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Generalized fractional calculus in Banach spaces and applications to existence results for boundary value problems

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Abstract

In this paper, we present the definitions of fractional integrals and fractional derivatives of a Pettis integrable function with respect to another function. This concept follows the idea of Stieltjes-type operators and should allow us to study fractional integrals using methods known from measure differential equations in abstract spaces. We will show that some of the well-known properties of fractional calculus for the space of Lebesgue integrable functions also hold true in abstract function spaces. In particular, we prove a general Goebel–Rzymowski lemma for the De Blasi measure of weak noncompactness and our fractional integrals.

We suggest a new definition of the Caputo fractional derivative with respect to another function, which allows us to investigate the existence of solutions to some Caputo-type fractional boundary value problems. As we deal with some Pettis integrable functions, the main tool utilized in our considerations is based on the technique of measures of weak noncompactness and Mönch's fixed-point theorem. Finally, to encompass the full scope of this research, some examples illustrating our main results are given.

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1 Preliminaries

The domain of fractional calculus is a very rich field because of its applications, for instance, in wave propagation in viscoelastic horns, sound-wave propagation or fractional models and controls (see [5, 16, 22]). There are several definitions for fractional integrals and for fractional derivatives [19, 36]. We are interested in the most general form of such operators. Till now, the most general known definition of the fractional operators seems to be the fractional integrals and derivatives of a Lebesgue function f with respect to another function g (see [36, Sect. 18.2], [19, Sect. 2.5] and [5]). However, let us mention that this definition allows us to operate only on real-valued functions. In the past decades, this general definition has proven its applicability in many and different natural situations, for instance, in [5], starting with the exponential growth model, the same problem was described by a fractional differential equation, and we shall see that the choice of the function g determines the accuracy of the model.

Our goal is to expand the applications of such an approach for vector-valued functions. Recently, considerable attention has been paid to the theory of fractional calculus in abstract spaces, which is more complicated and different from the classical fractional calculus of real-valued functions. This is due to the fact that some of the long-known properties of the real-valued function do not carry over into arbitrary Banach spaces. For instance, the classical fundamental theorem of calculus in Banach spaces is more complicated than the standard one. In addition, the weak absolute continuity of Banach-valued functions does not necessary imply strong or everywhere weak differentiability.

The aim of this paper is two-fold. On the one hand, we define and discuss the properties of the generalized form of the new fractional operators applied for the class of Pettis integrable functions that seems to be interesting in itself. On the other hand, we apply those results in order to ensure the existence of weakly continuous solutions for some boundary value problems of fractional order.

We should at least briefly recall why we discuss as one topic the fractional calculus with Orlicz spaces. This goes back to the origin of fractional calculus and fractional operators in function spaces. It is motivated by some applications to integral equations or partial differential equations [24, 27]. On the other hand, Pettis integrability is also strictly related to some weak integrability conditions in Orlicz spaces ([38], for instance).

However, our results complement some of those obtained in [1, 3, 4, 11, 12, 29–31, 35] or [39]), dealing with the properties of the fractional integral and differential operators when acting on the space of Pettis integrable functions.

Let us recall that a function $\psi:\mathbb{R}^+\to\mathbb{R}^+$ is said to be a Young function if ψ is increasing, even, convex, and continuous with $\psi(0)=0$ and $\lim_{u\to\infty}\psi(u)=\infty$). For any Young function ψ, the function $\widetilde{\psi}:\mathbb{R}^+\to\mathbb{R}^+$ defined by $\sup_{v\geq 0}\{v|u|-\psi(v)\}$ is called the Young complement of ψ and it is well known that $\widetilde{\psi}$ is a Young-type function as well.

The Orlicz space $L_\psi=L_\psi([a,b],\mathbb{R})$ consists of all (classes of) measurable functions $x:[a,b]\to\mathbb{R}$ for which

$$\|x\|_\psi := \inf\left\{k>0:\int_a^b \psi\left(\frac{|x(s)|}{k}\right)ds<1\right\}, \tag{1}$$

is finite (see, e.g., [20]). The particular choice $\psi(u)=\psi_p(u):=\frac{1}{p}|u|^p$, $p\in[1,\infty)$ leads to the Lebesgue space $L_p=L_p([a,b],\mathbb{R})$, $p\in[1,\infty)$. In this case, it can be easily seen that $\tilde{\psi}_p=\psi_{\tilde{p}}$ with $\frac{1}{p}+\frac{1}{\tilde{p}}=1$ for $p>1$.

In this connection, it is worth recalling that, for any Young function ψ, we have $\psi(u-v)\leq\psi(u)-\psi(v)$ and $\psi(\rho u)\leq\rho\psi(u)$ hold for any $u,v\in\mathbb{R}$ and $\rho\in[0,1]$. Also, for the nontrivial Young function ψ, $L_\infty\subset L_\psi\subset L_\psi$. For further properties of Young functions and Orlicz spaces generated by such functions we refer the reader to [2, 20, 35].

In the forthcoming pages E will be considered as a Banach space with norm $\|\cdot\|$ and with its dual space E^*. Also, E_w denotes the space E when endowed with its weak topology $\sigma(E,E^*)$. Let $C[I,E]$ denote the Banach space of (strongly) continuous functions $x:I\to E$ endowed by the norm $\|x\|_0=\sup_{t\in I}\|x(t)\|$. By $C[I,E_w]$ we denote the Banach space of all weakly continuous functions $x:I\to E$ with its weak topology (i.e., generated by continuous linear functionals on E).

Throughout this paper, we let g be a positive increasing function on an interval $I := [a,b]$, having a positive continuous derivative, with $g(a) = 0$ (see, e.g., [19, Sect. 2.5] or [36, Sect. 18.2]).

In this paper, we will have one more important class of functions. Namely, we let $\vartheta : \mathbb{R}^+ \to \mathbb{R}^+$ be a Hölderian function, i.e., ϑ is increasing and continuous with $\vartheta(0) = 0$. The (generalized) Hölder space $\mathcal{C}^{\vartheta}[I,E]$ consists, by definition, of all $x \in C[I,E]$ satisfying

$$\left\| x(t) - x(s) \right\| \leq L\vartheta\left(\left|g(t) - g(s)\right|\right), \quad L > 0.$$

Equipped with the norm

$$\|x\|_{\vartheta} := \max_{t\in I}\left\|x(t)\right\| + [x]_{\vartheta}, \quad \text{where } [x]_{\vartheta} := \sup_{t\neq s} \frac{\|x(t) - x(s)\|}{\vartheta(|g(t) - g(s)|)},$$

the space $\mathcal{C}_g^{\vartheta}[I,E]$ becomes a Banach space. Elements of $\mathcal{C}_g^{\vartheta}[I,E]$ are called generalized Hölderian functions.

The particular choice $g(t) = t$, $\vartheta(t) = t^{\alpha}$, $\alpha \in (0,1]$ leads, of course, to the classical Hölder space.

Let $\mathcal{C}_g^{\vartheta}[I,E_w]$ denote the Banach space of generalized Hölderian functions $x : I \to E$, with its weak topology (i.e., generated by continuous linear functionals on E).

Recall that the map $T : X \to Y$, X and Y are Banach spaces and said to be weakly–weakly sequentially continuous (ww-sequentially continuous) if and only it maps weakly convergent sequences (x_n) to $x \in E$ into sequences $(T(x_n))$ that are weakly convergent to $T(x)$ in Y.

Definition 1 ([13]) Let $\mathcal{M}_E$ be a family of all bounded subsets of E and B_1 denotes the unit ball of E. The De Blasi measure of weak noncompactness is the mapping

$$\boldsymbol{\mu} : \mathcal{M}_E \to [0,\infty)$$

defined by

$$\boldsymbol{\mu}(X) := \inf\{\epsilon > 0 : \text{there exists a weakly compact subset } \Omega \text{ of } E : X \subset \epsilon B_1 + \Omega\}.$$

For the properties of $\boldsymbol{\mu}$ see [13]. The following important Ambrosetti-type lemma will be used in the paper:

Lemma 1 ([23]) *Let $V \subset C[I,E]$ be bounded and strongly equicontinuous. Then,*

1. $t \mapsto \boldsymbol{\mu}(V(t)) \in C[I,\mathbb{R}^+]$, *where* $V(t) := \{v(t) : v \in V, t \in I\}$;
2. $\boldsymbol{\mu}_C(V) = \sup_{t\in I} \boldsymbol{\mu}(V(t)) = \boldsymbol{\mu}(V(I))$,

where $\boldsymbol{\mu}_C$ denotes the De Blasi measure of weak noncompactness in $C[I,E]$.

For our purpose, we will need the following Mönch fixed-point theorem whose foundations of use for the weak topology we can find in [6]

Theorem 1 ([21]) *Let $\mathcal{Q}$ be a nonempty, closed, convex, and equicontinuous subset of a metrizable locally convex vector space $C(I,E)$ such that $0 \in \mathcal{Q}$. Suppose $T : \mathcal{Q} \to \mathcal{Q}$ is* *weakly–weakly sequentially continuous. If the implication*

$$\overline{V} = \overline{conv}\big(\{0\} \cup T(V)\big) \Rightarrow V \quad \textit{is relatively weakly compact} \tag{2}$$

holds for every subset $V \subset \mathcal{Q}$, then the operator T has a fixed point in $\mathcal{Q}$.

The following definition goes back to Pettis [28]

Definition 2 (Pettis integral) A weakly measurable function $x : I \to E$ is said to be Pettis integrable on I if

1. x is Dunford integrable on I, that is, $\varphi x \in L_1$ for every $\varphi \in E^*$;
2. for any measurable $A \subset I$ there exists an element in E denoted by $\int_A x(s)\,ds$ such that

$$\varphi\left(\int_A x(s)\,ds\right) = \int_A \varphi x(s)\,ds \quad \text{for every } \varphi \in E^*.$$

By $P[I,E]$ denote the space of E-valued Pettis integrable functions on I. In particular, the space $P[I,\mathbb{R}] = L_1[I,\mathbb{R}]$. We need to introduce more function spaces. For convenience, we recall the following:

Definition 3 ([8, 28]) For any Young function ψ we define a class $\mathcal{H}^\psi(E)$ as

$$\mathcal{H}^\psi(E) := \big\{x : I \to E : x \text{ weakly measurable satisfying } \varphi x \in L_\psi(I) \text{ for every } \varphi \in E^*\big\}.$$

As its subspace let us consider

$$\widetilde{\mathcal{H}}^\psi(E) := \big\{x : I \to E : x \text{ strongly measurable satisfying } \varphi x \in L_\psi(I) \text{ for every } \varphi \in E^*\big\}.$$

Moreover, the class $\mathcal{H}_0^\psi(E)$ (resp., $\widetilde{\mathcal{H}}_0^\psi(E)$) is defined to be the subspace of $\mathcal{H}^\psi(E)$ (resp., $\widetilde{\mathcal{H}}^\psi(E)$) composed of Pettis integrable functions on I, that is

$$\mathcal{H}_0^\psi(E) := \big\{x \in \mathcal{H}^\psi(E) : x \in P[I,E]\big\}, \qquad \widetilde{\mathcal{H}}_0^\psi(E) := \big\{x \in \widetilde{\mathcal{H}}^\psi(E) : x \in P[I,E]\big\}.$$

In particular, the well-known class $\mathcal{H}_0^p(E)$ denotes the class $\mathcal{H}_0^\psi(E)$ for the particular choice $\psi \equiv \frac{|\cdot|^p}{p}$.

Obviously, $\widetilde{\mathcal{H}}_0^\psi(E) \subseteq \mathcal{H}_0^\psi(E) \subseteq \mathcal{H}^\psi(E)$ and $\widetilde{\mathcal{H}}_0^\psi(E) \equiv \mathcal{H}_0^\psi(E)$ holds true whenever E is separable (cf. [28, Corollary 1.11]). Some special facts about these spaces are known (cf. [14, 28, 38]):

Proposition 1

(1) *If E is reflexive, then $\mathcal{H}^1(E) \equiv \mathcal{H}_0^1(E)$.*

(2) *For any Young function ψ with $\lim_{u\to\infty} \psi(u)/u \to \infty$, $\widetilde{\mathcal{H}}^\psi(E) \subseteq \mathcal{H}_0^\psi(E)$. In particular, $\widetilde{\mathcal{H}}^p(E) \subseteq \mathcal{H}_0^p(E)$ holds true for any $p > 1$. If, additionally, E is weakly complete or even more generally, contains no isomorphic copy of c_0, it is also true for any Young function ψ. That is, $\widetilde{\mathcal{H}}^1(E) \subseteq \mathcal{H}_0^1(E)$ whenever E satisfies this additional condition.*

Clearly, since the weak continuity implies a strong measurability (see [18, page 73]), in view of Proposition 1 it implies that:

Corollary 1 *For any nontrivial Young function ψ the space $C[I, E_w]$ is a proper subset of $\widetilde{\mathcal{H}}_0^{\psi}(E)$.*

Let us stress that the connection between the Pettis integrability and Orlicz spaces is much deeper than presented in [38] (see [7]). In the following, we will integrate vector-valued functions with respect to some real-valued ones. For this reason we recall the results that complement some of those from [28, 35], dealing with the integrability of Pettis integrable functions multiplied by real-valued ones.

Proposition 2 ([11, Proposition 5]) *If $x \in \mathcal{H}_0^{\psi}(E)$, then $x(\cdot)y(\cdot) \in P[I,E]$ for every $y \in L_{\tilde{\psi}}$.*

Let us stress that y cannot be vector valued, unless the space E is a Banach algebra. Now, we should state an immediate, but important, consequence of Proposition 2:

Proposition 3 (cf. [28, Corollary 3.41]) *If $x \in P[I,E]$, then $x(\cdot)y(\cdot) \in P[I,E]$ for every $y \in L_\infty[I]$.*

Let us recall necessary definitions and known facts about weak-type derivatives in Banach spaces. Let us collect all of them that are applied for problems described in the paper.

Definition 4 ([14, 28]) Consider a vector-valued function $x : I \to E$. If for every $\varphi \in E^*$ functions φx are differentiable almost everywhere on I and if there exists a function $y : I \to E$ such that for every $\varphi \in E^*$ there exists a null set $N(\varphi) \subset I$ with

$$\big(\varphi x(t)\big)' = \varphi y(t), \quad \text{for every } t \in I \setminus N(\varphi),$$

then the function x is said to be pseudodifferentiable on I.

In this above definition, y is called a pseudoderivative of x. If the null set independent of φ, then x is said to be a.e. weakly differentiable on I and y (in this case) is called a weak derivative of x and exists almost everywhere on I. In particular, when $E = \mathbb{R}$ it is clear that the pseudo- and a.e. weak derivatives coincide with the classical derivatives of real-valued functions.

Let $\mathfrak{D}_p$ denote the pseudodifferential operator (*resp.*, $\mathfrak{D}_\omega$ for the weak one). The best result for a descriptive definition of the Pettis integral is that given by Pettis in [28, Sect. 8] (see also [25, Theorem 5.1] and [18, 23]).

Lemma 2

(1) *The indefinite integral of Pettis integrable (resp., weakly continuous) function is weakly absolutely continuous and it is pseudo- (resp., weakly) differentiable with respect to the right endpoint of the integration interval and its pseudo- (resp., weak) derivative equals the integrand at that point.*

(2) *A function $x : I \to E$ is an indefinite Pettis integral if and only if x is weakly absolutely continuous and has a pseudoderivative $\mathfrak{D}_p x$ on I. In this case, $\mathfrak{D}_p x \in P[I,E]$ and*

$$x(t) = x(a) + \int_a^t \mathfrak{D}_p x(s)\,ds, \quad t \in I.$$

Before embarking on the next section, we remark that it is natural to assume that the space E has total dual, i.e., a countable determining set. In fact, if E is separable, then both E and E^* have total dual, so even spaces like $BV(I)$ or $L_\infty(I)$ have this property. In this connections, all considered pseudoderivatives of a function from I to E, will be uniquely determined up to a set of measure zero. Deep results concerning this problem can be found in [26, Corollary 3.4, Theorem 3.6].

We also recall the following facts: for any continuous $g : I \to \mathbb{R}$ having a positive, continuous derivative g' on I, Proposition 3 may be combined with Corollary 1 in order to assure that $x(\cdot)g'(\cdot) \in P[I,E]$ (resp. $x(\cdot)g'(\cdot) \in C[I,E_w]$) holds true for every $x \in \mathcal{H}_0^{\psi}(E)$ (resp., $x \in C[I,E_w]$). From which, in view of Lemma 2, it follows that

$$\begin{cases} (\frac{1}{g'(t)}\mathfrak{D}_\omega)\Im_a^{1,g}x(t) = (\frac{1}{g'(t)}\mathfrak{D}_\omega)\int_a^t x(s)g'(s)\,ds = x, \text{ holds for any } x \in C[I,E_w], \quad (\maltese), \\ (\frac{1}{g'(t)}\mathfrak{D}_p)\Im_a^{1,g}x(t) = (\frac{1}{g'(t)}\mathfrak{D}_p)\int_a^t x(s)g'(s)\,ds = x, \text{holds for any } x \in P[I,E], \quad (\diamond). \end{cases}$$

Remark 1 Let us note that

- The fact that the indefinite Pettis integral of a function $x \in P[I,E]$ does not enjoy the strong property of being a.e. weakly differentiable (see [15]), tells us that ($\maltese$) does not necessarily hold for arbitrary $x \in P[I,E]$.
- The formula ($\diamond$) is not uniquely determined unless E has total dual E^*. Evidently, according to (e.g., [37, page 2] and [10]), it may happen that $(\frac{1}{g'(t)}\mathfrak{D}_\omega)\Im_a^{1,g}x = y$, with y being weakly equivalent to x (but they need not be necessarily a.e. equal).

2 Generalized fractional integrals

Various modifications and generalizations of classical fractional integration operators are known and are widely used both in theory and applications. In this section, we dwell on such modifications such as fractional integrals of a given function x with respect to another function g.

Definition 5 (cf. [5, 19, 36]) The generalized fractional (or g-fractional) integral of a given function $x : [a,b] \to E$ of order α is defined by

$$\Im_a^{\alpha,g}x(t) := \frac{1}{\Gamma(\alpha)}\int_a^t \frac{x(s)}{(g(t)-g(s))^{1-\alpha}}g'(s)\,ds, \quad (-\infty \le a < b \le \infty), \alpha > 0. \tag{3}$$

For completeness, we define $\Im_a^{\alpha,g}x(a) := 0$. In the preceding definition the sign "$\int$" stands for the Pettis integral (in particular, the Lebesgue integral when $E = \mathbb{R}$).

It should be noted that, for the real-valued function $x \in L_1[a,b]$, it is well known that (see, e.g., [5, 36]) $\Im_a^{\alpha,g}x$ makes sense a.e. on I and $\Im_a^{\alpha,g}\Im_a^{\beta,g}x = \Im_a^{\beta,g}\Im_a^{\alpha,g}x = \Im_a^{\alpha+\beta,g}x$ holds true for any $\alpha, \beta > 0$. We also remark that, in a special case $g(t) = t$, $t \in [a,b]$ or $g(t) = \ln t$,

$t \in [1,e]$ we obtain two classical fractional integral operators: the Riemann-Louville and the Hadamard ones.

Definition 5 allows us to unify different fractional integral for vector-valued functions and consequently, in a unified manner, to solve some boundary value problems with different types of fractional integrals and derivatives. Clearly, it is not only a unification, we extend existing results too.

Example 2.1 Let $\alpha > 0$ and $J \subset I$ be a set of positive measure. Consider the Banach space $E = B[I]$ of bounded real-valued functions on I. Define a weakly measurable function $x : I \to B[I]$ by

$$x(t) := \begin{cases} \chi_{\{t\}}(\cdot), & t \in J, \\ \theta, & t \notin J. \end{cases}$$

Obviously $x \in P[I, B[I]]$. To see this, let us remark that any $\varphi \in B^*[I]$ may be identified with a countable additive measure ζ defined on the σ-algebra on I. More precisely, every bounded linear functional on $B[I]$ is of the form $x \longmapsto \int_I x(t)\,d\zeta$ for some countable additive measure ζ. Thus, for every measurable $\Sigma \subset I$ we have

$$\int_\Sigma \varphi\bigl(x(s)\bigr)\,ds = \int_\Sigma \left(\int_J \chi_{\{s\}}\,d\zeta \right) ds = \varphi(\theta).$$

From which, by the definition of the Pettis integral, we conclude that $x \in P[I, B[I]]$ as claimed. Now, we will show that $\Im_a^{\alpha,g} x$ exists on I with $\Im_a^{\alpha,g} x = \theta$: Evidently, for every measurable $\Sigma \subset I$ we have

$$\frac{1}{\Gamma(\alpha)} \int_\Sigma \varphi\left(\frac{x(s)}{(g(t)-g(s))^{1-\alpha}} g'(s) \right) ds = \frac{1}{\Gamma(\alpha)} \int_\Sigma \frac{\varphi(x(s))}{(g(t)-g(s))^{1-\alpha}} g'(s)\,ds = 0 = \varphi(\theta).$$

That is, by the definition of the Pettis integral, $\Im_a^{\alpha,g} x$ exists on I and $\Im_a^{\alpha,g} x = \theta$.

Remark 2 For any $\alpha \geq 1$, $\Im_a^{\alpha,g} x$ exists for any $x \in \mathcal{H}_0^1(E)$. This is a direct consequence of Proposition 3, as we obtain $s \to (g(t)-g(s))^{\alpha-1} g'(s) \in L_\infty[a,t]$ for a.e. $t \in [a,b]$.

We sometimes considered some special cases of spaces E. Let us present one useful one:

Lemma 3 *Let $\alpha \in (0,1]$ and assume that E has no isomorphic copy of c_0. Then, $\Im_a^{\alpha,g} : \widetilde{\mathcal{H}}_0^1(E) \to P[I,E]$.*

Proof Let $x \in \widetilde{\mathcal{H}}_0^1(E)$. By virtue of the fact that the strong measurability is preserved under a multiplication operation of functions (*cf.* e.g., [18]), the product $(g(t)-g(\cdot))^{\alpha-1} g'(\cdot) x(\cdot) : [a,t] \to E$ is strongly measurable on $[0,t]$ for almost every $t \in I$. Consequently, by Young's inequality, it can be shown that for every $\varphi \in E^*$, the real-valued function $s \mapsto \varphi((g(t) - g(s))^{\alpha-1} g'(s) x(s)) = (g(t)-g(s))^{\alpha-1} g'(s) \varphi(x(s))$ is Lebesgue integrable on $[a,t]$, for almost every $t \in I$. Hence, the existence of $\Im_a^{\alpha,g} x$ follows from [17, Theorem 22].

Now, we proceed in order to show that $\Im_a^{\alpha,g} : \widetilde{\mathcal{H}}_0^1(E) \to P[I,E]$. To see this, let $x \in \widetilde{\mathcal{H}}_0^1(E)$, define $y := \Im_a^{\alpha,g} x$ and note that $y \in \mathcal{H}^1(E)$. Thus, for any interval $[c,d] \subseteq I$, and any $\varphi \in E^*$

we have

$$\int_c^d \varphi\big(y(t)\big)\,dt = \int_c^d \Im_a^{\alpha,g}\varphi\big(x(t)\big)\,dt = \varphi(x_{[c,d]}),$$

where

$$x_{[c,d]} = \frac{1}{\Gamma(1+\alpha)}\int_a^d x(s)\big(g(d)-g(s)\big)^{\alpha} g'(s)\,ds - \frac{1}{\Gamma(1+\alpha)}\int_a^c x(s)\big(g(c)-g(s)\big)^{\alpha} g'(s)\,ds.$$

Since $x \in P[I,E]$, then owing to Proposition 3, we have that $x(\cdot)(g(c)-g(\cdot))^{\alpha}g'(s)$ and $x(\cdot)(g(b)-g(\cdot))^{\alpha}g'(\cdot)$ are Pettis integrable on I and so $x_{[c,d]} \in E$. A combination of these results yields $y \in \mathcal{H}^1(E)$ and there exists an element $x_{[c,d]} \in E$ such that $\varphi(x_{[c,d]}) = \int_c^d \varphi(y(t))\,dt$, for every $\varphi \in E^*$ and any $[c,d] \subseteq I$. Since E has no copy of c_0, it follows in view of [17, Theorem 23] that $y \in P[I,E]$. The lemma is thus proved. □

In what follows, we outline and prove some aspects of a g−fractional integral in Banach spaces and weak topologies. The following theorem complements similar results in [32, Lemma 1] and [11, Theorem 2] dealing with the statements revealing how much the fractional integral $\Im_a^{\alpha,g}x$ is "better", in the sense of space inclusions, than the function x.

Theorem 2 *Let $\alpha \in (0,1]$. For any Young function ψ with its complementary Young function $\tilde{\psi}$ satisfying*

$$\int_0^t \widetilde{\psi}\left(s^{\alpha-1}\right)ds < \infty, \quad t > 0, \tag{4}$$

the operator $\Im_a^{\alpha,g}$ maps the space $\mathcal{H}_0^{\psi}(E)$ into the (generalized) Hölder space $C_g^{\widetilde{\Psi}_\alpha}[I,E_w]$. Also, for any $x \in \mathcal{H}_0^{\psi}(E)$ there is $\varphi \in E^*$, with $\|\varphi\| = 1$ such that

$$\left\|\Im_a^{\alpha,g}x\right\|_{\widetilde{\Psi}_\alpha} \le \frac{4}{\Gamma(\alpha)}\left\|\varphi(x)\right\|_{\psi}\left(1+\widetilde{\Psi}_\alpha\left(\|g\|\right)\right).$$

In particular, $\Im_a^{\alpha,g} : C[I,E_w] \to C_g^{\widetilde{\Psi}_\alpha}[I,E_w]$. Here, $\widetilde{\Psi}_\alpha : \mathbb{R}^+ \to \mathbb{R}^+$ is defined as

$$\widetilde{\Psi}_\alpha(t) := \inf\left\{k>0 : k^{\frac{1}{\alpha-1}}\int_0^{tk^{\frac{1}{1-\alpha}}} \widetilde{\psi}\left(s^{\alpha-1}\right)ds \le 1\right\}, \quad t \ge 0. \tag{5}$$

To make the proof of Theorem 2 simpler we split it into several stages, providing the following lemmas:

Lemma 4 ([11, Proposition 2]) *For any $\alpha \in (0,1]$, the function $\widetilde{\Psi}_\alpha$ defined as in (5) is a Hölderian-type function, i.e., $\widetilde{\Psi}_\alpha$ is well defined, increasing, and continuous with $\widetilde{\Psi}_\alpha(0) = 0$. In other words, the space $C_g^{\widetilde{\Psi}_\alpha}[I,E_w]$ is a Hölderian-type space.*

Proof It is clear that for any $t > 0$, the function

$$u_t(\sigma) := \sigma - \int_0^{t\sigma} \widetilde{\psi}\left(s^{\alpha-1}\right)ds$$

has a positive derivative for sufficiently large $\sigma > 0$ (because $\widetilde{\psi}(u) \to 0$ as $u \to 0$). Consequently, for any $t > 0$, there is $\sigma > 0$ such that $u_t(\sigma) > 0$ and then for any $t > 0$ the set

$$\left\{\sigma > 0 : \sigma^{-1} \int_0^{t\sigma} \widetilde{\psi}\left(s^{\alpha-1}\right) ds \le 1\right\} \ne \emptyset. \tag{6}$$

Together with $\widetilde{\Psi}_\alpha(0) = 0$, this implies that $\widetilde{\Psi}_\alpha$ is well defined on I. In view of the definition of $\widetilde{\Psi}_\alpha$, for $0 \le t \le s$ we have

$$\int_0^{(\widetilde{\Psi}_\alpha(s))^{\frac{1}{\alpha-1}} t} \widetilde{\psi}\left(s^{\alpha-1}\right) ds \le \int_0^{(\widetilde{\Psi}_\alpha(s))^{\frac{1}{\alpha-1}} s} \widetilde{\psi}\left(s^{\alpha-1}\right) ds \le \left(\widetilde{\Psi}_\alpha(s)\right)^{\frac{1}{\alpha-1}}.$$

Thus, we may put $k = \widetilde{\Psi}_\alpha(s)$ in (5), which implies $\widetilde{\Psi}_\alpha(t) \le \widetilde{\Psi}_\alpha(s)$, as required for the monotonicity of $\widetilde{\Psi}_\alpha$. Finally, the continuity of $\widetilde{\Psi}_\alpha$ follows from the continuity and concavity of $t \mapsto \int_0^t \widetilde{\psi}(s^{\alpha-1})\, ds$. □

Lemma 5 *Let $\alpha \in (0,1]$. For any Young function ψ with its Young complement $\tilde{\psi}$ satisfying (4), the integral $\Im_a^{\alpha,g} x$ exists (is convergent) for any $x \in \mathcal{H}_0^\psi(E)$. Moreover, it is true for every $x \in \widetilde{\mathcal{H}}^\psi(E)$ provided ψ satisfies the additional property that $\lim_{u\to\infty} \psi(u)/u \to \infty$.*

In particular, if E is reflexive (resp., weakly complete), $\Im_a^{\alpha,g} x$, $x \in \mathcal{H}^\psi(E)$ (resp., $x \in \widetilde{\mathcal{H}}^\psi(E)$) exists for any nontrivial Young function ψ.

Proof First, let us define $u : I \to \mathbb{R}^+$ by

$$u(s) := \begin{cases} (g(t) - g(s))^{\alpha-1} g'(s), & s \in [a,t], t > a, \\ 0, & \text{otherwise} \end{cases}$$

and observe that for any $t \in I$ the function

$$u_t(\eta) := \eta - \frac{1}{\|g'\|} \int_0^{\eta g(t)} \widetilde{\psi}\left(s^{\alpha-1}\right) ds,$$

has a positive derivative for some sufficiently large $\eta > 0$ (because $\widetilde{\psi}(u) \to 0$ as $u \to 0$). Consequently, for any $t \in I$ there is a sufficiently large $\eta > 0$ such that $u_t(\eta) > 0$ and thus for any $t \in I$

$$\left\{k > 0 : \frac{1}{\|g'\|} \int_0^{(\frac{k}{\|g'\|})^{\frac{1}{1-\alpha}} g(t)} \widetilde{\psi}\left(s^{\alpha-1}\right) ds \le \left(\frac{k}{\|g'\|}\right)^{\frac{1}{1-\alpha}}\right\} \ne \emptyset. \tag{7}$$

This is in line with the following observations that they give:

$$\begin{aligned} &\int_a^b \widetilde{\psi}\left(\frac{|u(s)|}{k}\right) ds \\ &\quad = \int_a^t \widetilde{\psi}\left(\frac{|(g(t) - g(s))^{\alpha-1}|g'(s)}{k}\right) ds = \int_a^t \widetilde{\psi}\left(\frac{|(g(t) - g(s))^{\alpha-1}| \|g'\|}{k} \frac{g'(s)}{\|g'\|}\right) ds \\ &\quad \le \int_a^t \widetilde{\psi}\left(\frac{|(g(t) - g(s))^{\alpha-1}| \|g'\|}{k}\right) \frac{g'(s)\, ds}{\|g'\|} = \frac{(\frac{k}{\|g'\|})^{\frac{1}{\alpha-1}}}{\|g'\|} \int_0^{(\frac{k}{\|g'\|})^{\frac{1}{1-\alpha}} g(t)} \widetilde{\psi}\left(s^{\alpha-1}\right) ds, \end{aligned}$$

hold for any $k > 0$, so $u \in L_{\widetilde{\psi}}(I)$. The assertion of our lemma follows directly from Proposition 2.

Now, we claim that $\Im_a^{\alpha,g} x$ exists for any $x \in \widetilde{\mathcal{H}}^{\psi}(E)$ with ψ satisfying the additional property $\lim_{u\to\infty} \psi(u)/u \to \infty$. In view of the above observation, it follows from part (2) of Proposition 1.

Next, let us assume that E is weakly complete, $x \in \widetilde{\mathcal{H}}^{\psi}(E)$ for arbitrary ψ and note that in this case $\widetilde{\mathcal{H}}^{\psi}(E) \subset \widetilde{\mathcal{H}}^{1}(E)$. Since the strong measurability is preserved under a multiplication operation, the pointwise product of strongly measurable functions $(g(t) - g(\cdot))^{\alpha-1} g'(\cdot)x(\cdot) : [a,t] \to E$ is strongly measurable on $[a,t]$, $t \in I$. In view of Young's inequality, we know that for every $\varphi \in E^*$, the real-valued function, $\varphi((g(t) - g(\cdot))^{\alpha-1} g'(\cdot)x(\cdot)) = (g(t) - g(\cdot))^{\alpha-1} g'(\cdot)\varphi x(\cdot)$ is Lebesgue integrable on $[a,t]$ for every $t \in I$. Hence, the result is a consequence of part (2) of Proposition 1.

Similarly, when E is reflexive, the result follows from part (1) of Proposition 1. In this case indeed, as for any nontrivial ψ we have $\mathcal{H}^{\psi}(E) \subseteq \mathcal{H}^{1}(E)$. Consequently, for any $x \in \mathcal{H}^{\psi}(E)$ and every $\varphi \in E^*$ the measurable real-valued function $\varphi((g(t) - g(\cdot))^{\alpha-1} g'(\cdot)x(\cdot)) = (g(t) - g(\cdot))^{\alpha-1} g'(\cdot)\varphi x(\cdot)$ is Lebesgue integrable on $[a,t]$ for every $t \in I$, and hence is weakly measurable. The fact that in reflexive spaces any weakly measurable $u : I \to E$ is Pettis integrable if and only if $\varphi u \in L_1$ holds for every $\varphi \in E^*$ (cf. Lemma 1 part (1)), guarantees the existence of $\Im_a^{\alpha,g} x$ on I. □

Remark 3 According to the assertion of Lemma 5, the function $(g(t) - g(\cdot))^{\alpha-1} g'(\cdot)x(\cdot) \in P[[a,t],E]$ for every $t \in I$ and any $x \in \mathcal{H}_0^{\psi}(E)$. Consequently, accordingly to the definition of a Pettis integral for any $t \in I$ there exists an element of E denoted by $\Im_a^{\alpha,g} x(t)$ such that

$$\begin{aligned} \varphi\left(\Im_a^{\alpha,g} x(t)\right) &= \frac{1}{\Gamma(\alpha)} \int_a^t \varphi\left(\frac{x(s)g'(s)}{(g(t)-g(s))^{1-\alpha}}\right) ds \\ &= \frac{1}{\Gamma(\alpha)} \int_a^t \frac{\varphi(x(s))g'(s)\, ds}{(g(t)-g(s))^{1-\alpha}} = \Im_a^{\alpha,g} \varphi\left(x(t)\right) \end{aligned} \tag{8}$$

holds true for every $\varphi \in E^*$.

Remark 4 We should remark that, if $\Im_a^{\alpha,g} x$ does not exist for some $x \in \mathcal{H}_0^{\psi}(E)$, then it cannot exist if we "enlarge" the space E into F. To see this, we argue by contradiction assuming that $\Im_a^{\alpha,g} x$ (when we consider x as a function from $\mathcal{H}_0^{\psi}(F)$) exists. In this case, for the particular choice for the functional $\varphi \in F^*$ having $\varphi|_E = \theta$ we conclude, in view of (8) and $x(I) \subseteq E$, that $\varphi(\Im_a^{\alpha,g} x(t)) = \Im_a^{\alpha,g} \varphi(x(t)) = 0$, from which $\Im_a^{\alpha,g} x(t) \in E$. This would lead to a contradiction.

Remark 5 Let a Young function ψ be such that the integral in (4) is finite. For any $\alpha \in (0,1)$, the assertion of Theorem 2 is still valid if at least one of the following cases holds true:

1. $x \in \widetilde{\mathcal{H}}^{\psi}(E)$, where ψ satisfies the additional property $\lim_{u\to\infty} \psi(u)/u \to \infty$;
2. E is weakly complete and $x \in \widetilde{\mathcal{H}}^{\psi}(E)$;
3. E is reflexive and $x \in \mathcal{H}^{\psi}(E)$.

Evidently, it follows from Theorem 2, as in view of Lemma 5, in all of the above cases we have $\widetilde{\mathcal{H}}^{\psi}(E) \subseteq \mathcal{H}^{\psi}(E) \subseteq \mathcal{H}_0^{\psi}(E)$.

We are now ready to provide the proof of Theorem 2.

Proof of Theorem 2. Let $a \le t_1 \le t_2 \le b$ and $x \in \mathcal{H}_0^{\psi}(E)$. According to Lemma 5 and by the definition of the indefinite Pettis integral, we ensure that $\Im_a^{\alpha,g}x$ is well defined. In view of Remark 3, it allows us to state the following chain of inequalities

$$\begin{aligned}
&\left|\varphi\left(\Im_a^{\alpha,g}x(t_2) - \Im_a^{\alpha,g}x(t_1)\right)\right| \\
&\quad = \left|\Im_a^{\alpha,g}\varphi\left(x(t_2)\right) - \Im_a^{\alpha,g}\varphi\left(x(t_1)\right)\right| \\
&\quad \le \frac{1}{\Gamma(\alpha)}\left(\int_a^{t_1}\left|\left(g(t_2)-g(s)\right)^{\alpha-1} - \left(g(t_1)-g(s)\right)^{\alpha-1}\right|\left|g'(s)\right|\left|\varphi\left(x(s)\right)\right| ds\right. \\
&\qquad \left. + \int_{t_1}^{t_2}\left(g(t_2)-g(s)\right)^{\alpha-1} g'(s)\left|\varphi\left(x(s)\right)\right| ds\right) \\
&\quad = \frac{1}{\Gamma(\alpha)}\int_a^b \left[h_1(s)+h_2(s)\right]\left|\varphi\left(x(s)\right)\right| ds,
\end{aligned}$$

where

$$h_1(s) := \begin{cases} |(g(t_2)-g(s))^{\alpha-1} - (g(t_1)-g(s))^{\alpha-1}|g'(s) & s \in [a,t_1], \\ 0 & \text{otherwise} \end{cases}$$

and

$$h_2(s) := \begin{cases} (g(t_2)-g(s))^{\alpha-1}g'(s) & s \in [t_1,t_2], \\ 0 & \text{otherwise.} \end{cases}$$

We claim that $h_i \in L_{\widetilde{\psi}}(I)$, $(i = 1,2)$. Once our claim is established, in view of the Hölder inequality in Orlicz spaces, we conclude that

$$\left|\varphi\left(\Im_a^{\alpha,g}x(t_2) - \Im_a^{\alpha,g}x(t_1)\right)\right| \le \frac{2[\|h_1\|_{\widetilde{\psi}} + \|h_2\|_{\widetilde{\psi}}]}{\Gamma(\alpha)}\left\|\varphi(x)\right\|_{\psi}. \tag{9}$$

It remains to prove our claim by showing that $h_i \in L_{\widetilde{\psi}}(I)$, $i = 1,2$. To see this, fix $k > 0$. An appropriate substitution, using some properties of Young functions, leads to the following estimation

$$\begin{aligned}
&\int_a^b \widetilde{\psi}\left(\frac{|h_1(s)|}{k}\right) ds \\
&\quad = \int_a^{t_1} \widetilde{\psi}\left(\frac{|(g(t_2)-g(s))^{\alpha-1} - (g(t_1)-g(s))^{\alpha-1}|\,\|g'\|}{k}\frac{g'(s)}{\|g'\|}\right) ds \\
&\quad \le \int_a^{t_1} \widetilde{\psi}\left(\frac{[(g(t_1)-g(s))^{\alpha-1} - (g(t_2)-g(s))^{\alpha-1}]\,\|g'\|}{k}\right)\frac{g'(s)}{\|g'\|} ds \\
&\quad \le \int_a^{t_1}\left[\widetilde{\psi}\left(\frac{(g(t_1)-g(s))^{\alpha-1}\|g'\|}{k}\right) - \widetilde{\psi}\left(\frac{(g(t_2)-g(s))^{\alpha-1}]\,\|g'\|}{k}\right)\right]\frac{g'(s)}{\|g'\|} ds \\
&\quad \le \frac{(\frac{k}{\|g'\|})^{\frac{1}{\alpha-1}}}{\|g'\|}\left[\int_0^{(\frac{k}{\|g'\|})^{\frac{1}{1-\alpha}}g(t_1)} \widetilde{\psi}\left(s^{\alpha-1}\right) ds - \int_{(\frac{k}{\|g'\|})^{\frac{1}{1-\alpha}}(g(t_2)-g(t_1))}^{(\frac{k}{\|g'\|})^{\frac{1}{1-\alpha}}g(t_2)} \widetilde{\psi}\left(s^{\alpha-1}\right) ds\right] \\
&\quad = \frac{(\frac{k}{\|g'\|})^{\frac{1}{\alpha-1}}}{\|g'\|}\left[\int_0^{(\frac{k}{\|g'\|})^{\frac{1}{1-\alpha}}g(t_1)} \widetilde{\psi}\left(s^{\alpha-1}\right) ds - \int_0^{(\frac{k}{\|g'\|})^{\frac{1}{1-\alpha}}g(t_2)} \widetilde{\psi}\left(s^{\alpha-1}\right) ds\right.
\end{aligned}$$

$$
\begin{aligned}
&+\int_0^{(\frac{k}{\|g'\|})^{\frac{1}{1-\alpha}}(g(t_2)-g(t_1))} \widetilde{\psi}\left(s^{\alpha-1}\right) ds\Bigg] \\
&\le \frac{(\frac{k}{\|g'\|})^{\frac{1}{\alpha-1}}}{\|g'\|} \int_0^{(\frac{k}{\|g'\|})^{\frac{1}{1-\alpha}}(g(t_2)-g(t_1))} \widetilde{\psi}\left(s^{\alpha-1}\right) ds.
\end{aligned}
$$

In view of (6), the above observations guarantee the existence of $k > 0$ for which $\int_a^b \widetilde{\psi}(\frac{|h_1(s)|}{k})\, ds \le 1$. Then, we can conclude that $h_1 \in L_{\widetilde{\psi}}(I)$. Moreover, our definitions of $\widetilde{\Psi}_\alpha$ and the norm in Orlicz spaces, along with the above observations, give us

$$
\begin{aligned}
\|h_1\|_{\widetilde{\psi}} &= \inf\left\{k > 0 : \int_a^b \widetilde{\psi}\left(\frac{|h_1(s)|}{k}\right) ds \le 1\right\} \\
&= \|g'\| \inf\left\{\frac{k}{\|g'\|} > 0 : \int_a^b \widetilde{\psi}\left(\frac{|h_1(s)|}{k}\right) ds \le 1\right\} \\
&\le \widetilde{\Psi}_\alpha\left(\left|g(t_2) - g(t_1)\right|\right).
\end{aligned}
$$

Arguing similarly as above, we can show that

$$
h_2 \in L_{\widetilde{\psi}}(I), \quad \text{and} \quad \|h_2\|_{\widetilde{\psi}} \le \widetilde{\Psi}_\alpha\left(\left|g(t_2) - g(t_1)\right|\right).
$$

Thus, for any $\varphi \in E^*$ equation (9) takes the form

$$
\left|\varphi\left(\Im_a^{\alpha,g} x(t_2) - \Im_a^{\alpha,g} x(t_1)\right)\right| \le \frac{4\widetilde{\Psi}_\alpha(|g(t_2) - g(t_1)|)}{\Gamma(\alpha)} \left\|\varphi(x)\right\|_\psi. \tag{10}
$$

This may be combined along with the Hahn–Banach theorem, in order to assure that

$$
\left\|\Im_a^{\alpha,g} x(t_2) - \Im_a^{\alpha,g} x(t_1)\right\| \le \frac{4\widetilde{\Psi}_\alpha(|g(t_2) - g(t_1)|)}{\Gamma(\alpha)} \left\|\varphi(x)\right\|_\psi
$$

holds true for some $\varphi \in E^*$ with $\|\varphi\| = 1$. Hence, $\Im_a^{\alpha,g} : \mathcal{H}_0^\psi(E) \to C_g^{\widetilde{\Psi}_\alpha}[I, E_w]$. Also,

$$
\left[\Im_a^{\alpha,g} x\right]_{\widetilde{\Psi}_\alpha} \le \frac{4}{\Gamma(\alpha)} \left\|\varphi(x)\right\|_\psi.
$$

Moreover, in view of our definition $\Im_a^{\alpha,g} x(a) := 0$, we observe that

$$
\left\|\Im_a^{\alpha,g} x(t)\right\| = \left\|\Im_a^{\alpha,g} x(t) - \Im_a^{\alpha,g} x(a)\right\| \le \widetilde{\Psi}_\alpha\left(\|g\|\right)\left[\Im_a^{\alpha,g} x\right]_{\widetilde{\Psi}_\alpha}.
$$

We finally obtain

$$
\left\|\Im_a^{\alpha,g} x\right\|_{\widetilde{\Psi}_\alpha} \le \frac{4}{\Gamma(\alpha)} \left\|\varphi(x)\right\|_\psi \left(1 + \widetilde{\Psi}_\alpha\left(\|g\|\right)\right). \tag{11}
$$

In this connection, the particular case follows from Corollary 1 and the theorem is then proved. □

Example 2.2 Let $\alpha \in (0,1)$ and $\psi(u) = \psi_p(u) := \frac{1}{p}|u|^p$, $p \in (1,\infty)$. In this case, we have $\tilde{\psi}_p = \psi_{\tilde{p}}$ with $\frac{1}{p} + \frac{1}{\tilde{p}} = 1$. It can be easily seen that (4) holds true if and only if $p > \frac{1}{\alpha}$. From which we conclude that $\Im_a^{\alpha,g}$ maps the Bochner space $L_p[I,E]$, $p > \frac{1}{\alpha}$ into the Hölder space $\mathcal{C}_g^{\widetilde{\Psi}_\alpha}[I,E_w]$, where

$$\widetilde{\Psi}_\alpha(t) = \frac{t^{\alpha-\frac{1}{p}}}{\sqrt[\tilde{p}]{\tilde{p}[1-\tilde{p}(1-\alpha)]}}, \quad t \in \mathbb{R}^+.$$

For instance, in view of the above observation, $\Im_a^{\alpha,g} : L_2[I,\mathbb{R}] \to \mathcal{C}^{\widetilde{\Psi}_\alpha}[I,\mathbb{R}]$ for $\alpha \in (0.5,1)$ with $\widetilde{\Psi}_\alpha(t) = \frac{t^{\alpha-\frac{1}{2}}}{\sqrt{4\alpha-2}}$.

Remark 6 Theorem 2 may be combined with [11, Example 1] in order to assure the existence of a Young function ψ (for instance, $\psi(u) := e^{|u|} - |u| - 1$) for which $\Im_a^{\alpha,g}$ maps $\mathcal{H}_0^{\psi}(E)$ into $\mathcal{C}_g^{\widetilde{\Psi}_\alpha}[I,I,E_w]$ "for all" $\alpha \in (0,1]$. According to Example 2.2, this interesting phenomenon has no analog in the case of Lebesgue spaces $L_p[I,\mathbb{R}]$).

Example 2.3 Let $\alpha > 0$ and $a,b \in \mathbb{R}^+$ such that $b - a = 1$. Define a strongly measurable function $x : [a,b] \to L_2[a,b]$ by

$$x(t) := \sum_{n=1}^{\infty} e_n \cdot \chi_{I_n}(t) = \begin{cases} e_n(\cdot), & t \in I_n, \\ 0, & \text{otherwise}, \end{cases}$$

where $\{e_n\}$ is an orthonormal system in $L_2[a,b]$ and I_n's are the pairwise disjoint subintervals of $[a,b]$ defined by $I_n = (a + 1/2^n, a + 1/2^n + 1/4^n)$, $n \in \mathbb{N}$. Since

$$\left(\int_a^b \left|\varphi\big(x(t)\big)\right|^2 dt\right)^{\frac{1}{2}} = \left(\sum_{n=1}^{\infty} \frac{|\varphi(e_n)|^2}{4^n}\right)^{\frac{1}{2}} \le \left(\sum_{n=1}^{\infty} \left|\varphi(e_n)\right|^2\right)^{\frac{1}{2}} \le \|\varphi\|_{L_2}$$

holds true for every $\varphi \in L_2[a,b]^* = L_2[a,b]$, we obtain $\varphi x \in L_2[a,b]$ for every $\varphi \in L_2[a,b]^*$. Hence, $x \in P[[a,b],L_2[a,b]]$ (by applying Proposition 1). More precisely, $x \in \mathcal{H}_0^{\psi_2}(L_2[a,b])$. Since $L_2[a,b]$ is reflexive, the integral $\Im_a^{\alpha,g}x$ exists for any $\alpha > 0$ (cf. Remark 2 when $\alpha \ge 1$ and Remark 5 when $\alpha \in (0,1)$). Moreover, in view of Example 2.2, we know that $\Im_a^{\alpha,g}x \in \mathcal{C}_g^{\widetilde{\Psi}_\alpha}[[a,b],(L_2[a,b])_\omega]$, with $\widetilde{\Psi}_\alpha(t) = \frac{t^{\alpha-\frac{1}{2}}}{\sqrt{4\alpha-2}}$ holds for any $\alpha \in (0.5,1)$.

Example 2.4 Let $\alpha \in (0,1]$ and define $x : [0,1] \to L_1[0,1]$ by

$$x(t) := \frac{1}{\Gamma(1-\alpha)}\big(g(t) - g(\cdot)\big)^{-\alpha}\chi_{[a,t]}(\cdot), \quad t \in [0,1].$$

This function is weakly continuous on $I = [0,1]$. Indeed, if $\phi \in L_\infty \cong L_1^*$ corresponds to $\varphi \in L_1^*$, then $\varphi(x(t)) = \Im_a^{1-\alpha,g}(\frac{\phi(t)}{g'(t)})$. Since $\Im_a^{1-\alpha,g}$ maps $C[I,\mathbb{R}]$ into itself, we can conclude that $\varphi x \in C[I,\mathbb{R}]$ for every $\varphi \in L_1^*$ that gives a reason to believe that x is weakly continuous on I. Consequently, in view of Theorem 2, it follows that $\Im_a^{\alpha,g}x$ exists on I. In this context, we can show that

$$\Im_a^{\alpha,g}x(t)(\cdot) = \chi_{[a,t]}(\cdot), \quad \text{holds for any } \alpha \in (0,1]. \tag{12}$$

This is easy to demonstrate because, by letting $\phi \in L_\infty$ corresponding to $\varphi \in L_1^*$ and carrying out the necessary calculations using the substitution $s = \frac{g(s)-g(\xi)}{g(t)-g(\xi)}$, it can be verified that

$$\begin{aligned}
&\int_a^t \frac{\varphi([g(t)-g(s)]^{\alpha-1}g'(s)x(s))}{\Gamma(\alpha)}\,ds \\
&\quad = \frac{1}{\Gamma(\alpha)}\int_a^t \big[g(t)-g(s)\big]^{\alpha-1}g'(s)\varphi\big(x(s)\big)\,ds \\
&\quad = \int_a^t \frac{[g(t)-g(s)]^{\alpha-1}g'(s)}{\Gamma(\alpha)}\int_a^s \frac{\phi(\xi)[g(s)-g(\xi)]^{-\alpha}}{\Gamma(1-\alpha)}\,d\xi\,ds \\
&\quad = \int_a^t \phi(\xi)\int_\xi^t \frac{[g(t)-g(s)]^{\alpha-1}}{\Gamma(\alpha)}\frac{[g(s)-g(\xi)]^{-\alpha}}{\Gamma(1-\alpha)}g'(s)\,ds\,d\xi \\
&\quad = \int_a^t \phi(\xi)\,d\xi = \int_a^b \phi(\xi)\chi_{(a,t]}(\xi)\,d\xi = \varphi(\chi_{[a,t]}),
\end{aligned}$$

as needed for (12).

In view of the semigroup property of $\Im_a^{\alpha,g}$ in Lebesgue spaces, an analogous reasoning as in [11, Lemma 2] gives us the following:

Lemma 6 *Let $\alpha,\beta\in(0,1]$. If $x\in\mathcal{H}_0^{\psi}(E)$, where ψ is a Young function with its complement $\widetilde{\psi}$ satisfying*

$$\int_0^t \widetilde{\psi}\big(s^{-\nu}\big)\,ds<\infty,\qquad t>0, \text{where } \nu:=\max\{1-\alpha,1-\beta\}, \tag{13}$$

then

$$\Im_a^{\beta,g}\Im_a^{\alpha,g}x=\Im_a^{\beta+\alpha,g}x=\Im_a^{\alpha,g}\Im_a^{\beta,g}x\quad \text{on } I. \tag{14}$$

In particular, the property (14) holds true for every $x\in C[I,E_w]$.

Let us investigate some important properties of generalized fractional integrals with Pettis integrals and measures of weak noncompactness. We need to prove a Goebel–Rzymowski lemma that is important in our considerations and very useful in many similar problems. We follow the idea from [9].

Lemma 7 *Let $\boldsymbol{\mu}$ be the De Blasi measure of weak noncompactness. For any $\alpha\in(0,1]$, $t\in I$ and any bounded strongly equicontinuous set $V\subset C[I,E_w]$*

$$\boldsymbol{\mu}\big(\Im_a^{\alpha,g}V(t)\big):=\boldsymbol{\mu}\big(\Im_a^{\alpha,g}v(t):v\in V,t\in I\big)\le\Im_a^{\alpha,g}\boldsymbol{\mu}\big(V(t)\big)\le\frac{\|g\|^\alpha}{\Gamma(1+\alpha)}\cdot\boldsymbol{\mu}_C(V).$$

Proof At the beginning, we note, in view of Theorem 2, that $\Im_a^{\alpha,g}v$ exists and weakly continuous on I. Hence, $\boldsymbol{\mu}(\Im_a^{\alpha,g}V(t))$ makes sense. Next, define a function $G:I\times I\to\mathbb{R}^+$ by

$$G(t,s):=\frac{1}{\Gamma(\alpha)}\begin{cases}\frac{g'(s)}{(g(t)-g(s))^{1-\alpha}}, & s\in[a,t], t>a,\\ 0, & \text{otherwise.}\end{cases}$$

From the above definition we have $\Im_a^{\alpha,g}x(t) = \int_a^t G(t,s)x(s)\,ds$. From the properties of the Pettis integral for arbitrary $w \in P[I,E]$ and $t \in I$ we have

$$\int_a^{t-\tau} w(s)\,ds + \int_{t-\tau}^t w(s)\,ds = \int_a^t w(s)\,ds, \quad \text{for some sufficiently small } \tau.$$

As V is equicontinuous, the set $\{G(t,\cdot)V(\cdot)\}$ is Pettis uniformly integrable on I, so for any $x \in V$ the set $\{\varphi(G(t,\cdot)x(\cdot)) : \varphi \in E^*, \|\varphi\| \leq 1\}$ is equiintegrable. Then, for any $\varepsilon > 0$ there exists (sufficiently small) τ such that

$$\left\| \int_{t-\tau}^t G(t,s)V(s)\,ds \right\| < \varepsilon. \tag{15}$$

Thus, we can cover the set $\{\int_{t-\tau}^t G(t,s)v(s)\,ds : s \in [t-\tau,t], v \in V\}$ by balls with radius less than ε and then

$$\boldsymbol{\mu}\left(\left\{ \int_{t-\tau}^t G(t,s)v(s)\,ds : s \in [t-\tau,t], v \in V \right\} \right) < \varepsilon.$$

Now, let us estimate the set of integrals on $[a,t-\tau]$. Put $v(\cdot) = \boldsymbol{\mu}(V(\cdot))$. In view of Lemma 1, v is a continuous function. Note that from our assumption it follows that $s \to G(t,s)v(s)$ is continuous on $[a,t-\tau]$, and hence uniformly continuous.

Thus, there exists $\delta > 0$ such that

$$\left| G(t,\eta)v(q) - G(t,s)v(s) \right| < \varepsilon, \tag{16}$$

provided that $|q-s| < \delta$ and $|\eta - s| < \delta$ with $\eta, s, q \in [a,t-\tau]$.

Divide the interval $[a,t-\tau]$ into n parts $a = t_0 < t_1 < \cdots < t_n = t-\tau$ such that $|t_i - t_{i-1}| < \delta$ for $i = 1,2,\dots,n$. Put $T_i = [t_{i-1},t_i]$. As v is uniformly continuous, there exists $s_i \in T_i$ such that $v(s_i) = \beta(V(T_i))$ $(i = 1,2,\dots,n)$.

As

$$\begin{aligned} &\left\{ \int_a^{t-\tau} G(t,s)x(s)\,ds : s \in [a,t-\tau], x \in V \right\} \\ &\quad \subset \sum_{i=1}^n \left\{ \int_{T_i} G(t,s)x(s)\,ds : s \in [a,t-\tau], x \in V \right\}, \end{aligned}$$

by the mean value theorem for the Pettis integral

$$\int_{T_i} G(t,s)V(s)\,ds \in meas(T_i) \cdot \overline{\text{conv}}\{G(t,s)V(s) : s \in T_i\}.$$

Hence,

$$\begin{aligned} &\boldsymbol{\mu}\left(\left\{ \int_a^{t-\tau} G(t,s)x(s)\,ds : s \in [a,t-\tau], x \in V \right\} \right) \\ &\quad \leq \sum_{i=1}^n \boldsymbol{\mu}\left(\left\{ \int_{T_i} G(t,s)x(s)\,ds : s \in [a,t-\tau], x \in V \right\} \right) \end{aligned}$$

$$\begin{aligned}
&\le \sum_{i=1}^{b} meas(T_i)\cdot \boldsymbol{\mu}\bigl(\overline{\operatorname{conv}}\bigl\{G(t,s)V(s) : s\in T_i\bigr\}\bigr)\\
&\le \sum_{i=1}^{b} meas(T_i)\cdot \max_{s\in T_i} G(t,s)\cdot \boldsymbol{\mu} V(T_i)\\
&\le \sum_{i=1}^{b} meas(T_i)\cdot G(t,t_i)\cdot \boldsymbol{\mu} V(T_i)) \le \sum_{i=1}^{b} meas(T_i)\cdot G(t,t_i)\cdot \nu(s_i).
\end{aligned}$$

Note that from (16) it follows that

$$\sum_{i=1}^{b} meas(T_i)\cdot G(t,t_i)\cdot \nu(s_i) \le \int_a^{t-\tau} G(t,s)\nu(s)\,ds + (t-\tau)\cdot\varepsilon.$$

Then,

$$\begin{aligned}
\left\{\int_a^t G(t,s)x(s)\,ds : s\in[a,t-\tau], x\in V\right\} \subset{}& \left\{\int_a^t G(t,s)x(s)\,ds : s\in[a,t-\tau], x\in V\right\}\\
&+\left\{\int_{t-\tau}^t G(t,s)x(s)\,ds : s\in[a,t-\tau], x\in V\right\}
\end{aligned}$$

and

$$\boldsymbol{\mu}\left(\left\{\int_a^t G(t,s)x(s)\,ds : s\in[a,t], x\in V\right\}\right) \le \int_a^{t-\tau} G(t,s)\nu(s)\,ds + (t-\tau)\cdot\varepsilon + \varepsilon.$$

As ε is arbitrarily small, we obtain

$$\boldsymbol{\mu}\left(\left\{\int_a^t G(t,s)x(s)\,ds : s\in[a,t], x\in V\right\}\right) \le \int_a^{t-\tau} G(t,s)\nu(s)\,ds,$$

i.e.,

$$\boldsymbol{\mu}\bigl(\Im_a^{\alpha,g} V(t)\bigr) \le \Im_a^{\alpha,g}\boldsymbol{\mu}\bigl(V(t)\bigr).$$

It remains to prove the second estimation. Let us observe that

$$\frac{\partial}{\partial s}\left(\frac{(g(t)-g(s))^{\alpha}}{\alpha}\right) = \frac{-g'(s)}{(g(t)-g(s))^{1-\alpha}}.$$

As $g(a)=0$,

$$\int_a^t G(t,s)\,ds = \frac{(g(t))^{\alpha}}{\alpha}.$$

Thus, $\Im_a^{\alpha,g}\boldsymbol{\mu}(V(t)) \le \frac{(g(t))^{\alpha}}{\alpha\cdot\Gamma(\alpha)}\cdot\boldsymbol{\mu}_c(V) \le \frac{\|g\|^{\alpha}}{\Gamma(1+\alpha)}\cdot\boldsymbol{\mu}_c(V)$. □

3 Generalized fractional derivatives

From now, the definitions of the g-fractional derivatives of x become a natural requirement.

Definition 6 ([5, 19, 36]) The g–Caputo fractional-pseudo- (*resp.*, weak) derivative of a given function x of order $\alpha \in (m, m+1]$, $m \in \mathbb{N} := \{0, 1, 2, \ldots\}$ is defined by

$$\frac{d_p^{\alpha,g}}{dt^\alpha} x := \Im_a^{m+1-\alpha,g} \delta_p^{m+1} x, \qquad \left(resp., \ \frac{d_\omega^{\alpha,g}}{dt^\alpha} x := \Im_a^{m+1-\alpha,g} \delta_\omega^{m+1} x \right), \quad t \in I. \tag{17}$$

Here, δ_p and δ_ω are defined as

$$\delta_p := \frac{1}{g'(t)} \mathfrak{D}_p \quad \text{and} \quad \delta_\omega := \frac{1}{g'(t)} \mathfrak{D}_\omega.$$

Remark 7 It is worthwhile to remark here that $\frac{d_p^{\alpha,g}}{dt^\alpha} x$ (if exists), does not depend on the choice of the mth pseudoderivatives of x. Evidently, if $\delta_p^m x = y_1$, $\delta_p^m x = y_2$, we know that y_1, y_2 are weakly equivalent on I. It follows that

$$\varphi\big(\Im_a^{m-\alpha,g} y_1(t)\big) = \Im_a^{m-\alpha,g} \varphi\big(y_1(t)\big) = \Im_a^{m-\alpha,g} \varphi\big(y_2(t)\big) = \varphi\big(\Im_a^{m-\alpha,g} y_2(t)\big), \quad \text{for any } \varphi \in E^*.$$

Hence, $\Im_a^{m-\alpha,g} y_1(t) = \Im_a^{m-\alpha,g} y_2$ as needed.

This is a good place to remark that the conditions required for the existence of g-Caputo fractional derivative are very restrictive. A very rough condition that ensures the existence of $\frac{d_p^{\alpha,g}}{dt^\alpha} x$ is that $x \in AC^{m-1}[[a,b], E_w]$. In other words, the g–Caputo-type fractional derivative has the disadvantage that it completely loses its meaning if $D^{m-1}x$ fails to be (almost everywhere) differentiable on $[a,b]$. Unfortunately, even in the Hölder spaces, outside of the space of absolutely continuous functions, the g-Caputo-type fractional differential operator does not enjoy the "nice" behavior of being left inverse of the corresponding g-fractional integral operator. In other words, outside of the space of absolutely continuous functions, the equivalence of the g-fractional integral equations and the corresponding g-Caputo fractional differential problem is no longer necessarily true even in the Hölder spaces. This goes back to the well-known fact that the Riemann-Louville fractional integral operator $\Im_0^{\alpha,t}$ is a continuous mapping from Hölder spaces "onto" Hölder spaces (which, of course, contains also continuous nowhere differentiable functions), see, e.g., [36, Theorem 13.13]. Indeed, in what follows, we will show that even in the context of real-valued Hölderian functions the converse implication from the fractional integral equations to the corresponding Caputo-type differential form is no longer necessarily true.

To see this, let us consider a particular form of the fractional integral operator $\Im_a^{\beta,g}$, $\alpha \in (0,1)$ with $g(t) = t$, $t \in [0,1]$, $E = \mathbb{R}$. Let x be a Hölderian (but nowhere differentiable on $[0,1]$) function of some critical order $\gamma < 1$. According to [36, Theorem 13.13] we know that there is $\alpha \in (0,1)$ depending only on γ and a Hölderian function $y \notin AC[0,1]$ such that $\Im_0^{\alpha,t} y = x$. From this we can conclude that $\frac{d_p^{\alpha,t}}{dt^\alpha} \Im_0^{\alpha,t} y = \frac{d_p^{\alpha,t}}{dt^\alpha} x$ is "meaningless". This gives a reason to believe that even on Hölder spaces (but out of the space of absolutely continuous functions), the operator $\frac{d^\alpha}{dt^\alpha}$ has no left inverse of $\Im_0^{\alpha,t} y$ as required. For more examples revealing the lack of equivalence between differential and integral forms of the Caputo-type fractional problems, we refer the reader to [12]. It will be clarified later how to avoid such a phenomenon (see formula ($\diamondsuit$) and Lemma 8 below).

However, the following example shows that on the space $C[I, E_w]$, but still outside of the space of weakly absolutely continuous functions, it is no longer necessarily true that $\frac{d_p^{\alpha,g}}{dt^\alpha}$ is a left inverse of $\Im_a^{\alpha,g}$ for any $\alpha > 0$.

Example 3.1 Let $\alpha \in (m, m+1]$, $m \in \mathbb{N}$. Define $x : [0,1] \to L_1[0,1]$ by

$$x(t) := \frac{1}{\Gamma(1+m-\alpha)} \big(g(t) - g(\cdot)\big)^{m-\alpha} \chi_{[a,t]}(\cdot), \quad t \in I = [0,1].$$

Reasoning as in Example 2.4, we can ensure that this function is weakly (but not weakly absolutely) continuous on I having a g-Caputo fractional integral of order $\alpha - m \in (0,1]$ on I and that $\Im_a^{\alpha-m,g} x(t)(\cdot) = \chi_{[a,t]}(\cdot)$. In this connection, in view of the continuity of x in Theorem 2 and Lemma 6, implies that

$$\Im_a^{\alpha,g} x(t) = \Im_a^{m,g} \Im_a^{\alpha-m,g} x(t) = \Im_a^{m,g} \chi_{[a,t]}.$$

By the aid of ($\diamondsuit$), it follows that $\delta_p^m \Im_a^{\alpha,g} x = \chi_{[a,t]}$. Since $\chi_{[a,t]}(\cdot)$, $t \in I$ is weakly absolutely continuous and have no pseudo- (so trivially no weak) derivatives on I (see [33, Theorem 3]), we conclude that the g–Caputo fractional pseudo- (trivially weak) derivative is "meaningless". Namely, $\frac{d_p^{\alpha,g}}{dt^\alpha} \Im_a^{\alpha,g} x \neq x$ as required.

In order to avoid such a problem with the equivalence of the g-Caputo-type boundary value problem of fractional orders $\alpha > 1$ and the corresponding integral form, we are, similarly as in [12], going to modify (slightly) our definition of the g–Caputo-type fractional differential operator into a more suitable one

Definition 7 The modified g–Caputo fractional pseudo- (*resp.*, weak) derivative "briefly MCFPD (*resp.*, MCFWD)" of order $m + \alpha$, $m \in \mathbb{N}$, $\alpha \in (0,1)$ applied to the function $x \in P[I,E]$ is defined as

$$\frac{d_p^{m+\alpha,g}}{dt^{m+\alpha}} x := \delta_p^m \Im_a^{1-\alpha,g} \delta_p x, \qquad \left(resp.\ \frac{d_\omega^{m+\alpha,g}}{dt^{m+\alpha}} x := \delta_\omega^m \Im_a^{1-\alpha,g} \delta_\omega x \right). \tag{18}$$

Obviously, Definition 7 coincides with the usual definition of the g–Caputo-type fractional differential operators when $m = 0$. Also, unless the space E has total dual E^* (cf. [10]), a g–Caputo fractional pseudoderivative of x is not necessary uniquely determined.

In what follows we will show that the results obtained in Example 3.1 have no analog in the case of MCFPD whenever $\alpha > 1$. Evidently, arguing similarly as in [12, proof of Lemma 7], we can prove the following:

Lemma 8 *Let $\alpha > 1$. Assume that $\alpha = m + \eta$, where $m \geq 1$ with some $\eta \in (0,1)$. If ψ is a Young function with its complementary function $\tilde{\psi}$ satisfying*

$$\int_0^t \tilde{\psi}\left(s^{-\nu}\right) ds < \infty, \quad t > 0, \nu := \max\{\eta, 1-\eta\}, \tag{19}$$

then $\frac{d_p^{\alpha,g}}{dt^\alpha} \Im_a^{\alpha,g}$ is well defined on $\mathcal{H}_0^\psi(E)$. If, additionally, the space E has total dual, then $\frac{d_p^{\alpha,g}}{dt^\alpha}$ is the left-inverse of $\Im_a^{\alpha,g}$, where the fractional differential operator is taken in the sense of Definition 7.

Remark 8 Let us remark that, in view of (✠), the assertion of Lemma 8 is still valid even in the case of applying the operator (MCFWD) provided $x \in C[I, E_w]$.

The following example shows that our assumption that E has total dual is essential in Lemma 8 and cannot be omitted even if x is weakly absolutely continuous on I. Out of the context of such spaces we should assume instead that our derivatives should be of strongly bounded variation (cf. [26]).

Example 3.2 Let $\alpha \in (m, m+1]$, $m \geq 1$ and assume that $B[I]$ is the Banach space of bounded real-valued functions on I. Define a Pettis integrable function $x : I \to B[I]$ as in Example 2.1. The choice of this space is not accidental, because $B[I]$ has no total dual.

Note that $g'(\cdot)x(\cdot) \in P[I, B[I]]$ (cf. Proposition 3). Bearing in mind the existence of $\Im_a^{\alpha,g}x$ on I and arguing similarly as in [12, proof of Lemma 7] there is no difficulty in proving that

$$\frac{d_p^{\alpha,g}}{dt^\alpha}\Im_a^{\alpha,g}x = \frac{1}{g'(t)}\mathfrak{D}_p\int_a^t g'(s)x(s)\,ds, \quad t \in I. \tag{20}$$

On the one hand, reasoning as in Example 2.1, we know that

$$\int_a^t g'(s)x(s)\,ds = \theta, \quad t \in I.$$

From which, by definition of $\mathfrak{D}_p$, it can be easily seen that $\mathfrak{D}_p\int_a^t g'(s)x(s)\,ds = \theta$ on I. On the other hand, in view of Lemma 2, we conclude that $\mathfrak{D}_p\int_a^t g'(s)x(s)\,ds = g'(t)x(t)$ on I. In this connection, we deduce that the function $t \mapsto \int_a^t g'(s)x(s)\,ds$ has two pseudoderivatives $g'(t)x(t)$ and θ on I that differ on a set of positive measures. Consequently, we infer by the aid of (20) that on such a set $\frac{d_p^{\alpha,g}}{dt^\alpha}\Im_a^{\alpha,g}x \neq x$.

In the remaining part of this paper, all g–Caputo fractional pseudo- (trivially weak) derivatives are taken in the sense of Definition 7.

Now, we are in the position to investigate the existence of solutions to the following g–Caputo fractional boundary value problem

$$\frac{d_p^{\alpha,g}}{dt^\alpha}x(t) = \lambda f\left(t, x(t), \frac{d_p^{\beta,g}}{dt^\beta}x(t)\right), \quad \beta \in (0,1), \alpha \in (1,2), t \in I, \lambda \in \mathbb{R} \tag{21}$$

combined with the nonlocal three-point boundary conditions

$$x(a) = 0, \qquad x(b) - px(\xi) = c, \qquad a < \xi, < b, \qquad p \in [0, \infty), \qquad c \in E. \tag{22}$$

Here, $\frac{d_p^{\alpha,g}}{dt^\alpha}$ denotes the g–Caputo fractional pseudodifferential operators defined as in (18). It is absolutely necessary to start from the definition of a solution of this problem. Let us introduce the following:

Definition 8 The function $x \in C[I, E_w]$ is called a pseudo- (*resp.*, weak) solution to the problem ((21) and (22) if x admits a g–Caputo fractional pseudo- (*resp.*, weak) derivative of order $\alpha \in (1,2)$ and satisfies (22) together with

$$\frac{d_p^{\alpha,g}}{dt^\alpha}x(t) = \lambda f\left(t, x(t), \frac{d_p^{\beta,g}}{dt^\beta}x(t)\right)$$

and, respectively, for the weak derivative

$$\frac{d_{\omega}^{\alpha,g}}{dt^{\alpha}}x(t) = \lambda f\left(t, x(t), \frac{d_{\omega}^{\beta,g}}{dt^{\beta}}x(t)\right) \quad \text{a.e. on } I.$$

Let us present some proposed integral form for the differential problem (21). Assuming very natural conditions, we always have the following relationship of their solutions:

Lemma 9 *Let $\alpha \in (1,2)$, $\beta \in (0,1)$, $p \in [0,\infty)$ and $\xi \in I$ be such that $\|g\| \neq p^{\frac{1}{\alpha-1}} g(\xi)$. For any $f \in P[I,E]$, the integral equation modeled off the problem ((21) and (22)) in the form*

$$u(t) = \frac{c_1}{\Gamma(\alpha-\beta)}\big(g(t)\big)^{\alpha-\beta} + \lambda \Im_a^{\alpha-\beta,g} f\big(t, u(t), \Im_a^{\beta,g} u(t)\big), \tag{23}$$

where $x = \Im_a^{\beta,g} u$ and

$$\begin{aligned} c_1 = {} & \frac{\Gamma(\alpha)}{\|g\|^{\alpha-1} - p(g(\xi))^{\alpha-1}} \\ & \times \big[c + \lambda\big(p\Im_a^{\alpha,g} f\big(\xi, u(\xi), \Im_a^{\beta,g} u(\xi)\big) - \Im_a^{\alpha,g} f\big(b, u(b), \Im_a^{\beta,g} u(b)\big)\big)\big] \end{aligned} \tag{24}$$

has a solution $u \in C[I,E_w]$ provided $x = \Im_a^{\beta,g} u$ is a solution of BVP (21) and (22).

Proof Let $x \in C[I,E_w]$ satisfy the problem ((21) and (22)) and define a function $u := \frac{d_p^{\beta,g}}{dt^{\beta}} x = \Im_a^{1-\beta,g}\delta_p x$. By virtue of our boundary condition $x(a) = 0$, using Lemma 2 we arrive at $x = \Im_a^{\beta,g} u$. Also, the boundary condition $x(b) - px(\xi) = c$ is transformed into $\Im_a^{\beta,g} u(b) - p\Im_a^{\beta,g} u(\xi) = c$. In this case, the differential equation (21) reads as

$$\frac{d_p^{\alpha,g}}{dt^{\alpha}} \Im_a^{\beta,g} u(t) = \lambda f\big(t, u(t), \Im_a^{\beta,g} u(t)\big), \quad \beta \in (0,1), \alpha \in (1,2), t \in I, \lambda \in \mathbb{R}. \tag{25}$$

Now, in view of Definition 7 of MCFPD and letting

$$\frac{d^{\alpha,g}}{dt^{\alpha}} \Im_a^{\beta,g} u(t) = \delta_p \Im_a^{2-\alpha,g} \delta_p \Im_a^{\beta,g} u(t) = \lambda f\big(t, u(t), \Im_a^{\beta,g} u(t)\big), \quad t \in I,$$

means that

$$\mathfrak{D}_p \Im_a^{2-\alpha} \delta_p \Im_a^{\beta,g} u(t) = \lambda g'(t) f\big(t, u(t), \Im_a^{\beta,g} u(t)\big), \quad t \in I.$$

Thus, "formally" we obtain

$$\Im_a^{2-\alpha,g} \delta_p \Im_a^{\beta,g} u(t) = c_1 + \lambda \int_a^t g'(s) f\big(s, u(s), \Im_a^{\beta,g} u(s)\big)\, ds = c_1 + \lambda \Im_a^{1,g} f\big(t, u(t), \Im_a^{\beta,g} u(t)\big).$$

Operating by $\Im_a^{\alpha,g}$ yields

$$\begin{aligned} \Im_a^{2,g} \delta_p \Im_a^{\beta,g} u(t) &= c_1 \frac{(g(t))^{\alpha}}{\Gamma(1+\alpha)} + \lambda \Im_a^{\alpha+1,g} f\big(t, u(t), \Im_a^{\beta,g} u(t)\big) \\ &= c_1 \frac{(g(t))^{\alpha}}{\Gamma(1+\alpha)} + \lambda \int_a^t g'(s)\big(\Im_a^{\alpha,g} f\big(t, u(t), \Im_a^{\beta,g} u(t)\big)\big)\, ds. \end{aligned}$$

Now, differentiating δ_p both sides twice, we arrive at

$$\delta_p \Im_a^{\beta,g} u(t) = c_1 \frac{(g(t))^{\alpha-2}}{\Gamma(\alpha-1)} + \lambda \Im_a^{\alpha-1,g} f\big(t, u(t), \Im_a^{\beta,g} u(t)\big). \tag{26}$$

From which (still "formally"), we obtain

$$\Im_a^{\beta,g} u(t) = c_0 + \frac{c_1}{\Gamma(\alpha)} \big(g(t)\big)^{\alpha-1} + \lambda \Im_a^{\alpha,g} f\big(t, u(t), \Im_a^{\beta,g} u(t)\big), \tag{27}$$

with some (presently unknown) quantities $c_0, c_1 \in E$. Since $x(a) = \Im_a^{\beta,g} u(a) = 0$ and $\Im_a^{\beta,g} u(b) - p\Im_a^{\beta,g} u(\xi) = c$, it can be easily seen that $c_0 = 0$ and

$$c_1 = \frac{\Gamma(\alpha)}{\|g\|^{\alpha-1} - p(g(\xi))^{\alpha-1}} \big[c + \lambda \big(p\Im_a^{\alpha,g} f\big(\xi, u(\xi), \Im_a^{\beta,g} u(\xi)\big) - \Im_a^{\alpha,g} f\big(b, u(b), \Im_a^{\beta,g} u(b)\big)\big)\big].$$

Operating by $\Im_a^{1-\beta,g}$ on both sides of (27) yields

$$\Im_a^{1,g} u(t) = \frac{c_1}{\Gamma(1+\alpha-\beta)} \big(g(t)\big)^{\alpha-\beta} + \lambda \Im_a^{1+\alpha-\beta,g} f\big(t, u(t), \Im_a^{\beta,g} u(t)\big).$$

In this connection, we conclude that

$$u(t) = \frac{c_1}{\Gamma(\alpha-\beta)} \big(g(t)\big)^{\alpha-\beta-1} + \lambda \Im_a^{\alpha-\beta,g} f\big(t, u(t), \Im_a^{\beta,g} u(t)\big), \quad t \in I. \tag{28}$$

Now, inserting c_1 into (28) yields ("formally") the integral equation. This completes the proof. □

We should answer the question when the two problems are equivalent. To do this we need to present a precise definition of the solutions for (23).

Definition 9 By a weak solution of (23) we mean a function $u \in C[I, E_w]$ satisfying

$$\varphi\big(u(t)\big) = \varphi\left(\frac{c_1}{\Gamma(\alpha-\beta)} \big(g(t)\big)^{\alpha-\beta-1} + \lambda \Im_a^{\alpha-\beta,g} f\big(t, u(t), \Im_a^{\beta,g} u(t)\big)\right), \quad t \in I, \text{ for all } \varphi \in E^*.$$

Let us recall that if we are studying pseudosolutions, some negligible sets D_φ, where the equation is not satisfied, are excluded and they are dependent on $\varphi \in E^*$. Such a set does not affect the calculated fractional Pettis integrals (cf. Remark 7).

Since the space of all Pettis integrable functions is not complete, not all methods of the proofs of the existence of solutions to the integral equation (23) are allowed and we cannot follow many ideas taken from the case of the strong topology. We restrict our attention to the case of weakly continuous solutions of the mentioned integral equation and then to pseudosolutions of the problem (23).

Now, we are ready to present the following theorem that will allow us to introduce the assertions that provide conditions under which we ensure the existence of weakly continuous solutions to the integral equation (23).

Theorem 3 *Let $\beta \in (0,1)$, $\alpha \in (1,2)$ such that $\alpha \geq 1+\beta$. Assume that*

A) *ψ is a Young function such that its complementary Young function $\widetilde{\psi}$ satisfies*

$$\int_0^t \widetilde{\psi}\left(s^{-\nu}\right) ds < \infty, \quad t > 0, \nu := \max\{2+\beta-\alpha, \alpha-\beta-1\}; \tag{29}$$

B) *Let $f : I \times E \times E \to E$ satisfy the following assumptions:*

(1) *For every $t \in I$, $f(t,\cdot,\cdot)$ is ww-sequentially continuous;*

(2) *For every $x, y \in C[I,E_w]$, $f(\cdot,x(\cdot),y(\cdot)) \in P[I,E]$;*

(3) *For any $r > 0$ and each $\varphi \in E^*$ there exists an $L_\psi(I,\mathbb{R})$-integrable function $M_r^\varphi : I \to \mathbb{R}^+$ such that $|\varphi(f(t,x,y))| \leq M_r^\varphi(t)$ for a.e. $t \in I$ and all $x, y \in C[I,E_w]$ whenever $\max\{\|y\|,\|x\|\} \leq r$. Moreover, there exists a continuous nondecreasing function $\Omega : \mathbb{R}^+ \to \mathbb{R}^+$ and such that for all $\varphi \in E^*$ with $\|\varphi\| \leq 1$, $\|M_r^\varphi\|_\psi < \Omega(r)$ and $\int_0^\infty \frac{dr}{\|M_r^\varphi\|_\psi} = \infty$;*

(4) *There exists a positive constant k such that for arbitrary bounded sets $B_1, B_2 \subset E$, we have*

$$\boldsymbol{\mu}\left(f(I,B_1,B_2)\right) \leq k\left[\boldsymbol{\mu}(B_1) + \boldsymbol{\mu}(B_2)\right]. \tag{30}$$

Then, there is $\rho > 0$ such that for any $\lambda \in \mathbb{R}$ with $|\lambda| \leq \rho$, the integral equation (23) has at least one weak solution $u \in C[I,E_w]$.

Remark 9

1. In [34, Lemma 19] one can find some sufficient conditions to satisfy assumption B) (2).
2. The integral in (29) is convergent, so in view of [11, Proposition 1], we also have

$$\int_0^t \widetilde{\psi}\left(s^{\alpha-\beta-2}\right) ds < \infty \quad \text{and} \quad \int_0^t \widetilde{\psi}\left(s^{1+\beta-\alpha}\right) ds < \infty \quad \text{for any } t \in I. \tag{31}$$

Before embarking on the proof of the above theorem, let us define a constant

$$\Delta := \frac{4\widetilde{\Psi}_{\alpha-\beta-1}(\|g\|)}{\Gamma(\alpha-\beta-1)}\left[1 + \frac{\|g\|^\alpha(1+p)\Gamma(\alpha)}{\Gamma(\alpha-\beta)\Gamma(2+\beta)|\|g\|^{\alpha-1} - p(g(\xi))^{\alpha-1}|}\right].$$

Moreover, let us define a positive real number ρ by

$$\rho := \begin{cases} \min\{\frac{1}{H}, \frac{1}{L}\} & H \neq 0, \\ \frac{1}{L} & H = 0, \end{cases} \tag{32}$$

where

$$H := \Delta \frac{\max\{1, \|g\|^\beta\}}{\Gamma(1+\beta)} \limsup_{r>0} \frac{\Omega(r)}{r},$$

$$L := k\|g\|^{\alpha-\beta}\left(\frac{(1+p)\|g\|^{\alpha-1}}{\alpha\Gamma(\alpha-\beta)|\|g\|^{\alpha-1} - p(g(\xi))^{\alpha-1}|} + \frac{1}{\Gamma(1+\alpha-\beta)}\right)\left[1 + \frac{\|g\|^\beta}{\Gamma(1+\beta)}\right].$$

In this case, for any $\lambda \in \mathbb{R}$ with $|\lambda| < \rho$, we have

$$|\lambda|\Delta \limsup_{r>0} \frac{\Omega(r)}{r} \le \frac{\Gamma(1+\beta)}{\max\{1, \|g\|^{\beta}\}}.$$

In this connection, reasoning as in [34, proof of inequality (44)], we can show that there exists $R_0 > 0$ such that for any $R > R_0$ we have

$$\frac{\|g\|^{\alpha-\beta-1}\|c\|\Gamma(\alpha)}{\Gamma(\alpha-\beta)|\|g\|^{\alpha-1} - p(g(\xi))^{\alpha-1}|} + |\lambda|\Delta\Omega(R) \le \frac{\Gamma(1+\beta)}{\max\{1, \|g\|^{\beta}\}} R. \tag{33}$$

For brevity and to allow a generalization, let us keep in the following a symbol R_0. We are assured that under our assumptions for sufficiently small λ we have global solutions, i.e., functions defined on the interval I.

Proof of Theorem 3 Define an operator T on $C[I, E_w]$ generating the right-hand side of the differential equation in BVP, i.e., of the form

$$Tu(t) := \frac{c_1(g(t))^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} + \lambda \Im_a^{1,g} Ux(t), \tag{34}$$

where $\beta \in (0,1)$, $\alpha \in (1,2)$, with $\alpha \ge 1+\beta$, $t \in I$ and

$$Uu(t) := \Im_a^{\alpha-\beta-1,g} f\big(t, u(t), \Im_a^{\beta,g} u(t)\big) \tag{35}$$

and c_1 is defined by (24).

I. First, we note that the operators U and T are well defined on $C[I, E_w]$. To see this, let us observe that for any $u \in C[I, E_w]$, by Lemma 5 $\Im_a^{\beta,g} u$ is well defined and by Theorem 2, we have $\Im_a^{\beta,g} u \in C[I, E_w]$.

Under assumption B) (2), for any $u \in C[I, E_w]$ the superposition $F(u) := f(\cdot, u(\cdot), \Im_a^{\beta,g} u(\cdot))$ is weakly measurable, Pettis integrable on I. Hence, in view of Remark 2, we obtain the existence of $\Im_a^{\alpha,g} F(u)$ for any $u \in C[I, E_w]$. This means that the element $c_1 \in E$ defined by (24) is well defined, so is the first component of T.

Moreover, by assumption B) (3) we have

$$\int_a^b \psi\left(\frac{|\varphi(f(t, u(t), \Im_a^{\beta,g} u(t)))|}{\|M_r^{\varphi}\|_{\psi}}\right) dt \le \int_a^b \psi\left(\frac{M_r^{\varphi}(t)}{\|M_r^{\varphi}\|_{\psi}}\right) dt \le 1,$$

for any $r \ge \max\{\|u\|, \|\Im_a^{\beta,g} u\|\}$ and for every $\varphi \in E^*$.

Then, $F(u)(\cdot) \in \mathcal{H}_0^{\psi}(E)$ and $\|Fu\|_{\psi} \le \|M_r^{\varphi}\|_{\psi}$. Bearing in mind (29), it follows in view of Theorem 2 that U is a well-defined operator on $C[I, E_w]$ with its values in $C[I, E_w]$. In this connection, Corollary 1 and Proposition 2 yield that $Uu(\cdot)g'(\cdot) \in P[I, E]$ holds true for any $u \in C[I, E_w]$. Since $\alpha - \beta \ge 1$, in view of (31), by applying Lemma 6 and using Remark 2, it follows that $\Im_a^{\alpha-\beta,g} f(\cdot, u(\cdot), \Im_a^{\beta,g} u(\cdot)) = \Im_a^{1,g} Uu(\cdot))$ and hence the operator T is defined on the space $C[I, E_w]$.

Moreover, as $\frac{c_1(g(t))^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}$ is continuous with values in E and we just proved that $U : C[I, E_w] \to C[I, E_w]$, then the same property holds true for T.

II. Now, let us construct an invariant domain for T, which is required by Theorem 1. Define a convex and closed subset $Q \subset C[I, E_w]$ by

$$\mathcal{Q} = \Big\{ u \in [I, E_w] : \|u\| \le R_0, \text{ and } \big\|u(t) - u(s)\big\| \\ \le \frac{\|c_1\|(|(g(t))^{\alpha-\beta-1} - (g(s))^{\alpha-\beta-1}|)}{\Gamma(\alpha-\beta)} + \frac{4|\lambda|}{\Gamma(\alpha-\beta-1)} \int_{g(s)}^{g(t)} \widetilde{\Psi}_{\alpha-\beta-1}(\zeta)\, d\zeta, t, s \in I \Big\}.$$

Lemma 4 implies that $\mathcal{Q}$ is a strongly equicontinuous subset of $C[I, E_w]$.

Observe that $\beta \in (0,1)$ and define $r_0 > R_0$ by $r_0 := \frac{R_0}{\Gamma(1+\beta)} \max\{1, \|g\|^{\beta}\}$. For any $u \in \mathcal{Q}$, we have $\|u\| \le r_0$ and $\|\Im_a^{\beta,g} u\| \le r_0$.

We proved in Theorem 2 that for any $\theta \in (0,1]$ and $x \in \mathcal{H}_0^{\psi}(E)$ we have an estimation

$$\big\|\Im_a^{\theta,g} x(t)\big\|_{\widetilde{\Psi}_\theta} \le \frac{4\widetilde{\Psi}_\theta(|g(t)|)}{\Gamma(\theta)} \big\|\varphi(x)\big\|_{\psi}$$

for some $\varphi \in E^*$ with $\|\varphi\| \le 1$. Then, for $\theta = \alpha - \beta - 1$ this inequality is fulfilled. Take an arbitrary $u \in \mathcal{Q}$. By applying the estimation from B) (3) and as $M_{r_0}^{\varphi} \in L_{\psi}$, for any $\varphi \in E^*$ we have

$$\big|\Im_a^{\alpha-\beta-1,g}\big|\varphi\big(f\big(t, u(t), \Im_a^{\beta,g} u(t)\big)\big)\big|\big| \le \big|\Im_a^{\alpha-\beta-1,g} M_{r_0}^{\varphi}(t)\big| \\ \le \frac{4\widetilde{\Psi}_{\alpha-\beta-1}(g(t))}{\Gamma(\alpha-\beta-1)} \big\|M_{r_0}^{\varphi}\big\|_{\psi} \le \frac{4\widetilde{\Psi}_{\alpha-\beta-1}(\|g\|)}{\Gamma(\alpha-\beta-1)} \big\|M_{r_0}^{\varphi}\big\|_{\psi}.$$

We can also estimate $\varphi(c_1)$

$$\begin{aligned}
\big|\varphi(c_1)\big| &\le \frac{\Gamma(\alpha)}{|\|g\|^{\alpha-1} - p(g(\xi))^{\alpha-1}|} \big[\varphi(c) + |\lambda|\big(p\Im_a^{\alpha,g}\big|\varphi\big(f\big(\xi, u(\xi), \Im_a^{\beta,g} u(\xi)\big)\big)\big| \\
&\quad + \Im_a^{\alpha,g}|\varphi\big(f\big(b, u(b), \Im_a^{\beta,g} u(b)|\big)\big)\big] \\
&\le \frac{\Gamma(\alpha)}{|\|g\|^{\alpha-1} - p(g(\xi))^{\alpha-1}|} \big[\varphi(c) + |\lambda|\big(p\Im_a^{1+\beta,g}\Im_a^{\alpha-\beta-1,g} M_{r_0}^{\varphi}(\xi) \\
&\quad + \Im_a^{1+\beta,g}\Im_a^{\alpha-\beta-1,g} M_{r_0}^{\varphi}(b)\big)\big] \\
&\le \frac{\Gamma(\alpha)}{|\|g\|^{\alpha-1} - p(g(\xi))^{\alpha-1}|} \bigg[\varphi(c) + \frac{4|\lambda|(p+1)\widetilde{\Psi}_{\alpha-\beta-1}(\|g\|)\|M_{r_0}^{\varphi}\|_{\psi}}{\Gamma(2+\beta)\Gamma(\alpha-\beta-1)} \|g\|^{1+\beta}\bigg].
\end{aligned}$$

By taking the supremum over all $\varphi \in E^*$ with $\|\varphi\| \le 1$ in the above inequality and by applying the Hahn–Banach theorem we obtain

$$\|c_1\| \le \frac{\Gamma(\alpha)}{|\|g\|^{\alpha-1} - p(g(\xi))^{\alpha-1}|} \bigg[\|c\| + \frac{4|\lambda|(p+1)\widetilde{\Psi}_{\alpha-\beta-1}(\|g\|)\Omega(r_0)}{\Gamma(2+\beta)\Gamma(\alpha-\beta-1)} \|g\|^{1+\beta}\bigg]. \tag{36}$$

Moreover, again as a consequence of the Hahn–Banach theorem, for any $u \in \mathcal{Q}$ and any $t \in I$ there exists $\varphi \in E^*$ with $\|\varphi\| = 1$ such that $\|T(u)(t)\| = \varphi(T(u)(t))$. Using (36), there is no difficulty in showing that

$$\big\|T(u)(t)\big\| = \big|\varphi(Tu)(t)\big|$$

$$\leq \frac{\|c_1\|\|g\|^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} + \frac{4|\lambda|\widetilde{\Psi}_{\alpha-\beta-1}(\|g\|)}{\Gamma(\alpha-\beta-1)}\Omega(r_0)$$
$$\leq \frac{\|g\|^{\alpha-\beta-1}\|c\|\Gamma(\alpha)}{\Gamma(\alpha-\beta)|\|g\|^{\alpha-1}-p(g(\xi))^{\alpha-1}|} + |\lambda|\Delta\Omega(r_0).$$

Since $r_0 > R_0$, it follows in view of (33) that

$$\|Tx\| = \max_{t\in I}\|Tu(t)\| \leq \frac{\Gamma(1+\beta)}{\max\{1,\|g\|^{\beta}\}}r_0 = R_0. \tag{37}$$

Also, for any $t,s\in I$ and any $u\in\mathcal{Q}$, it can be easily seen that

$$\|T(u)(t)-T(u)(s)\| \leq \frac{\|c_1\|(|(g(t))^{\alpha-\beta-1}-(g(s))^{\alpha-\beta-1}|)}{\Gamma(\alpha-\beta)} + \frac{4|\lambda|}{\Gamma(\alpha-\beta-1)}\int_{g(s)}^{g(t)}\widetilde{\Psi}_{\alpha-\beta-1}(\zeta)\,d\zeta.$$

III. Now, we need to prove that T is weakly–weakly sequentially continuous. Let $\{u_n\}$ be a sequence in $\mathcal{Q}$ and let $u_n\to u$ in $C[I,E_w]$. Recall, that weak convergence in $C[I,E_w]$ means exactly its boundedness and weak pointwise convergence for any $t\in I$. The first condition is assured by the definition of $\mathcal{Q}$.

Fix an arbitrary $t\in I$. Consider now the operator U and observe that

$$U(u_n)(t) = \Im_a^{\alpha-\beta-1,g}f\big(t,u_n(t),\Im_a^{\beta,g}u_n(t)\big).$$

From the dominated convergence theorem for the Pettis integral applied to $\Im_a^{\beta,g}$ we obtain convergence of $\Im_a^{\beta,g}u_n(t)$ to $\Im_a^{\beta,g}u(t))$. Hence, assumption B) (1) implies that the sequence $f(t,u_n(t),\Im_a^{\beta,g}u_n(t))$ converges weakly to $f(t,u(t),\Im_a^{\beta,g}u(t))$. This implies that $\Im_a^{1,g}Uu_n(t)\to\Im_a^{1,g}Uu(t)$ and finally $(Tx_n)(t)$ converges weakly to $(Tu)(t)$ in (E,w) for each $t\in I$, which means that $T:\mathcal{Q}\to\mathcal{Q}$ is weakly–weakly sequentially continuous in $\mathcal{Q}$.

IV. Let us verify condition (2) in Theorem 1.

Let V be a subset of $\mathcal{Q}$ satisfying $\bar{V}=\overline{conv}((TV)\cup\{0\})$. Obviously, $V(t)\subset\overline{conv}((TV)(t)\cup\{0\})$, $t\in I$. Since $T(\mathcal{Q})$ is uniformly bounded and strongly equicontinuous in $C[I,E_w]$, it follows that V is also bounded and equicontinuous. Taking into account our Lemma 1, the function $v(t):=\boldsymbol{\mu}(V(t))$ is continuous on I, $V(t):=\{v(t):v\in V\}$ and

$$TV(t) = \{Tv(t):u\in V\} = \left\{\frac{c_1(g(t))^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} + \lambda\Im_a^{1,g}Uu(t):u\in V\right\}.$$

Arguing similarly as in [11, Step 3 of the proof of Theorem 3] (see also Lemma 7), we can show that

$$\boldsymbol{\mu}\big(\{\Im_a^{\alpha-\beta-1,g}f\big(t,u(t),\Im_a^{\beta,g}u(t)\big):u\in V\}\big) \leq \Im_a^{\alpha-\beta-1,g}\boldsymbol{\mu}\big(\{f\big(t,u(t),\Im_a^{\beta,g}u(t)\big):u\in V\}\big).$$

Also, by the aid of properties of $\boldsymbol{\mu}$ (see [13, 21]), in view of Lemma 7, we obtain that $\boldsymbol{\mu}(\{\Im_a^{1,g}UV(t)\}\leq\Im_a^{1,g}\boldsymbol{\mu}(\{UV(t)\}$. Thus,

$$\boldsymbol{\mu}\big(TV(t)\big) \leq \boldsymbol{\mu}\left(\frac{c_1(V)(g(t))^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}\right) + |\lambda|\boldsymbol{\mu}\big(\Im_a^{1,g}UV(t)\big)$$

$$= \boldsymbol{\mu}\big(c_1(V)\big) \cdot \frac{(g(t))^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} + |\lambda| \Im_a^{1,g} \boldsymbol{\mu}\big(\big\{\Im_a^{\alpha-\beta-1,g} f\big(t, u(t), \Im_a^{\beta,g} u(t)\big) : u \in V\big\}\big)$$
$$\leq \boldsymbol{\mu}\big(c_1(V)\big) \cdot \frac{(g(t))^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} + |\lambda| \Im_a^{\alpha-\beta,g} \boldsymbol{\mu}\big(\big\{f\big(t, u(t), \Im_a^{\beta,g} u(t)\big) : u \in V\big\}\big). \quad (38)$$

By applying assumption B) (4), we ensure that

$$\boldsymbol{\mu}\big(TV(t)\big) \leq \boldsymbol{\mu}\big(c_1(V)\big) \cdot \frac{(g(t))^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} + k|\lambda| \cdot \Im_a^{\alpha-\beta,g}\big[\boldsymbol{\mu}\big(V(t)\big) + \boldsymbol{\mu}\big(\Im_a^{\beta,g} V(t)\big)\big]$$
$$\leq \boldsymbol{\mu}\big(c_1(V)\big) \cdot \frac{\|g\|^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} + \frac{|\lambda| k \|g\|^{\alpha-\beta}}{\Gamma(1+\alpha-\beta)}\left[1 + \frac{\|g\|^{\beta}}{\Gamma(1+\beta)}\right]\boldsymbol{\mu}_C(V). \quad (39)$$

An analogous reasoning leads to the estimate

$$\boldsymbol{\mu}\big(\big\{\Im_a^{\alpha,g} f\big(t, u(t), \Im_a^{\beta,g} u(t)\big) : u \in V\big\}\big) \leq \frac{k\|g\|^{\alpha}}{\Gamma(1+\alpha)}\left[1 + \frac{\|g\|^{\beta}}{\Gamma(1+\beta)}\right]\boldsymbol{\mu}_C(V).$$

Since

$$\boldsymbol{\mu}\left(\left\{\frac{\|g\|^{\alpha-\beta-1}\Gamma(\alpha)}{\Gamma(\alpha-\beta)|\|g\|^{\alpha-1} - p(g(\xi))^{\alpha-1}|} c\right\}\right) = 0,$$

it follows in view of the definition of c_1

$$\boldsymbol{\mu}\big(c_1(V)\big)\frac{\|g\|^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} \leq \frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)|\|g\|^{\alpha-1} - p(g(\xi))^{\alpha-1}|} \frac{|\lambda| k(1+p)\|g\|^{2\alpha-\beta-1}}{\Gamma(1+\alpha)}$$
$$\times \left[1 + \frac{\|g\|^{\beta}}{\Gamma(1+\beta)}\right]\boldsymbol{\mu}_C(V). \quad (40)$$

From the definition of the set V and by applying properties of measures of weak noncompactness we obtain

$$\boldsymbol{\mu}\big(V(t)\big) = \boldsymbol{\mu}\big(conv\big(\{0\} \cup TV(t)\big)\big) = \boldsymbol{\mu}\big(TV(t)\big)$$
$$\leq |\lambda| k \|g\|^{\alpha-\beta}\left(\frac{(1+p)\|g\|^{\alpha-1}}{\alpha\Gamma(\alpha-\beta)|\|g\|^{\alpha-1} - p(g(\xi))^{\alpha-1}|} + \frac{1}{\Gamma(1+\alpha-\beta)}\right)$$
$$\times \left[1 + \frac{\|g\|^{\beta}}{\Gamma(1+\beta)}\right]\boldsymbol{\mu}_C(V).$$

Hence, we can take the supremum over all $t \in I$

$$\boldsymbol{\mu}_C(V) \leq |\lambda| L \boldsymbol{\mu}_C(V).$$

Taking into account that $|\lambda| L \leq 1$, immediately, we obtain $\boldsymbol{\mu}_C(V) = 0$, so V should be relatively weakly compact in $C[I, E]$.

Finally, Theorem 1 implies that T has a fixed point being a pseudosolution to the integral equation (23). □

We point out that if E is reflexive then the implication (2) of Theorem 1 is automatically satisfied, as subsets of reflexive Banach spaces are weakly compact if and only they are weakly closed and norm bounded. In this situation, it is no longer necessary to assume any compactness hypothesis imposed on the nonlinearity of f to assure the existence of solutions to the fractional integral equation (23).

In addition to giving a conditions under which the integral equation (23) admits a solutions in the space $C[I, E_w]$, Theorem 3 may be used to obtain a result concerning the existence of solutions to the boundary value problem (21) and (22).

Now, we are in the position to state and prove the following existence theorem:

Theorem 4 *Let $\beta \in (0,1)$, $\alpha \in (1,2)$ such that $\alpha \geq 1+\beta$. Assume that ψ is a Young function such that its complementary Young function $\widetilde{\psi}$ satisfies*

$$\int_0^t \widetilde{\psi}\left(s^{-\nu}\right) ds < \infty, \quad t > 0, \nu := \max\{2+\beta-\alpha, \alpha-\beta-1\}. \tag{41}$$

Assume that E has a total dual. If $f : I \times E \to E$ is a function fulfilling all assumptions B) from Theorem 3, then the problem (21) and (22) admits at least one pseudosolution $x \in C[I, E_w]$.

Proof At the beginning, we note that if $u \in C[I, E_w]$ solves the integral equation (23) then, obviously u is weakly absolutely continuous function having integrable pseudoderivative (cf. Lemma 2). Indeed, we have

$$\delta u(t) = \frac{1}{g'(t)} \frac{d_p}{dt} u(t) = \frac{c_1}{\Gamma(\alpha-\beta-1)} \big(g(t)\big)^{\alpha-\beta-2} + \lambda \Im_a^{\alpha-\beta-1,g} f\big(t, u(t), \Im_a^{\beta,g} u(t)\big).$$

Hence, by the definition of the pseudoderivative we have

$$\frac{1}{g'(t)} \frac{d\varphi u(t)}{dt} \in L_p,$$

for some $p \in (1, \frac{1}{2+\beta-\alpha})$. Now, since $\Im_a^{\alpha-\beta-1,g} f(\cdot, u(\cdot), \Im_a^{\beta,g} u(\cdot)) \in C[I, E_w]$, it follows that

$$\Im_a^{\beta,g} \Im_a^{\alpha-\beta-1,g} f\big(\cdot, u(\cdot), \Im_a^{\beta,g} u(\cdot)\big) \in C[I, E_w].$$

In view of Lemma 6 we conclude that

$$\begin{aligned}\Im_a^{1-\beta,g} \Im_a^{\beta,g} \Im_a^{\alpha-\beta-1,g} f\big(\cdot, u(\cdot), \Im_a^{\beta,g} u(\cdot)\big) &= \Im_a^{1,g} \Im_a^{\alpha-\beta-1,g} f\big(\cdot, u(\cdot), \Im_a^{\beta,g} u(\cdot)\big) \\ &= \Im_a^{\alpha-\beta,g} f\big(t, u(t), \Im_a^{\beta,g} u(t)\big).\end{aligned}$$

Consequently,

$$\Im_a^{1-\beta,g} \Im_a^{\beta,g} \delta_p u(t) = \frac{c_1}{\Gamma(\alpha-\beta)} \big(g(t)\big)^{\alpha-\beta-1} + \lambda \Im_a^{\alpha-\beta,g} f\big(t, u(t), \Im_a^{\beta,g} u(t)\big) = u.$$

Seeing that $u(a) = 0$, we have, in view of Lemma 2, $\Im_a^{1,g} \delta_p u(t) = \int_a^t \mathfrak{D}_p u(s)\, ds = u(t)$, $t \in I$. Now, let us define $x := \Im_a^{\beta,g} u \in C[I, E_w]$. Further,

$$x = \Im_a^{\beta,g} \big(\Im_a^{1-\beta,g} \Im_a^{\beta,g} \delta_p u(t)\big).$$

Therefore, for any $\varphi \in E^*$, by the commutative property of the g-fractional integral operator, we have

$$\varphi x = \Im_a^{\beta,g}\left(\Im_a^{1-\beta,g}\Im_a^{\beta,g}\varphi\left(\delta_p u(t)\right)\right) = \Im_a^{1,g}\Im_a^{\beta,g}\varphi\left(\delta_p u(t)\right) \quad \Rightarrow \quad \frac{1}{g'(t)}\frac{d\varphi x}{dt} = \Im_a^{\beta,g}\varphi\left(\delta_p u(t)\right).$$

Accordingly,

$$\Im_a^{\beta,g}\left(\frac{1}{g'(t)}\frac{d\varphi x}{dt}\right) = \Im_a^{1,g}\varphi\left(\delta_p u(t)\right) = \varphi(u)$$

and

$$\frac{d_p^{\beta,g}}{dt^\alpha}x = u.$$

Thus, if $u \in C[I, E_w]$ solves (23) then for all $t \in I$ we obtain

$$\Im_a^{\beta,g}u(t) = \frac{c_1}{\Gamma(\alpha-\beta)}\Im_a^{\beta,g}\left(g(t)\right)^{\alpha-\beta-1} + \lambda\Im_a^{\beta,g}\Im_a^{\alpha-\beta,g}f\left(t, u(t), \Im_a^{\beta,g}u(t)\right)).$$

Hence, by applying the above equality we obtain

$$x(t) = \frac{\Gamma(\alpha-\beta)c_1}{\Gamma(\alpha)\Gamma(\alpha-\beta)}\left(g(t)\right)^{\alpha-1} + \lambda\Im_a^{\alpha,g}f(\left(t, x(t), \frac{d_p^{\beta,g}}{dt^\alpha}x(t)\right), \quad t \in I.$$

By the definition of our generalized fractional derivative $\frac{d_p^{\alpha,g}}{dt^\alpha}$ we infer

$$\frac{d_p^{\alpha,g}}{dt^\alpha}\left[\frac{c_1}{\Gamma(\alpha)}\left(g(t)\right)^{\alpha-1}\right] = 0,$$

and by applying Lemma 8, we conclude that

$$\frac{d_p^{\alpha,g}}{dt^\alpha}x(t) = \lambda f\left(t, x(t), \frac{d_p^{\beta,g}}{dt^\alpha}x(t)\right).$$

On the other hand, there is no difficulty in showing that x satisfies

$$\begin{cases} x(a) = 0, \\ x(b) - px(\xi) = c, \quad a < \xi < b, p \in \mathbb{R}^+, c \in E. \end{cases} \tag{42}$$

This means that x is a pseudosolution x of the problem ((21) and (22)). □

Some examples of the use of our theorem to the problem ((21) and (22)) can be found in [12, Sect. 4]. However, it is worth noting that:

1. Our assumption that E has total dual is essential in Theorem 4 and cannot be omitted even if f is weakly absolutely continuous on I. Evidently, if we define $f : I \times B[I] \times B[I] \to B[I]$ by

$$f\left(t, x(t), y(t)\right) := \begin{cases} \chi_{\{t\}}(\cdot), & t \in J, x, y \in B[I], \\ \theta, & t \notin J. \end{cases}$$

Then, using similar arguments as in the proof of Theorem 4, by the aid of Example 3.2 we can show that

$$\frac{d_p^{\alpha,g}}{dt^\alpha}x(t) \neq \lambda f\left(t, x(t), \frac{d_p^{\beta,g}}{dt^\alpha}x(t)\right)$$

holds true on a subset of I of positive measure.

2. By virtue of the fact that the indefinite Pettis integral of a function $f \in P[I,E]$ does not enjoy the strong property of being a.e. weakly differentiable, it is immediately clear that the result obtained in Theorem 4 has no analog if we replace $\frac{d_p^{\alpha,g}}{dt^\alpha}$ by $\frac{d_\omega^{\alpha,g}}{dt^\alpha}$.
3. Arguing similarly as in the [11, Theorem 5], we are to consider the multivalued case of the problem ((21) and (22)).

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