(\mathrm{u})$ is given by

$$\begin{aligned}&\big({}^{R}\mathfrak{D}^{\alpha,\beta,\gamma,\mu}_{\iota,\sigma,\omega,\phi}\,\mathfrak{g}\big)(\mathrm{u})\\&\quad=\frac{\Delta(\alpha)}{1-\alpha}\frac{1}{\omega(\mathrm{u})\phi'(\mathrm{u})}\frac{d}{d\mathrm{u}}\int_{\iota}^{\mathrm{u}}\phi'(\mathrm{v})\mathbb{E}^{\gamma,\sigma}_{\beta,\mu}\big(\lambda_\alpha,\big(\phi(\mathrm{u})-\phi(\mathrm{v})\big)\big)(\omega\mathfrak{g})(\mathrm{v})\,d\mathrm{v},\end{aligned} \tag{3.1}$$

where $\Delta(\alpha)$ is the normalization function with $\Delta(0)=\Delta(1)=1$, $\omega\in C^1(\iota,\tau)$ is a weight function such that $\omega>0$ on $[\iota,\tau]$, $\lambda_\alpha=\frac{-\alpha}{1-\alpha}$, and $\mathbb{E}^{\gamma,\sigma}_{\beta,\mu}$ is a modified version of the generalized Mittag–Leffler function defined in (2.4).

Remark 3.2 Note that for any specific values for the parameters $\alpha,\beta,\gamma,\mu,\sigma,\phi,\omega$, we obtain new and known definitions in the literature as follows:

1. If $\mu=\sigma=1$, and $\phi(\mathrm{u})=\mathrm{u}$ in the new Def. 3.1, we obtain the Hattaf–Riemann–Liouville fractional derivative [26], which is given by

$$\big({}^{R}\mathfrak{D}^{\alpha,\beta,\gamma,1}_{\iota,1,\omega,\mathrm{u}}\,\mathfrak{g}\big)(\mathrm{u})=\frac{\Delta(\alpha)}{1-\alpha}\frac{1}{\omega(\mathrm{u})}\frac{d}{d\mathrm{u}}\int_{\iota}^{\mathrm{u}}\mathbb{E}^{\gamma,1}_{\beta,1}\big(\lambda_\alpha,(\mathrm{u}-\mathrm{v})\big)(\omega\mathfrak{g})(\mathrm{v})\,d\mathrm{v}.$$

2. If $\alpha=\beta=\gamma, \phi(\mathrm{u})=\mathrm{u}$, and $\omega(\mathrm{u})=1$ in the new Def. 3.1, we obtain the generalized *ABR* fractional derivative with the generalized Mittag–Leffler function [23], which is given by

$$\big({}^{R}\mathfrak{D}^{\alpha,\alpha,\alpha,\mu}_{\iota,\sigma,1,\mathrm{u}}\,\mathfrak{g}\big)(\mathrm{u})=\frac{\Delta(\alpha)}{1-\alpha}\frac{d}{d\mathrm{u}}\int_{\iota}^{\mathrm{u}}\mathbb{E}^{\alpha,\sigma}_{\alpha,\mu}\big(\lambda_\alpha,(\mathrm{u}-\mathrm{v})\big)\mathfrak{g}(\mathrm{v})\,d\mathrm{v}.$$

3. If $\alpha=\beta=\gamma$, $\mu=\sigma=1$, and $\omega(\mathrm{u})=1$ in the new Def. 3.1, we obtain the *ABR* fractional derivative of $\mathfrak{g}$ with respect to another function $\phi(\mathrm{u})$ [21], which is given by

$$\big({}^{R}\mathfrak{D}^{\alpha,\alpha,\alpha,1}_{\iota,1,1,\phi}\,\mathfrak{g}\big)(\mathrm{u})=\frac{\Delta(\alpha)}{1-\alpha}\frac{1}{\phi'(\mathrm{u})}\frac{d}{d\mathrm{u}}\int_{\iota}^{\mathrm{u}}\phi'(\mathrm{v})\mathbb{E}^{\alpha,1}_{\alpha,1}\big(\lambda_\alpha,\big(\phi(\mathrm{u})-\phi(\mathrm{v})\big)\big)\mathfrak{g}(\mathrm{v})\,d\mathrm{v}.$$

4. If $\alpha = \beta = \gamma$, $\mu = \sigma = 1, \phi(\mathrm{u}) = \mathrm{u}$, and $\omega(\mathrm{u}) = 1$ in the new Def. 3.1, we obtain the *ABR* fractional derivative [19], which is given by

$$\left({}^{R}\mathfrak{D}^{\alpha,\alpha,\alpha,1}_{\iota,1,1,\mathrm{u}}\,\mathfrak{g}\right)(\mathrm{u}) = \frac{\Delta(\alpha)}{1-\alpha}\frac{d}{d\mathrm{u}}\int_{\iota}^{\mathrm{u}} \mathbb{E}^{\alpha,1}_{\alpha,1}\left(\lambda_\alpha,(\mathrm{u}-\mathrm{v})\right)\mathfrak{g}(\mathrm{v})\,d\mathrm{v}.$$

5. If $\beta = \gamma = \mu = \sigma = 1$, in the new Def. 3.3, we obtain the weighted *CFR* fractional derivative of $\mathfrak{g}$ with respect to another function $\phi(\mathrm{u})$ [24], which is given by

$$\left({}^{R}\mathfrak{D}^{\alpha,1,1,1}_{\iota,1,\omega,\phi}\,\mathfrak{g}\right)(\mathrm{u}) = \frac{\Delta(\alpha)}{1-\alpha}\frac{1}{\omega(\mathrm{u})\phi'(\mathrm{u})}\frac{d}{d\mathrm{u}}\int_{\iota}^{\mathrm{u}} \phi'(\mathrm{v})\exp\left(\lambda_\alpha\left(\phi(\mathrm{u})-\phi(\mathrm{v})\right)\right)(\omega\mathfrak{g})(\mathrm{v})\,d\mathrm{v}.$$

6. If $\beta = \gamma = \mu = \sigma = 1$, and $\omega(\mathrm{u}) = 1$ in the new Def. 3.1, we obtain the *CFR* fractional derivative of $\mathfrak{g}$ with respect to another function $\phi(\mathrm{u})$ [24], which is given by

$$\left({}^{R}\mathfrak{D}^{\alpha,1,1,1}_{\iota,1,1,\phi}\,\mathfrak{g}\right)(\mathrm{u}) = \frac{\Delta(\alpha)}{1-\alpha}\frac{1}{\phi'(\mathrm{u})}\frac{d}{d\mathrm{u}}\int_{\iota}^{\mathrm{u}} \phi'(\mathrm{v})\exp\left(\lambda_\alpha\left(\phi(\mathrm{u})-\phi(\mathrm{v})\right)\right)\mathfrak{g}(\mathrm{v})\,d\mathrm{v}.$$

7. If $\beta = \gamma = \mu = \sigma = 1, \phi(\mathrm{u}) = \mathrm{u}$, and $\omega(\mathrm{u}) = 1$ in the new Def. 3.1, we obtain the *CFR* fractional derivative [18], which is given by

$$\left({}^{R}\mathfrak{D}^{\alpha,1,1,1}_{\iota,1,1,\mathrm{u}}\,\mathfrak{g}\right)(\mathrm{u}) = \frac{\Delta(\alpha)}{1-\alpha}\frac{d}{d\mathrm{u}}\int_{\iota}^{\mathrm{u}} \exp\left(\lambda_\alpha(\mathrm{u}-\mathrm{v})\right)\mathfrak{g}(\mathrm{v})\,d\mathrm{v}.$$

8. The limiting process $\alpha,\beta,\gamma,\mu,\sigma \to 1, \phi(\mathrm{u}) = \mathrm{u}$, and $\omega(\mathrm{u}) = 1$, in the new Def. 3.1, we obtain the first ordinary derivative. Furthermore, the new Def. 3.1, has a singular kernel for $\mu \in (0,1)$.

Now, we define a generalized weighted fractional derivative in the Caputo sense of a function $\mathfrak{g}$ with respect to another function $\phi(\mathrm{u})$.

Definition 3.3 Consider $\phi : [\iota,\tau] \to \mathbb{R}$ to be an increasing function on $[\iota,\tau]$. Let $\alpha \in (0,1], \beta,\gamma > 0, \mathrm{Re}(\mu) > 0, \sigma \in \mathbb{R}$ and $\mathfrak{g} \in \mathcal{H}^1(\iota,\tau)$. The new αth left-sided generalized weighted fractional derivative in the Caputo sense of a function $\mathfrak{g}$ with respect to another function $\phi(\mathrm{u})$ is given by

$$\left({}^{C}\mathfrak{D}^{\alpha,\beta,\gamma,\mu}_{\iota,\sigma,\omega,\phi}\,\mathfrak{g}\right)(\mathrm{u}) = \frac{\Delta(\alpha)}{1-\alpha}\frac{1}{\omega(\mathrm{u})}\int_{\iota}^{\mathrm{u}} \mathbb{E}^{\gamma,\sigma}_{\beta,\mu}\left(\lambda_\alpha,\left(\phi(\mathrm{u})-\phi(\mathrm{v})\right)\right)(\omega\mathfrak{g})'(\mathrm{v})\,d\mathrm{v}, \tag{3.2}$$

where $\Delta(\alpha)$ is the normalization function with $\Delta(0) = \Delta(1) = 1$, $\omega \in C^1(\iota,\tau)$ is a weight function such that $\omega,\omega' > 0$ on $[\iota,\tau]$, $\lambda_\alpha = \frac{-\alpha}{1-\alpha}$, and $\mathbb{E}^{\gamma,\sigma}_{\beta,\mu}$ is the modified version of the generalized Mittag–Leffler function defined in (2.4).

Remark 3.4 Note that for any specific values for the parameters $\alpha,\beta,\gamma,\mu,\sigma,\phi,\omega$, we obtain new and known definitions in the literature as follows:

1. If $\mu = \sigma = 1$ and $\phi(\mathrm{u}) = \mathrm{u}$ in the new Def. 3.3, we obtain the Hattaf–Caputo fractional derivative [26], which is given by

$$\left({}^{C}\mathfrak{D}^{\alpha,\beta,\gamma,1}_{\iota,1,\omega,\mathrm{u}}\,\mathfrak{g}\right)(\mathrm{u}) = \frac{\Delta(\alpha)}{1-\alpha}\frac{1}{\omega(\mathrm{u})}\int_{\iota}^{\mathrm{u}} \mathbb{E}^{\gamma,1}_{\beta,1}\left(\lambda_\alpha,(\mathrm{u}-\mathrm{v})\right)(\omega\mathfrak{g})'(\mathrm{v})\,d\mathrm{v}, \quad \mathrm{u} \in [\iota,\tau].$$

2. If $\alpha = \beta = \gamma, \phi(\mathrm{u}) = \mathrm{u}$, and $\omega(\mathrm{u}) = 1$ in the new Def. 3.3, we obtain the generalized *ABC* fractional derivative with the generalized Mittag–Leffler function [23], which is given by

$$\left({}^{C}\mathfrak{D}^{\alpha,\alpha,\alpha,\mu}_{\iota,\sigma,1,\mathrm{u}}\,\mathfrak{g}\right)(\mathrm{u}) = \frac{\Delta(\alpha)}{1-\alpha}\int_{\iota}^{\mathrm{u}} \mathbb{E}^{\alpha,\sigma}_{\alpha,\mu}\left(\lambda_\alpha, (\mathrm{u}-\mathrm{v})\right)\mathfrak{g}'(\mathrm{v})\,d\mathrm{v}, \quad \mathrm{u}\in[\iota,\tau].$$

3. If $\alpha = \beta = \gamma$, $\mu = \sigma = 1$, and $\phi(\mathrm{u}) = \mathrm{u}$ in the new Def. 3.3, we obtain the weighted *ABC* fractional derivative [25], which is given by

$$\left({}^{C}\mathfrak{D}^{\alpha,\alpha,\alpha,1}_{\iota,1,\omega,\mathrm{u}}\,\mathfrak{g}\right)(\mathrm{u}) = \frac{\Delta(\alpha)}{1-\alpha}\frac{1}{\omega(\mathrm{u})}\int_{\iota}^{\mathrm{u}} \mathbb{E}^{\alpha,1}_{\alpha,1}\left(\lambda_\alpha, (\mathrm{u}-\mathrm{v})\right)(\omega\mathfrak{g})'(\mathrm{v})\,d\mathrm{v}, \quad \mathrm{u}\in[\iota,\tau].$$

4. If $\alpha = \beta = \gamma$, $\mu = \sigma = 1$, and $\omega(\mathrm{u}) = 1$ in the new Def. 3.3, we obtain the *ABC* fractional derivative of $\mathfrak{g}$ with respect to another function $\phi(\mathrm{u})$ [21], which is given by

$$\left({}^{C}\mathfrak{D}^{\alpha,\alpha,\alpha,1}_{\iota,1,1,\phi}\,\mathfrak{g}\right)(\mathrm{u}) = \frac{\Delta(\alpha)}{1-\alpha}\int_{\iota}^{\mathrm{u}} \mathbb{E}^{\alpha,1}_{\alpha,1}\left(\lambda_\alpha, \left(\phi(\mathrm{u})-\phi(\mathrm{v})\right)\right)\mathfrak{g}'(\mathrm{v})\,d\mathrm{v}, \quad \mathrm{u}\in[\iota,\tau].$$

5. If $\alpha = \beta = \gamma$, $\mu = \sigma = 1, \phi(\mathrm{u}) = \mathrm{u}$, and $\omega(\mathrm{u}) = 1$ in the new Def. 3.3, we obtain the *ABC* fractional derivative [19], which is given by

$$\left({}^{C}\mathfrak{D}^{\alpha,\alpha,\alpha,1}_{\iota,1,1,\mathrm{u}}\,\mathfrak{g}\right)(\mathrm{u}) = \frac{\Delta(\alpha)}{1-\alpha}\int_{\iota}^{\mathrm{u}} \mathbb{E}^{\alpha,1}_{\alpha,1}\left(\lambda_\alpha, (\mathrm{u}-\mathrm{v})\right)\mathfrak{g}'(\mathrm{v})\,d\mathrm{v}, \quad \mathrm{u}\in[\iota,\tau].$$

6. If $\beta = \gamma = \mu = \sigma = 1$ in the new Def. 3.3, we obtain the weighted *CFC* fractional derivative of $\mathfrak{g}$ with respect to another function $\phi(\mathrm{u})$ [24], which is given by

$$\left({}^{C}\mathfrak{D}^{\alpha,1,1,1}_{\iota,1,\omega,\phi}\,\mathfrak{g}\right)(\mathrm{u}) = \frac{\Delta(\alpha)}{1-\alpha}\frac{1}{\omega(\mathrm{u})}\int_{\iota}^{\mathrm{u}} \exp\left(\lambda_\alpha\left(\phi(\mathrm{u})-\phi(\mathrm{v})\right)\right)(\omega\mathfrak{g})'(\mathrm{v})\,d\mathrm{v}, \quad \mathrm{u}\in[\iota,\tau].$$

7. If $\beta = \gamma = \mu = \sigma = 1$ and $\omega(\mathrm{u}) = 1$ in the new Def. 3.3, we obtain the *CFC* fractional derivative of $\mathfrak{g}$ with respect to another function $\phi(\mathrm{u})$ [24], which is given by

$$\left({}^{C}\mathfrak{D}^{\alpha,1,1,1}_{\iota,1,1,\phi}\,\mathfrak{g}\right)(\mathrm{u}) = \frac{\Delta(\alpha)}{1-\alpha}\int_{\iota}^{\mathrm{u}} \exp\left(\lambda_\alpha\left(\phi(\mathrm{u})-\phi(\mathrm{v})\right)\right)\mathfrak{g}'(\mathrm{v})\,d\mathrm{v}, \quad \mathrm{u}\in[\iota,\tau].$$

8. If $\beta = \gamma = \mu = \sigma = 1, \phi(\mathrm{u}) = \mathrm{u}$, and $\omega(\mathrm{u}) = 1$ in the new Def. 3.3, we obtain the *CFC* fractional derivative [18], which is given by

$$\left({}^{C}\mathfrak{D}^{\alpha,1,1,1}_{\iota,1,1,\mathrm{u}}\,\mathfrak{g}\right)(\mathrm{u}) = \frac{\Delta(\alpha)}{1-\alpha}\int_{\iota}^{\mathrm{u}} \exp\left(\lambda_\alpha(\mathrm{u}-\mathrm{v})\right)\mathfrak{g}'(\mathrm{v})\,d\mathrm{v}, \quad \mathrm{u}\in[\iota,\tau].$$

9. The limiting process $\alpha,\beta,\gamma,\mu,\sigma \to 1, \phi(\mathrm{u}) = \mathrm{u}$, and $\omega(\mathrm{u}) = 1$, in the new Def. 3.3, tends to the first ordinary derivative. Furthermore, the new Def. 3.3, has a singular kernel for $\mu \in (0,1)$.

Proposition 3.5 *Let $\alpha \in (0,1], \beta,\gamma > 0, \mathrm{Re}(\mu) > 0, \sigma \in \mathbb{R}$, and $\mathfrak{g} \in \mathcal{H}^1(\iota,\tau)$. The new αth left-sided generalized fractional derivatives that are given in* (3.1) *and* (3.2) *can be reformulated in series forms as follows*:

$$\left({}^{R}\mathfrak{D}^{\alpha,\beta,\gamma,\mu}_{\iota,\sigma,\omega,\phi}\,\mathfrak{g}\right)(\mathrm{u}) = \frac{\Delta(\alpha)}{1-\alpha}\sum_{k=0}^{\infty}(\sigma)_k\frac{\lambda_\alpha^k\Gamma(\gamma k+\mu)}{k!\Gamma(\beta k+\mu)}\left({}^{R}\mathfrak{J}^{k\gamma+\mu-1}_{\iota,\omega,\phi}\,\mathfrak{g}\right)(\mathrm{u}),$$

$$\left({}^{C}\mathfrak{D}^{\alpha,\beta,\gamma,\mu}_{\iota,\sigma,\omega,\phi}\mathfrak{g}\right)(\mathrm{u}) = \frac{\Delta(\alpha)}{1-\alpha}\sum_{k=0}^{\infty}(\sigma)_k\frac{\lambda_\alpha^k\Gamma(\gamma k+\mu)}{k!\Gamma(\beta k+\mu)}{}^{R}\mathfrak{J}^{k\gamma+\mu}_{\iota,\omega,\phi}\left(\frac{(\omega\mathfrak{g})'}{\omega\phi'}\right)(\mathrm{u}),$$

respectively, where ${}^{R}\mathfrak{J}^{k\gamma+\mu-1}_{\iota,\omega,\phi}$ *and* ${}^{R}\mathfrak{J}^{k\gamma+\mu}_{\iota,\omega,\phi}$ *are defined as in* (2.1).

Proof The modified version of the generalized Mittag–Leffler function is given as follows:

$$\mathbb{E}^{\gamma,\sigma}_{\beta,\mu}\big(\lambda_\alpha,\big(\phi(\mathrm{u})-\phi(\mathrm{v})\big)\big) = \sum_{k=0}^{\infty}(\sigma)_k\frac{\lambda_\alpha^k(\phi(\mathrm{u})-\phi(\mathrm{v}))^{k\gamma+\mu-1}}{k!\Gamma(\beta k+\mu)},$$

this series being locally uniformly convergent on the entire complex plane. Hence, the Riemann–Liouville type is given by

$$\begin{aligned}
&\left({}^{R}\mathfrak{D}^{\alpha,\beta,\gamma,\mu}_{\iota,\sigma,\omega,\phi}\mathfrak{g}\right)(\mathrm{u})\\
&\quad= \frac{\Delta(\alpha)}{1-\alpha}\frac{1}{\omega(\mathrm{u})\phi'(\mathrm{u})}\frac{d}{d\mathrm{u}}\int_\iota^{\mathrm{u}}\phi'(\mathrm{v})\mathbb{E}^{\gamma,\sigma}_{\beta,\mu}\big(\lambda_\alpha,\big(\phi(\mathrm{u})-\phi(\mathrm{v})\big)\big)(\omega\mathfrak{g})(\mathrm{v})\,d\mathrm{v}\\
&\quad= \frac{\Delta(\alpha)}{1-\alpha}\frac{1}{\omega(\mathrm{u})\phi'(\mathrm{u})}\frac{d}{d\mathrm{u}}\int_\iota^{\mathrm{u}}\phi'(\mathrm{v})\sum_{k=0}^{\infty}(\sigma)_k\frac{\lambda_\alpha^k(\phi(\mathrm{u})-\phi(\mathrm{v}))^{k\gamma+\mu-1}}{k!\Gamma(\beta k+\mu)}(\omega\mathfrak{g})(\mathrm{v})\,d\mathrm{v}\\
&\quad= \frac{\Delta(\alpha)}{1-\alpha}\frac{1}{\phi'(\mathrm{u})}\frac{d}{d\mathrm{u}}\sum_{k=0}^{\infty}(\sigma)_k\frac{\lambda_\alpha^k\Gamma(\gamma k+\mu)}{k!\Gamma(\beta k+\mu)}\frac{1}{\omega(\mathrm{u})\Gamma(\gamma k+\mu)}\\
&\qquad\times\int_\iota^{\mathrm{u}}\phi'(\mathrm{v})\big(\phi(\mathrm{u})-\phi(\mathrm{v})\big)^{k\gamma+\mu-1}(\omega\mathfrak{g})(\mathrm{v})\,d\mathrm{v}\\
&\quad= \frac{\Delta(\alpha)}{1-\alpha}\frac{1}{\phi'(\mathrm{u})}\frac{d}{d\mathrm{u}}\sum_{k=0}^{\infty}(\sigma)_k\frac{\lambda_\alpha^k\Gamma(\gamma k+\mu)}{k!\Gamma(\beta k+\mu)}\left({}^{R}\mathfrak{J}^{k\gamma+\mu}_{\iota,\omega,\phi}\mathfrak{g}\right)(\mathrm{u})\\
&\quad= \frac{\Delta(\alpha)}{1-\alpha}\sum_{k=0}^{\infty}(\sigma)_k\frac{\lambda_\alpha^k\Gamma(\gamma k+\mu)}{k!\Gamma(\beta k+\mu)}\left({}^{R}\mathfrak{J}^{k\gamma+\mu-1}_{\iota,\omega,\phi}\mathfrak{g}\right)(\mathrm{u}).
\end{aligned}$$

Also, the Caputo type is given by

$$\begin{aligned}
&\left({}^{C}\mathfrak{D}^{\alpha,\beta,\gamma,\mu}_{\iota,\sigma,\omega,\phi}\mathfrak{g}\right)(\mathrm{u})\\
&\quad= \frac{\Delta(\alpha)}{1-\alpha}\frac{1}{\omega(\mathrm{u})}\int_\iota^{\mathrm{u}}\mathbb{E}^{\gamma,\sigma}_{\beta,\mu}\big(\lambda_\alpha,\big(\phi(\mathrm{u})-\phi(\mathrm{v})\big)\big)(\omega\mathfrak{g})'(\mathrm{v})\,d\mathrm{v}\\
&\quad= \frac{\Delta(\alpha)}{1-\alpha}\frac{1}{\omega(\mathrm{u})}\int_\iota^{\mathrm{u}}\phi'(\mathrm{v})\sum_{k=0}^{\infty}(\sigma)_k\frac{\lambda_\alpha^k(\phi(\mathrm{u})-\phi(\mathrm{v}))^{k\gamma+\mu-1}}{k!\Gamma(\beta k+\mu)}\frac{(\omega\mathfrak{g})'(\mathrm{v})}{\phi'(\mathrm{v})}\,d\mathrm{v}\\
&\quad= \frac{\Delta(\alpha)}{1-\alpha}\sum_{k=0}^{\infty}(\sigma)_k\frac{\lambda_\alpha^k\Gamma(\gamma k+\mu)}{k!\Gamma(\beta k+\mu)}\frac{1}{\omega(\mathrm{u})\Gamma(\gamma k+\mu)}\\
&\qquad\times\int_\iota^{\mathrm{u}}\phi'(\mathrm{v})\big(\phi(\mathrm{u})-\phi(\mathrm{v})\big)^{k\gamma+\mu-1}\omega(\mathrm{v})\frac{(\omega\mathfrak{g})'(\mathrm{v})}{(\omega\phi')(\mathrm{v})}\,d\mathrm{v}\\
&\quad= \frac{\Delta(\alpha)}{1-\alpha}\sum_{k=0}^{\infty}(\sigma)_k\frac{\lambda_\alpha^k\Gamma(\gamma k+\mu)}{k!\Gamma(\beta k+\mu)}{}^{R}\mathfrak{J}^{k\gamma+\mu}_{\iota,\omega,\phi}\left(\frac{(\omega\mathfrak{g})'}{\omega\phi'}\right)(\mathrm{u}).
\end{aligned}$$

□

Proposition 3.6 *Let* $\alpha\in(0,1],\beta,\gamma>0,\mathrm{Re}(\mu)>0,\sigma\in\mathbb{R}$, *and* $\mathfrak{g}\in\mathcal{H}^1(\iota,\tau)$. *The weighted Laplace transforms with respect to another function* ϕ *of* $({}^{R}\mathfrak{D}^{\alpha,\beta,\gamma,\mu}_{\iota,\sigma,\omega,\phi}\mathfrak{g})(\mathrm{u})$ *and*

$({}^{C}\mathfrak{D}^{\alpha,\beta,\gamma,\mu}_{\iota,\sigma,\omega,\phi}\mathfrak{g})(\mathrm{u})$, *are given by*

$$\mathbb{L}_{\omega,\phi}\big\{\big({}^{R}\mathfrak{D}^{\alpha,\beta,\gamma,\mu}_{\iota,\sigma,\omega,\phi}\mathfrak{g}\big)(\mathrm{u})\big\}(s) = \frac{\Delta(\alpha)}{1-\alpha}\sum_{k=0}^{\infty}(\sigma)_k\frac{\lambda_\alpha^k\Gamma(\gamma k+\mu)}{k!\Gamma(\beta k+\mu)s^{k\gamma+\mu-1}}\mathbb{L}_{\omega,\phi}\big\{\mathfrak{g}(\mathrm{u})\big\}(s),$$

$$\mathbb{L}_{\omega,\phi}\big\{\big({}^{C}\mathfrak{D}^{\alpha,\beta,\gamma,\mu}_{\iota,\sigma,\omega,\phi}\mathfrak{g}\big)(\mathrm{u})\big\}(s) = \frac{\Delta(\alpha)}{1-\alpha}\sum_{k=0}^{\infty}(\sigma)_k\frac{\lambda_\alpha^k\Gamma(\gamma k+\mu)}{k!\Gamma(\beta k+\mu)s^{k\gamma+\mu}}\big[s\mathbb{L}_{\omega,\phi}\big\{\mathfrak{g}(\mathrm{u})\big\}(s)-(\omega\mathfrak{g})(\iota)\big].$$

Proof By using Proposition 3.5 and Lemma 2.2, we have

$$\begin{aligned}\mathbb{L}_{\omega,\phi}\big\{\big({}^{R}\mathfrak{D}^{\alpha,\beta,\gamma,\mu}_{\iota,\sigma,\omega,\phi}\mathfrak{g}\big)(\mathrm{u})\big\}(s) &= \frac{\Delta(\alpha)}{1-\alpha}\sum_{k=0}^{\infty}(\sigma)_k\frac{\lambda_\alpha^k\Gamma(\gamma k+\mu)}{k!\Gamma(\beta k+\mu)}\mathbb{L}_{\omega,\phi}\big\{{}^{R}\mathfrak{J}^{k\gamma+\mu-1}_{\iota,\omega,\phi}\mathfrak{g}(\mathrm{u})\big\}(s)\\ &= \frac{\Delta(\alpha)}{1-\alpha}\sum_{k=0}^{\infty}(\sigma)_k\frac{\lambda_\alpha^k\Gamma(\gamma k+\mu)}{k!\Gamma(\beta k+\mu)s^{k\gamma+\mu-1}}\mathbb{L}_{\omega,\phi}\big\{\mathfrak{g}(\mathrm{u})\big\}(s).\end{aligned}$$

Also,

$$\begin{aligned}\mathbb{L}_{\omega,\phi}\big\{\big({}^{C}\mathfrak{D}^{\alpha,\beta,\gamma,\mu}_{\iota,\sigma,\omega,\phi}\mathfrak{g}\big)(\mathrm{u})\big\}(s) &= \frac{\Delta(\alpha)}{1-\alpha}\sum_{k=0}^{\infty}(\sigma)_k\frac{\lambda_\alpha^k\Gamma(\gamma k+\mu)}{k!\Gamma(\beta k+\mu)}\mathbb{L}_{\omega,\phi}\left\{{}^{R}\mathfrak{J}^{k\gamma+\mu}_{\iota,\omega,\phi}\left(\frac{(\omega\mathfrak{g})'}{\omega\phi'}\right)(\mathrm{u})\right\}(s)\\ &= \frac{\Delta(\alpha)}{1-\alpha}\sum_{k=0}^{\infty}(\sigma)_k\frac{\lambda_\alpha^k\Gamma(\gamma k+\mu)}{k!\Gamma(\beta k+\mu)s^{k\gamma+\mu}}\mathbb{L}_{\omega,\phi}\big\{\big(D^1_{\omega,\phi}\mathfrak{g}\big)(\mathrm{u})\big\}(s)\\ &= \frac{\Delta(\alpha)}{1-\alpha}\sum_{k=0}^{\infty}(\sigma)_k\frac{\lambda_\alpha^k\Gamma(\gamma k+\mu)}{k!\Gamma(\beta k+\mu)s^{k\gamma+\mu}}\big[s\mathbb{L}_{\omega,\phi}\big\{\mathfrak{g}(\mathrm{u})\big\}(s)-(\omega\mathfrak{g})(\iota)\big],\end{aligned}$$

where $(D^1_{\omega,\phi}\mathfrak{g})(\mathrm{u}) = (\frac{(\omega\mathfrak{g})'}{\omega\phi'})(\mathrm{u})$, [14]. □

Proposition 3.7 *Let* $\alpha\in(0,1],\beta,\gamma>0,\sigma\in\mathbb{R},\mathrm{Re}(\mu)>0,k\gamma+\mu>1,$ *for* $k=0,1,2,\ldots,$ *and the weight function* $\omega\in C^1(\iota,\tau)$ *with* $\omega>0$ *on* $[\iota,\tau]$, *we have*

$$\mathbb{L}_{\omega,\phi}\big\{\omega^{-1}(\mathrm{u})\mathbb{E}^{\gamma,\sigma}_{\beta,\mu}\big(\lambda_\alpha,\big(\phi(\mathrm{u})-\phi(\iota)\big)\big)\big\}(s) = \sum_{k=0}^{\infty}(\sigma)_k\frac{\lambda_\alpha^k\Gamma(k\gamma+\mu)}{k!\Gamma(\beta k+\mu)s^{k\gamma+\mu}}.$$

Proof By Lemma (2.2), part (*iii*), we obtain

$$\begin{aligned}&\mathbb{L}_{\omega,\phi}\big\{\omega^{-1}(\mathrm{u})\mathbb{E}^{\gamma,\sigma}_{\beta,\mu}\big(\lambda_\alpha,\big(\phi(\mathrm{u})-\phi(\iota)\big)\big)\big\}(s)\\ &\quad= \sum_{k=0}^{\infty}(\sigma)_k\frac{\lambda_\alpha^k\mathbb{L}_{\omega,\phi}\{\omega^{-1}(\mathrm{u})(\phi(\mathrm{u})-\phi(\iota))^{k\gamma+\mu-1}\}(s)}{k!\Gamma(\beta k+\mu)}\\ &\quad= \sum_{k=0}^{\infty}(\sigma)_k\frac{\lambda_\alpha^k\Gamma(k\gamma+\mu)}{k!\Gamma(\beta k+\mu)s^{k\gamma+\mu}}.\end{aligned}$$

□

Lemma 3.8 *Let* $\alpha\in(0,1],\beta,\gamma>0,\sigma\in\mathbb{R},\mathrm{Re}(\mu)>0,$ *and the weight function* $\omega\in C^1(\iota,\tau)$ *with* $\omega>0$ *on* $[\iota,\tau]$. *Then, the relation between Caputo and Riemann–Liouville generalized weighted fractional derivatives of a function* $\mathfrak{g}$ *with respect to another function* $\phi(\mathrm{u})$, *is given* *by*

$$\left({}^{C}\mathfrak{D}^{\alpha,\beta,\gamma,\mu}_{\iota,\sigma,\omega,\phi}\mathfrak{g}\right)(\mathrm{u}) = \left({}^{R}\mathfrak{D}^{\alpha,\beta,\gamma,\mu}_{\iota,\sigma,\omega,\phi}\mathfrak{g}\right)(\mathrm{u}) - \frac{\Delta(\alpha)}{1-\alpha}\frac{\mathbb{E}^{\gamma,\sigma}_{\beta,\mu}(\lambda_\alpha,(\phi(\mathrm{u})-\phi(\iota)))}{\omega(\mathrm{u})}(\omega\mathfrak{g})(\iota). \tag{3.3}$$

Proof In view of Propositions 3.6 and 3.7, we obtain

$$\begin{aligned}
&\mathbb{L}_{\omega,\phi}\left\{\left({}^{C}\mathfrak{D}^{\alpha,\beta,\gamma,\mu}_{\iota,\sigma,\omega,\phi}\mathfrak{g}\right)(\mathrm{u})\right\}(s)\\
&= \frac{\Delta(\alpha)}{1-\alpha}\sum_{k=0}^{\infty}(\sigma)_k\frac{\lambda_\alpha^k\Gamma(\gamma k+\mu)\mathbb{L}_{\omega,\phi}\{\mathfrak{g}(\mathrm{u})\}(s)}{k!\Gamma(\beta k+\mu)s^{k\gamma+\mu-1}}\\
&\quad - \frac{\Delta(\alpha)}{1-\alpha}\sum_{k=0}^{\infty}(\sigma)_k\frac{\lambda_\alpha^k\Gamma(\gamma k+\mu)}{k!\Gamma(\beta k+\mu)s^{k\gamma+\mu}}(\omega\mathfrak{g})(\iota)\\
&= \mathbb{L}_{\omega,\phi}\left\{\left({}^{R}\mathfrak{D}^{\alpha,\beta,\gamma,\mu}_{\iota,\sigma,\omega,\phi}\mathfrak{g}\right)(\mathrm{u}\right\}(s) - \frac{\Delta(\alpha)}{1-\alpha}(\omega\mathfrak{g})(\iota)\mathbb{L}_{\omega,\phi}\left\{\omega^{-1}(\mathrm{u})\mathbb{E}^{\gamma,\sigma}_{\beta,\mu}\left(\lambda_\alpha,\left(\phi(\mathrm{u})-\phi(\iota)\right)\right)\right\}(s).
\end{aligned}$$

By applying the inverse Laplace, the proof is finished. □

3.2 The new corresponding fractional integral

In this subsection, we introduce a new generalized weighted fractional integral with respect to another function, and we discuss some properties relating to new fractional derivatives and integral definitions.

Proposition 3.9 *Let* $\alpha \in (0,1]$, $\beta = \gamma > 0$, $\mathrm{Re}(\mu) > 0$, *and* $\sigma = 1,2,\dots,n$, *where* $n \in \mathbb{N}$. *Then, the fractional differential equation* $({}^{R}\mathfrak{D}^{\alpha,\beta,\beta,\mu}_{\iota,\sigma,\omega,\phi}\mathfrak{g})(\mathrm{u}) = f(\mathrm{u})$ *has a unique solution given as*

$$\mathfrak{g}(\mathrm{u}) = \sum_{i=0}^{\sigma}\binom{\sigma}{i}\frac{\alpha^i}{(1-\alpha)^{i-1}\Delta(\alpha)}\left({}^{R}\mathfrak{I}^{i\beta-\mu+1}_{\iota,\omega,\phi}f\right)(\mathrm{u}). \tag{3.4}$$

Proof In view of Proposition 3.6, for $\gamma = \beta$, we have

$$\mathbb{L}_{\omega,\phi}\left\{\left({}^{R}\mathfrak{D}^{\alpha,\beta,\beta,\mu}_{\iota,\sigma,\omega,\phi}\mathfrak{g}\right)(\mathrm{u})\right\}(s) = \frac{\Delta(\alpha)}{1-\alpha}\sum_{k=0}^{\infty}\frac{(\sigma)_k\lambda_\alpha^k}{k!s^{k\beta+\mu-1}}\mathbb{L}_{\omega,\phi}\left\{\mathfrak{g}(\mathrm{u})\right\}(s). \tag{3.5}$$

Then, for $\sigma = 1$, and applying the Laplace transform on both sides of $({}^{R}\mathfrak{D}^{\alpha,\beta,\beta,\mu}_{\iota,1,\omega,\phi}\mathfrak{g})(\mathrm{u}) = f(\mathrm{u})$ and by using (3.5), we find

$$\begin{aligned}
F(s) = \mathbb{L}_{\omega,\phi}\left\{f(\mathrm{u})\right\}(s) &= \mathbb{L}_{\omega,\phi}\left\{\left({}^{R}\mathfrak{D}^{\alpha,\beta,\beta,\mu}_{\iota,1,\omega,\phi}\mathfrak{g}\right)(\mathrm{u})\right\}(s)\\
&= \frac{\Delta(\alpha)}{1-\alpha}\sum_{k=0}^{\infty}\left(\lambda_\alpha^k s^{-k\beta-\mu+1}\right)\mathbb{L}_{\omega,\phi}\left\{\mathfrak{g}(\mathrm{u})\right\}(s)\\
&= \frac{\Delta(\alpha)}{1-\alpha}\left(s^{-\mu+1}+\lambda_\alpha s^{-\beta-\mu+1}+\lambda_\alpha^2 s^{-2\beta-\mu+1}+\lambda_\alpha^3 s^{-3\beta-\mu+1}+\dots\right)G(s)\\
&= \frac{\Delta(\alpha)}{1-\alpha}s^{-\mu+1}\left(1+\lambda_\alpha s^{-\beta}+\lambda_\alpha^2 s^{-2\beta}+\lambda_\alpha^3 s^{-3\beta}+\dots\right)G(s)\\
&= \frac{\Delta(\alpha)}{1-\alpha}s^{-\mu+1}\left(1-\lambda_\alpha s^{-\beta}\right)^{-1}G(s),
\end{aligned}$$

where, $F(s) = \mathbb{L}_{\omega,\phi}\{f(\mathrm{u})\}(s)$ and $G(s) = \mathbb{L}_{\omega,\phi}\{\mathfrak{g}(\mathrm{u})\}(s)$, which yields that

$$G(s) = \frac{1-\alpha}{\Delta(\alpha)} s^{\mu-1}\left(1-\lambda_\alpha s^{-\beta}\right)F(s) = \frac{1-\alpha}{\Delta(\alpha)} s^{\mu-1}F(s) + \frac{\alpha}{\Delta(\alpha)} s^{-\beta+\mu-1}F(s).$$

By taking the inverse Laplace transform and using Lemma 2.2, we obtain

$$\mathfrak{g}(\mathrm{u}) = \frac{1-\alpha}{\Delta(\alpha)}\left({}^{R}\mathfrak{I}^{1-\mu}_{\iota,\omega,\phi} f\right)(\mathrm{u}) + \frac{\alpha}{\Delta(\alpha)}\left({}^{R}\mathfrak{I}^{\beta-\mu+1}_{\iota,\omega,\phi} f\right)(\mathrm{u}).$$

Now, for $\sigma = 2$, we have

$$\begin{aligned}
F(s) &= \mathbb{L}_{\omega,\phi}\left\{f(\mathrm{u})\right\}(s) = \mathbb{L}_{\omega,\phi}\left\{\left({}^{R}\mathfrak{D}^{\alpha,\beta,\beta,\mu}_{\iota,2,\omega,\phi}\mathfrak{g}\right)(\mathrm{u})\right\}(s) \\
&= \frac{\Delta(\alpha)}{1-\alpha}\sum_{k=0}^{\infty}\frac{\Gamma(2+k)}{\Gamma(2)k!}\left(\lambda_\alpha^k s^{-k\beta-\mu+1}\right)\mathbb{L}_{\omega,\phi}\left\{\mathfrak{g}(\mathrm{u})\right\}(s) \\
&= \frac{\Delta(\alpha)}{1-\alpha}\sum_{k=0}^{\infty}\left(\lambda_\alpha^k(1+k)s^{-k\beta-\mu+1}\right)\mathbb{L}_{\omega,\phi}\left\{\mathfrak{g}(\mathrm{u})\right\}(s) \\
&= \frac{\Delta(\alpha)}{1-\alpha}\left(s^{-\mu+1} + 2\lambda_\alpha s^{-\beta-\mu+1} + 3\lambda_\alpha^2 s^{-2\beta-\mu+1} + 4\lambda_\alpha^3 s^{-3\beta-\mu+1} + \ldots\right)G(s) \\
&= \frac{\Delta(\alpha)}{1-\alpha} s^{-\mu+1}\left(1 + 2\lambda_\alpha s^{-\beta} + 3\lambda_\alpha^2 s^{-2\beta} + 4\lambda_\alpha^3 s^{-3\beta} + \ldots\right)G(s) \\
&= \frac{\Delta(\alpha)}{1-\alpha} s^{-\mu+1}\left(1-\lambda_\alpha s^{-\beta}\right)^{-2}G(s).
\end{aligned}$$

Thus, it implies that

$$\begin{aligned}
G(s) &= \frac{1-\alpha}{\Delta(\alpha)} s^{\mu-1}\left(1-\lambda_\alpha s^{-\beta}\right)^2 F(s) \\
&= \frac{1-\alpha}{\Delta(\alpha)} s^{\mu-1}F(s) + \frac{2\alpha}{\Delta(\alpha)} s^{-\beta+\mu-1}F(s) + \frac{\alpha}{(1-\alpha)\Delta(\alpha)} s^{-2\beta+\mu-1}F(s).
\end{aligned}$$

Due to the inverse Laplace transform and Lemma 2.2, we obtain

$$\mathfrak{g}(\mathrm{u}) = \frac{1-\alpha}{\Delta(\alpha)}\left({}^{R}\mathfrak{I}^{1-\mu}_{\iota,\omega,\phi} f\right)(\mathrm{u}) + \frac{2\alpha}{\Delta(\alpha)}\left({}^{R}\mathfrak{I}^{\beta-\mu+1}_{\iota,\omega,\phi} f\right)(\mathrm{u}) + \frac{\alpha}{(1-\alpha)\Delta(\alpha)}\left({}^{R}\mathfrak{I}^{2\beta-\mu+1}_{\iota,\omega,\phi} f\right)(\mathrm{u}).$$

Hence, by the same manner upto $\sigma = n$, one has

$$\mathfrak{g}(\mathrm{u}) = \sum_{i=0}^{\sigma}\binom{\sigma}{i}\frac{\alpha^i}{(1-\alpha)^{i-1}\Delta(\alpha)}\left({}^{R}\mathfrak{I}^{i\beta-\mu+1}_{\iota,\omega,\phi} f\right)(\mathrm{u}),$$

which completes the proof. □

Definition 3.10 Consider $\mathfrak{g}(\mathrm{u})$ to be a continuous function on $[\iota,\tau]$. Let $\alpha \in (0,1], \beta > 0, \sigma \in \mathbb{R}$ and $\mathrm{Re}(\mu) > 0$. The new αth left-sided generalized weighted fractional integral of a function $\mathfrak{g}$ with respect to another function $\phi(\mathrm{u})$ is given as follows:

$$\left(\mathfrak{I}^{\alpha,\beta,\mu}_{\iota,\sigma,\omega,\phi}\mathfrak{g}\right)(\mathrm{u}) = \sum_{i=0}^{\infty}\binom{\sigma}{i}\frac{\alpha^i}{(1-\alpha)^{i-1}\Delta(\alpha)}\left({}^{R}\mathfrak{I}^{i\beta-\mu+1}_{\iota,\omega,\phi}\mathfrak{g}\right)(\mathrm{u}), \tag{3.6}$$

where ${}^{R}\mathfrak{I}^{i\beta-\mu+1}_{\iota,\omega,\phi}$ is defined in (2.1).

Remark 3.11 We observe that, for any specific values of the parameters $\alpha, \beta, \mu, \phi, \omega$, we obtain new and known definitions in the literature as follows:

1. If $\mu = 1, \sigma = 1$, and $\phi(\mathrm{u}) = \mathrm{u}$ in the new Def. 3.10, we obtain the weighted Hattaf–Riemann–Liouville fractional integral [26], which is given by

$$\left(\mathfrak{J}^{\alpha,\beta,1}_{\iota,1,\omega,\mathrm{u}}\mathfrak{g}\right)(\mathrm{u}) = \frac{1-\alpha}{\Delta(\alpha)}\mathfrak{g}(\mathrm{u}) + \frac{\alpha}{\Delta(\alpha)}\left({}^{R}\mathfrak{J}^{\beta}_{\iota,\omega}\mathfrak{g}\right)(\mathrm{u}).$$

2. If $\alpha = \beta, \phi(\mathrm{u}) = \mathrm{u}$, and $\omega(\mathrm{u}) = 1$ in the new Def. 3.10, we obtain the fractional integral corresponding to the generalized *AB* fractional derivative with the generalized Mittag–Leffler function [27], which is given by

$$\left(\mathfrak{J}^{\alpha,\alpha,\mu}_{\iota,\sigma,1,\mathrm{u}}\mathfrak{g}\right)(\mathrm{u}) = \sum_{i=0}^{\infty}\binom{\sigma}{i}\frac{\alpha^{i}}{(1-\alpha)^{i-1}\Delta(\alpha)}\left({}^{R}\mathfrak{J}^{i\alpha-\mu+1}_{\iota}\mathfrak{g}\right)(\mathrm{u}).$$

3. If $\alpha = \beta, \sigma = 1, \phi(\mathrm{u}) = \mathrm{u}$, and $\omega(\mathrm{u}) = 1$ in the new Def. 3.10, we obtain the fractional integral corresponding to the generalized *AB* fractional derivative with the generalized Mittag–Leffler function [23], which is given by

$$\left(\mathfrak{J}^{\alpha,\alpha,\mu}_{\iota,1,1,\mathrm{u}}\mathfrak{g}\right)(\mathrm{u}) = \frac{1-\alpha}{\Delta(\alpha)}\left({}^{R}\mathfrak{J}^{1-\mu}_{\iota}\mathfrak{g}\right)(\mathrm{u}) + \frac{\alpha}{\Delta(\alpha)}\left({}^{R}\mathfrak{J}^{\alpha-\mu+1}_{\iota}\mathfrak{g}\right)(\mathrm{u}).$$

4. If $\alpha = \beta$, $\sigma = \mu = 1$, and $\phi(\mathrm{u}) = \mathrm{u}$ in the new Def. 3.10, we obtain the weighted *AB* fractional integral [25], which is given by

$$\left(\mathfrak{J}^{\alpha,\alpha,1}_{\iota,1,\omega,\mathrm{u}}\mathfrak{g}\right)(\mathrm{u}) = \frac{1-\alpha}{\Delta(\alpha)}\mathfrak{g}(\mathrm{u}) + \frac{\alpha}{\Delta(\alpha)}\left({}^{R}\mathfrak{J}^{\alpha}_{\iota,\omega}\mathfrak{g}\right)(\mathrm{u}).$$

5. If $\alpha = \beta$, $\sigma = \mu = 1$, and $\omega(\mathrm{u}) = 1$ in the new Def. 3.10, we obtain the *AB* fractional integral of $\mathfrak{g}$ with respect to another function $\phi(\mathrm{u})$ [28], which is given by

$$\left(\mathfrak{J}^{\alpha,\alpha,1}_{\iota,1,1,\phi}\mathfrak{g}\right)(\mathrm{u}) = \frac{1-\alpha}{\Delta(\alpha)}\mathfrak{g}(\mathrm{u}) + \frac{\alpha}{\Delta(\alpha)}\left({}^{R}\mathfrak{J}^{\alpha}_{\iota,\phi}\mathfrak{g}\right)(\mathrm{u}).$$

6. If $\alpha = \beta$, $\sigma = \mu = 1, \phi(\mathrm{u}) = \mathrm{u}$, and $\omega(\mathrm{u}) = 1$ in the new Def. 3.10, we obtain the *AB* fractional integral [19], which is given by

$$\left(\mathfrak{J}^{\alpha,\alpha,1}_{\iota,1,1,\mathrm{u}}\mathfrak{g}\right)(\mathrm{u}) = \frac{1-\alpha}{\Delta(\alpha)}\mathfrak{g}(\mathrm{u}) + \frac{\alpha}{\Delta(\alpha)}\left({}^{R}\mathfrak{J}^{\alpha}_{\iota}\mathfrak{g}\right)(\mathrm{u}).$$

7. If $\beta = \sigma = \mu = 1$ in the new Def. 3.10, we obtain the weighted *CF* fractional integral of $\mathfrak{g}$ with respect to another function $\phi(\mathrm{u})$ [24], which is given by

$$\left(\mathfrak{J}^{\alpha,1,1}_{\iota,1,\omega,\phi}\mathfrak{g}\right)(\mathrm{u}) = \frac{1-\alpha}{\Delta(\alpha)}\mathfrak{g}(\mathrm{u}) + \frac{\alpha}{\Delta(\alpha)}\frac{1}{\omega(\mathrm{u})}\int_{\iota}^{\mathrm{u}}\phi'(\mathrm{v})\omega(\mathrm{v})\mathfrak{g}(\mathrm{v})\,d\mathrm{v}.$$

8. If $\beta = \sigma = \mu = 1, \phi(\mathrm{u}) = \mathrm{u}$, and $\omega(\mathrm{u}) = 1$ in the new Def. 3.10, we obtain the *CF* fractional integral [18], which is given by

$$\left(\mathfrak{J}^{\alpha,1,1}_{\iota,1,1,\mathrm{u}}\mathfrak{g}\right)(\mathrm{u}) = \frac{1-\alpha}{\Delta(\alpha)}\mathfrak{g}(\mathrm{u}) + \frac{\alpha}{\Delta(\alpha)}\int_{\iota}^{\mathrm{u}}\mathfrak{g}(\mathrm{v})\,d\mathrm{v}.$$

9. The limiting process $\alpha, \beta, \mu \to 1, \sigma = 1, \phi(\mathrm{u}) = \mathrm{u}$, and $\omega(\mathrm{u}) = 1$, in the new Def. 3.10, reduces to the first ordinary integral.

Proposition 3.12 *Let $\phi : [\iota, \tau] \to \mathbb{R}$ be an increasing function with $\phi'(\mathrm{u}) \neq 0$, $\forall \mathrm{u} \in [\iota, \tau]$. For $\alpha \in (0,1], \beta > 0, \mathrm{Re}(\mu) > 0, \sigma \in \mathbb{R}$ and weight function $\omega \in C^1(\iota, \tau)$, such that $\omega, \omega' > 0$ on $[\iota, \tau]$, the following relations hold:*

(i) $\mathfrak{J}^{\alpha,\beta,\mu}_{\iota,\sigma,\omega,\phi}({}^{R}\mathfrak{D}^{\alpha,\beta,\beta,\mu}_{\iota,\sigma,\omega,\phi}\mathfrak{g})(\mathrm{u}) = \mathfrak{g}(\mathrm{u})$;

(ii) ${}^{R}\mathfrak{D}^{\alpha,\beta,\beta,\mu}_{\iota,\sigma,\omega,\phi}(\mathfrak{J}^{\alpha,\beta,\mu}_{\iota,\sigma,\omega,\phi}\mathfrak{g})(\mathrm{u}) = \mathfrak{g}(\mathrm{u})$;

(iii) $\mathfrak{J}^{\alpha,\beta,\mu}_{\iota,\sigma,\omega,\phi}({}^{C}\mathfrak{D}^{\alpha,\beta,\beta,\mu}_{\iota,\sigma,\omega,\phi}\mathfrak{g})(\mathrm{u}) = \frac{1}{\omega(\mathrm{u})}((\omega\mathfrak{g})(\mathrm{u}) - (\omega\mathfrak{g})(\iota))$;

(iv)

$$
{}^{C}\mathfrak{D}^{\alpha,\beta,\beta,\mu}_{\iota,\sigma,\omega,\phi}\left(\mathfrak{J}^{\alpha,\beta,\mu}_{\iota,\sigma,\omega,\phi}\mathfrak{g}\right)(\mathrm{u}) = \begin{cases} \mathfrak{g}(\mathrm{u}) - \frac{\mathbb{E}^{\beta,\sigma}_{\beta,1}(\lambda_\alpha,(\phi(\mathrm{u})-\phi(\iota)))}{\omega(\mathrm{u})}(\omega\mathfrak{g})(\iota), & \textit{for } \mu = 1; \\ \mathfrak{g}(\mathrm{u}), \quad \textit{for } \mathrm{Re}(\beta - \mu + 1) > 0, \mathrm{Re}(1-\mu) > 0, \textit{ and } \mu \neq 1. \end{cases}
$$

Proof

(i) By using Def. 3.10, Proposition 3.5, and (Lemma (3), [27]), we obtain

$$
\begin{aligned}
&\mathfrak{J}^{\alpha,\beta,\mu}_{\iota,\sigma,\omega,\phi}\left({}^{R}\mathfrak{D}^{\alpha,\beta,\beta,\mu}_{\iota,\sigma,\omega,\phi}\mathfrak{g}\right)(\mathrm{u}) \\
&= \sum_{i=0}^{\infty}\binom{\sigma}{i}\frac{\alpha^i}{(1-\alpha)^{i-1}\Delta(\alpha)}\left({}^{R}\mathfrak{J}^{i\beta-\mu+1}_{\iota,\omega,\phi}\left({}^{R}\mathfrak{D}^{\alpha,\beta,\beta,\mu}_{\iota,\sigma,\omega,\phi}\mathfrak{g}\right)(\mathrm{u})\right) \\
&= \sum_{i=0}^{\infty}\binom{\sigma}{i}\frac{\alpha^i}{(1-\alpha)^{i-1}\Delta(\alpha)}{}^{R}\mathfrak{J}^{i\beta-\mu+1}_{\iota,\omega,\phi}\left(\frac{\Delta(\alpha)}{1-\alpha}\sum_{k=0}^{\infty}\frac{(\sigma)_k\lambda_\alpha^k}{k!}\left({}^{R}\mathfrak{J}^{k\beta+\mu-1}_{\iota,\omega,\phi}\mathfrak{g}\right)(\mathrm{u})\right) \\
&= \sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^i\binom{\sigma}{i}(\sigma)_k\lambda_\alpha^{k+i}}{k!}\left({}^{R}\mathfrak{J}^{(k+i)\beta}_{\iota,\omega,\phi}\mathfrak{g}\right)(\mathrm{u}) \\
&= \sum_{m=0}^{\infty}\lambda_\alpha^m\left({}^{R}\mathfrak{J}^{m\beta}_{\iota,\omega,\phi}\mathfrak{g}\right)(\mathrm{u})\sum_{i=0}^{m}\frac{(-1)^i\binom{\sigma}{i}(\sigma)_{m-i}}{(m-i)!} \\
&= \mathfrak{g}(\mathrm{u}) + \sum_{m=1}^{\infty}\lambda_\alpha^m\left({}^{R}\mathfrak{J}^{m\beta}_{\iota,\omega,\phi}\mathfrak{g}\right)(\mathrm{u})\sum_{i=0}^{m}\frac{(-1)^i\binom{\sigma}{i}(\sigma)_{m-i}}{(m-i)!} \\
&= \mathfrak{g}(\mathrm{u}).
\end{aligned}
$$

(ii) Using Def. 3.10, Proposition 3.5, and (Lemma (3), [27]), we obtain

$$
\begin{aligned}
&{}^{R}\mathfrak{D}^{\alpha,\beta,\beta,\mu}_{\iota,\sigma,\omega,\phi}\left(\mathfrak{J}^{\alpha,\beta,\mu}_{\iota,\sigma,\omega,\phi}\mathfrak{g}\right)(\mathrm{u}) \\
&= {}^{R}\mathfrak{D}^{\alpha,\beta,\beta,\mu}_{\iota,\sigma,\omega,\phi}\left(\sum_{i=0}^{\infty}\binom{\sigma}{i}\frac{\alpha^i}{(1-\alpha)^{i-1}\Delta(\alpha)}\left({}^{R}\mathfrak{J}^{i\beta-\mu+1}_{\iota,\omega,\phi}\mathfrak{g}(\mathrm{u})\right)\right) \\
&= \frac{\Delta(\alpha)}{1-\alpha}\sum_{k=0}^{\infty}\frac{(\sigma)_k\lambda_\alpha^k}{k!}{}^{R}\mathfrak{J}^{k\beta+\mu-1}_{\iota,\omega,\phi}\left(\sum_{i=0}^{\infty}\binom{\sigma}{i}\frac{\alpha^i}{(1-\alpha)^{i-1}\Delta(\alpha)}{}^{R}\mathfrak{J}^{i\beta-\mu+1}_{\iota,\omega,\phi}(\mathfrak{g}(\mathrm{u}))\right) \\
&= \sum_{k=0}^{\infty}\sum_{i=0}^{\infty}\frac{(-1)^i\binom{\sigma}{i}(\sigma)_k\lambda_\alpha^{k+i}}{k!}\left({}^{R}\mathfrak{J}^{(k+i)\beta}_{\iota,\omega,\phi}\mathfrak{g}\right)(\mathrm{u}) \\
&= \sum_{m=0}^{\infty}\lambda_\alpha^m\left({}^{R}\mathfrak{J}^{m\beta}_{\iota,\omega,\phi}\mathfrak{g}\right)(\mathrm{u})\sum_{i=0}^{m}\frac{(-1)^i\binom{\sigma}{i}(\sigma)_{m-i}}{(m-i)!}
\end{aligned}
$$

$$= \mathfrak{g}(\mathrm{u}) + \sum_{m=1}^{\infty} \lambda_\alpha^m \left({}^R\mathfrak{I}_{\iota,\omega,\phi}^{m\beta}\mathfrak{g}\right)(\mathrm{u}) \sum_{i=0}^{m} \frac{(-1)^i \binom{\sigma}{i} (\sigma)_{m-i}}{(m-i)!}$$

$$= \mathfrak{g}(\mathrm{u}).$$

(iii) By using Def. 3.10, Proposition 3.5, and (Lemma (3), [27]), we have

$$\begin{aligned}
&\mathfrak{I}_{\iota,\sigma,\omega,\phi}^{\alpha,\beta,\mu}\left({}^C\mathfrak{D}_{\iota,\sigma,\omega,\phi}^{\alpha,\beta,\beta,\mu}\mathfrak{g}\right)(\mathrm{u}) \\
&= \sum_{i=0}^{\infty}\binom{\sigma}{i}\frac{\alpha^i}{(1-\alpha)^{i-1}\Delta(\alpha)}\left({}^R\mathfrak{I}_{\iota,\omega,\phi}^{i\beta-\mu+1}\left({}^C\mathfrak{D}_{\iota,\sigma,\omega,\phi}^{\alpha,\beta,\beta,\mu}\mathfrak{g}\right)(\mathrm{u})\right) \\
&= \sum_{i=0}^{\infty}\binom{\sigma}{i}\frac{\alpha^i}{(1-\alpha)^{i-1}\Delta(\alpha)}{}^R\mathfrak{I}_{\iota,\omega,\phi}^{i\beta-\mu+1}\left(\frac{\Delta(\alpha)}{1-\alpha}\sum_{k=0}^{\infty}(\sigma)_k\frac{\lambda_\alpha^k}{k!}{}^R\mathfrak{I}_{\iota,\omega,\phi}^{k\beta+\mu}\left(\frac{(\omega\mathfrak{g})'}{\omega\phi'}\right)(\mathrm{u})\right) \\
&= \sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^i\binom{\sigma}{i}(\sigma)_k\lambda_\alpha^{k+i}}{k!}{}^R\mathfrak{I}_{\iota,\omega,\phi}^{(k+i)\beta+1}\left(\frac{(\omega\mathfrak{g})'}{\omega\phi'}\right)(\mathrm{u}) \\
&= \sum_{m=0}^{\infty}\lambda_\alpha^{m}{}^R\mathfrak{I}_{\iota,\omega,\phi}^{m\beta+1}\left(\frac{(\omega\mathfrak{g})'}{\omega\phi'}\right)(\mathrm{u})\sum_{i=0}^{m}\frac{(-1)^i\binom{\sigma}{i}(\sigma)_{m-i}}{(m-i)!} \\
&= {}^R\mathfrak{I}_{\iota,\omega,\phi}^{1}\left(\frac{(\omega\mathfrak{g})'}{\omega\phi'}\right)(\mathrm{u}) + \sum_{m=1}^{\infty}\lambda_\alpha^{m}{}^R\mathfrak{I}_{\iota,\omega,\phi}^{m\beta+1}\left(\frac{(\omega\mathfrak{g})'}{\omega\phi'}\right)(\mathrm{u})\sum_{i=0}^{m}\frac{(-1)^i\binom{\sigma}{i}(\sigma)_{m-i}}{(m-i)!} \\
&= {}^R\mathfrak{I}_{\iota,\omega,\phi}^{1}\left(\frac{(\omega\mathfrak{g})'}{\omega\phi'}\right)(\mathrm{u}) + 0 = \frac{1}{\omega(\mathrm{u})}\int_{\iota}^{\mathrm{u}}(\omega\mathfrak{g})'(\mathrm{v})\,d\mathrm{v} \\
&= \frac{1}{\omega(\mathrm{u})}\left((\omega\mathfrak{g})(\mathrm{u}) - (\omega\mathfrak{g})(\iota)\right).
\end{aligned}$$

(iv) In view of part (*ii*), Eqs. (3.3), (2.4), and Def. 3.10, we obtain

$$\begin{aligned}
&\left({}^C\mathfrak{D}_{\iota,\sigma,\omega,\phi}^{\alpha,\beta,\beta,\mu}\left(\mathfrak{I}_{\iota,\sigma,\omega,\phi}^{\alpha,\beta,\mu}\mathfrak{g}\right)\right)(\mathrm{u}) \\
&= \left({}^R\mathfrak{D}_{\iota,\sigma,\omega,\phi}^{\alpha,\beta,\beta,\mu}\left(\mathfrak{I}_{\iota,\sigma,\omega,\phi}^{\alpha,\beta,\mu}\mathfrak{g}\right)\right)(\mathrm{u}) - \frac{\Delta(\alpha)}{1-\alpha}\frac{\mathbb{E}_{\beta,\mu}^{\beta,\sigma}(\lambda_\alpha,(\phi(\mathrm{u})-\phi(\iota)))}{\omega(\mathrm{u})}\omega(\iota)\left(\mathfrak{I}_{\iota,\sigma,\omega,\phi}^{\alpha,\beta,\mu}\mathfrak{g}\right)(\iota) \\
&= \mathfrak{g}(\mathrm{u}) - \frac{\Delta(\alpha)}{1-\alpha}\omega(\iota)\frac{\mathbb{E}_{\beta,\mu}^{\beta,\sigma}(\lambda_\alpha,(\phi(\mathrm{u})-\phi(\iota)))}{\omega(\mathrm{u})} \\
&\quad\times\sum_{i=0}^{\infty}\binom{\sigma}{i}\frac{\alpha^i}{(1-\alpha)^{i-1}\Delta(\alpha)}\left({}^R\mathfrak{I}_{\iota,\omega,\phi}^{i\beta-\mu+1}\mathfrak{g}\right)(\iota).
\end{aligned}$$

Clearly, for $\mathrm{Re}(i\beta-\mu+1)>0, \mathrm{Re}(1-\mu)>0$ and $\mu\neq 1$, we have $({}^R\mathfrak{I}_{\iota,\omega,\phi}^{i\beta-\mu+1}\mathfrak{g})(\iota)=0$. Thus, we conclude that

$$\left({}^C\mathfrak{D}_{\iota,\sigma,\omega,\phi}^{\alpha,\beta,\beta,\mu}\left(\mathfrak{I}_{\iota,\sigma,\omega,\phi}^{\alpha,\beta,\mu}\mathfrak{g}\right)\right)(\mathrm{u}) = \mathfrak{g}(\mathrm{u}).$$

Moreover, for $\mu=1$, we obtain

$$\begin{aligned}
&\left({}^C\mathfrak{D}_{\iota,\sigma,\omega,\phi}^{\alpha,\beta,\beta,1}\left(\mathfrak{I}_{\iota,\sigma,\omega,\phi}^{\alpha,\beta,1}\mathfrak{g}\right)\right)(\mathrm{u}) \\
&= \mathfrak{g}(\mathrm{u}) - \frac{\mathbb{E}_{\beta,1}^{\beta,\sigma}(\lambda_\alpha,(\phi(\mathrm{u})-\phi(\iota)))}{\omega(\mathrm{u})}(\omega\mathfrak{g})(\iota)
\end{aligned}$$

$$
\begin{aligned}
&- \frac{\Delta(\alpha)}{1-\alpha}\omega(\iota)\frac{\mathbb{E}^{\beta,\sigma}_{\beta,1}(\lambda_\alpha,(\phi(\mathrm{u})-\phi(\iota)))}{\omega(\mathrm{u})}\sum_{i=1}^{\infty}\binom{\sigma}{i}\frac{\alpha^i}{(1-\alpha)^{i-1}\Delta(\alpha)}\left({}^{R}\mathfrak{J}^{i\beta}_{\iota,\omega,\phi}\mathfrak{g}\right)(\iota) \\
&= \mathfrak{g}(\mathrm{u}) - \frac{\mathbb{E}^{\beta,\sigma}_{\beta,1}(\lambda_\alpha,(\phi(\mathrm{u})-\phi(\iota)))}{\omega(\mathrm{u})}(\omega\mathfrak{g})(\iota).
\end{aligned}
$$

□

Hence, the proof is completed.

Lemma 3.13 *Consider $\phi:[\iota,\tau]\to\mathbb{R}$ to be an increasing function with $\phi'(\mathrm{u})\neq 0, \forall \mathrm{u}\in[\iota,\tau]$. Let $\alpha\in(0,1], \beta,\gamma,\varrho>0, \sigma\in\mathbb{R}, \mathrm{Re}(\mu)>0$, and $\omega\in C^1(\iota,\tau)$ is the weight function such that $\omega,\omega'>0$ on $[\iota,\tau]$. Then, the following relations hold*:

(i) $\mathfrak{J}^{\alpha,\beta,\mu}_{\iota,\sigma,\omega,\phi}[\frac{[\phi(\mathrm{u})-\phi(\iota)]^{\varrho}}{\omega(\mathrm{u})}] = \sum_{i=0}^{\infty}\binom{\sigma}{i}\frac{\alpha^i}{(1-\alpha)^{i-1}\Delta(\alpha)}\frac{\Gamma(\varrho+1)}{\Gamma(i\beta-\mu+\varrho+2)}[\frac{[\phi(\mathrm{u})-\phi(\iota)]^{i\beta-\mu+\varrho+1}}{\omega(\mathrm{u})}]$;

(ii) ${}^{C}\mathfrak{D}^{\alpha,\beta,\gamma,\mu}_{\iota,\sigma,\omega,\phi}[\frac{[\phi(\mathrm{u})-\phi(\iota)]^{\varrho}}{\omega(\mathrm{u})}] = \frac{\Delta(\alpha)}{1-\alpha}\sum_{k=0}^{\infty}(\sigma)_k\frac{\lambda_\alpha^k\Gamma(\gamma k+\mu)}{k!\Gamma(\beta k+\mu)}\frac{\Gamma(\varrho+1)}{\Gamma(k\gamma+\mu+\varrho)}\frac{[\phi(\mathrm{u})-\phi(\iota)]^{k\gamma+\mu+\varrho-1}}{\omega(\mathrm{u})}$;

(iii) $({}^{C}\mathfrak{D}^{\alpha,\beta,\gamma,\mu}_{\iota,\sigma,1,\phi}C)(\mathrm{u})=0$, *where $\omega(\mathrm{u})=1$ and C is constant.*

Proof

(i) For this, we apply Def. 3.10 and Eq. (2.2), and obtain

$$
\begin{aligned}
&\mathfrak{J}^{\alpha,\beta,\mu}_{\iota,\sigma,\omega,\phi}\left[\frac{[\phi(\mathrm{u})-\phi(\iota)]^{\varrho}}{\omega(\mathrm{u})}\right] \\
&= \sum_{i=0}^{\infty}\binom{\sigma}{i}\frac{\alpha^i}{(1-\alpha)^{i-1}\Delta(\alpha)}{}^{R}\mathfrak{J}^{i\beta-\mu+1}_{\iota,\omega,\phi}\left[\frac{[\phi(\mathrm{u})-\phi(\iota)]^{\varrho}}{\omega(\mathrm{u})}\right] \\
&= \sum_{i=0}^{\infty}\binom{\sigma}{i}\frac{\alpha^i}{(1-\alpha)^{i-1}\Delta(\alpha)}\frac{\Gamma(\varrho+1)}{\Gamma(i\beta-\mu+\varrho+2)}\left[\frac{[\phi(\mathrm{u})-\phi(\iota)]^{i\beta-\mu+\varrho+1}}{\omega(\mathrm{u})}\right].
\end{aligned}
$$

(ii) Due to Proposition 3.5 and Eq. (2.2), we find

$$
\begin{aligned}
&{}^{C}\mathfrak{D}^{\alpha,\beta,\gamma,\mu}_{\iota,\sigma,\omega,\phi}\left[\frac{[\phi(\mathrm{u})-\phi(\iota)]^{\varrho}}{\omega(\mathrm{u})}\right] \\
&= \frac{\Delta(\alpha)}{1-\alpha}\sum_{k=0}^{\infty}(\sigma)_k\frac{\lambda_\alpha^k\Gamma(\gamma k+\mu)}{k!\Gamma(\beta k+\mu)}{}^{R}\mathfrak{J}^{k\gamma+\mu}_{\iota,\omega,\phi}\left(\frac{[[\phi(\mathrm{u})-\phi(\iota)]^{\varrho}]'}{(\omega\phi')(\mathrm{u})}\right) \\
&= \frac{\Delta(\alpha)}{1-\alpha}\sum_{k=0}^{\infty}(\sigma)_k\frac{\lambda_\alpha^k\Gamma(\gamma k+\mu)}{k!\Gamma(\beta k+\mu)}{}^{R}\mathfrak{J}^{k\gamma+\mu}_{\iota,\omega,\phi}\left(\frac{\varrho[\phi(\mathrm{u})-\phi(\iota)]^{\varrho-1}}{\omega(\mathrm{u})}\right) \\
&= \frac{\Delta(\alpha)}{1-\alpha}\sum_{k=0}^{\infty}(\sigma)_k\frac{\lambda_\alpha^k\Gamma(\gamma k+\mu)}{k!\Gamma(\beta k+\mu)}\frac{\Gamma(\varrho+1)}{\Gamma(k\gamma+\mu+\varrho)}\frac{[\phi(\mathrm{u})-\phi(\iota)]^{k\gamma+\mu+\varrho-1}}{\omega(\mathrm{u})}.
\end{aligned}
$$

□

(iii) This follows from Def. 3.3 directly.

Lemma 3.14 *Let $\alpha\in(0,1], \beta,\gamma>0, \mathrm{Re}(\mu)>0, \sigma\in\mathbb{R}, \lambda_\alpha=\frac{-\alpha}{1-\alpha}$ and $\omega>0$. Then,*

$$
\begin{aligned}
&{}^{C}\mathfrak{D}^{\alpha,\beta,\gamma,\mu}_{\iota,\sigma,\omega,\phi}\left[\frac{\mathbb{E}^{\gamma,\sigma}_{\beta,\mu}(\lambda_\alpha,(\phi(\mathrm{u})-\phi(\iota)))}{\omega(\mathrm{u})}\right] \\
&= \frac{\Delta(\alpha)}{1-\alpha}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\sigma)_k(\sigma)_n\lambda_\alpha^{k+n}}{\Gamma(\gamma(k+n)+2\mu-1)}\frac{\Gamma(k\gamma+\mu)}{k!n!\Gamma(k\beta+\mu)}\frac{[\phi(\mathrm{u})-\phi(\iota)]^{\gamma(k+n)+2\mu-2}}{\omega(\mathrm{u})}.
\end{aligned}
$$

In particular, for $\beta = \gamma$, we have

$$ {}^{C}\mathfrak{D}^{\alpha,\beta,\beta,\mu}_{\iota,\sigma,\omega,\phi}\left[\frac{\mathbb{E}^{\beta,1}_{\beta,\mu}(\lambda_\alpha,(\phi(\mathrm{u})-\phi(\iota)))}{\omega(\mathrm{u})}\right] = \frac{\Delta(\alpha)}{1-\alpha}\left[\frac{\mathbb{E}^{\beta,\sigma^2}_{\beta,2\mu-1}(\lambda_\alpha,(\phi(\mathrm{u})-\phi(\iota)))}{\omega(\mathrm{u})}\right]. $$

Proof By using Lemma 3.13, we have

$$ \begin{aligned} &{}^{C}\mathfrak{D}^{\alpha,\beta,\gamma,\mu}_{\iota,\sigma,\omega,\phi}\left[\frac{\mathbb{E}^{\gamma,\sigma}_{\beta,\mu}(\lambda_\alpha,(\phi(\mathrm{u})-\phi(\iota)))}{\omega(\mathrm{u})}\right] \\ &= \sum_{k=0}^{\infty}\frac{(\sigma)_k\lambda_\alpha^k}{k!\Gamma(k\beta+\mu)}\,{}^{C}\mathfrak{D}^{\alpha,\beta,\gamma,\mu}_{\iota,\sigma,\omega,\phi}\left[\frac{(\phi(\mathrm{u})-\phi(\iota))^{k\gamma+\mu-1}}{\omega(\mathrm{u})}\right] \\ &= \frac{\Delta(\alpha)}{1-\alpha}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\sigma)_k(\sigma)_n\lambda_\alpha^{k+n}}{\Gamma(\gamma(k+n)+2\mu-1)}\frac{\Gamma(k\gamma+\mu)}{k!n!\Gamma(k\beta+\mu)}\frac{[\phi(\mathrm{u})-\phi(\iota)]^{\gamma(k+n)+2\mu-2}}{\omega(\mathrm{u})}. \end{aligned} $$

If $\beta = \gamma$, then we obtain

$$ \begin{aligned} &{}^{C}\mathfrak{D}^{\alpha,\beta,\beta,\mu}_{\iota,\sigma,\omega,\phi}\left[\frac{\mathbb{E}^{\beta,\sigma}_{\beta,\mu}(\lambda_\alpha,(\phi(\mathrm{u})-\phi(\iota)))}{\omega(\mathrm{u})}\right] \\ &= \frac{\Delta(\alpha)}{1-\alpha}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\sigma)_k(\sigma)_n\lambda_\alpha^{k+n}}{\Gamma(\beta(k+n)+2\mu-1)}\frac{1}{k!n!}\frac{[\phi(\mathrm{u})-\phi(\iota)]^{\beta(k+n)+2\mu-2}}{\omega(\mathrm{u})} \\ &= \frac{\Delta(\alpha)}{1-\alpha}\left[\frac{\mathbb{E}^{\beta,\sigma^2}_{\beta,2\mu-1}(\lambda_\alpha,(\phi(\mathrm{u})-\phi(\iota)))}{\omega(\mathrm{u})}\right]. \end{aligned} $$

Then, the proof is finished. □

Example 3.15 For $\alpha = 0.85, 0.75, 0.65, 0.55$, we find the value of

$$ {}^{C}\mathfrak{D}^{\alpha,2,2,\frac{3}{2}}_{1,1,\omega,\phi}\left[\frac{\mathbb{E}^{2,1}_{2,\frac{3}{2}}(\lambda_\alpha,(\phi(\mathrm{u})-\phi(1)))}{\omega(\mathrm{u})}\right], $$

where $\omega(\mathrm{u}) = \mathrm{u}$ and $\phi(\mathrm{u}) = \log(\mathrm{u}), \forall \mathrm{u} \in [1,2]$.

According to Lemma 3.14, we have

$$ \begin{aligned} {}^{C}\mathfrak{D}^{\alpha,2,2,\frac{3}{2}}_{1,1,\omega,\phi}\left[\frac{\mathbb{E}^{2,1}_{2,\frac{3}{2}}(\lambda_\alpha,(\phi(\mathrm{u})-\phi(\iota)))}{\omega(\mathrm{u})}\right] &= \frac{\Delta(\alpha)}{1-\alpha}\left[\frac{\mathbb{E}^{2,1}_{2,2}(\lambda_\alpha,\log(\mathrm{u}))}{\mathrm{u}}\right] \\ &= \frac{\Delta(\alpha)}{1-\alpha}\sum_{k=0}^{\infty}\frac{\lambda_\alpha^k(\log(\mathrm{u}))^{2k+1}}{\Gamma(2k+2)\mathrm{u}}. \end{aligned} $$

Figure 1 shows the effectiveness of the generalized weighted Caputo fractional derivatives with respect to another function $\phi(\mathrm{u}) = \log(\mathrm{u})$, of various values $\alpha = 0.85, 0.75, 0.65, 0.55$, for Example 3.15.

4 Conclusions

Fractional operators with Mittag–Leffler kernels, whose integer order as a limiting case by means of delta Dirac functions, have played an important role in developing the theory

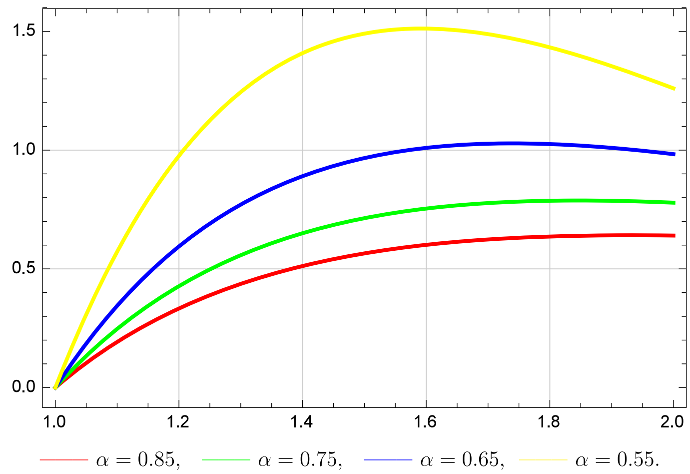

Figure 1 Plots of Example 3.15, for various values of α.

of fractional calculus. The corresponding fractional integrals of the ABC and ABR fractional operators with one parameter α consist of certain linear combination of the function itself and a Riemann–Liouville-type fractional integral that forces their linear fractional equations with constant coefficients to have a trivial solution. Adding a Mittag–Leffler function with three parameters inside the kernel of ABC enables us to obtain nontrivial solutions. In addition, the kernel may be singular in some cases. In this article, we have defined and investigated the weighted fractional operators, both derivatives and integrals, with respect to another function and with modified three-parameter Mittag–Leffler kernels. Such operators allow us to obtain operators with exponential kernels as special cases. In addition, several new and existing operators of nonsingular and singular kernels were obtained as special cases. Further, we have studied many important properties related to our new operators, such as series versions involving Riemann–Liouville fractional integrals and weighted Laplace transform with respect to another function.

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