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Normalized solutions for the discrete Schrödinger equations

Qilin Xie[1] and Huafeng Xiao[2,3*]

*Correspondence:
huafeng@gzhu.edu.cn
[2]School of Mathematics and Information Science, Guangzhou University, Guangzhou 510006, Guangdong, China
[3]Guangzhou Center for Applied Mathematics, Guangzhou University, Guangzhou 510006, Guangdong, China
Full list of author information is available at the end of the article

Abstract

In the present paper, we consider the existence of solutions with a prescribed l^2-norm for the following discrete Schrödinger equations,

$$\begin{cases} -\Delta^2 u_{k-1} - f(u_k) = \lambda u_k & k \in \mathbb{Z}, \\ \sum_{k\in\mathbb{Z}} |u_k|^2 = \alpha^2, \end{cases}$$

where $\Delta^2 u_{k-1} = u_{k+1} + u_{k-1} - 2u_k$, $f \in C(\mathbb{R})$, α is a fixed constant, and $\lambda \in \mathbb{R}$ arises as a Lagrange multiplier. To get the solutions, we investigate the corresponding minimizing problem with the l^2-norm constraint:

$$E_\alpha = \inf\left\{\frac{1}{2}\sum|\Delta u_{k-1}|^2 - \sum F(u_k) : \sum|u_k|^2 = \alpha^2\right\}.$$

An elaborative analysis on a minimizing sequence with respect to E_α is obtained. We prove that there is a constant $\alpha_0 \geq 0$ such that there exists a global minimizer if $\alpha > \alpha_0$, and there exists no global minimizer if $\alpha < \alpha_0$. It seems that it is the first time to consider the solution with a prescribed l^2-norm of the discrete Schrödinger equations.

MSC: 35Q51; 39A12; 39A70

Keywords: Normalized solutions; Lagrange multiplier; Non-vanishing; Minimizer

1 Introduction and main results

In the present paper, we consider the following discrete Schrödinger equations

$$\begin{cases} -\Delta^2 u_{k-1} - f(u_k) = \lambda u_k & k \in \mathbb{Z}, \\ \sum_{k\in\mathbb{Z}} |u_k|^2 = \alpha^2, \end{cases} \tag{P_α}$$

where $f \in C(\mathbb{R})$, $\alpha > 0$ is a given constant, and $\lambda \in \mathbb{R}$ arises as a Lagrange multiplier. Here $\Delta u_{k-1} = u_k - u_{k-1}$ and $\Delta^2 = \Delta(\Delta)$ is the one dimensional discrete Laplacian operator.

Discrete Schrödinger equations play an important role in many areas, such as nonlinear optics [9], biomolecular chains [12], and Bose-Einstein condensates [16]. For more applications, we refer to [10, 11] and references therein.

Many authors concentrated on the periodic case of the equations, such as [6, 18–20, 22, 24, 28, 31–34]. As to the nonperiodic case, in Ma and Guo [17] and Zhang and Pankov [29], the authors derived a discrete version of compact embedding theorem and obtained the nontrivial solution of discrete Schrödinger equations with a coercive potential by calculus of variations. In [8], Chen et al. investigated the sign-changing ground state solutions for a class of discrete nonlinear Schrödinger equations. In Lin et al. [13], the authors obtained the existence of the homoclinic solutions when the nonlinearity is asymptotically linear at infinity. We refer the readers to [7, 21, 26, 27, 30] for related results.

The main feature of equation (P_α) is that the desired solution have a priori prescribed l^2-norm. The solutions with this type are usually referred as normalized solutions. This kind of normalized solutions have been widely studied in the Schrödinger equations, the continuous case. We refer the readers to [1, 2, 4, 5, 23]. However, little results have been known concerning normalized solutions with respect to the discrete Schrödinger equations.

This kind of discrete Schrödinger equations actually has been studied in the past twenty year. Weinstein [25] considered excitation thresholds for ground state localized modes, sometimes referred to as 'breathers', for the wave equations of nonlinear Schrödinger type. Excitation thresholds are rigorously characterized by variational methods. The excitation threshold is related to the optimal constant in a class of discrete interpolation inequalities related to the Hamiltonian energy.

In this paper, we will investigate the solutions $(u_\alpha, \lambda_\alpha)$ with a priori prescribed l^2-norm of equation (P_α) by variational methods. More precisely, we consider a constrained variational problem as follows. Under a general assumption (f_1) on the nonlinearity,

(f_1) $f \in C(\mathbb{R},\mathbb{R})$, and there exist $C > 0$ and $p > 2$ such that

$$\bigl|f(t)\bigr| \le C\bigl(|t| + |t|^{p-1}\bigr) \quad \text{for any } t \in \mathbb{R},$$

it is possible to define a C^1 functional $I: l^2 \to \mathbb{R}$ by

$$I(u) = \frac{1}{2}\sum_{k\in\mathbb{Z}} |\Delta u_{k-1}|^2 - \sum_{k\in\mathbb{Z}} F(u_k),$$

where $u = (u_k)_{k\in\mathbb{Z}}$ and $F(t) = \int_0^t f(s)\,ds$. Then the solutions of (P_α) can be characterized as critical points of I restrained on the constraint,

$$\mathcal{M}_\alpha = \left\{u \in l^2 : \sum_{k\in\mathbb{Z}} |u_k|^2 = \alpha^2\right\}.$$

If u_α is a critical point of I on $\mathcal{M}_\alpha$, then u_α is a solution of equation (P_α), where λ_α is determined as the Lagrange multiplier. It is evident to check that I is bounded from below on $\mathcal{M}_\alpha$. Thus, the existence of a minimizer of the following well-defined infimum is expected,

$$E_\alpha = \inf_{u\in\mathcal{M}_\alpha} I(u). \tag{1.1}$$

To get some suitable properties on E_α, we need the following assumptions on f,

(f_2) $f(t) = o(t)$ as $t \to 0$.

(f_3) $2F(t) < f(t)t$ for any $t \in \mathbb{R} \setminus \{0\}$.

Therefore, there exists a $\alpha_0 \geq 0$ such that (see (2.12) in the proof of Theorem 1.2),

$$E_\alpha = 0 \quad \text{if } 0 < \alpha \leq \alpha_0, \qquad E_\alpha < 0 \quad \text{if } \alpha > \alpha_0. \tag{1.2}$$

We have an accurate description of the minimizing sequence on $\mathcal{M}_\alpha$ with respect to E_α. More precisely, our main results are stated as follows.

Theorem 1.1 *Assume that* (f_1)-(f_3) *and* $\alpha > 0$. *If* $\{u^n\}_{n\in\mathbb{N}} \subset \mathcal{M}_\alpha$ *is a minimizing sequence with respect to* E_α, *then one of the following cases holds*:

(i) (*vanishing*) $u^n \to 0$ *in* l^q *for* $q \in (2, \infty]$ *as* $n \to \infty$.

(ii) (*nonvanishing*) *there exist* $u_\alpha \in \mathcal{M}_\alpha$ *and a family* $\{k_n\}_{n\in\mathbb{N}} \subset \mathbb{N}$ *such that*

$$k_n * u^n \to u_\alpha \quad \text{in } l^2$$

as $n \to \infty$, *where we denote that* $j * u \equiv (u_{k+j})$.

Theorem 1.2 *Assume that* (f_1)-(f_3) *hold. There exists* $\alpha_0 \geq 0$ *satisfying* (1.2) *such that the following statements hold*:

(i) *if* $0 < \alpha < \alpha_0$, *there is no minimizer with respect to* E_α;

(ii) *if* $\alpha > \alpha_0$, *there exists a minimizer with respect to* E_α. *Moreover, there exists a couple of solution* $(u_\alpha, \lambda_\alpha) \in \mathcal{M}_\alpha \times \mathbb{R}^-$ *satisfying the following equation*:

$$-\Delta^2 u_{k-1} - f(u_k) = \lambda u_k, \quad k \in \mathbb{Z}.$$

Actually, we prove that there exists a constant $\alpha_0 \geq 0$ such that there exists a global minimizer if $\alpha > \alpha_0$, and there exists no global minimizer if $\alpha < \alpha_0$ in Theorem 1.2. It is natural to consider if $\alpha_0 = 0$ or not. The following theorem shows that it heavily depends on the behavior of f near 0.

Theorem 1.3 *Assume that* (f_1)-(f_3) *hold. Then*

(i) *the strict subadditivity property holds, i.e., for any* $\alpha + \beta > \alpha_0$,

$$E_{\alpha+\beta} < E_\alpha + E_\beta.$$

(ii) *if* $\lim_{t\to 0} \frac{F(t)}{t^4} = +\infty$, *then* $\alpha_0 = 0$.

Remark 1.4 An analysis of the behavior of a minimizing sequence with respect to E_α is obtained in Theorem 1.1. As to the continuous Schrödinger equations, it is a classical result so-called concentration-compactness principle from Lions [14, 15]. Some excitation thresholds for ground state localized results have been known in the discrete Schrödinger equations. Weinstein has considered the special class of the nonlinearities $|u|^{p-1}u$ in his early work [25]. In our Theorem 1.2, we research the normalized solutions to the discrete Schrödinger equations with a general nonlinearity. We find that whether the minimizing sequence vanishing or not, it heavily depends on the priori prescribed l^2-norm.

Remark 1.5 Lastly, a so-called strict subadditivity property is obtained in Theorem 1.3(i), which is similar to the celebrated results in Lions [14, 15]. We conclude that the reason for the existence of a minimizer only for $\alpha > \alpha_0$ is that the strict subadditivity holds for $\alpha > \alpha_0$.

2 Proof of the main results

In the following, we denote the universal positive constants by C. As usual, the standard real sequence space l^q, $q \in [1,\infty]$, endowed with the norm

$$\|u\|_q = \left(\sum_{k\in\mathbb{Z}} |u_k|^q\right)^{1/q}, \quad q \in [1,\infty), \qquad \|u\|_\infty = \sup_{k\in\mathbb{Z}} |u_k|,$$

where $u = (u_k)_{k\in\mathbb{Z}}$. The following embedding is well known,

$$l^{q_1} \subset l^{q_2}, \qquad \|u\|_{q_2} \le \|u\|_{q_1}, \quad 1 \le q_1 \le q_2 \le \infty.$$

For simplicity of writing, we define that $\|\Delta u\|_2 := (\sum_{k\in\mathbb{Z}} |\Delta u_k|^2)^{1/2}$.

Lemma 2.1 *Let $\{u^n\}_{n\in\mathbb{N}}$ be a sequence in l^2 satisfying $\lim_{n\to\infty} \|u^n\|_2 = \alpha$. If we set that $\widetilde{u}^n = \frac{\alpha}{\|u^n\|_2} u^n := a_n u^n$, the following fact holds*:

$$\widetilde{u}^n \in \mathcal{M}_\alpha \quad \text{and} \quad \lim_{n\to\infty} \bigl(I(\widetilde{u}^n) - I(u^n)\bigr) = 0.$$

Proof It is clear that $\lim_{n\to\infty} a_n = 1$. Moreover, by a direct computation, it follows that

$$\begin{aligned} I(\widetilde{u}^n) - I(u^n) &= \frac{a_n^2 - 1}{2} \|\Delta u^n\|_2^2 - \sum_{k\in\mathbb{Z}} \bigl(F(a_n u_k^n) - F(u_k^n)\bigr) \\ &= \frac{a_n^2 - 1}{2} \|\Delta u^n\|_2^2 - \sum_{k\in\mathbb{Z}} \int_0^1 f\bigl(u_k^n + (a_n - 1)s u_k^n\bigr)(a_n - 1)s u_k^n \, ds \\ &= \frac{a_n^2 - 1}{2} \|\Delta u^n\|_2^2 - (a_n - 1)\sum_{k\in\mathbb{Z}} \int_0^1 f\bigl(u_k^n + (a_n - 1)s u_k^n\bigr) s u_k^n \, ds. \end{aligned} \tag{2.1}$$

Under the assumption (f_1), we have

$$\begin{aligned} &\left|\sum_{k\in\mathbb{Z}} \int_0^1 f\bigl(u_k^n + (a_n - 1)s u_k^n\bigr) s u_k^n \, ds\right| \\ &\quad\le \sum_{k\in\mathbb{Z}} \int_0^1 C\Bigl(\bigl(|a_n| + 2\bigr)\bigl|u_k^n\bigr|^2 + \bigl(|a_n| + 2\bigr)^{p-1}\bigl|u_k^n\bigr|^p\Bigr) \, ds \\ &\quad\le \sum_{k\in\mathbb{Z}} C\Bigl(\bigl(|a_n| + 2\bigr)\bigl|u_k^n\bigr|^2 + \bigl(|a_n| + 2\bigr)^{p-1}\bigl|u_k^n\bigr|^p\Bigr) \\ &\quad\le C\Bigl(\bigl(|a_n| + 2\bigr)\|u^n\|_2^2 + \bigl(|a_n| + 2\bigr)^{p-1}\|u^n\|_p^p\Bigr). \end{aligned} \tag{2.2}$$

Since (2.1), (2.2), $\lim_{n\to\infty} a_n = 1$ and the boundedness of u^n in l^2, we achieve our conclusion. □

Lemma 2.2 *Under the assumptions* (f_1)-(f_3), *the following statements hold*:

(i) $-\infty < E_\alpha \le 0$ *for any* $\alpha > 0$.

(ii) $E_{\alpha+\beta} \le E_\alpha + E_\beta$ *for any* $\alpha, \beta > 0$.

(iii) $E_\alpha < 0$ *for sufficiently large* α.

(iv) $\alpha \mapsto E_\alpha$ *is nonincreasing and continuous*.

Proof (i) It follows from (f_1) that

$$\left|F(u)\right| \le C\left(|u|^2 + |u|^p\right).$$

For any $u \in \mathcal{M}_\alpha$, one has that

$$\begin{aligned} I(u) &= \frac{1}{2}\sum_{k\in\mathbb{Z}} |\Delta u_k|^2 - \sum_{k\in\mathbb{Z}} F(u_k) \\ &\ge -C\|u\|_2^2 - C\|u\|_p^p \\ &\ge -C\|u\|_2^2 - C\|u\|_2^p = -C_1 > -\infty, \end{aligned}$$

where $C_1 = C\alpha^2 + C\alpha^p$. Here we have used the fact that

$$\|u\|_q \le \|u\|_2 \quad \text{for any } u \in l^2 \text{ and } q \in (2, \infty].$$

Let $u^N = (\ldots, 0, \underbrace{(\frac{\alpha}{N^{1/2}}), \ldots, (\frac{\alpha}{N^{1/2}})}_{N}, 0, \ldots)$ and $\|u^N\|_2 = \alpha$. Moreover, it is easy to check that

$$\left\|u^N\right\|_p^p = \frac{\alpha^p}{N^{(p-2)/2}} \quad \text{and} \quad \left\|\Delta u^N\right\|_2^2 = \frac{2\alpha^2}{N}. \tag{2.3}$$

Combining (f_1) with (f_2), for any ε, there exists $C_\varepsilon > 0$ such that

$$\left|F(u)\right| \le \varepsilon|u|^2 + C_\varepsilon|u|^p. \tag{2.4}$$

For any $\delta > 0$, setting that $\varepsilon = \delta/(2\alpha^2)$ in (2.4) and $N_0 = [(2C_\varepsilon\alpha^p\delta^{-1})^{2/(p-2)}] + 1$. Thus, for any positive integer $N > N_0$, the following holds:

$$\left|\sum_{k\in\mathbb{Z}} F\left(u_k^N\right)\right| \le \varepsilon\left\|u^N\right\|_2^2 + C_\varepsilon\left\|u^N\right\|_p^p \le \varepsilon\alpha^2 + C_\varepsilon\frac{\alpha^p}{N^{(p-2)/2}} < \frac{\delta}{2} + \frac{\delta}{2} = \delta.$$

By the arbitrariness of $\delta > 0$, we obtain

$$\lim_{N\to\infty}\left|\sum_{k\in\mathbb{Z}} F\left(u_k^N\right)\right| = 0. \tag{2.5}$$

It follows from (2.3) and (2.5) that $\lim_{N\to\infty} I(u^N) = 0$. Here $E_\alpha \le 0$ follows directly. Then (i) holds.

(ii) Set that

$$\mathcal{C} = \left\{u = (u_k)_{k\in\mathbb{Z}} \in l^2 : \left\{k : |u_k| > 0\right\} \text{ is a finite set}\right\}.$$

Therefore, we know that $\mathcal{C}$ is dense in l^2. By the definition of E_α and E_β, for any $\varepsilon > 0$, there exist $u \in \mathcal{M}_\alpha \cap \mathcal{C}$ and $v \in \mathcal{M}_\beta \cap \mathcal{C}$ such that

$$I(u) \leq E_\alpha + \varepsilon, \qquad I(v) \leq E_\beta + \varepsilon,$$

respectively. Since $u, v \in \mathcal{C}$, by translation, there exist $n_0, r \in \mathbb{N}$ and $r < n_0$ such that

$$\operatorname{supp} u \subset B(-n_0, r), \qquad \operatorname{supp} v \subset B(n_0, r),$$

where $B(x, r)$ is a ball center at x with the radius r in the integer $\mathbb{Z}$. Thus, $u + v \in \mathcal{M}_{\alpha+\beta}$. Moreover, there is

$$E_{\alpha+\beta} \leq I(u+v) = I(u) + I(v) \leq E_\alpha + E_\beta + 2\varepsilon.$$

Then $E_{\alpha+\beta} \leq E_\alpha + E_\beta$ follows from the arbitrariness of ε.

(iii) For any $s \in \mathbb{R}$, let $u^{s,N} = (\ldots, 0, \underbrace{s, \ldots, s}_{N}, 0, \ldots)$. Then, $\|u^N\|_2^2 = Ns^2$ and

$$I\left(u^{s,N}\right) = \frac{1}{2} \sum_{k \in \mathbb{Z}} \left| \Delta u_k^{s,N} \right| - \sum_{k \in \mathbb{Z}} F\left(u_k^{s,N}\right) = s^2 - NF(s).$$

Taking s_0 such that $F(s_0) > 0$ by (f_3) and $N_0 := [s_0^2/F(s_0)] + 1$. Thus, for any $N > N_0$, we have

$$E_\alpha \leq I\left(u^{s_0,N}\right) = s_0^2 - NF(s_0) < 0.$$

Here one obtains that $E_\alpha < 0$ for $\alpha > N_0^{1/2} s_0$.

(iv) It follows from (i) and (ii) that

$$E_{\alpha+\beta} \leq E_\alpha + E_\beta \leq E_\alpha,$$

for any $\alpha, \beta > 0$. Thus, $\alpha \mapsto E_\alpha$ is nonincreasing. Fix $\alpha > 0$, we know that $E_{\alpha-\delta}$ and $E_{\alpha+\delta}$ are monotonic and bounded as $\delta \to 0^+$. Moreover, $E_{\alpha-\delta} \geq E_\alpha \geq E_{\alpha+\delta}$ and

$$\lim_{\delta \to 0^+} E_{\alpha-\delta} \geq E_\alpha \geq \lim_{\delta \to 0^+} E_{\alpha+\delta}.$$

To prove the continuous, it remains to prove the inverse inequalities.

(a) $\lim_{\delta \to 0^+} E_{\alpha-\delta} \leq E_\alpha$. If $E_\alpha = 0$, $\lim_{\delta \to 0^+} E_{\alpha-\delta} \leq E_\alpha$ holds. We consider the case $E_\alpha < 0$. Let $u \in \mathcal{M}_\alpha$ and $u_\delta = (1 - \delta/\alpha)u$ with $\delta \in (0, \alpha)$. It is easy to check that $\|u_\delta\|_2 = \alpha - \delta$ and $u_\delta \to u$ as $\delta \to 0^+$ in l^2. Therefore,

$$\lim_{\delta \to 0^+} E_{\alpha-\delta} \leq \lim_{\delta \to 0^+} I(u_\delta) = I(u).$$

By the arbitrariness of $u \in \mathcal{M}_\alpha$, $\lim_{\delta \to 0^+} E_{\alpha-\delta} \leq E_\alpha$ holds.

(b) $\lim_{\delta \to 0^+} E_{\alpha+\delta} \geq E_\alpha$. Since the left-hand side converges, it sufficient to consider the case $\delta = \frac{1}{n}$ with $n \in \mathbb{N}$. Let $u^n \in \mathcal{M}_{\alpha+1/n}$ and $I(u^n) \leq E_{\alpha+1/n} + \frac{1}{n}$ for any $n \in \mathbb{N}$. Thus,

$$\lim_{n \to \infty} I\left(u^n\right) = \lim_{\delta \to 0^+} E_{\alpha+\delta}.$$

Let $v^n = u^n/(1 + 1/(\alpha n))$ for any $n \in \mathbb{N}$. Moreover,

$$\|v^n\|_2 = \frac{\|u^n\|_2}{1 + 1/(\alpha n)} = \frac{\alpha + 1/n}{1 + 1/(\alpha n)} = \alpha,$$

which implies $v^n \in \mathcal{M}_\alpha$. By Lemma 2.1, we obtain

$$E_\alpha \le I(v^n) = I(u^n) + o(1)$$

as $n \to \infty$. Thus, $\lim_{\delta\to 0^+} E_{\alpha+\delta} = \lim_{n\to\infty} I(u^n) \ge E_\alpha$. The proof is completed. □

Lemma 2.3 *Under the assumptions* (f_1)-(f_3), *the following statements hold*:

(i) *Suppose that u is a minimizer on* $\mathcal{M}_\alpha$ *with respect to* E_α. *Then* $E_\beta < E_\alpha$ *for any* $\beta > \alpha$.

(ii) *Suppose that u and v are two minimizers on* $\mathcal{M}_\alpha$ *and* $\mathcal{M}_\beta$ *with respect to* E_α *and* E_β, *respectively. Then* $E_{\alpha+\beta} < E_\alpha + E_\beta$.

Proof (i) Suppose that u is a minimizer on $\mathcal{M}_\alpha$ with respect to E_α and $\beta > \alpha$. Consider the following function

$$I(tu) - t^2 I(u) = t^2 \sum_{k\in\mathbb{Z}} \left(F(u_k) - \frac{F(tu_k)}{t^2} \right) := t^2 g(t), \quad t \ge 1,$$

where $g(t) = \sum_{k\in\mathbb{Z}} (F(u_k) - \frac{F(tu_k)}{t^2})$ for $t \in [1, +\infty)$. Clearly, $g(1) = 0$. By (f_1), it is not difficult to prove that $g(t) \in C^1((1, +\infty))$ and

$$g'(t) = \frac{1}{t^3} \sum_{k\in\mathbb{Z}} \bigl(2F(tu_k) - f(tu_k)tu_k\bigr).$$

Since $u \in \mathcal{M}_\alpha$, there exists $k_0 \in \mathbb{Z}$ such that $u_{k_0} \neq 0$. Therefore,

$$g'(t) = \frac{1}{t^3} \sum_{k\in\mathbb{Z}} \bigl(2F(tu_k) - f(tu_k)tu_k\bigr) \le \frac{1}{t^3} \bigl(2F(tu_{k_0}) - f(tu_{k_0})tu_{k_0}\bigr) < 0,$$

for any $t > 1$. Combining with the above inequality and $g(1) = 0$, we obtain that $g(t) < 0$ for any $t > 1$. Set $\theta = \beta/\alpha$, it follows from $\theta u \in \mathcal{M}_\beta$ and $I(u) \le 0$ that

$$E_\beta \le I(\theta u) < \theta^2 I(u) = \theta^2 E_\alpha \le \theta E_\alpha \le E_\alpha. \tag{2.6}$$

(ii) Without loss of generality, taking $0 < \alpha \le \beta$ in the above inequalities, we obtain

$$E_{\alpha+\beta} < \frac{\alpha+\beta}{\beta} E_\beta = \frac{\alpha}{\beta} E_\beta + E_\beta \le E_\alpha + E_\beta.$$

This proof is completed. □

Lemma 2.4 *Assume that* $\{u^n\}_{n\in\mathbb{N}}$ *is bounded in* l^2 *and* $u^n \rightharpoonup u$ *weakly in* l^2. *Then there holds*,

$$\sum_{k\in\mathbb{Z}} F(u_k^n) = \sum_{k\in\mathbb{Z}} F(u_k^n - u_k) + \sum_{k\in\mathbb{Z}} F(u_k) + o(1), \tag{2.7}$$

as $n \to \infty$.

Proof This proof follows from the Brezis–Lieb Lemma (see [3]), for completeness, we state it here. Let $v^n = u^n - u$, then $v^n \rightharpoonup 0$ weakly in l^2 and $v_k^n \to 0$ for any $k \in \mathbb{Z}$ as $n \to \infty$. It follows from (f_1), the mean value theorem, and the Young inequality that

$$\begin{aligned}
\left|F(u_k^n) - F(v_k^n) - F(u_k)\right| &\le \left|F(u_k^n) - F(v_k^n)\right| + \left|F(u_k)\right| \\
&= \left|F(v_k^n + u_k) - F(v_k^n)\right| + \left|F(u_k)\right| \\
&= \left|f(v_k^n + \tau u_k)u_k\right| + \left|F(u_k)\right| \\
&\le C\big(\big(\left|v_k^n\right| + |u_k|\big) + \big(\left|v_k^n\right| + |u_k|\big)^{p-1}\big)|u_k| + \left|F(u_k)\right| \\
&\le C\big(\big(\left|v_k^n\right| + 2^p\left|v_k^n\right|^{p-1} + |u_k| + 2^p|u_k|^{p-1}\big)|u_k| + \left|F(u_k)\right| \\
&\le \varepsilon\phi_\varepsilon\big(\left|v_k^n\right|\big) + \psi_\varepsilon\big(|u_k|\big) + \left|F(u_k)\right|,
\end{aligned}$$

where $\tau \in [0,1]$, $\phi(|v_k^n|) = C(|v_k^n|^2 + |2v_k^n|^p)$ and

$$\psi_\varepsilon\big(|u_k|\big) = C\big(\big(1 + \varepsilon^{-1}\big)|u_k|^2 + \big(1 + \varepsilon^{1-p}\big)|2u_k|^p\big).$$

It follows that $\sum_{k\in\mathbb{Z}} \psi_\varepsilon(|u_k|) < \infty$ and $\sum_{k\in\mathbb{Z}} \phi_\varepsilon(|v_k^n|) < C < \infty$ for some constant C, independent on ε and n. Set

$$W_k^n = \max\big\{\left|F(u_k^n) - F(v_k^n) - F(u_k)\right| - \varepsilon\phi_\varepsilon\big(\left|v_k^n\right|\big), 0\big\},$$

then $W_k^n \le \psi_\varepsilon(|u_k|) + F(u_k)$. By the dominated convergence theorem, we have $\sum_{k\in\mathbb{Z}} W_k^n \to 0$ as $n \to \infty$. Therefore,

$$\left|F(u_k^n) - F(v_k^n) - F(u_k)\right| \le W_k^n + \varepsilon\phi_\varepsilon\big(\left|v_k^n\right|\big)$$

which implies that

$$\sum_{k\in\mathbb{Z}} \left|F(u_k^n) - F(v_k^n) - F(u_k)\right| \le C\varepsilon.$$

The result follows from the arbitrariness of $\varepsilon > 0$. This proof is completed. □

The proof Theorem 1.1 Assume that $\{u^n\}_{n\in\mathbb{N}} \subset \mathcal{M}_\alpha$ is a minimizing sequence with respect to E_α. If $\{u^n\}$ does not satisfy (i), we can assume that $u^n \nrightarrow 0$ in l^∞. In fact, if there exists $q_0 \in (2,\infty)$ such that $u^n \nrightarrow 0$ in l^{q_0}, then there exists a $\xi > 0$ such that

$$0 < \xi \le \left\|u^n\right\|_{q_0}^{q_0} \le \left\|u^n\right\|_\infty^{q_0-2}\left\|u^n\right\|_2^2 = \alpha^2\left\|u^n\right\|_\infty^{q_0-2},$$

which implies $u^n \nrightarrow 0$ in l^∞. There exist $\delta > 0$ and a family of $\{k_n\}_{n\in\mathbb{N}} \subset \mathbb{N}$ such that

$$\left|u_{k_n}^n\right| \ge \delta.$$

Set $k_n * u^n = (u_{k+k_n}^n)$ for any $n \in \mathbb{N}$. Here, $\|k_n * u^n\|_2 = \|u^n\|_2 = \alpha$. We assume that $k_n * u^n \rightharpoonup u_\alpha(\ne 0)$ in l^2. In the rest part, we try to prove $u_\alpha \in \mathcal{M}_\alpha$. Arguing indirectly, set

$$v^n = k_n * u^n - u_\alpha.$$

By Lemma 2.4, one has

$$\sum_{k\in\mathbb{Z}} F\left(u^n_{k+k_n}\right) = \sum_{k\in\mathbb{Z}} F(u_{\alpha,k}) + \sum_{k\in\mathbb{Z}} F\left(v^n_k\right) + o(1),$$

as $n \to \infty$. It is evident to check that

$$\left\|k_n * u^n\right\|_2^2 = \|u_\alpha\|_2^2 + \left\|v^n\right\|_2^2 + o(1),$$

and

$$\left\|\Delta\left(k_n * u^n\right)\right\|_2^2 = \|\Delta u_\alpha\|_2^2 + \left\|\Delta v^n\right\|_2^2 + o(1),$$

as $n \to \infty$. Combining with the above three inequalities, it obtains that

$$I\left(u^n\right) = I\left(k_n * u^n\right) = I(u_\alpha) + I\left(v^n\right) + o(1). \tag{2.8}$$

We claim that

$$v^n \to 0 \quad \text{in } l^p \text{ for any } p \in (2,\infty]. \tag{2.9}$$

Arguing indirectly, we can assume $v^n \nrightarrow 0$ in l^∞ similarly. By the boundedness of $\{v^n\}$ in l^2, there exist a family $\{z_n\} \subset \mathbb{Z}$ and $v \in l^2$ satisfying $v \neq 0$ such that $z_n * v^n \rightharpoonup v$ in l^2. Set $w^n = z_n * v^n - v$, then

$$\left\|k_n * v^n\right\|_2^2 = \|v\|_2^2 + \left\|w^n\right\|_2^2 + o(1),$$

and

$$\begin{aligned}&\left\|\Delta\left(k_n * v^n\right)\right\|_2^2 = \|\Delta v\|_2^2 + \left\|\Delta w^n\right\|_2^2 + o(1),\\ &I\left(v^n\right) = I\left(k_n * v^n\right) = I(v) + I\left(w^n\right) + o(1),\end{aligned}$$

as $n \to \infty$. Let $\|u_\alpha\|_2 = c_1$ and $\|v\|_2 = c_2$ and $\delta^2 = \alpha^2 - c_1^2 - c_2^2$. Thus $\lim_{n\to\infty} \|w^n\|_2 = \delta \geq 0$.

If $\delta > 0$, setting $\widetilde{w}^n = a_n w^n$ and $a_n = \delta/\|w^n\|_2$ in Lemma 2.1, we have $\widetilde{w}^n \in \mathcal{M}_\delta$ and $I(\widetilde{w}^n) = I(w^n) + o(1)$. Thus,

$$\begin{aligned}I\left(u^n\right) &= I(u_\alpha) + I(v) + I\left(w^n\right) + o(1)\\ &= I(u_\alpha) + I(v) + I\left(\widetilde{w}^n\right) + o(1)\\ &\geq I(u_\alpha) + I(v) + E_\delta + o(1),\end{aligned}$$

as $n \to \infty$, which implies that

$$E_\alpha \geq I(u_\alpha) + I(v) + E_\delta \geq E_{c_1} + E_{c_2} + E_\delta \geq E_{c_1+c_2+\delta} = E_\alpha. \tag{2.10}$$

Hence u_α and ν are two minimizers on $\mathcal{M}_{c_1}$ and $\mathcal{M}_{c_1}$ with respect to E_{c_1} and E_{c_2}. By Lemma 2.2, we have

$$E_{c_1} + E_{c_2} > E_{c_1+c_2}.$$

It contradicts (2.10).

If $\delta = 0$, then $\alpha = c_1 + c_2$ and $\lim_{n\to\infty} \|w^n\|_2 = 0$. Similar to the proof of (2.5), we can prove that $\lim_{n\to\infty} \sum_{k\in\mathbb{Z}} F(w_k^n) = 0$ and $\liminf_{n\to\infty} I(w^n) \geq 0$. It follows that

$$E_\alpha \geq I(u_\alpha) + I(\nu) \geq E_{c_1} + E_{c_2} \geq E_{c_1+c_2} = E_\alpha,$$

which implies that u_α and ν are two minimizers on $\mathcal{M}_{c_1}$ and $\mathcal{M}_{c_1}$ with respect to E_{c_1} and E_{c_2}. By Lemma 2.3, one obtains

$$E_{c_1} + E_{c_2} > E_{c_1+c_2},$$

which is a contradiction. Thus, our claim (2.9) holds.

Lastly, we complete the proof by getting $\lim_{n\to\infty} \|\nu^n\|_2 = 0$, that is $\|u_\alpha\|_2 = \alpha$. It is sufficient to prove that $c_1 = \alpha$. Otherwise, $c_1 < \alpha$ holds. By (2.4) and (2.9), one has $\lim_{n\to\infty} \sum_{k\in\mathbb{Z}} F(\nu_k^n) = 0$ and

$$\liminf_{n\to\infty} I\left(\nu^n\right) \geq 0.$$

Combining with the above inequality and taking the limit in (2.8), we obtain $E_\alpha \geq I(u_\alpha)$. Then it follows from Lemma 2.2 and $u_\alpha \in \mathcal{M}_{c_1}$ that

$$E_\alpha \geq I(u_\alpha) \geq E_{c_1} \geq E_\alpha, \tag{2.11}$$

which implies $E_{c_1} = E_\alpha$. Moreover, u_α is a minimizer with respect to E_{c_1}. By Lemma 2.3(i), we obtain $E_{c_1} > E_\alpha$ for $c_1 < \alpha$. It contradicts (2.11). Then the desired result $\|u_\alpha\|_2 = \alpha$ and (ii) hold. This completes the proof. □

The proof of Theorem 1.2 Define that

$$\alpha_0 = \inf\{\alpha > 0 : E_\alpha < 0\}. \tag{2.12}$$

By Lemma 2.2, α_0 is well defined and the following fact holds:

$$E_\alpha = 0 \quad \text{if } 0 < \alpha \leq \alpha_0, \qquad E_\alpha < 0 \quad \text{if } \alpha > \alpha_0.$$

(i) Arguing indirectly, if $0 < \alpha < \alpha_0$, there exists a minimizer with respect to E_α. By the definition of α_0, we have $E_\alpha = 0$. It follows from Lemma 2.3(i) that

$$0 = E_\alpha > E_{\alpha_0},$$

which is impossible for $E_{\alpha_0} = 0$.

(ii) If $\alpha > \alpha_0$, $E_\alpha < 0$. Assume that $\{u^n\}_{n\in\mathbb{N}} \subset \mathcal{M}_\alpha$ is a minimizing sequence with respect to E_α. It is sufficient to show that $\{u^n\}$ satisfies Theorem 1.1(ii). Arguing indirectly, if Theorem 1.1(i) holds, that is $u^n \to 0$ in l^q for any $q \in (2,\infty]$. Thus, we can prove that

$$\lim_{n\to\infty}\sum_{k\in\mathbb{Z}} F\left(u_k^n\right) = 0.$$

It follows that

$$E_\alpha = \lim_{n\to\infty} I\left(u^n\right) \geq -\lim_{n\to\infty}\sum_{k\in\mathbb{Z}} F\left(u_k^n\right) = 0,$$

which is impossible for $E_\alpha < 0$. So $k_n * u^n \to u_\alpha$ in l^2 and u_α is a minimizer on $\mathcal{M}_\alpha$ with respect to E_α. Therefore, there exists $\lambda_\alpha \in \mathbb{R}$ such that $I'(u) - \lambda_\alpha u_\alpha = 0$, i.e., $(u_\alpha, \lambda_\alpha)$ is a couple of solution to the following equation

$$-\Delta^2 u_{k-1} - f(u_k) = \lambda u_k, \quad k \in \mathbb{Z}.$$

Moreover,

$$\lambda\alpha^2 = \left(I'(u_\alpha), u_\alpha\right) = \|\Delta u_\alpha\|_2^2 - \sum_{k\in\mathbb{Z}} f(u_{\alpha,k})u_{\alpha,k} < \|\Delta u_\alpha\|_2^2 - \sum_{k\in\mathbb{Z}} 2F(u_{\alpha,k}) = 2E_\alpha < 0,$$

which implies $\lambda < 0$. The proof is completed. □

The proof of Theorem 1.3 (i) Without loss of generality, we assume $0 < \alpha \leq \beta$. We divide into three cases: (1) $E_\alpha = E_\beta = 0$; (2) $E_\alpha = 0, E_\beta < 0$; (3) $E_\alpha < 0, E_\beta < 0$. If Case (1), it is evident that $E_\alpha + E_\beta = 0 > E_{\alpha+\beta}$ for $\alpha + \beta > \alpha$. If Case (2), there exists a minimizer with respect to E_β by Theorem 1.2(ii). Then by Lemma 2.3(i), we obtain

$$E_\beta > E_{\alpha+\beta}.$$

Lastly, in case (3), there exist two minimizers with respect to E_α and E_β by Theorem 1.2(ii), respectively. Our conclusion follows from Lemma 2.3(ii).

(ii) For any fixed $\alpha > 0$, take $u^{\alpha,N} = (\ldots, 0, \underbrace{(\frac{\alpha}{N^{1/2}}), \ldots, (\frac{\alpha}{N^{1/2}})}_{N}, 0, \ldots)$, then $u^{\alpha,N} \in \mathcal{M}_\alpha$. It follows from $\lim_{t\to 0}\frac{F(t)}{t^4} = +\infty$ that for $M > \alpha^{-2}$ there exists $\delta > 0$ such that $|t| < \delta$,

$$F(t) \geq Mt^4.$$

Let N be large such that $\frac{\alpha}{N^{1/2}} < \delta$, then

$$\begin{aligned} E_\alpha &\leq I\left(u^{\alpha,N}\right) \\ &= \frac{1}{2}\sum_{k\in\mathbb{Z}}\left|\Delta u_k^{\alpha,N}\right| - \sum_{k\in\mathbb{Z}} F\left(u_k^{\alpha,N}\right) \\ &= \frac{\alpha^2}{N} - NF\left(\frac{\alpha}{N^{1/2}}\right) \end{aligned}$$

$$\leq \frac{\alpha^2}{N} - NM\left(\frac{\alpha}{N^{1/2}}\right)^4 = \frac{\alpha^2}{N}\left(1 - M\alpha^2\right) < 0.$$

Thus, $E_\alpha < 0$ for any $\alpha > 0$, which implies that $\alpha_0 = 0$. The proof is completed. □

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Xie wrote the main manuscript text. All authors reviewed the manuscript.

Author details
[1]School of Mathematics and Statistics, Guangdong University of Technology, Guangzhou, Guangdong 510520, China. [2]School of Mathematics and Information Science, Guangzhou University, Guangzhou 510006, Guangdong, China. [3]Guangzhou Center for Applied Mathematics, Guangzhou University, Guangzhou 510006, Guangdong, China.

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References

1. Bartsch, T., De Valeriola, S.: Normalized solutions of nonlinear Schrödinger equations. Arch. Math. **100**, 75–83 (2013)
2. Bartsch, T., Soave, N.: A natural constraint approach to normalized solutions on nonlinear Schrödinger equations and systems. J. Funct. Anal. **272**, 4998–5037 (2017)
3. Brezis, H., Lieb, E.: A relation between pointwise convergence of functions and convergence of functionals. Proc. Am. Math. Soc. **88**, 486–490 (1983)
4. Cazenave, T.: Semilinear Schrödinger Equations. Courant Lecture Notes in Mathematics, vol. 10. Am. Math. Soc., Providence (2003)
5. Cazenave, T., Lions, P.L.: Orbital stablity of standing waves for some nonlinear Schrödinger equations. Commun. Math. Phys. **85**, 549–561 (1982)
6. Chen, G., Ma, S.W.: Homoclinic solutions of discrete nonlinear Schrödinger equations with asymptotically or super linear terms. Appl. Math. Comput. **232**, 787–798 (2014)
7. Chen, G., Schechter, M.: Non-periodic discrete Schrödinger equations: ground state solutions. Z. Angew. Math. Phys. **67**(3), 72 (2016)
8. Chen, S.T., Tang, X.H., Yu, J.: Sign-changing ground state solutions for discrete nonlinear Schrödinger equations. J. Differ. Equ. Appl. **25**(2), 202–218 (2019)
9. Christodoulides, D.N., Lederer, F., Silberberg, Y.: Discretizing light behaviour in linear and nonlinear waveguide lattices. Nature **424**, 817–823 (2003)
10. Kevreides, P.G., Rasmussen, K., Bishop, A.R.: The discrete nonlinear Schrödinger equation: a survey of recent results. Int. J. Mod. Phys. B **15**, 2883–2900 (2001)
11. Kivshar, Y.S., Agrawal, G.P.: Optical Solitons: From Fibers to Photonic Crystals. Academic Press, San Diego (2003)
12. Kopidakis, G., Aubry, S., Tsironis, G.P.: Targented enery transfer through discrete breathers in nonlinear systems. Phys. Rev. Lett. **87**, 165501 (2001)
13. Lin, G.H., Zhou, Z., Yu, J.: Ground state solutions of discrete asymptotically linear Schrödinger equations with bounded and non-periodic potentials. J. Dyn. Differ. Equ. **32**(2), 527–555 (2020)
14. Lions, P.-L.: The concentration-compactness principle in the calculus of variations. The locally compact case, part 1. Ann. Inst. Henri Poincaré **1**, 109–145 (1984)
15. Lions, P.-L.: The concentration-compactness principle in the calculus of variations. The locally compact case, part 2. Ann. Inst. Henri Poincaré **1**, 223–283 (1984)
16. Livi, R., Franzosi, R., Oppo, G.-L.: Self-localization of Bose-Einstein condensates in optical lattices via boundary dissipation. Phys. Rev. Lett. **97**, 060401 (2006)
17. Ma, M., Guo, Z.: Homoclinic orbits for second order self-adjoint difference equations. J. Math. Anal. Appl. **323**(1), 513–521 (2006)
18. Ma, M., Guo, Z.: Homoclinic orbits and subharmonics for nonlinear second order difference equations. Nonlinear Anal. **67**(6), 1737–1745 (2007)
19. Pankov, A.: Gap solitons in periodic discrete nonlinear Schrödinger equations. Nonlinearity **19**, 27–40 (2006)
20. Pankov, A.: Gap solitons in periodic discrete nonlinear Schrödinger equations II: a generalized Nehari manifold approach. Discrete Contin. Dyn. Syst. **19**, 419–430 (2007)

21. Pankov, A., Zhang, G.: Standing wave solutions for discrete nonlinear Schrödinger equations with unbounded potentials and saturable nonlinearity. J. Math. Sci. **177**(1), 71–82 (2011)
22. Shi, H., Zhang, H.: Existence of gap solitons in periodic discrete nonlinear Schrödinger equations. J. Math. Anal. Appl. **361**, 411–419 (2010)
23. Shibata, M.: Stable standing waves of nonlinear Schrödinger equations with a general nonlinear term. Manuscr. Math. **143**, 221–237 (2014)
24. Tang, X.H.: Non-Nehari manifold method for periodic discrete superlinear Schrödinger equation. Acta Math. Sin. Engl. Ser. **32**(4), 463–473 (2016)
25. Weintein, M.: Excitation thresholds for nonlinear localized modes on lattices. Nonlinearity **12**, 673–691 (1999)
26. Xie, Q.L.: Multiple solutions for the nonhomogeneous discrete nonlinear Schrödinger equation. Appl. Math. Lett. **91**, 144–150 (2019)
27. Xie, Q.L.: Solutions for discrete Schrödinger equations with a nonlocal term. Appl. Math. Lett. **114**, 106930 (2021)
28. Yu, J., Guo, Z., Zou, X.: Periodic solutions of second order self-adjoint difference equations. J. Lond. Math. Soc. **71**(1), 146–160 (2005)
29. Zhang, G., Pankov, A.: Standing waves of the discrete nonlinear Schrödinger equations with growing potentials. Commun. Math. Anal. **5**(2), 38–49 (2008)
30. Zhang, G., Pankov, A.: Standing wave solutions of the discrete non-linear Schrödinger equations with unbounded potentials, II. Appl. Anal. **89**(9), 1541–1557 (2010)
31. Zhou, Z., Yu, J.: On the existence of homoclinic solutions of a class of discrete nonlinear periodic systems. J. Differ. Equ. **249**, 1199–1212 (2010)
32. Zhou, Z., Yu, J., Chen, Y.: On the existence of gap solitons in a periodic discrete nonlinear Schrödinger equation with saturable nonlinearity. Nonlinearity **23**, 1727–1740 (2010)
33. Zhou, Z., Yu, J., Chen, Y.: Homoclinic solutions in periodic difference equations with saturable nonlinearity. Sci. China Math. **54**, 83–93 (2011)
34. Zhou, Z., Yu, J., Guo, Z.: Periodic solutions of higher-dimensional discrete systems. Proc. R. Soc. Edinb., Sect. A **134**(5), 1013–1022 (2004)

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