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Global classical solutions to the viscous two-phase flow model with slip boundary conditions in 3D exterior domains

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Abstract

We consider the two-phase flow model in 3D exterior domains with slip boundary conditions. We establish the global existence of classical solutions of this system, provided that the initial energy is suitably small. Furthermore, we prove that the pressure has large oscillations and contains vacuum states when the initial pressure allows large oscillations and a vacuum. Finally, we also obtain the large-time behavior of the classical solutions.

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1 Introduction

The two-fluid flow models widely used in the petroleum industry to describe the production and transport of oil and gas through long pipelines or wells can be written as (see [19–21, 34, 42])

$$\begin{cases} \rho_t + \operatorname{div}(\rho u) = 0, \\ m_t + \operatorname{div}(mu) = 0, \\ ((\rho + m)u)_t + \operatorname{div}[(\rho + m)u \otimes u] + \nabla P(\rho, m) = \mu\Delta u + (\lambda + \mu)\nabla \operatorname{div} u, \end{cases} \tag{1.1}$$

where $(x,t) \in \Omega \times (0,T]$, Ω is a domain in $\mathbb{R}^3$. $\rho \geq 0$, $m \geq 0$, $u = (u_1, u_2, u_3)$, and $P(\rho,m) = \rho^\gamma + m^\alpha$ ($\gamma > 1, \alpha \geq 1$) are the unknown two-phase flow model's fluid density, velocity, and pressure, respectively. The constants μ and λ are the shear viscosity and bulk coefficients, respectively, satisfying the following physical restrictions:

$$\mu > 0, \qquad 2\mu + 3\lambda \geq 0. \tag{1.2}$$

In this paper, the domain Ω is the exterior of a simply connected bounded domain D in $\mathbb{R}^3$, and its boundary $\partial\Omega$ is smooth. In addition, the system is studied subject to the given initial data

$$\begin{aligned} &\rho(x,0)=\rho_0(x), \qquad m(x,0)=m_0(x),\\ &(\rho+m)u(x,0)=(\rho_0+m_0)u_0(x), \quad x\in\Omega, \end{aligned} \tag{1.3}$$

and the slip boundary condition

$$u\cdot n=0, \qquad \operatorname{curl} u\times n=0 \quad \text{on } \partial\Omega, \tag{1.4}$$

with the far field behavior

$$u(x,t)\to 0, \qquad \rho(x,t)\to\rho_\infty\ge 0, \qquad m(x,t)\to m_\infty\ge 0, \quad \text{as } |x|\to\infty, \tag{1.5}$$

where $n=(n^1,n^2,n^3)$ is the unit outward normal vector to the boundary $\partial\Omega$ pointing outside Ω, ρ_∞ and m_∞ are the nonnegative constants.

The first condition in (1.4) is the non-penetration boundary condition, while the second one is also known in the form

$$\big(D(u)n\big)_\tau=-\kappa_\tau u_\tau,$$

where $D(u)=(\nabla u+(\nabla u)^{\mathrm{tr}})/2$ is the deformation tensor, κ_τ is the corresponding normal curvature of $\partial\Omega$ in the τ direction, and the symbol u_τ represents the projection of tangent plane of the vector u on $\partial\Omega$. This type of boundary condition was originally introduced by Navier [27] in 1823, which was followed by many applications, numerical studies, and analyses for various fluid mechanical problems, see, for instance, [8, 21, 31] and the references therein.

Many models are related to the two-phase model (1.1), especially the case of $\alpha=1$ corresponds to the hydrodynamic equations, which was derived as the asymptotic limit of Vlasov–Fokker–Planck equations coupled with compressible Navier–Stokes equations, see [7, 26]. The case of $\alpha=2$ is associated with a compressible Oldroyd-B type model with stress diffusion, see [2]. Furthermore, if we let $m\equiv 0$, then the viscous liquid-gas two-flow model (1.1) reduces to the classical isentropic compressible Navier–Stokes equations. Compared with the isentropic compressible Navier–Stokes equations, the main difference is that the pressure law $P(\rho,m)=\rho^\gamma+m^\alpha$ depends on two different variables from the continuity equations.

Before stating our main result, we briefly recall some previous known results on the viscous two-fluid model. For the one-dimensional case, Evje and Karlsen [10] obtained the first global existence result on weak solutions with large initial data subject to the domination conditions. Later, the domination condition was removed by Evje–Wen–Zhu [11] using the decomposition of the pressure term, which allows transition to each single-phase flow. Recently, Gao–Guo–Li [13] considered the Cauchy problem of the 1D viscous two-fluid model and established the global existence of strong solutions with a large initial value and vacuum. For more related results, please refer to [9, 10, 37, 38] and the references therein. For the multi-dimensional case, Yao–Zhang–Zhu [36] proved the global existence of weak solutions to the 2D Cauchy problem case when the initial energy is small and both of the initial densities are positive. Hao and Li [15] obtained the existence and uniqueness of the global strong solutions to the Cauchy problem in $\mathbb{R}^d$ with $d \geq 2$ in the framework of Besov spaces, where the possible vacuum state is included in the equilibrium state for the gas component at far field. Zhang and Zhu [40] considered the 3D Cauchy problem and proved the global existence of a strong solution when H^2-norm of the initial perturbation around a constant state is sufficiently small. When both phases contain a vacuum initially, Guo–Yang–Yao [14] proved the global existence of strong solutions to the 3D Cauchy problem under the assumption that initial energy is sufficiently small. Very recently, the domination condition was removed by Yu [39] for the global existence of the strong solution to the 3D case when the initial energy is small. For large initial data cases, Vasseur–Wen–Yu [32] obtained the global existence of weak solutions to the Dirichlet boundary value problem of (1.1) in $\mathbb{R}^3$ with the pressure $P(\rho, m) = \rho^r + m^\alpha$ ($r > 1$, $\alpha \geq 1$) and the domination conditions. Novotný and Pokorný [29] extended the domination condition to the case that both γ and α can touch $\frac{9}{5}$, where more general pressure laws covering the cases of $P(\rho, m) = \rho^r + m^\alpha$ ($r > 1$, $\alpha \geq 1$) were considered. Wen [35] obtained the global existence of weak solutions to 3D Dirichlet problem of compressible two-fluid model without any domination conditions. However, there are few results about classical solutions to compressible two-fluid model for general bounded domains, which is one of our main motivations of the present paper.

When we take $m = 0$ in (1.1), the two-phase flow model (1.1) changes into the compressible Navier–Stokes equations. In the last several decades, significant progress on the compressible Navier–Stokes equations has been achieved by many authors in the analysis of the well-posedness and large-time behavior. We only briefly review some results related to the existence of strong or classical solutions. The global classical solutions were first obtained by Matsumura–Nishida [25] for initial data close to a nonvacuum equilibrium in $H^3(\mathbb{R}^3)$. It is worth mentioning that their results have been improved by Huang–Li–Xin [18] and Li–Xin [23], in which the global existence of classical solutions is obtained with smooth initial data that are of small energy but possibly large oscillations. Very recently, for the 3D bounded domain (or 3D Exterior Domains) with slip boundary conditions, Cai–Li [5] (or Cai–Li–Lv [6]) proved the existence and large-time behavior of global classical solutions to the compressible Navier–Stokes equations. And Cai–Huang–Shi [4] proved the global existence and exponential growth of classical solutions subject to large potential forces with slip boundary condition in 3D bounded domains. For 3D bounded Domains with Non-Slip Boundary Conditions, Fan–Li [12] proved global classical solutions to the compressible Navier–Stokes system with a vacuum.

Before stating the main results, we introduce some notations and conventions used in this paper. We denote

$$\int f\,dx \triangleq \int_\Omega f\,dx.$$

For integer k and $1 \leq q < +\infty$, the standard homogeneous Sobolev spaces are denoted as follows:

$$D_0^{k,q}(\Omega) \triangleq \bigl\{u \in L^1_{\mathrm{loc}}(\Omega) \mid \bigl\|\nabla^k u\bigr\|_{L^q(\Omega)} < +\infty\bigr\}, \qquad \|\nabla u\|_{D^{k,q}(\Omega)} \triangleq \bigl\|\nabla^k u\bigr\|_{L^q(\Omega)}.$$

We also denote

$$D^k(\Omega) = D^{k,2}(\Omega), \qquad H^k(\Omega) = W^{k,2}(\Omega), \qquad W^{k,q}(\Omega) = L^q(\Omega) \cap D^{k,p}(\Omega)$$

with the norm $\|u\|_{W^{k,q}(\Omega)} \triangleq (\sum_{|m|\leq k} \|\nabla^m u\|^q_{L^q(\Omega)})^{\frac{1}{q}}$.

Simply, $L^q(\Omega)$, $D^{k,q}(\Omega)$, $D^k(\Omega)$, $W^{k,q}(\Omega)$, and $H^k(\Omega)$ can be denoted by L^q, $D^{k,q}$, D^k, $W^{k,q}$ and H^k, respectively and set that

$$B_R \triangleq \{x \in \mathbb{R}^3 \mid |x| < R\}.$$

For two 3×3 matrices $A = \{a_{ij}\}$, $B = \{b_{ij}\}$, the symbol $A : B$ represents the trace of AB,

$$A : B \triangleq \sum_{i,j=1}^{3} a_{ij} b_{ji}.$$

Define the initial total energy of (1.1) by:

$$C_0 \triangleq \int_\Omega \left(\frac{1}{2}(\rho_0 + m_0)|u_0|^2 + G(\rho_0, m_0) \right) dx \tag{1.6}$$

with

$$G(\rho, m) = \rho \int_{\rho_\infty}^{\rho} \frac{P(s, m) - P(\rho_\infty, m)}{s^2}\, ds + m \int_{m_\infty}^{m} \frac{P(\rho, s) - P(\rho, m_\infty)}{s^2}\, ds.$$

Finally, for $v = (v^1, v^2, v^3)$, we set $\nabla_j v = (\partial_j v^1, \partial_j v^2, \partial_j v^3)$, for $j = 1, 2, 3$, $P_0 = P(\rho_0, m_0)$, and $P_\infty = P(\rho_\infty, m_\infty)$.

Our first result is stated below:

Theorem 1.1 *Let Ω be the exterior of a simply connected bounded domain in $\mathbb{R}^3$ with smooth boundary $\partial\Omega$. For $M \geq 1$, $\bar{\rho} \geq \rho_\infty + 1$, $\bar{m} \geq m_\infty + 1$ and some $q \in (3, 6)$, assume that the initial data (ρ_0, m_0, u_0) satisfy the following condition*:

$$u_0 \in \{\bar{f} \in D^1 \cap D^2 : \bar{f} \cdot n = 0, \operatorname{curl} \bar{f} \times n = 0 \text{ on } \partial\Omega\}, \tag{1.7}$$

$$(\rho_0 - \rho_\infty, m_0 - m_\infty, P(\rho_0, m_0) - P(\rho_\infty, m_\infty)) \in H^2 \cap W^{2,q}, \tag{1.8}$$

$$0 \leq \rho_0 \leq \bar{\rho}, \qquad 0 \leq m_0 \leq \bar{m}, \qquad \mu \|\operatorname{curl} u_0\|^2_{L^2} + (\lambda + 2\mu)\|\operatorname{div} u_0\|^2_{L^2} \triangleq M, \tag{1.9}$$

and the compatibility condition

$$-\mu \Delta u_0 - (\lambda + \mu)\nabla \operatorname{div} u_0 + \nabla P_0 = (\rho_0 + m_0)^{1/2} g, \tag{1.10}$$

for some $g \in L^2$. Then there exists a positive constant ε depending only on λ, μ, γ, α, Ω, M, $\bar{\rho}$ and $\bar{m}$ such that if

$$C_0 \leq \varepsilon, \tag{1.11}$$

then the slip problem (1.1)–(1.5) *has a unique global classical solution* (ρ, m, u) *in* $\Omega \times (0,\infty)$ *satisfying*

$$0 \le \rho(x,t) \le 2\bar{\rho}, \qquad 0 \le m(x,t) \le 2\bar{m}, \tag{1.12}$$

$$\begin{cases} (\rho-\rho_\infty, m-m_\infty, P-P_\infty) \in C([0,\infty); H^2 \cap W^{2,q}), \\ \nabla u \in C([0,\infty); H^1) \cap L^\infty_{\mathrm{loc}}(0,\infty; W^{2,q}), \\ u_t \in L^\infty_{\mathrm{loc}}(0,\infty; D^1 \cap D^2) \cap H^1_{\mathrm{loc}}(0,\infty; D^1), \\ (\rho+m)^{\frac{1}{2}} u_t \in L^\infty(0,\infty; L^2). \end{cases} \tag{1.13}$$

In addition, the following large-time behavior

$$\lim_{t\to\infty} \int \left(|P-P_\infty|^q + (\rho+m)^{\frac{1}{2}}|u|^4 + |\nabla u|^2\right)(x,t)\,dx = 0 \tag{1.14}$$

holds for any $2 < q < \infty$.

With (1.14) at hand, we are able to obtain the following large-time behavior of the gradient of the pressure when vacuum states initially appear. It was just a parrel result, which was first established by Li and his collaborators in [6].

Theorem 1.2 *Under the conditions of Theorem* 1.1*, further assume that* $P_\infty > 0$ *and there exists some point* $x_0 \in \Omega$ *such that* $P_0(x_0) = 0$. *Then the unique global classical solution* (ρ, m, u) *to the problem* (1.1)–(1.5) *obtained in Theorem* 1.1 *has to blow up as* $t \to \infty$ *in the sense that for any* $3 < r < \infty$,

$$\lim_{t\to\infty} \left\| \nabla P(\cdot,t) \right\|_{L^r} = \infty. \tag{1.15}$$

Remark 1.1 *When* $\alpha \le 1$ *and* $\gamma > 1$, *it is easy to show that there exist* $0 < C_i < 1$ $(i = 1,2)$ *depending on* $\bar{\rho}$, $\bar{m}$, ρ_∞ *and* m_∞, *such that the following formula holds*

$$C_1(m-m_\infty)^2 + C_2(\rho-\rho_\infty)^2 \le m\int_{m_\infty}^{m} \frac{s-m_\infty}{s^2}\,ds + \rho\int_{\rho_\infty}^{\rho} \frac{s-\rho_\infty}{s^2}\,ds.$$

Now, we give some comments on the analysis of this paper. Compared with the bounded domains, because the domain is unbounded, we need to overcome two additional difficulties. First, thanks to [33] (see Lemma 2.6), we can control ∇u by means of $\operatorname{div} u$ and $\operatorname{curl} u$, the other one is how to control the boundary integrals, especially (see (3.23)),

$$-\int_{\partial\Omega} \sigma^h F u \cdot \left(\nabla n + (\nabla n)^{\mathrm{tr}}\right)^i \left(u^\perp \times n \cdot \nabla u^i\right) ds.$$

In fact, thanks to

$$\nabla \cdot (g \times h) = \nabla \times g \cdot h - \nabla \times h \cdot g, \qquad \nabla \times (\nabla g) = 0$$

and divergence theorem, we can control it.

Next, denote by

$$\dot{v} \triangleq v_t + u \cdot \nabla v, \tag{1.16}$$

and

$$F \triangleq (\lambda + 2\mu)\operatorname{div} u - (P - P_\infty), \tag{1.17}$$

the material derivative of v and the effective viscous flux, respectively. Then the equation $(1.1)_3$ can be written as

$$(\rho + m)\dot{u} = \nabla F - \mu \nabla \times \operatorname{curl} u, \tag{1.18}$$

which together with the boundary condition (1.4) implies that one can treat $(1.1)_3$ as a Helmholtz–Wyle decomposition of $(\rho + m)\dot{u}$, which makes it possible to estimate ∇F and $\nabla \operatorname{curl} u$. Finally, whereas $u \cdot n = 0$ on $\partial\Omega$, we have

$$u \cdot \nabla u \cdot n = -u \cdot \nabla n \cdot u, \tag{1.19}$$

which together with $\operatorname{curl} u \times n = 0$ on $\partial\Omega$ is the key to estimating the integrals on the boundary $\partial\Omega$.

2 Preliminaries

In this section, we will recall some known facts and elementary inequalities which will be used frequently later. First, we can get the local existence of strong and classical solutions (see [17]).

Lemma 2.1 *Suppose that Ω satisfies the condition of Theorem 1.1, and (ρ_0, m_0, u_0) satisfies (1.7), (1.8) and (1.10). Then there exists a small time $T_0 > 0$ and a unique strong solution (ρ, m, u) to the problem (1.1)–(1.5) on $\Omega \times (0, T_0]$ satisfying for any $\tau \in (0, T_0)$,*

$$\begin{cases} (\rho - \rho_\infty, m - m_\infty, P - P_\infty) \in C([0,\infty); H^2 \cap W^{2,q}), \\ u \in C([0,\infty); D^1 \cap D^2), \nabla u \in L^2(0,T;H^2) \cap L^{p_0}(0,T;W^{2,q}), \\ \nabla u \in L^\infty(\tau, T; H^2 \cap W^{2,q}), \\ u_t \in L^\infty(\tau, T; D^1 \cap D^2) \cap H^1(\tau, T; D^1), \\ \sqrt{\rho + m} u_t \in L^\infty(0,T;L^2), \end{cases}$$

where $q \in (3,6)$ and $p_0 = \frac{9q-6}{10q-12} \in (1, \frac{7}{6})$.

Second, the following Gagliardo–Nirenberg inequality (see [28]) will be used frequently later.

Lemma 2.2 *Let Ω be the exterior of a simply connected domain D in $\mathbb{R}^3$. For any $f \in H^1(\Omega)$ and $g \in L^q(\Omega) \cap D^{1,r}(\Omega)$, there exist some generic constants $C > 0$, which may depend on p, q, and r such that*

$$\|f\|_{L^p(\Omega)} \le C\|f\|_{L^2}^{\frac{6-p}{2p}} \|\nabla f\|_{L^2}^{\frac{3p-6}{2p}}, \tag{2.1}$$

$$\|g\|_{C(\bar{\Omega})} \le C\|g\|_{L^q}^{q(r-3)/(3r+q(r-3))}\|\nabla g\|_{L^r}^{3r/(3r+q(r-3))}, \tag{2.2}$$

for $p \in [2,6]$, $q \in (1,\infty)$, *and* $r \in (3,\infty)$.

Then, to get the uniform (in time) upper bound of the density ρ and m, we need the following Zlotnik inequality, which was first used in Huang–Li–Xin [17].

Lemma 2.3 ([41]) *For* $g \in C(R)$ *and* $y, b \in W^{1,1}(0,T)$, *assume that the function* y *satisfies*

$$y'(t) = g(y) + b'(t) \quad \text{on } [0,T], \qquad y(0) = y^0.$$

If $g(\infty) = -\infty$ *and*

$$b(t_2) - b(t_1) \le N_0 + N_1(t_2 - t_1) \tag{2.3}$$

for all $0 \le t_1 < t_2 \le T$ *with some* $N_0 \ge 0$ *and* $N_1 \ge 0$, *then*

$$y(t) \le \max\{y^0, \hat{\zeta}\} + N_0 < \infty \quad \text{on } [0,T],$$

where $\hat{\zeta}$ *is a constant such that*

$$g(\zeta) \le -N_1 \quad \text{for } \zeta \ge \hat{\zeta}. \tag{2.4}$$

Next, thanks to [1, 33], we have the following two lemmas.

Lemma 2.4 *Assume that* $D \subset \mathbb{R}^3$ *is a simply connected bounded domain with* $C^{k+1,1}$ *boundary* ∂D, $1 < q < +\infty$ *and a integer* $k \ge 0$, *then for* $v \in W^{k+1,q}(D)$ *with* $v \cdot n = 0$ *on* ∂D, *there exists a constant* $C = C(q,k,D)$ *such that*

$$\|v\|_{W^{k+1,q}(D)} \le C\big(\|\operatorname{div} v\|_{W^{k,q}(D)} + \|\operatorname{curl} v\|_{W^{k,q}(D)}\big). \tag{2.5}$$

If $k = 0$, *it holds that*

$$\|\nabla v\|_{L^q(D)} \le C\big(\|\operatorname{div} v\|_{L^q(D)} + \|\operatorname{curl} v\|_{L^q(D)}\big). \tag{2.6}$$

Lemma 2.5 *Assume that* $D \subset \mathbb{R}^3$ *is a bounded domain, and its* $C^{k+1,1}$ *boundary only has a finite number of two-dimensional connected components. For the integer* $k \ge 0$ *and* $1 < q < \infty$, *and for* $v \in W^{k+1,q}(D)$ *with* $v \times n = 0$ *on* ∂D, *then exists a positive constant* C *depending only on* q, k, Ω *such that*

$$\|v\|_{W^{k+1,q}(D)} \le C\big(\|\operatorname{div} v\|_{W^{k,q}(D)} + \|\operatorname{curl} v\|_{W^{k,q}(D)} + \|v\|_{L^q(D)}\big).$$

If D *has no holes, then*

$$\|v\|_{W^{k+1,q}(D)} \le C\big(\|\operatorname{div} v\|_{W^{k,q}(D)} + \|\operatorname{curl} v\|_{W^{k,q}(D)}\big).$$

The following conclusion is shown in [1, 33].

Lemma 2.6 *Assume that Ω is the exterior of a simply connected domains $D \subset \mathbb{R}^3$ with $C^{1,1}$ boundary. Then for $v \in D^{1,q}(\Omega)$ with $v \cdot n = 0$ on $\partial\Omega$, it holds that*

$$\|\nabla v\|_{L^q(\Omega)} \le C\big(\|\operatorname{div} v\|_{L^q(\Omega)} + \|\operatorname{curl} v\|_{L^q(\Omega)}\big) \quad \text{for any } 1 < q < 3, \tag{2.7}$$

and

$$\|\nabla v\|_{L^q(\Omega)} \le C\big(\|\operatorname{div} v\|_{L^q(\Omega)} + \|\operatorname{curl} v\|_{L^q(\Omega)} + \|\nabla v\|_{L^2(\Omega)}\big) \quad \text{for any } 3 \le q < +\infty. \tag{2.8}$$

Due to [24], we obtain the following fact.

Lemma 2.7 *Suppose that Ω satisfies the conditions in Lemma 2.6, for any $v \in W^{1,q}(\Omega)$ $(1 < q < +\infty)$ with $v \times n = 0$ on $\partial\Omega$, it holds that*

$$\|\nabla v\|_{L^q(\Omega)} \le C\big(\|v\|_{L^q(\Omega)} + \|\operatorname{div} v\|_{L^q(\Omega)} + \|\operatorname{curl} v\|_{L^q(\Omega)}\big).$$

By Lemmas 2.4–2.7, we can get the following result (see [6]).

Lemma 2.8 *Let Ω be the exterior of a simply connected domain $D \subset \mathbb{R}^3$ with smooth boundary. For any $p \in [2,6]$ and integer $k \ge 0$, and every $v \in \{D^{k+1,p(\Omega)} \cap D^{1,2}(\Omega) \mid v(x,t) \to 0 \text{ as } |x| \to \infty\}$ with $v \cdot n|_{\partial\Omega} = 0$ or $v \times n|_{\partial\Omega} = 0$, then there exists some positive constant C depending only on p, k, and D such that*

$$\|\nabla v\|_{W^{k,q}(\Omega)} \le C\big(\|\operatorname{div} v\|_{W^{k,q}(\Omega)} + \|\operatorname{curl} v\|_{W^{k,q}(\Omega)} + \|\nabla v\|_{L^2(\Omega)}\big). \tag{2.9}$$

Then we recall the following Beale–Kato–Majda-type inequality with respect to the slip boundary condition (1.4), which was first proved in [3, 22] when $\operatorname{div} u \equiv 0$, it can estimate $\|\nabla u\|_{L^\infty}$.

Lemma 2.9 ([6]) *Assume that $u \cdot n = 0$, $\operatorname{curl} u \times n = 0$, $\nabla u \in W^{1,q}$, for $3 < q < \infty$, then there exists a constant $C = C(q)$ such that the following estimate holds*

$$\|\nabla u\|_{L^\infty} \le C\big(\|\operatorname{div} u\|_{L^\infty} + \|\operatorname{curl} u\|_{L^\infty}\big)\ln\big(e + \|\nabla^2 u\|_{L^q}\big) + C\|\nabla u\|_{L^2} + C. \tag{2.10}$$

Consider the Neumann boundary value problem

$$\begin{cases} -\triangle v = \operatorname{div} f, & \text{in } \Omega, \\ \frac{\partial v}{\partial n} = -f \cdot n, & \text{on } \partial\Omega, \\ \nabla v \to 0, & \text{as } |x| \to \infty. \end{cases} \tag{2.11}$$

Indeed, the problem is equivalent to

$$\int \nabla v \cdot \nabla \eta \, dx = \int f \cdot \nabla \eta \, dx, \quad \forall \eta \in C_0^\infty(\mathbb{R}^3).$$

Lemma 2.10 ([6, 30]) *For systems (2.11), we have*

(1) *For some* $f \in L^q$, $q \in (1,\infty)$, *then there exists a unique* (*modulo constants*) *solution* $v \in D^{1,q}$ *such that*

$$\|\nabla v\|_{L^q(\Omega)} \le C(q,\Omega)\|f\|_{L^q}.$$

(2) *For some* $f \in W^{k,q}$, $q \in (1,\infty)$, $k > 1$, *then* $\nabla F \in W^{k,q}$ *and*

$$\|\nabla v\|_{W^{k,q}} \le C\|f\|_{W^{k,q}}.$$

Finally, we give the following conclusions for F and $\operatorname{curl} u$, whose proof is in [6]. We sketch it here for completeness.

Lemma 2.11 *Let* $\Omega \subset \mathbb{R}^3$ *be an exterior domain of some simply connected bounded domain with smooth boundary. For any* $2 \le p \le 6$ *and* $q \in (1,\infty)$, *suppose that* (ρ, m, u) *is a smooth solution of* (1.1) *with the boundary condition* (1.4), *then there exists a positive constant* C *depending only on* p, q, λ, μ, *and* Ω *such that*

$$\|\nabla F\|_{L^q} \le C\big\|(\rho+m)\dot{u}\big\|_{L^q}, \tag{2.12}$$

$$\|\nabla \operatorname{curl} u\|_{L^p} \le C\big(\big\|(\rho+m)\dot{u}\big\|_{L^p} + \big\|(\rho+m)\dot{u}\big\|_{L^2} + \|\nabla u\|_{L^2}\big), \tag{2.13}$$

$$\|F\|_{L^p} \le C\big\|(\rho+m)\dot{u}\big\|_{L^2}^{\frac{3p-6}{2p}}\big(\|\nabla u\|_{L^2} + \|P-P_\infty\|_{L^2}\big)^{\frac{6-p}{2p}}. \tag{2.14}$$

Moreover,

$$\|\operatorname{curl} u\|_{L^p} \le C\big\|(\rho+m)\dot{u}\big\|_{L^2}^{\frac{3p-6}{2p}}\|\nabla u\|_{L^2}^{\frac{6-p}{2p}} + C\|\nabla u\|_{L^2}, \tag{2.15}$$

$$\|\nabla u\|_{L^p} \le C\big(\big\|(\rho+m)\dot{u}\big\|_{L^2} + \|P-P_\infty\|_{L^6}\big)^{\frac{3p-6}{2p}}\|\nabla u\|_{L^2}^{\frac{6-p}{2p}} + C\|\nabla u\|_{L^2}. \tag{2.16}$$

Proof First, due to $(1.1)_3$, it is easy to find that F satisfies

$$\begin{cases} -\triangle F = \operatorname{div}((\rho+m)\dot{u}), & \text{in } \Omega,\\ \frac{\partial F}{\partial n} = -((\rho+m)\dot{u})\cdot n, & \text{on } \partial\Omega,\\ \nabla F \to 0, & \text{as } |x| \to \infty, \end{cases} \tag{2.17}$$

It follows from Lemma 2.10 that

$$\|\nabla F\|_{L^q} \le C(q,\Omega)\big\|(\rho+m)\dot{u}\big\|_{L^q}, \tag{2.18}$$

and

$$\|\nabla F\|_{W^{k,q}} \le C\big\|(\rho+m)\dot{u}\big\|_{W^{k,q}}. \tag{2.19}$$

Due to (1.18) and (1.4), from $\operatorname{div}\operatorname{curl} u = 0$, Lemma 2.7 and (2.18), we get

$$\begin{aligned} \|\nabla \operatorname{curl} u\|_{L^q} &\le C\|\operatorname{curl} u\|_{L^q} + \|\operatorname{div}\operatorname{curl} u\|_{L^q} + \|\nabla\times\operatorname{curl} u\|_{L^q}\\ &\le C\big(\|\operatorname{curl} u\|_{L^q} + \big\|(\rho+m)\dot{u}\big\|_{L^q} + \|\nabla F\|_{L^q}\big)\\ &\le C\big(\big\|(\rho+m)\dot{u}\big\|_{L^q} + \|\operatorname{curl} u\|_{L^q}\big). \end{aligned} \tag{2.20}$$

By virtue of Lemma 2.8, (1.18), (2.19), and (2.20), it indicates that

$$
\begin{aligned}
\|\nabla \operatorname{curl} u\|_{W^{k,q}} &\le C\|\operatorname{div}\operatorname{curl} u\|_{W^{k,q}} + \|\operatorname{curl}\operatorname{curl} u\|_{W^{k,q}} + \|\nabla \operatorname{curl} u\|_{L^2} \\
&\le C\big(\|\nabla F\|_{W^{k,q}} + \big\|(\rho+m)\dot{u}\big\|_{W^{k,q}} + \|\operatorname{curl} u\|_{L^2} + \big\|(\rho+m)\dot{u}\big\|_{L^2}\big) \\
&\le C\big(\big\|(\rho+m)\dot{u}\big\|_{W^{k,q}} + \big\|(\rho+m)\dot{u}\big\|_{L^2} + \|\nabla u\|_{L^2}\big).
\end{aligned} \tag{2.21}
$$

By (2.1) and (2.20), we can obtain

$$
\begin{aligned}
\|\nabla \operatorname{curl} u\|_{L^p} &\le C\big(\big\|(\rho+m)\dot{u}\big\|_{L^p} + \|\operatorname{curl} u\|_{L^p}\big) \\
&\le C\big(\big\|(\rho+m)\dot{u}\big\|_{L^p} + \|\operatorname{curl} u\|_{L^2} + \|\nabla \operatorname{curl} u\|_{L^2}\big) \\
&\le C\big(\big\|(\rho+m)\dot{u}\big\|_{L^p} + \big\|(\rho+m)\dot{u}\big\|_{L^2} + \|\nabla u\|_{L^2}\big),
\end{aligned} \tag{2.22}
$$

for any $2 \le p \le 6$.

Employing (1.17), (2.1), (2.18) and (2.22), one has

$$
\|F\|_{L^p} \le C\big(\|F\|_{L^2}^{\frac{6-p}{2p}} \|\nabla F\|_{L^2}^{\frac{3p-6}{2p}}\big) \tag{2.23}
$$

$$
\le C\big(\|\nabla u\|_{L^2} + \|P-P_\infty\|_{L^2}\big)^{\frac{6-p}{2p}} \big\|(\rho+m)\dot{u}\big\|_{L^2}^{\frac{3p-6}{2p}}, \tag{2.24}
$$

and

$$
\begin{aligned}
\|\operatorname{curl} u\|_{L^p} &\le C\big(\|\operatorname{curl} u\|_{L^2}^{\frac{6-p}{2p}} \|\nabla \operatorname{curl} u\|_{L^2}^{\frac{3p-6}{2p}}\big) \\
&\le C\|\nabla u\|_{L^2}^{\frac{6-p}{2p}} \big(\big\|(\rho+m)\dot{u}\big\|_{L^2} + \|\nabla u\|_{L^2}\big)^{\frac{3p-6}{2p}} \\
&\le C\|\nabla u\|_{L^2}^{\frac{6-p}{2p}} \big\|(\rho+m)\dot{u}\big\|_{L^2}^{\frac{3p-6}{2p}} + C\|\nabla u\|_{L^2}.
\end{aligned} \tag{2.25}
$$

Combining Lemma 2.6 with (2.1), (2.18), and (2.25) gives that

$$
\begin{aligned}
\|\nabla u\|_{L^p} &\le C\|\nabla u\|_{L^2}^{\frac{6-p}{2p}} \|\nabla u\|_{L^6}^{\frac{3p-6}{2p}} \\
&\le C\|\nabla u\|_{L^2}^{\frac{6-p}{2p}} \big(\|\operatorname{div} u\|_{L^6} + \|\operatorname{curl} u\|_{L^6} + \|\nabla u\|_{L^2}\big)^{\frac{3p-6}{2p}} \\
&\le C\|\nabla u\|_{L^2}^{\frac{6-p}{2p}} \big(\|F\|_{L^6} + \|P-P_\infty\|_{L^6} + \big\|(\rho+m)\dot{u}\big\|_{L^2} + \|\nabla u\|_{L^2}\big)^{\frac{3p-6}{2p}} \\
&\le C\|\nabla u\|_{L^2}^{\frac{6-p}{2p}} \big(\big\|(\rho+m)\dot{u}\big\|_{L^2} + \|P-P_\infty\|_{L^6} + \|\nabla u\|_{L^2}\big)^{\frac{3p-6}{2p}} \\
&\le C\|\nabla u\|_{L^2}^{\frac{6-p}{2p}} \big(\big\|(\rho+m)\dot{u}\big\|_{L^2} + \|P-P_\infty\|_{L^6}\big)^{\frac{3p-6}{2p}} + C\|\nabla u\|_{L^2}.
\end{aligned} \tag{2.26}
$$

Thus, (2.18), (2.22), (2.23), (2.25) and (2.26) yields the desired result of Lemma 2.11. □

3 A priori estimates (i): lower order estimates

Assume that Ω is the exterior of a simply connected domain $D \subset \mathbb{R}^3$. Choosing a positive real number R such that $\bar{D} \subset B_R$, one can extend the unit outer normal n to Ω as

$$
n \in C^3(\bar{\Omega}), \qquad n \equiv 0 \quad \text{on } \mathbb{R}^3 \backslash B_{2R}. \tag{3.1}
$$

We will establish some necessary a priori bounds for smooth solutions of the problem (1.1)–(1.5) to extend the local classical solution guaranteed by Lemma 2.1. Thus, let $T > 0$ be a fixed time and (ρ, m, u) be the smooth solution to (1.1)–(1.5) on $\Omega \times (0, T]$ with smooth initial data (ρ_0, m_0, u_0) satisfying (1.9) and (1.10). To get the estimates of the obtained solution, set $\sigma(t) \triangleq \min\{1, t\}$ and define

$$A_1(T) \triangleq \sup_{0\le t\le T} \left(\sigma \|\nabla u\|_{L^2}^2\right) + \int_0^T \int \sigma(\rho + m)|\dot{u}|^2 \, dx \, dt, \tag{3.2}$$

$$A_2(T) \triangleq \sup_{0\le t\le T} \sigma^3 \int (\rho + m)|\dot{u}|^2 \, dx + \int_0^T \int \sigma^3 |\nabla \dot{u}|^2 \, dx \, dt, \tag{3.3}$$

and

$$A_3(T) \triangleq \sup_{0\le t\le T} \int |\nabla u|^2 \, dx. \tag{3.4}$$

Then, to get the existence of a global classical solution of (1.1)–(1.5), we can get the following proposition.

Proposition 3.1 *Under the conditions of Theorem 1.1, there exists a positive constant ε depending only on λ, μ, γ, α, Ω, $\bar{\rho}$, $\bar{m}$, and M such that if (ρ, m, u) is a smooth solution of (1.1)–(1.5) on $\Omega \times (0, T]$ satisfying*

$$\begin{aligned} &\sup_{\Omega\times[0,T]} \rho \le 2\bar{\rho}, \qquad \sup_{\Omega\times[0,T]} m \le 2\bar{m}, \\ &A_1(T) + A_2(T) \le 2C_0^{\frac{1}{2}}, \qquad A_3\big(\sigma(T)\big) \le 2M, \end{aligned} \tag{3.5}$$

then

$$\begin{aligned} &\sup_{\Omega\times[0,T]} \rho \le 7\bar{\rho}/4, \qquad \sup_{\Omega\times[0,T]} m \le 7\bar{m}/4, \\ &A_1(T) + A_2(T) \le C_0^{\frac{1}{2}}, \qquad A_3\big(\sigma(T)\big) \le M, \end{aligned} \tag{3.6}$$

provided $C_0 \le \varepsilon$.

Proof Proposition 3.1 is deduced from Lemmas 3.4–3.7. □

First, we start with the standard energy estimate of (ρ, m, u).

Lemma 3.2 *Suppose that (ρ, m, u) is a smooth solution of (1.1)–(1.5) on $\Omega \times (0, T]$. Then there is a positive constant C depending only on λ, μ, and Ω such that*

$$\sup_{0\le t\le T} \int \big((\rho + m)|u|^2 + G(\rho, m)\big) \, dx + \int_0^T \|\nabla u\|_{L^2}^2 \, dt \le CC_0. \tag{3.7}$$

Proof First, due to $-\Delta u = -\nabla \operatorname{div} u + \nabla \times \operatorname{curl} u$, we rewrite the third equation of (1.1) as

$$(\rho + m)\dot{u} - (\lambda + 2\mu)\nabla \operatorname{div} u + \mu \nabla \times \operatorname{curl} u + \nabla P = 0. \tag{3.8}$$

Multiplying (3.8) by u and integrating the resultant equation over Ω, we obtain that

$$\frac{1}{2}\left(\int(\rho+m)|u|^2\,dx\right)_t+(\lambda+2\mu)\|\operatorname{div}u\|_{L^2}^2+\mu\|\operatorname{curl}u\|_{L^2}^2+\int u\cdot\nabla P\,dx=0. \tag{3.9}$$

Multiplying $(1.1)_1$ by $(\int_{\rho_\infty}^{\rho}\frac{P(s,m)-P(\rho_\infty,m)}{s^2}\,ds+\frac{P(\rho,m)-P(\rho_\infty,m)}{\rho})$ and using (1.4), we have

$$\left(\int\rho\int_{\rho_\infty}^{\rho}\frac{P(s,m)-P(\rho_\infty,m)}{s^2}\,ds\,dx\right)_t+\int\big(P(\rho,m)-P(\rho_\infty,m)\big)\operatorname{div}u\,dx=0. \tag{3.10}$$

By the same way, $(1.1)_2$ shows that

$$\left(\int m\int_{m_\infty}^{m}\frac{P(\rho,s)-P(\rho,m_\infty)}{s^2}\,ds\,dx\right)_t+\int\big(P(\rho,m)-P(\rho,m_\infty)\big)\operatorname{div}u\,dx=0. \tag{3.11}$$

Combining (3.9), (3.10), and (3.11), we have

$$\left(\int\left(\frac{1}{2}(\rho+m)|u|^2+G(\rho,m)\right)dx\right)_t+(\lambda+2\mu)\|\operatorname{div}u\|_{L^2}^2+\mu\|\operatorname{curl}u\|_{L^2}^2=0. \tag{3.12}$$

Integrating (3.12) over $(0,T]$ and using (2.7), we find (3.7). This completes the proof of Lemma 3.2. □

Lemma 3.3 *Suppose (ρ,m,u) is a smooth solution of (1.1)–(1.5) satisfying (3.5) on $\Omega\times(0,T]$. Then there is a positive constant C depending on λ, μ, γ, α, $\bar\rho$, $\bar m$, M, and Ω such that*

$$A_1(T)\le CC_0+C\int_0^T\int\sigma|\nabla u|^3\,dx\,dt, \tag{3.13}$$

and

$$A_2(T)\le CC_0+CA_1(T)+C\int_0^T\int\sigma^3|\nabla u|^4\,dx\,dt. \tag{3.14}$$

Proof Motivated by Hoff [16] and Cai–Li–Lü [6], for $h\ge 0$, multiplying $(1.1)_3$ by $\sigma^h\dot u$ and then integrating it over Ω lead to

$$\begin{aligned}\int\sigma^h(\rho+m)|\dot u|^2\,dx&=(\lambda+2\mu)\int\nabla\operatorname{div}u\cdot\sigma^h\dot u\,dx\\&\quad-\mu\int\nabla\times\operatorname{curl}u\cdot\sigma^h\dot u\,dx-\int\nabla P\cdot\sigma^h\dot u\,dx\\&=:I_1+I_2+I_3.\end{aligned} \tag{3.15}$$

Using (1.19) and the fact that $\operatorname{div}(u\cdot\nabla u)=\nabla u:\nabla u+u\cdot\nabla\operatorname{div}u$, a direct calculation gives

$$\begin{aligned}
I_1 &= (\lambda+2\mu)\int \nabla\operatorname{div}u\cdot\sigma^h\dot{u}\,dx\\
&= -(\lambda+2\mu)\int\sigma^h\operatorname{div}u\operatorname{div}\dot{u}\,dx+(\lambda+2\mu)\int_{\partial\Omega}\sigma^h\operatorname{div}u\dot{u}\cdot n\,ds\\
&= -(\lambda+2\mu)\int\left[\sigma^h\operatorname{div}u\operatorname{div}u_t+\sigma^h\operatorname{div}u\operatorname{div}(u\cdot\nabla u)\right]dx\\
&\quad+(\lambda+2\mu)\int_{\partial\Omega}\sigma^h\operatorname{div}uu\cdot\nabla u\cdot n\,ds\\
&\le -\frac{\lambda+2\mu}{2}\left(\int\sigma^h(\operatorname{div}u)^2\,dx\right)_t+Ch\sigma^{h-1}\sigma'\int(\operatorname{div}u)^2\,dx\\
&\quad-(\lambda+2\mu)\int\sigma^h\operatorname{div}u\nabla u:\nabla u\,dx\\
&\quad-\frac{\lambda+2\mu}{2}\int\sigma^h u\cdot\nabla(\operatorname{div}u)^2\,dx-(\lambda+2\mu)\int_{\partial\Omega}\sigma^h\operatorname{div}uu\cdot\nabla n\cdot u\,ds\\
&\le -\frac{\lambda+2\mu}{2}\left(\int\sigma^h(\operatorname{div}u)^2\,dx\right)_t+\delta\left\|(\rho+m)^{\frac12}\dot{u}\right\|_{L^2}^2+Ch\sigma^{h-1}\sigma'\|\nabla u\|_{L^2}^2\\
&\quad+C\sigma^h\left(\|\nabla u\|_{L^2}^2+\|\nabla u\|_{L^2}^4+\|\nabla u\|_{L^3}^3\right). \qquad (3.16)
\end{aligned}$$

For the boundary term in the first inequality on the right-hand side of (3.16), it follows from (1.17), (2.12), (3.5), and Young's inequality that

$$\begin{aligned}
-(\lambda+2\mu)\int_{\partial\Omega}\operatorname{div}uu\cdot\nabla n\cdot u\,ds &= -\int_{\partial\Omega}Fu\cdot\nabla n\cdot u\,ds-\int_{\partial\Omega}(P-P_\infty)u\cdot\nabla n\cdot u\,ds\\
&\le C\left(\|F\|_{L^2(\partial\Omega)}\|u\|_{L^4(\partial\Omega)}^2+\|u\|_{L^2(\partial\Omega)}^2\right)\\
&\le C\left(\|\nabla F\|_{L^2}\|\nabla u\|_{L^2}^2+\|\nabla u\|_{L^2}^2\right)\\
&\le \delta\left\|(\rho+m)^{\frac12}\dot{u}\right\|_{L^2}^2+C\left(\|\nabla u\|_{L^2}^2+\|\nabla u\|_{L^2}^4\right).
\end{aligned}$$

Notice that $\operatorname{curl}(u\cdot\nabla u)=\nabla u^i\times\nabla_i u+u\cdot\nabla\operatorname{curl}u$, by (1.4), we have

$$\begin{aligned}
I_2 &= -\mu\int\sigma^h\nabla\times\operatorname{curl}u\cdot\dot{u}\,dx\\
&= -\mu\sigma^h\int\operatorname{curl}u\cdot\operatorname{curl}u_t\,dx-\mu\sigma^h\int\operatorname{curl}u\cdot\operatorname{curl}(u\cdot\nabla u)\,dx\\
&\quad-\mu\sigma^h\int_{\partial\Omega}\operatorname{curl}u\times\dot{u}\cdot n\,ds\\
&= -\frac{\mu}{2}\left(\sigma^h\int(\operatorname{curl}u)^2\,dx\right)_t+\frac{\mu}{2}h\sigma^{h-1}\sigma'\int(\operatorname{curl}u)^2\,dx\\
&\quad-\mu\sigma^h\int\operatorname{curl}u\cdot\left(\nabla u^i\times\nabla_i u\right)dx\\
&\quad-\mu\int u\cdot\nabla\left(\frac{(\operatorname{curl}u)^2}{2}\right)dx\\
&\le -\frac{\mu}{2}\left(\sigma^h\int(\operatorname{curl}u)^2\,dx\right)_t+Ch\sigma^{h-1}\sigma'\|\nabla u\|_{L^2}^2+C\sigma^h\|\nabla u\|_{L^3}^3. \qquad (3.17)
\end{aligned}$$

Finally, a direct calculation leads to

$$
\begin{aligned}
I_3 &= -\int \sigma^h \dot{u} \cdot \nabla P \, dx \\
&= \int \sigma^h (P - P_\infty) \operatorname{div} u_t \, dx + \int \sigma^h (P - P_\infty) \operatorname{div}(u \cdot \nabla u) \, dx \\
&\quad - \int_{\partial\Omega} \sigma^h (P - P_\infty) u \cdot \nabla u \cdot n \, ds \\
&= \left(\sigma^h \int (P - P_\infty) \operatorname{div} u \, dx \right)_t - h\sigma^{h-1}\sigma' \int (P - P_\infty) \operatorname{div} u \, dx \\
&\quad - \sigma^h \int P_t \operatorname{div} u \, dx + \int \sigma^h (P - P_\infty)(\nabla u : \nabla u + u \cdot \nabla \operatorname{div} u) \, dx \\
&\quad + \int_{\partial\Omega} \sigma^h (P - P_\infty) u \cdot \nabla n \cdot u \, ds \\
&= \left(\sigma^h \int (P - P_\infty) \operatorname{div} u \, dx \right)_t - h\sigma^{h-1}\sigma' \int (P - P_\infty) \operatorname{div} u \, dx \\
&\quad + \int \sigma^h (P - P_\infty) \nabla u : \nabla u \, dx \\
&\quad + \sigma^h \int \big((\gamma - 1)\rho^\gamma + (\alpha - 1) m^\alpha + P_\infty \big) (\operatorname{div} u)^2 \, dx \\
&\quad + \int_{\partial\Omega} \sigma^h (P - P_\infty) u \cdot \nabla n \cdot u \, ds \\
&\le \left(\sigma^h \int (P - P_\infty) \operatorname{div} u \, dx \right)_t + Ch\sigma^{h-1}\sigma' \big(\|P - P_\infty\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 \big) \\
&\quad + C\sigma^h \|\nabla u\|_{L^2}^2,
\end{aligned} \tag{3.18}
$$

where we have used the fact that

$$
\begin{aligned}
-\sigma^h \int P_t \operatorname{div} u \, dx &= \sigma^h \int \big(\operatorname{div}(Pu) + (\gamma - 1)\rho^\gamma \operatorname{div} u + (\alpha - 1) m^\alpha \operatorname{div} u \big) \operatorname{div} u \, dx \\
&= \sigma^h \int \operatorname{div}(Pu) \operatorname{div} u \, dx + \sigma^h \int (\gamma - 1)\rho^\gamma (\operatorname{div} u)^2 \, dx \\
&\quad + \sigma^h \int (\alpha - 1) m^\alpha (\operatorname{div} u)^2 \, dx.
\end{aligned}
$$

Combining (3.15) and (3.16)–(3.18) gives that for enough small δ.

$$
\begin{aligned}
&\left(\frac{\lambda + 2\mu}{2} \sigma^h \int (\operatorname{div} u)^2 \, dx + \frac{\mu}{2} \sigma^h \int (\operatorname{curl} u)^2 \, dx \right)_t \\
&\qquad + \sigma^h \int (\rho + m) |\dot{u}|^2 \, dx - \left(\sigma^h \int (P - P_\infty) \operatorname{div} u \, dx \right)_t \\
&\quad \le \delta \sigma^h \big\| (\rho + m)^{\frac{1}{2}} \dot{u} \big\|_{L^2}^2 + Ch\sigma^{h-1}\sigma' \big(\|\nabla u\|_{L^2}^2 + \|P - P_\infty\|_{L^2}^2 \big) \\
&\qquad + C\sigma^h \big(\|\nabla u\|_{L^2}^2 + \|\nabla u\|_{L^2}^4 + \|\nabla u\|_{L^3}^3 \big).
\end{aligned} \tag{3.19}
$$

Integrating (3.19) over $(0,T]$, by Lemma 2.6, (3.5) and (3.7), for $h\geq 1$, we have

$$\begin{aligned}&\sigma^h\|\nabla u\|_{L^2}^2+\int_0^T\sigma^h\int(\rho+m)|\dot u|^2\,dx\\&\quad\leq CC_0+C\int_0^T\sigma^h\|\nabla u\|_{L^3}^3\,dt+C\int_0^T\sigma^h\|\nabla u\|_{L^2}^4\,dt,\end{aligned}$$

where we have used $\int_0^T h\sigma^{h-1}\sigma'\|P-P_\infty\|_{L^2}^2\,dt\leq CC_0$. Choosing $h=1$ and using (3.5) and (3.7), we get (3.13).

Now, we prove (3.14). Applying the operator $\sigma^h\dot u^j[\partial/\partial t+\operatorname{div}(u\cdot)]$ to $(1.18)^j$, summing all the equalities with respect to j and integrating over Ω, we obtain

$$\begin{aligned}&\frac12\left(\sigma^h\int(\rho+m)|\dot u|^2\,dx\right)_t-\frac12 h\sigma^{h-1}\sigma'\int(\rho+m)|\dot u|^2\,dx\\&\quad=\int\sigma^h\big[\dot u\cdot\nabla F_t+\dot u^j\operatorname{div}(u\partial_jF)\big]\,dx\\&\qquad-\mu\int\sigma^h\big[\dot u\cdot\nabla\times\operatorname{curl}u_t+\dot u^j\operatorname{div}\big(u(\nabla\times\operatorname{curl}u)^j\big)\big]\,dx\\&\quad=:J_1+J_2.\end{aligned}\tag{3.20}$$

For the term J_1, a direct computation shows that

$$\begin{aligned}J_1&=\int\sigma^h\big[\dot u\cdot\nabla F_t+\dot u^j\operatorname{div}(u\partial_jF)\big]\,dx\\&=-\int\sigma^h\operatorname{div}\dot uF_t\,dx+\int\sigma^h\dot u\cdot\nabla\operatorname{div}(uF)\,dx-\int\sigma^h\dot u^j\operatorname{div}(\partial_juF)\,dx\\&\quad+\int_{\partial\Omega}\sigma^hF_tu\cdot\nabla u\cdot n\,ds\\&=-(2\mu+\lambda)\int\sigma^h\operatorname{div}\dot u\operatorname{div}u_t\,dx+\int\sigma^h\operatorname{div}\dot uP_t\,dx-\int\sigma^h\operatorname{div}\dot u\operatorname{div}(uF)\,dx\\&\quad+\int_{\partial\Omega}\sigma^h\operatorname{div}(uF)\dot u\cdot n\,ds-\int\sigma^hF\dot u\cdot\nabla\operatorname{div}u\,dx-\int\sigma^h\dot u\cdot\nabla u\cdot\nabla F\,dx\\&\quad+\int_{\partial\Omega}\sigma^hF_tu\cdot\nabla u\cdot n\,ds\\&=-(2\mu+\lambda)\int\sigma^h(\operatorname{div}\dot u)^2\,dx+(2\mu+\lambda)\int\sigma^h\operatorname{div}\dot u\nabla u:\nabla u\,dx\\&\quad+\int\sigma^h\operatorname{div}\dot uu\cdot\nabla(F+P-P_\infty)\,dx-\int\sigma^h\operatorname{div}\dot u\big(u\cdot\nabla P+\gamma\rho^\gamma\operatorname{div}u+\alpha m^\alpha\operatorname{div}u\big)\,dx\\&\quad-\int\sigma^hF\operatorname{div}\dot u\operatorname{div}u\,dx-\int\sigma^h\operatorname{div}\dot uu\cdot\nabla F\,dx+\int_{\partial\Omega}\sigma^h\operatorname{div}(uF)u\cdot\nabla u\cdot n\,ds\\&\quad-\frac{1}{2\mu+\lambda}\int\sigma^hF\dot u\cdot\nabla F\,dx-\frac{1}{2\mu+\lambda}\int_{\partial\Omega}\sigma^h(P-P_\infty)F\dot u\cdot n dx\\&\quad+\frac{1}{2\mu+\lambda}\int\sigma^h(F\operatorname{div}\dot u+\dot u\cdot\nabla F)(P-P_\infty)\,dx+\int_{\partial\Omega}\sigma^hF_tu\cdot\nabla u\cdot n\,ds\\&\quad-\int\sigma^h\dot u\cdot\nabla u\cdot\nabla F\,dx\end{aligned}$$

$$
\begin{aligned}
&= -(2\mu+\lambda)\int \left[\sigma^h(\operatorname{div}\dot u)^2 - \sigma^h \operatorname{div}\dot u \nabla u : \nabla u\right] dx \\
&\quad - \int \sigma^h \operatorname{div}\dot u\big(\gamma\rho^\gamma \operatorname{div} u + \alpha m^\alpha \operatorname{div} u\big)\, dx - \int \sigma^h F \operatorname{div}\dot u \operatorname{div} u\, dx \\
&\quad + \int_{\partial\Omega} \sigma^h \operatorname{div}(uF) u\cdot\nabla u\cdot n\, ds - \frac{1}{2\mu+\lambda}\int \sigma^h F\dot u\cdot\nabla F\, dx \\
&\quad + \frac{1}{2\mu+\lambda}\int \sigma^h \dot u\cdot\nabla F(P-P_\infty)\, dx + \frac{1}{2\mu+\lambda}\int \sigma^h F\operatorname{div}\dot u(P-P_\infty)\, dx \\
&\quad - \frac{1}{2\mu+\lambda}\int_{\partial\Omega} \sigma^h (P-P_\infty)F\dot u\cdot n\, ds + \int_{\partial\Omega}\sigma^h F_t u\cdot\nabla u\cdot n\, ds \\
&\quad - \int \sigma^h \dot u\cdot\nabla u\cdot\nabla F\, dx.
\end{aligned}
\tag{3.21}
$$

Setting $u^\perp \triangleq -u\times n$, we have $u = u^\perp \times n$. Applying (2.12), (2.14), we can estimate the three boundary terms as

$$
\begin{aligned}
&\int_{\partial\Omega} \sigma^h F_t u\cdot\nabla u\cdot n\, ds \\
&\quad = -\left(\int_{\partial\Omega}\sigma^h F u\cdot\nabla n\cdot u\, ds\right)_t + h\sigma^{h-1}\sigma' \int_{\partial\Omega} F u\cdot\nabla n\cdot u\, ds \\
&\qquad + \int_{\partial\Omega}\sigma^h F u\cdot\big(\nabla n + (\nabla n)^{\mathrm{tr}}\big)\cdot u_t\, ds \\
&\quad = -\left(\int_{\partial\Omega}\sigma^h F u\cdot\nabla n\cdot u\, ds\right)_t + h\sigma^{h-1}\sigma' \int_{\partial\Omega} F u\cdot\nabla n\cdot u\, ds \\
&\qquad + \int_{\partial\Omega}\sigma^h F u\cdot\big(\nabla n + (\nabla n)^{\mathrm{tr}}\big)\cdot \dot u\, ds \\
&\qquad - \int_{\partial\Omega}\sigma^h F u\cdot\big(\nabla n + (\nabla n)^{\mathrm{tr}}\big)^i\big(u^\perp\times n\cdot\nabla u^i\big)\, ds \\
&\quad \le -\left(\int_{\partial\Omega}\sigma^h F u\cdot\nabla n\cdot u\, ds\right)_t \\
&\qquad + C\big(h\sigma^{h-1}\sigma'\|F\|_{L^2(\partial\Omega)}\|u\|^2_{L^4(\partial\Omega)} + \sigma^h\|F\|_{L^4(\partial\Omega)}\|u\|_{L^4(\partial\Omega)}\|\dot u\|_{L^2(\partial\Omega)}\big) \\
&\qquad + \int \big(|u|^2|\nabla u||\nabla F| + |\nabla u|^2|u||F| + |u|^2|\nabla u||F|\big)\, dx \\
&\quad \le -\left(\int_{\partial\Omega}\sigma^h F u\cdot\nabla n\cdot u\, ds\right)_t \\
&\qquad + C\big(h\sigma^{h-1}\sigma'\|\nabla F\|_{L^2}\|\nabla u\|^2_{L^2} + \sigma^h\|\nabla F\|_{L^2}\|\nabla u\|_{L^2}\|\nabla\dot u\|_{L^2}\big) \\
&\qquad + C\sigma^h\big(\|\nabla F\|_{L^6}\|\nabla u\|_{L^2}\|u\|^2_{L^6} + \|F\|_{L^6}\|\nabla u\|^2_{L^3}\|u\|^2_{L^6} + \|F\|_{L^6}\|\nabla u\|_{L^2}\|u\|^2_{L^6}\big) \\
&\quad \le -\left(\int_{\partial\Omega}\sigma^h F u\cdot\nabla n\cdot u\, ds\right)_t + Ch\sigma^{h-1}\sigma'\big\|(\rho+m)^{\frac12}\dot u\big\|_{L^2}\|\nabla u\|^2_{L^2} + \delta\sigma^h\|\nabla\dot u\|^2_{L^2} \\
&\qquad + C\sigma^h\big(\big\|(\rho+m)^{\frac12}\dot u\big\|^2_{L^2}\big(\|\nabla u\|^4_{L^2}+1\big) + \|\nabla u\|^2_{L^2} + \|\nabla u\|^6_{L^2} + \|\nabla u\|^4_{L^4}\big),
\end{aligned}
\tag{3.22}
$$

where we have used

$$
-\int_{\partial\Omega}\sigma^h F u\cdot\big(\nabla n + (\nabla n)^{\mathrm{tr}}\big)^i\big(u^\perp\times n\cdot\nabla u^i\big)\, ds
$$

$$
\begin{aligned}
&= \sigma^h \int \operatorname{div}\big(Fu \cdot \big(\nabla n + (\nabla n)^{\mathrm{tr}}\big)^i u^\perp \times \nabla u^i\big)\,dx \\
&= \sigma^h \int u^\perp \times \nabla u^i \cdot \nabla\big(Fu \cdot \big(\nabla n + (\nabla n)^{\mathrm{tr}}\big)^i\big)\,dx \\
&\quad + \sigma^h \int \big(\nabla \times u^\perp \cdot \nabla u^i\big)\big(Fu \cdot \big(\nabla n + (\nabla n)^{\mathrm{tr}}\big)^i\big)\,dx.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
&\int_{\partial\Omega} \sigma^h \operatorname{div}(uF) u \cdot \nabla u \cdot n\,ds - \frac{1}{2\mu+\lambda}\int_{\partial\Omega} \sigma^h (P - P_\infty) F \dot{u} \cdot n\,ds \\
&\quad = \int_{\partial\Omega} \sigma^h (u \cdot \nabla F)(u \cdot \nabla u \cdot n)\,ds + \frac{1}{2\mu+\lambda}\int_{\partial\Omega} \sigma^h F^2 u \cdot \nabla u \cdot n\,ds \\
&\quad = -\int_{\partial\Omega} \sigma^h \big(u^\perp \times n \cdot \nabla F\big)(u \cdot \nabla n \cdot u)\,ds - \frac{1}{2\mu+\lambda}\int_{\partial\Omega} \sigma^h F^2 u \cdot \nabla n \cdot u\,ds \\
&\quad = \int \sigma^h \operatorname{div}\big((u \cdot \nabla n \cdot u)\big(u^\perp \times \nabla F\big)\big)\,dx - \frac{1}{2\mu+\lambda}\int_{\partial\Omega} \sigma^h F^2 u \cdot \nabla n \cdot u\,ds \\
&\quad = \int \big[\sigma^h \nabla (u \cdot \nabla n \cdot u) \cdot \big(u^\perp \times \nabla F\big) + (u \cdot \nabla n \cdot u) \nabla \times u^\perp \cdot \nabla F\big]\,dx \\
&\qquad + \frac{1}{2\mu+\lambda}\int_{\partial\Omega} \sigma^h F^2 u \cdot \nabla n \cdot u\,ds \\
&\quad \le \int \sigma^h \big(|\nabla u||u|^2|\nabla F| + |u|^3|\nabla F|\big)\,dx + C\sigma^h \|F\|_{L^4(\partial\Omega)}^2 \|u\|_{L^4(\partial\Omega)}^2 \\
&\quad \le C\sigma^h \big(\|\nabla u\|_{L^2}\|u\|_{L^6}^2\|\nabla F\|_{L^6} + \|u\|_{L^6}^3\|\nabla F\|_{L^2} + \|\nabla F\|_{L^2}^2\|\nabla u\|_{L^2}^2\big) \\
&\quad \le \sigma^h \big(\big\|(\rho+m)^{\frac12}\dot{u}\big\|_{L^2}^2\big(\|\nabla u\|_{L^2}^2 + \|\nabla u\|_{L^2}^4\big) + \|\nabla u\|_{L^2}^2 + \|\nabla u\|_{L^2}^6\big) \\
&\qquad + \delta\sigma^h \|\nabla \dot{u}\|_{L^2}^2.
\end{aligned}
\tag{3.23}
$$

It follows from (2.12), (2.14), and (3.21)–(3.23) that

$$
\begin{aligned}
J_1 &\le -\bigg(\int_{\partial\Omega} \sigma^h F u \cdot \nabla n \cdot u\,ds\bigg)_t - (2\mu+\lambda)\int \sigma^h (\operatorname{div}\dot{u})^2\,dx \\
&\quad + Ch\sigma^{h-1}\sigma' \big\|(\rho+m)^{\frac12}\dot{u}\big\|_{L^2}\|\nabla u\|_{L^2}^2 + \delta\sigma^h\|\nabla\dot{u}\|_{L^2}^2 \\
&\quad + C\sigma^h \big(\big\|(\rho+m)^{\frac12}\dot{u}\big\|_{L^2}^2\big(\|\nabla u\|_{L^2}^2 + \|\nabla u\|_{L^2}^4\big) + \|\nabla u\|_{L^2}^2 + \|\nabla u\|_{L^2}^6 + \|\nabla u\|_{L^4}^4\big) \\
&\quad + C\sigma^h \big[\|\nabla\dot{u}\|_{L^2}\big(\|\nabla u\|_{L^4}^2 + \|\nabla u\|_{L^2}\big) \\
&\quad + \big(\|F\|_{L^2}\|\dot{u}\|_{L^6}\|\nabla F\|_{L^3} + \|\nabla F\|_{L^2}\|\dot{u}\|_{L^6}\|P - P_\infty\|_{L^3}\big] \\
&\quad + \|F\|_{L^6}\|\nabla\dot{u}\|_{L^2}\|P - P_\infty\|_{L^3} + \|\dot{u}\|_{L^6}\|\nabla F\|_{L^3}\|\nabla u\|_{L^2}\big) \\
&\le -\bigg(\int_{\partial\Omega} \sigma^h F u \cdot \nabla n \cdot u\,ds\bigg)_t \\
&\quad - (2\mu+\lambda)\int \sigma^h (\operatorname{div}\dot{u})^2\,dx + Ch\sigma^{h-1}\sigma' \big\|(\rho+m)^{\frac12}\dot{u}\big\|_{L^2}\|\nabla u\|_{L^2}^2 \\
&\quad + \delta\sigma^h\|\nabla\dot{u}\|_{L^2}^2 \\
&\quad + C\sigma^h \big(\big\|(\rho+m)^{\frac12}\dot{u}\big\|_{L^2}^2\big(1 + \|\nabla u\|_{L^2}^4\big) + \|\nabla u\|_{L^2}^2 + \|\nabla u\|_{L^2}^6 + \|\nabla u\|_{L^4}^4\big).
\end{aligned}
\tag{3.24}
$$

For the term J_2, a direct computation yields

$$\begin{aligned}
J_2 &= -\mu \int \sigma^h \dot{u} \cdot \nabla \times \operatorname{curl} u_t \, dx - \mu \int \sigma^h \dot{u}^j \operatorname{div}\big((\nabla \times \operatorname{curl} u)^j u\big)\, dx \\
&= -\mu \int \sigma^h \operatorname{curl} \dot{u} \operatorname{curl} u_t \, dx + \mu \int_{\partial\Omega} \sigma^h \operatorname{curl} u_t \times \dot{u} \cdot n \, ds \\
&\quad - \mu \int \sigma^h \dot{u} \cdot (\nabla \times \operatorname{curl} u) \operatorname{div} u \, dx - \mu \int \sigma^h u^i \dot{u} \cdot \nabla \times (\nabla_i \operatorname{curl} u)\, dx \\
&= -\mu \int \sigma^h (\operatorname{curl} \dot{u})^2 \, dx + \mu \int \sigma^h \operatorname{curl} \dot{u} \cdot \big(\nabla u^i \times \nabla^i u\big)\, dx \\
&\quad + \mu \int \sigma^h \operatorname{curl} \dot{u} \cdot (u \cdot \nabla \operatorname{curl} u)\, dx - \mu \int \sigma^h \operatorname{div} u \operatorname{curl} \dot{u} \cdot \operatorname{curl} u \, dx \\
&\quad - \mu \int \sigma^h \nabla \operatorname{div} u \times \dot{u} \cdot \operatorname{curl} u \, dx - \mu \int_{\partial\Omega} \sigma^h \operatorname{curl} u \times (\operatorname{div} u \dot{u}) \cdot n \, ds \\
&\quad - \mu \int \sigma^h u \cdot \nabla \operatorname{curl} u \cdot \operatorname{curl} \dot{u} \, dx - \mu \int \sigma^h \nabla u^i \times \dot{u} \cdot (\nabla_i \operatorname{curl} u)\, dx \\
&\quad - \mu \int_{\partial\Omega} \sigma^h \nabla_i \operatorname{curl} u \times \big(u^i \dot{u}\big) \cdot n \, ds \\
&= -\mu \int \sigma^h (\operatorname{curl} \dot{u})^2 \, dx + \mu \int \sigma^h \operatorname{curl} \dot{u} \cdot \big(\nabla u^i \times \nabla^i u\big)\, dx \\
&\quad - \mu \int \sigma^h \operatorname{div} u \operatorname{curl} \dot{u} \cdot \operatorname{curl} u \, dx - \mu \int \sigma^h \nabla \operatorname{div} u \times \dot{u} \cdot \operatorname{curl} u \, dx \\
&\quad - \mu \int \sigma^h \operatorname{curl} u \times \dot{u} \cdot \nabla \operatorname{div} u \, dx - \mu \int \sigma^h \operatorname{div} u \operatorname{div}(\operatorname{curl} u \times \dot{u})\, dx \\
&\quad - \mu \int \sigma^h \nabla u^i \times \dot{u} \cdot (\nabla_i \operatorname{curl} u)\, dx - \mu \int \sigma^h \nabla u^i \cdot (\nabla_i \operatorname{curl} u \times \dot{u})\, dx \\
&\quad - \mu \int \sigma^h u^i \operatorname{div}(\nabla_i \operatorname{curl} u \times \dot{u})\, dx \\
&= -\mu \int \sigma^h (\operatorname{curl} \dot{u})^2 \, dx + \mu \int \sigma^h \operatorname{curl} \dot{u} \cdot \big(\nabla u^i \times \nabla^i u\big)\, dx \\
&\quad - \mu \int \sigma^h \operatorname{div} u \operatorname{curl} \dot{u} \cdot \operatorname{curl} u \, dx - \mu \int \sigma^h \operatorname{div} u \operatorname{div}(\operatorname{curl} u \times \dot{u})\, dx \\
&\quad - \mu \int \sigma^h u^i \nabla^i \operatorname{div}(\operatorname{curl} u \times \dot{u})\, dx - \mu \int \sigma^h \nabla u^i \cdot (\operatorname{curl} u \times \nabla_i \dot{u})\, dx \\
&= -\mu \int \sigma^h (\operatorname{curl} \dot{u})^2 \, dx + \mu \int \sigma^h \operatorname{curl} \dot{u} \cdot \big(\nabla u^i \times \nabla^i u\big)\, dx \\
&\quad - \mu \int \sigma^h \operatorname{div} u \operatorname{curl} \dot{u} \cdot \operatorname{curl} u \, dx - \mu \int \sigma^h \nabla u^i \cdot (\operatorname{curl} u \times \nabla_i \dot{u})\, dx \\
&\le -\mu \int \sigma^h (\operatorname{curl} \dot{u})^2 \, dx + \delta \sigma^h \|\nabla \dot{u}\|_{L^2}^2 + C\sigma^h \|\nabla u\|_{L^4}^4.
\end{aligned} \tag{3.25}$$

Combining $u = u^\perp \times n$ and (1.4) gives

$$\big(\dot{u} - (u \cdot \nabla n) \times u^\perp\big) \cdot n = 0 \quad \text{on } \partial\Omega,$$

which together with (2.7) and (3.1) implies

$$\begin{aligned}
\|\nabla\dot u\|_{L^2}^2 &\le \big\|\dot u-(u\cdot\nabla n)\times u^\perp\big\|_{L^2}^2+\big\|(u\cdot\nabla n)\times u^\perp\big\|_{L^2}^2\\
&\le C\big(\|\operatorname{div}\dot u\|_{L^2}^2+\|\operatorname{curl}\dot u\|_{L^2}^2\\
&\quad+\big\|(u\cdot\nabla n)\times u^\perp\big\|_{L^2}^2+\big\|\nabla\big((u\cdot\nabla n)\times u^\perp\big)\big\|_{L^2}^2\big)\\
&\le C\big(\|\operatorname{div}\dot u\|_{L^2}^2+\|\operatorname{curl}\dot u\|_{L^2}^2+\|u\|_{L^4(B_{2R})}^4+\|\nabla u\|_{L^4(B_{2R})}^2\|u\|_{L^4(B_{2R})}^2\big)\\
&\le C(R)(\|\operatorname{div}\dot u\|_{L^2}^2+\|\operatorname{curl}\dot u\|_{L^2}^2+\|\nabla u\|_{L^4(B_{2R})}^4+\|\nabla u\|_{L^2(B_{2R})}^4.
\end{aligned} \tag{3.26}$$

Putting (3.24), (3.25), and (3.26) into (3.20), for a enough small δ, we obtain

$$\begin{aligned}
&\left(\int\sigma^h(\rho+m)|\dot u|^2\,dx\right)_t+\sigma^h\|\nabla\dot u\|_{L^2}^2+\left(\int_{\partial\Omega}\sigma^hFu\cdot\nabla n\cdot u\,dx\right)_t\\
&\quad\le C\sigma^{h-1}\sigma'\big\|(\rho+m)^{\frac12}\dot u\big\|_{L^2}\|\nabla u\|_{L^2}^2+C\sigma^h\big(\big\|(\rho+m)^{\frac12}\dot u\big\|_{L^2}^2\big(1+\|\nabla u\|_{L^2}^4\big)\\
&\qquad+\|\nabla u\|_{L^2}^2+\|\nabla u\|_{L^2}^6+\|\nabla u\|_{L^4}^4\big).
\end{aligned} \tag{3.27}$$

Integrating (3.27) over $(0,T]$ and using (2.12) and (3.5), when $h\ge 3$, we have

$$\begin{aligned}
&\sigma^h\int(\rho+m)|\dot u|^2\,dx+\int_0^T\sigma^h\|\nabla\dot u\|_{L^2}^2\,dt\\
&\quad\le-\int_{\partial\Omega}\sigma^hFu\cdot\nabla n\cdot u\,ds+CA_1(T)+CC_0+C\int_0^T\sigma^h\|\nabla u\|_{L^4}^4\,dt\\
&\quad\le\delta\sigma^h\big\|(\rho+m)^{\frac12}\dot u\big\|_{L^2}^2+CA_1(T)+CC_0+C\int_0^T\sigma^h\|\nabla u\|_{L^4}^4\,dt,
\end{aligned} \tag{3.28}$$

where in the last inequality, we have used

$$\begin{aligned}
-\int_{\partial\Omega}\sigma^hFu\cdot\nabla n\cdot u\,ds&\le C\sigma^h\|F\|_{L^2(\partial\Omega)}\|u\|_{L^4(\partial\Omega)}^2\le C\sigma^h\|\nabla F\|_{L^2}\|\nabla u\|_{L^2}^2\\
&\le\delta\sigma^h\big\|(\rho+m)^{\frac12}\dot u\big\|_{L^2}^2+CC_0.
\end{aligned}$$

Then taking $h=3$ and choosing enough small δ, we obtain (3.14). □

Lemma 3.4 *Suppose that (ρ,m,u) is a smooth solution of (1.1)–(1.5) on $\Omega\times(0,T]$ satisfying (3.5). Then there exists a positive constant C depending only on λ, μ, $\bar\rho$, $\bar m$, M, and Ω such that*

$$A_3\big(\sigma(T)\big)+\int_0^{\sigma(T)}\big\|(\rho+m)^{\frac12}\dot u\big\|_{L^2}^2\,dt\le M, \tag{3.29}$$

provided $C_0\le\varepsilon_1$.

Proof Taking $h = 0$ in (3.19), integrating over $(0, \sigma(T)]$, and using Lemma 2.6, (2.16), (3.5), and (3.7), we can obtain

$$\begin{aligned}
&\|\nabla u\|_{L^2}^2 + \int_0^{\sigma(T)} \big\|(\rho+m)^{\frac12}\dot u\big\|_{L^2}^2\,dt \\
&\quad\le \frac{M}{2} + C\int_0^{\sigma(T)} \big(\|\nabla u\|_{L^2}^2 + \|\nabla u\|_{L^2}^4 + \|\nabla u\|_{L^3}^3\big)\,dt + \delta\|\nabla u\|_{L^2}^2 \\
&\qquad + C\|P-P_\infty\|_{L^2}^2 + CC_0 \\
&\quad\le \frac{M}{2} + CC_0(1+M) + \delta\|\nabla u\|_{L^2}^2 \\
&\qquad + C\int_0^{\sigma(T)} \Big(\|\nabla u\|_{L^2}^{\frac32}\big(\big\|(\rho+m)\dot u\big\|_{L^2} + \|P-P_\infty\|_{L^6}\big)^{\frac32} + \|\nabla u\|_{L^2}^3\Big)\,dt \\
&\quad\le \frac{M}{2} + CC_0(1+M) + \delta\|\nabla u\|_{L^2}^2 + \delta\int_0^{\sigma(T)} \big\|(\rho+m)^{\frac12}\dot u\big\|_{L^2}^2\,dt \\
&\qquad + C\int_0^{\sigma(T)} \big(\|\nabla u\|_{L^2}^6 + \|P-P_\infty\|_{L^2}^2\big)\,dt \\
&\quad\le \frac{M}{2} + CC_0\big(1+M^2\big) + \delta\|\nabla u\|_{L^2}^2 + \delta\int_0^{\sigma(T)} \big\|(\rho+m)^{\frac12}\dot u\big\|_{L^2}^2\,dt.
\end{aligned} \tag{3.30}$$

Choosing δ small enough, (3.30) gives

$$A_3\big(\sigma(T)\big) + \int_0^{\sigma(T)} \big\|(\rho+m)^{\frac12}\dot u\big\|_{L^2}^2\,dt \le \frac{M}{2} + CC_0 + CM^2C_0 \le M,$$

provided $C_0 \le \varepsilon_1 \triangleq \{1, \frac{M}{4C}, \frac{1}{4MC}\}$. □

Lemma 3.5 *Suppose that (ρ, m, u) is a smooth solution of (1.1)–(1.5) on $\Omega \times (0, T]$ satisfying (3.5). Then there exists a positive constant C depending only on λ, μ, $\bar\rho$, $\bar m$, M and Ω such that*

$$\sup_{0\le t\le\sigma(T)} t\big\|(\rho+m)^{\frac12}\dot u\big\|_{L^2}^2 + \int_0^{\sigma(T)} t\|\nabla\dot u\|_{L^2}^2\,dt \le C. \tag{3.31}$$

Proof Taking $h = 1$ in (3.27), and integrating over $(0, \sigma(T)]$, by (3.5), (3.29) and (2.16), we have

$$\begin{aligned}
&\sup_{0\le t\le\sigma(T)} t\big\|(\rho+m)^{\frac12}\dot u\big\|_{L^2}^2 + \int_0^{\sigma(T)} t\|\nabla\dot u\|_{L^2}^2\,dt \\
&\quad\le C\int_0^{\sigma(T)} t\Big(\big\|(\rho+m)^{\frac12}\dot u\big\|_{L^2}^2\big(1+\|\nabla u\|_{L^2}^4\big) + \|\nabla u\|_{L^2}^2 + \|\nabla u\|_{L^6}^2 + \|\nabla u\|_{L^4}^4\Big)\,dt \\
&\qquad + C\int_0^{\sigma(T)} \big\|(\rho+m)^{\frac12}\dot u\big\|_{L^2}\|\nabla u\|_{L^2}^2\,dt + Ct\|F\|_{L^2(\partial\Omega)}\|u\|_{L^4(\partial\Omega)}^2 \\
&\quad\le C + C\int_0^{\sigma(T)} \Big[t\big(\big\|(\rho+m)\dot u\big\|_{L^2} + \|P-P_\infty\|_{L^6}\big)^3\|\nabla u\|_{L^2} + t\|\nabla u\|_{L^2}^4\Big]\,dt \\
&\qquad + Ct\|\nabla F\|_{L^2}\|\nabla u\|_{L^2}
\end{aligned}$$

$$\begin{aligned}
&\le C + C \sup_{0\le t\le\sigma(T)} \bigl(t\bigl\|(\rho+m)^{\frac{1}{2}}\dot{u}\bigr\|_{L^2}^2\bigr)^{\frac{1}{2}} \sup_{0\le t\le\sigma(T)} \bigl(\|\nabla u\|_{L^2}^2\bigr)^{\frac{1}{2}} \int_0^{\sigma(T)} \bigl\|(\rho+m)^{\frac{1}{2}}\dot{u}\bigr\|_{L^2}^2\,dt \\
&\quad + Ct\bigl\|(\rho+m)\dot{u}\bigr\|_{L^2}\|\nabla u\|_{L^2}^2 \\
&\le C + \delta \sup_{0\le t\le\sigma(T)} t\bigl\|(\rho+m)^{\frac{1}{2}}\dot{u}\bigr\|_{L^2}^2,
\end{aligned}$$

which gives (3.31) when we choose enough small δ. □

Lemma 3.6 *If (ρ, m, u) is a smooth solution of (1.1)–(1.5) on $\Omega \times (0,T]$ satisfying the assumption (3.5), then it holds that*

$$\int_0^T \sigma^3\|P-P_\infty\|_{L^4}^4\,dt \le CC_0. \tag{3.32}$$

Proof A direct computation shows that

$$\begin{aligned}
&-\Bigl(\int (P-P_\infty)^3\,dx\Bigr)_t \\
&\quad= -3\int (P-P_\infty)^2 P_t\,dx \\
&\quad= 3\int (P-P_\infty)^2\bigl(\gamma\rho^\gamma + \alpha m^\alpha\bigr)\operatorname{div} u\,dx + 3\int (P-P_\infty)^2 u\cdot\nabla P\,dx \\
&\quad= 3\int (P-P_\infty)^2\bigl(\gamma\rho^\gamma + \alpha m^\alpha\bigr)\operatorname{div} u\,dx - \int \operatorname{div} u(P-P_\infty)^3\,dx \\
&\quad= 3\int (P-P_\infty)^2\bigl(\gamma\rho^\gamma + \alpha m^\alpha\bigr)\operatorname{div} u\,dx - \int \frac{F}{2\mu+\lambda}(P-P_\infty)^3\,dx \\
&\qquad - \frac{1}{2\mu+\lambda}\int (P-P_\infty)^4\,dx,
\end{aligned}\tag{3.33}$$

which indicates that

$$\begin{aligned}
&\frac{1}{2\mu+\lambda}\sigma^3\int (P-P_\infty)^4\,dx \\
&\quad= \Bigl(\sigma^3\int (P-P_\infty)^3\,dx\Bigr)_t - 3\sigma^2\sigma'\int (P-P_\infty)^3\,dx \\
&\qquad + 3\sigma^3\int (P-P_\infty)^2\bigl(\gamma\rho^\gamma + \alpha m^\alpha\bigr)\operatorname{div} u\,dx - \frac{\sigma^3}{2\mu+\lambda}\int F(P-P_\infty)^3\,dx.
\end{aligned}\tag{3.34}$$

Combining (3.34), (2.14), and (3.5) with (3.7) implies that

$$\begin{aligned}
&\int_0^T \sigma^3\|P-P_\infty\|_{L^4}^4\,dt \\
&\quad\le \sigma^3\|P-P_\infty\|_{L^3}^3 + C\int_0^T \sigma'\|P-P_\infty\|_{L^2}^2\,dt + \delta\int_0^T \sigma^3\|P-P_\infty\|_{L^4}^4\,dt \\
&\qquad + C\int_0^T \bigl(\|\nabla u\|_{L^2}^2 + \sigma^3\|F\|_{L^4}^4\bigr)\,dt
\end{aligned}$$

$$
\begin{aligned}
&\le \delta \int_0^T \sigma^3 \|P-P_\infty\|_{L^4}^4\,dt + CC_0 \\
&\quad + C\int_0^T \sigma^3 \big(\|\nabla u\|_{L^2} + \|P-P_\infty\|_{L^2}\big) \big\|(\rho+m)^{\frac12}\dot u\big\|_{L^2}^3\,dt \\
&\le \delta \int_0^T \sigma^3 \|P-P_\infty\|_{L^4}^4\,dt + CC_0 \\
&\quad + C\int_0^{\sigma(T)} \big(\sigma^3 \big\|(\rho+m)^{\frac12}\dot u\big\|_{L^2}^2\big)^{\frac12} \big(\sigma\|\nabla u\|_{L^2}^2\big)^{\frac12} \sigma \big\|(\rho+m)^{\frac12}\dot u\big\|_{L^2}^2\,dt \\
&\quad + CC_0^{\frac12}\int_0^{\sigma(T)} \big(\sigma^3 \big\|(\rho+m)^{\frac12}\dot u\big\|_{L^2}^2\big)^{\frac12} \sigma \big\|(\rho+m)^{\frac12}\dot u\big\|_{L^2}^2\,dt \\
&\le \delta \int_0^T \sigma^3 \|P-P_0\|_{L^4}^4\,dt + CC_0,
\end{aligned}
$$

which yields (3.32) when we choose enough small δ. □

Lemma 3.7 *Assume that (ρ, m, u) is a smooth solution of (1.1)–(1.5) on $\Omega \times (0,T]$ satisfying (3.5). Then there exists a positive constant C depending only on λ, μ, γ, α, M, Ω, $\bar\rho$, and $\bar m$ such that*

$$
A_1(T) + A_2(T) \le C_0^{\frac12}, \tag{3.35}
$$

provided $C_0 \le \varepsilon_2$.

Proof By (2.16), (3.5), (3.7), and (3.32), it holds that

$$
\begin{aligned}
&\int_0^T \sigma^3 \|\nabla u\|_{L^4}^4\,dt \\
&\quad \le C\int_0^T \sigma^3 \|\nabla u\|_{L^2} \big(\big\|(\rho+m)^{\frac12}\dot u\big\|_{L^2} + \|P-P_\infty\|_{L^6}\big)^3\,dt \\
&\qquad + C\int_0^T \sigma^3 \|\nabla u\|_{L^2}^4\,dt \\
&\quad \le \int_0^T \big(\sigma^3 \big\|(\rho+m)^{\frac12}\dot u\big\|_{L^2}^2\big)^{\frac12} \big(\sigma\|\nabla u\|_{L^2}^2\big)^{\frac12} \sigma \big\|(\rho+m)^{\frac12}\dot u\big\|_{L^2}^2\,dt \\
&\qquad + C\int_0^T \sigma^3 \|P-P_\infty\|_{L^6}^6\,dt \\
&\qquad + C\big(1+C_0^{\frac12}\big)\int_0^T \|\nabla u\|_{L^2}^2\,dt \\
&\quad \le CC_0,
\end{aligned} \tag{3.36}
$$

which together with (3.13) and (3.14) gives

$$
A_1(T) + A_2(T) \le CC_0 + C\int_0^T \sigma\|\nabla u\|_{L^3}^3\,dt. \tag{3.37}
$$

It follows from (2.16), (3.5), (3.7), and (3.29) that

$$\begin{aligned}
&\int_0^{\sigma(T)} \sigma \|\nabla u\|_{L^3}^3 \,dt \\
&\leq C \int_0^{\sigma(T)} \sigma \|\nabla u\|_{L^2}^{\frac{3}{2}} \left(\left\|(\rho+m)^{\frac{1}{2}} \dot{u}\right\|_{L^2} + \|P-P_\infty\|_{L^6} \right)^{\frac{3}{2}} dt \\
&\quad + C \int_0^{\sigma(T)} \sigma \|\nabla u\|_{L^2}^3 \,dt \\
&\leq C \int_0^{\sigma(T)} \left(\|P-P_\infty\|_{L^6}^6 + \|\nabla u\|_{L^2}^2 + \sigma^{\frac{4}{3}} \left\|(\rho+m)^{\frac{1}{2}} \dot{u}\right\|_{L^2}^2 \|\nabla u\|_{L^2}^{\frac{2}{3}} + \|\nabla u\|_{L^2}^4 \right) dt \\
&\quad + CC_0 \\
&\leq CC_0^{\frac{2}{3}}.
\end{aligned} \tag{3.38}$$

On the other hand, using (3.7), (3.36) and Young's inequality, we can get

$$\begin{aligned}
\int_{\sigma(T)}^T \sigma \|\nabla u\|_{L^3}^3 \,dt &\leq C \int_{\sigma(T)}^T \sigma \|\nabla u\|_{L^2} \|\nabla u\|_{L^4}^2 \,dt \\
&\leq C \int_{\sigma(T)}^T \sigma \|\nabla u\|_{L^2}^2 \,dt + C \int_{\sigma(T)}^T \sigma^3 \|\nabla u\|_{L^4}^4 \,dt \\
&\leq CC_0.
\end{aligned} \tag{3.39}$$

By (3.36)–(3.39), we can obtain

$$A_1(T) + A_2(T) \leq C(\bar{\rho}, \bar{m}, M) C_0^{\frac{2}{3}} \leq C_0^{\frac{1}{2}}, \tag{3.40}$$

which gives (3.35) provided $C_0 \leq \varepsilon_2 \triangleq \{\varepsilon_1, (\frac{1}{C(\bar{\rho},\bar{m},M)})^6\}$. □

To get all the higher-order estimates and to extend the classical solution globally, we must derive a uniform (in time) upper bound of the density.

Lemma 3.8 *If (ρ, m, u) is a smooth solution of (1.1)–(1.5) on $\Omega \times (0, T]$ satisfying (3.5), then there exists a positive constant ε depending only on λ, μ, γ, α, ρ_∞, m_∞, Ω, M, $\bar{\rho}$, and $\bar{m}$ such that*

$$\sup_{0 \leq t \leq T} \left\|(\rho+m)(t)\right\|_{L^\infty} \leq \frac{7}{4}(\bar{\rho} + \bar{m}), \tag{3.41}$$

provided $C_0 \leq \varepsilon$.

Proof First, the equations $(1.1)_1$ and $(1.1)_2$ can be rewritten as

$$D_t(\rho+m) = g(\rho+m) + b'(t), \tag{3.42}$$

where $D_t(\rho+m) \triangleq (\rho+m)_t + u \cdot \nabla(\rho+m)$, $g(\rho+m) \triangleq -\frac{\rho+m}{2\mu+\lambda}(P-P_\infty)$, and $b(t) \triangleq -\frac{1}{2\mu+\lambda}\int_0^t (\rho+m) F \,d\tau$. On the one hand, for all $0 \leq t_1 \leq t_2 \leq \sigma(T)$, one deduces from (2.2), (2.12), (2.14), (3.5), (3.29), and (3.31) that

$$\begin{aligned}
&\big|b(t_2)-b(t_1)\big| \\
&\quad\le C\int_{t_1}^{t_2}\big|(\rho+m)F\big|\,dt \le C\int_0^{\sigma(T)}\|F\|_{L^\infty}\,dt \\
&\quad\le C\int_0^{\sigma(T)}\|F\|_{L^6}^{\frac12}\|\nabla F\|_{L^6}^{\frac12}\,dt \le C\int_0^{\sigma(T)}\big\|(\rho+m)^{\frac12}\dot{u}\big\|_{L^2}^{\frac12}\|\nabla\dot{u}\|_{L^2}^{\frac12}\,dt \\
&\quad\le C\int_0^{\sigma(T)}\big(t\big\|(\rho+m)^{\frac12}\dot{u}\big\|_{L^2}^2\big)^{\frac18}\big\|(\rho+m)^{\frac12}\dot{u}\big\|_{L^2}^{\frac14}\big(t\|\nabla\dot{u}\|_{L^2}^2\big)^{\frac14}t^{-\frac38}\,dt \\
&\quad\le C\bigg(\int_0^{\sigma(T)}t\big\|(\rho+m)^{\frac12}\dot{u}\big\|_{L^2}^2\,dt\bigg)^{\frac18}\bigg(\int_0^{\sigma(T)}\big\|(\rho+m)^{\frac12}\dot{u}\big\|_{L^2}^2\,dt\bigg)^{\frac18} \\
&\qquad\times\bigg(\int_0^{\sigma(T)}t\|\nabla\dot{u}\|_{L^2}^2\,dt\bigg)^{\frac14}\bigg(\int_0^{\sigma(T)}t^{-\frac34}\,dt\bigg)^{\frac12} \\
&\quad\le C(\bar{\rho}+\bar{m},M)C_0^{\frac{1}{16}}.
\end{aligned}$$

From Lemma 2.3, we choose $N_1=0$, $N_0=C(\bar{\rho}+\bar{m},M)C_0^{\frac{1}{16}}$, and $\hat{\zeta}=\bar{\rho}+\bar{m}$ and then we use (3.42) to get

$$\sup_{0\le t\le\sigma(T)}\|\rho+m\|_{L^\infty}\le\bar{\rho}+\bar{m}+C(\bar{\rho}+\bar{m},M)C_0^{\frac{1}{16}}\le\frac32(\bar{\rho}+\bar{m}), \tag{3.43}$$

provided

$$C_0\le\varepsilon_3\triangleq\min\bigg\{\varepsilon_2,\bigg(\frac{\bar{\rho}+\bar{m}}{2C(\bar{\rho}+\bar{m},M)}\bigg)^{16}\bigg\}.$$

On the other hand, for $\sigma(T)\le t_1\le t_2\le T$, it follow from (2.2), (2.12), (2.14), and (3.5) that

$$\begin{aligned}
\big|b(t_2)-b(t)_1\big| &\le\frac{1}{2\mu+\lambda}(t_2-t_1)+C\int_{t_1}^{t_2}\|F\|_{L^\infty}^4\,dt \\
&\le\frac{1}{2\mu+\lambda}(t_2-t_1)+C\int_{\sigma(T)}^{T}\|F\|_{L^6}^2\|\nabla F\|_{L^6}^2\,dt \\
&\le\frac{1}{2\mu+\lambda}(t_2-t_1)+C\int_{\sigma(T)}^{T}\big\|(\rho+m)^{\frac12}\dot{u}\big\|_{L^2}^2\|\nabla\dot{u}\|_{L^2}^2\,dt \\
&\le\frac{1}{2\mu+\lambda}(t_2-t_1)+CC_0.
\end{aligned}$$

Now, choosing $N_0=CC_0$, $N_1=\frac{1}{2\mu+\lambda}$ in (2.3) and setting $\hat{\zeta}=\bar{\rho}+\bar{m}$ in (2.4), it gives that for all $\zeta\ge\hat{\zeta}=\bar{\rho}+\bar{m}$,

$$g(\zeta)=-\frac{\zeta}{2\mu+\lambda}\big(P(\zeta)-P_\infty\big)\le-\frac{\bar{\rho}+\bar{m}}{2\mu+\lambda}\le-\frac{1}{2\mu+\lambda}=-N_1, \tag{3.44}$$

which together with Lemma 2.3, (3.43), and (3.44) implies

$$\sup_{t\in[\sigma(T),T]} \|\rho + m\|_{L^\infty} \le \frac{3}{2}(\bar{\rho} + \bar{m}) + CC_0 \le \frac{7}{4}(\bar{\rho} + \bar{m}), \tag{3.45}$$

provided

$$C_0 \le \varepsilon \triangleq \min\left\{\varepsilon_3, \frac{\bar{\rho} + \bar{m}}{4C}\right\}. \tag{3.46}$$

The combination of (3.43) with (3.45) completes the proof of Lemma 3.8. □

4 A priori estimates (ii): higher order estimates

Suppose that (ρ, m, u) is a smooth solution of (1.1)–(1.5). To extend the classical solution globally in time, assume that (3.46) holds, and the positive constant C may depend on

$$T, \quad \|g\|_{L^2}, \quad \|\nabla u_0\|_{H^1}, \quad \|\rho_0 - \rho_\infty\|_{H^2\cap W^{2,q}},$$
$$\|m_0 - m_\infty\|_{H^2\cap W^{2,q}}, \quad \left\|P(\rho_0, m_0) - P_\infty\right\|_{H^2\cap W^{2,q}},$$

for besides λ, μ, γ, α, M, Ω, M, $\bar{\rho}$, and $\bar{m}$, where $g \in L^2(\Omega)$ is gives as in (1.10), we can get some necessary higher-order estimates.

Lemma 4.1 *There exists a positive constant C, such that*

$$\sup_{0\le t\le T} \left\|(\rho + m)^{\frac{1}{2}}\dot{u}\right\|_{L^2} + \int_0^T \|\nabla\dot{u}\|_{L^2}^2\,dt \le C \quad \textit{and} \tag{4.1}$$

$$\sup_{0\le t\le T} \left(\|\nabla\rho\|_{L^2\cap L^6} + \|\nabla m\|_{L^2\cap L^6} + \|\nabla u\|_{H^1}\right) + \int_0^T \left(\|\nabla u\|_{L^\infty} + \left\|\nabla^2 u\right\|_{L^6}\right) dt \le C. \tag{4.2}$$

Proof By (3.19), (3.38), (3.39) and Lemma 2.6, it gives

$$\|\nabla u\|_{L^2}^2 + \int_0^T \left\|(\rho + m)^{\frac{1}{2}}\dot{u}\right\|_{L^2}^2 dt \le \int_0^T \|\nabla u\|_{L^2}^4\,dt + C,$$

which together with Growall's inequality yields that

$$\sup_{0\le t\le T} \|\nabla u\|_{L^2}^2 + \int_0^T \left\|(\rho + m)^{\frac{1}{2}}\dot{u}\right\|_{L^2}^2 dt \le C. \tag{4.3}$$

Choosing $h = 0$ in (3.27), we deduce from (2.12), (2.16), and (4.3) that

$$\begin{aligned}
&\sup_{0\le t\le T} \left\|(\rho + m)^{\frac{1}{2}}\dot{u}\right\|_{L^2}^2 + \int_0^T \|\nabla\dot{u}\|_{L^2}^2\,dt \\
&\quad\le C\int_{\partial\Omega} |Fu\cdot\nabla n\cdot u|\,ds - \int_{\partial\Omega} |F_0u_0\cdot\nabla n\cdot u_0|\,ds \\
&\qquad + C\int_0^T \left(\|\nabla u\|_{L^2}^2 + \|\nabla u\|_{L^2}^6 + \|\nabla u\|_{L^4}^4\right) dt \\
&\qquad + \int_0^T \left\|(\rho + m)^{\frac{1}{2}}\dot{u}\right\|_{L^2}^2\left(1 + \|\nabla u\|_{L^2}^4\right) dt
\end{aligned}$$

$$
\begin{aligned}
&\le C\|\nabla F\|_{L^2}\|\nabla u\|_{L^2}^2 + \|\nabla F_0\|_{L^2}\|\nabla u_0\|_{L^2}^2 + C \\
&\quad + \Big(\sup_{0\le t\le T}\|\nabla u\|_{L^2}^2\Big)^{\frac12}\Big(\sup_{0\le t\le T}\big\|(\rho+m)^{\frac12}\dot u\big\|_{L^2}^2\Big)^{\frac12}\int_0^T\big\|(\rho+m)^{\frac12}\dot u\big\|_{L^2}^2\,dt \\
&\le \delta\sup_{0\le t\le T}\big\|(\rho+m)^{\frac12}\dot u\big\|_{L^2}^2 + C.
\end{aligned}
$$

Then choosing δ small enough, it gives (4.1). Observe that for $2\le p\le 6$, it indicates that

$$
\begin{aligned}
&\big(\big|\nabla(\rho+m)\big|^p\big)_t + \operatorname{div}\big(\big|\nabla(\rho+m)\big|^p u\big) + (p-1)\big|\nabla(\rho+m)\big|^p\operatorname{div}u \\
&\quad + p\big|\nabla(\rho+m)\big|^{p-2}\big(\nabla(\rho+m)\big)^{\mathrm{tr}}\nabla u\big(\nabla(\rho+m)\big) + p(\rho+m)\big|\nabla(\rho+m)\big|^{p-2}\nabla(\rho+m) \\
&\quad\cdot\nabla\operatorname{div}u = 0.
\end{aligned}
$$

Integrating the above equality over Ω and using (2.12) imply that

$$
\begin{aligned}
\big(\big\|\nabla(\rho+m)\big\|_{L^p}\big)_t &\le C\big(1+\|\nabla u\|_{L^\infty}\big)\big\|\nabla(\rho+m)\big\|_{L^p} + \|\nabla F\|_{L^p} \\
&\le C\big(1+\|\nabla u\|_{L^\infty}\big)\big\|\nabla(\rho+m)\big\|_{L^p} + C\big\|(\rho+m)\dot u\big\|_{L^p}.
\end{aligned}\tag{4.4}
$$

Moreover, by Lemma 2.8, (1.17), (2.12), and (2.16), for any $2\le p\le 6$, we have that

$$
\begin{aligned}
\big\|\nabla^2 u\big\|_{L^p} &\le C\big(\|\operatorname{div}u\|_{W^{1,p}} + \|\operatorname{curl}u\|_{W^{1,p}} + \|\nabla u\|_{L^2}\big) \\
&\le C\big(\big\|(\rho+m)\dot u\big\|_{L^p} + \|\nabla P\|_{L^p} + \big\|(\rho+m)\dot u\big\|_{L^2} + \|\nabla u\|_{L^2} + \|P-P_\infty\|_{L^6}\big).
\end{aligned}\tag{4.5}
$$

Next, it follows from (2.2), (1.17), (2.12), (2.13), (3.14), and (4.1) that

$$
\begin{aligned}
&\|\operatorname{div}u\|_{L^\infty} + \|\operatorname{curl}u\|_{L^\infty} \\
&\quad\le C\big(\|F\|_{L^\infty} + \|P-P_\infty\|_{L^\infty}\big) + \|\operatorname{curl}u\|_{L^\infty} \\
&\quad\le C\big(\|F\|_{L^2} + \|\nabla F\|_{L^6} + \|\operatorname{curl}u\|_{L^2} + \|\nabla\operatorname{curl}u\|_{L^6} + 1\big) \\
&\quad\le C\big(\big\|(\rho+m)\dot u\big\|_{L^6} + \|P-P_\infty\|_{L^2} + \|\nabla u\|_{L^2} + \big\|(\rho+m)\dot u\big\|_{L^2} + 1\big) \\
&\quad\le C\big(\|\nabla\dot u\|_{L^2} + 1\big).
\end{aligned}\tag{4.6}
$$

By Lemma 2.9, (4.5) and (4.6), we get

$$
\begin{aligned}
\|\nabla u\|_{L^\infty} &\le C\big(\|\operatorname{div}u\|_{L^\infty} + \|\operatorname{curl}u\|_{L^\infty}\big)\ln\big(e+\big\|\nabla^2u\big\|_{L^6}\big) + C\big(\|\nabla u\|_{L^2}+1\big) \\
&\le C\big(\|\nabla\dot u\|_{L^2}+1\big)\ln\big(e+\big\|\nabla^2u\big\|_{L^6}\big) + C\big(\|\nabla u\|_{L^2}+1\big) \\
&\le C\big(\|\nabla\dot u\|_{L^2}+1\big)\ln\big(e+\big\|(\rho+m)\dot u\big\|_{L^6} + \|\nabla P\|_{L^6} + \|\nabla u\|_{L^2}\big) \\
&\qquad + C\big(\|\nabla u\|_{L^2}+1\big) \\
&\le C\big(\|\nabla\dot u\|_{L^2}+1\big)\big(\ln\big(e+\|\nabla\dot u\|_{L^2}\big) + \ln\big(e+\big\|\nabla(\rho+m)\big\|_{L^6}\big)\big) + C \\
&\le C\big(\|\nabla\dot u\|_{L^2}^2+1\big) + C\big(\|\nabla\dot u\|_{L^2}+1\big)\ln\big(e+\big\|\nabla(\rho+m)\big\|_{L^6}\big).
\end{aligned}\tag{4.7}
$$

Combining (4.7) with (4.4) yields

$$\begin{aligned}&\big(\ln\big(e+\big\|\nabla(\rho+m)\big\|_{L^6}\big)\big)_t\\&\quad\le C\big(1+\big(\|\nabla\dot{u}\|_{L^2}^2+1\big)\ln\big(e+\big\|\nabla(\rho+m)\big\|_{L^6}\big)\big)+C\big(\|\nabla\dot{u}\|_{L^2}+1\big).\end{aligned}$$

And then by Gronwall's inequality and (4.1), we obtain

$$\sup_{0\le t\le T}\big\|\nabla(\rho+m)\big\|_{L^6}\le C. \tag{4.8}$$

Moreover, (4.7) and (4.8) imply that

$$\int_0^T\|\nabla u\|_{L^\infty}\,dt\le C. \tag{4.9}$$

Using the above inequality, (4.4) and (4.9), when $p=2$, yields that

$$\sup_{0\le t\le T}\big\|\nabla(\rho+m)\big\|_{L^2}\le C,$$

which together with (4.1), (4.5), and (4.8) gives

$$\sup_{0\le t\le T}\big\|\nabla^2u\big\|_{L^2}\le C\quad\text{and}\quad\int_0^T\big\|\nabla^2u\big\|_{L^6}\,dt\le C.$$

Hence, we finish the proof of Lemma 4.1. □

Lemma 4.2 *There exists a constant C such that*

$$\sup_{0\le t\le T}\big\|(\rho+m)^{\frac12}u_t\big\|_{L^2}^2+\int_0^T\|\nabla u_t\|_{L^2}^2\,dt\le C, \tag{4.10}$$

$$\sup_{0\le t\le T}\big(\|\rho-\rho_\infty\|_{H^2}+\|m-m_\infty\|_{H^2}+\|P-P_\infty\|_{H^2}\big)\le C. \tag{4.11}$$

Proof By Lemma 4.1, a simple computation shows that

$$\begin{aligned}\big\|(\rho+m)^{\frac12}u_t\big\|_{L^2}^2&\le\big\|(\rho+m)^{\frac12}\dot{u}\big\|_{L^2}^2+\big\|(\rho+m)^{\frac12}u\cdot\nabla u\big\|_{L^2}^2\\&\le C+C\big\|(\rho+m)^{\frac12}u\big\|_{L^3}^2\|\nabla u\|_{L^6}^2\\&\le C+C\big\|(\rho+m)^{\frac12}u\big\|_{L^2}\|u\|_{L^6}\|\nabla u\|_{L^6}^2\le C\end{aligned}$$

and

$$\begin{aligned}\int_0^T\|\nabla u_t\|_{L^2}^2\,dt&\le\int_0^T\|\nabla\dot{u}\|_{L^2}^2\,dt+\int_0^T\big\|\nabla(u\cdot\nabla u)\big\|_{L^2}^2\,dt\\&\le C+\int_0^T\big(\|\nabla u\|_{L^4}^4+\|u\|_{L^\infty}^2\big\|\nabla^2u\big\|_{L^2}^2\big)\,dt\\&\le C+\int_0^T\big(\|\nabla u\|_{L^2}\|\nabla u\|_{L^6}^3+\|\nabla u\|_{H^1}^2\big)\,dt\le C,\end{aligned}$$

so we have (4.10). Using $(1.1)_1$, $(1.2)_2$, and (4.2), it shows that

$$\begin{aligned}\left(\left\|\nabla^2(\rho+m)\right\|_{L^2}\right)_t &\le C\left(1+\left\|\nabla^2 u\right\|_{L^6}+\|\nabla u\|_{L^\infty}\right)\left\|\nabla^2(\rho+m)\right\|_{L^2}+C\left\|\nabla^3 u\right\|_{L^2}\\ &\le C\left(1+\left\|\nabla^2 u\right\|_{L^6}+\|\nabla u\|_{L^\infty}\right)\left\|\nabla^2(\rho+m)\right\|_{L^2}\\ &\quad + C\left(\|\nabla\dot{u}\|_{L^2}^2+1\right),\end{aligned}\tag{4.12}$$

where in the last inequality,we have used the fact that

$$\begin{aligned}\left\|\nabla^3 u\right\|_{L^p} &\le C\left(\|\operatorname{div} u\|_{W^{2,p}}+\|\operatorname{curl} u\|_{W^{2,p}}+\|\nabla u\|_{L^2}\right)\\ &\le C\left(\left\|(\rho+m)\dot{u}\right\|_{W^{1,p}}+\|P-P_\infty\|_{W^{2,p}}+\|\nabla u\|_{L^2}\right.\\ &\quad \left.+\left\|(\rho+m)\dot{u}\right\|_{L^2}+\|P-P_\infty\|_{L^2}\right),\end{aligned}\tag{4.13}$$

for any $p\in[2,6]$ by (2.19)–(2.22) and (1.17).

Employing Gronwall's inequality, (4.1), (4.2), and (4.12) leads to

$$\sup_{0\le t\le T}\left\|\nabla^2(\rho+m)\right\|_{L^2}\le C.$$

Hence,

$$\left\|\nabla^2 P\right\|_{L^2}\le C\left\|\nabla^2(\rho+m)\right\|_{L^2}\le C.\tag{4.14}$$

Therefore, the proof of Lemma 4.2 is completed. □

Lemma 4.3 *There exists a constant C such that*

$$\sup_{0\le t\le T}\left(\|\rho_t\|_{H^1}+\|m_t\|_{H^1}+\|P_t\|_{H^1}\right)+\int_0^T\left(\|\rho_{tt}\|_{L^2}^2+\|m_{tt}\|_{L^2}^2+\|P_{tt}\|_{L^2}^2\right)dt\le C,\tag{4.15}$$

$$\sup_{0\le t\le T}\sigma\|\nabla u_t\|_{L^2}^2+\int_0^T\sigma\left\|(\rho+m)^{\frac{1}{2}}u_{tt}\right\|_{L^2}^2dt\le C.\tag{4.16}$$

Proof Using $(1.1)_1$, $(1.1)_2$, we have

$$(\rho+m)_t+u\cdot\nabla(\rho+m)+(\rho+m)\operatorname{div} u=0,\tag{4.17}$$

which together with (4.2) and (4.11) gives

$$\left\|(\rho+m)_t\right\|_{L^2}\le C\|u\|_{L^\infty}\left\|\nabla(\rho+m)\right\|_{L^2}+C\|\nabla u\|_{L^2}\le C\|\nabla u\|_{H^1}+C\le C.\tag{4.18}$$

Combining (4.17) with (4.2), (4.11) implies

$$\begin{aligned}\left\|\nabla(\rho_t+m_t)\right\|_{L^2} &\le C\|\nabla u\|_{L^4}\left\|\nabla(\rho+m)\right\|_{L^4}\\ &\quad + C\|u\|_{L^\infty}\left\|\nabla^2(\rho+m)\right\|_{L^2}+C\left\|\nabla^2 u\right\|_{L^2}\\ &\le C\|\nabla u\|_{H^1}\left\|\nabla(\rho+m)\right\|_{H^1}+C\|\nabla u\|_{H^1}+C\le C.\end{aligned}\tag{4.19}$$

Due to the fact that $P_t + u \cdot \nabla P + \gamma\rho^\gamma \operatorname{div} u + \alpha m^\alpha \operatorname{div} u = 0$, we have

$$\|P_t\|_{L^2} \le C\|u\|_{L^\infty}\|\nabla P\|_{L^2} + C\|\nabla u\|_{L^2} \le C\|\nabla u\|_{H^1} + C \le C \tag{4.20}$$

and

$$\begin{aligned}\|\nabla P_t\|_{L^2} &\le C\|\nabla\rho\|_{L^4}\|\rho_t\|_{L^4} + C\|\nabla\rho_t\|_{L^2} + C\|\nabla m\|_{L^4}\|m_t\|_{L^4} + C\|\nabla m_t\|_{L^2}\\ &\le C\|\nabla\rho\|_{H^1}\|\rho_t\|_{H^1} + C\|\nabla m\|_{H^1}\|m_t\|_{H^1} + C \le C.\end{aligned} \tag{4.21}$$

By applying (4.18)–(4.21), we get

$$\sup_{0\le t\le T}\left(\|\rho_t\|_{H^1} + \|m_t\|_{H^1} + \|P_t\|_{H^1}\right) \le C. \tag{4.22}$$

Differentiating $(1.1)_1$ and $(1.1)_2$ with respect to t implies

$$(\rho+m)_{tt} + u_t\cdot\nabla(\rho+m) + u\cdot\nabla(\rho_t+m_t) + (\rho_t+m_t)\operatorname{div} u + (\rho+m)\operatorname{div} u_t = 0. \tag{4.23}$$

Combining (4.23) with (4.2), (4.10), and (4.22) yields

$$\begin{aligned}\int_0^T \|(\rho+m)_{tt}\|_{L^2}^2\,dt &\le C\int_0^T \|u_t\|_{L^6}^2\|\nabla(\rho+m)\|_{L^3}^2\,dt + C\int_0^T\|u\|_{L^\infty}^2\|\nabla(\rho_t+m_t)\|_{L^2}^2\,dt\\ &\quad + C\int_0^T\|\rho_t+m_t\|_{L^3}^2\|\nabla u\|_{L^6}^2\,dt + C\int_0^T\|\nabla u_t\|_{L^2}^2\,dt\\ &\le C\int_0^T\|\nabla u_t\|_{L^2}^2\,dt + C\int_0^T\|\nabla^2 u\|_{L^2}^2\,dt + C \le C.\end{aligned}$$

Hence, it gives that

$$\begin{aligned}\int_0^T\|P_{tt}\|_{L^2}^2\,dt &\le C\int_0^T\left(\|\rho_t\|_{L^4}^4 + \|\rho_{tt}\|_{L^2}^2 + \|m_t\|_{L^4}^4 + \|m_{tt}\|_{L^2}^2\right)dt\\ &\le C\int_0^T\left(\|\rho_t\|_{H^1}^4 + \|m_t\|_{H^1}^4\right)dt + C \le C.\end{aligned}$$

So, we get (4.15).

Next, differentiating $(1.1)_3$ with respect to t and then multiplying by u_{tt} yield that

$$\begin{aligned}&\left(\frac{2\mu+\lambda}{2}\|\operatorname{div} u_t\|_{L^2}^2 + \frac{\mu}{2}\|\operatorname{curl} u_t\|_{L^2}^2\right)_t + \int(\rho+m)|u_{tt}|^2\,dx\\ &= -\int(\rho+m)_t u_t\cdot u_{tt}\,dx - \int(\rho+m)_t u\cdot\nabla u\cdot u_{tt}\,dx\\ &\quad - \int(\rho+m)u_t\cdot\nabla u\cdot u_{tt}\,dx - \int(\rho+m)u\cdot\nabla u_t\cdot u_{tt}\,dx\\ &\quad - \int\nabla P_t\cdot u_{tt}\,dx =: \sum_{i=1}^5 I_i.\end{aligned} \tag{4.24}$$

It follows from $(1.1)_1$, $(1.1)_2$, (4.2), (4.10), and (4.15) that

$$
\begin{aligned}
I_1 &= -\int (\rho+m)_t u_t \cdot u_{tt}\,dx \\
&= -\frac{1}{2}\left(\int (\rho+m)_t |u_t|^2\,dx\right)_t + \frac{1}{2}\int (\rho+m)_{tt}|u_t|^2\,dx \\
&= -\frac{1}{2}\left(\int (\rho+m)_t |u_t|^2\,dx\right)_t - \frac{1}{2}\int \bigl(\operatorname{div}(\rho u + mu)\bigr)_t |u_t|^2\,dx \\
&\le -\frac{1}{2}\left(\int (\rho+m)_t |u_t|^2\,dx\right)_t \\
&\quad + C\bigl(\bigl\|(\rho+m)_t\bigr\|_{L^3} + \bigl\|(\rho+m)u_t\bigr\|_{L^3}\bigr)\|\nabla u_t\|_{L^2}\|u_t\|_{L^6} \\
&\le -\frac{1}{2}\left(\int (\rho+m)_t |u_t|^2\,dx\right)_t + C\|\nabla u_t\|_{L^2}^2\bigl(\|\nabla u_t\|_{L^2}^2 + 1\bigr),
\end{aligned} \tag{4.25}
$$

$$
\begin{aligned}
I_2 &= -\int (\rho+m)_t u\cdot\nabla u\cdot u_{tt}\,dx \\
&= -\left(\int (\rho+m)_t u\cdot\nabla u\cdot u_t\,dx\right)_t + \int (\rho+m)_{tt} u\cdot\nabla u\cdot u_t\,dx \\
&\quad + \int (\rho+m)u_t\cdot\nabla u\cdot u_t\,dx + \int (\rho+m)u\cdot\nabla u_t\cdot u_t\,dx \\
&\le -\left(\int (\rho+m)_t u\cdot\nabla u\cdot u_t\,dx\right)_t + C\bigl\|(\rho+m)_{tt}\bigr\|_{L^2}\|u_t\|_{L^6}\|u\|_{L^6}\|\nabla u\|_{L^6} \\
&\quad + C\bigl\|(\rho+m)^{\frac{1}{2}}u_t\bigr\|_{L^2}\|u_t\|_{L^6}\|\nabla u\|_{L^3} + C\|\nabla u_t\|_{L^2}\|u_t\|_{L^6}\bigl\|(\rho+m)^{\frac{1}{2}}u\bigr\|_{L^3} \\
&\le -\left(\int (\rho+m)_t u\cdot\nabla u\cdot u_t\,dx\right)_t + C\bigl(\bigl\|(\rho+m)_{tt}\bigr\|_{L^2}^2 + \|\nabla u_t\|_{L^2}^2 + 1\bigr),
\end{aligned} \tag{4.26}
$$

$$
\begin{aligned}
I_3 + I_4 + I_5 &= -\int (\rho+m)u_t\cdot\nabla u\cdot u_{tt}\,dx - \int (\rho+m)u\cdot\nabla u_t\cdot u_{tt}\,dx \\
&\quad - \int \nabla P_t\cdot u_{tt}\,dx \\
&\le C\bigl\|(\rho+m)^{\frac{1}{2}}u_{tt}\bigr\|_{L^2}\|u_t\|_{L^6}\|\nabla u\|_{L^3} \\
&\quad + C\bigl\|(\rho+m)^{\frac{1}{2}}u_{tt}\bigr\|_{L^2}\|\nabla u_t\|_{L^2}\|u\|_{L^\infty} \\
&\quad + \left(\int P_t \operatorname{div} u_t\,dx\right)_t - \int P_{tt}\operatorname{div} u_t\,dx \\
&\le \delta\bigl\|(\rho+m)^{\frac{1}{2}}u_{tt}\bigr\|_{L^2}^2 + \left(\int P_t\operatorname{div} u_t\,dx\right)_t + C\bigl(\|P_{tt}\|_{L^2}^2 + \|\nabla u_t\|_{L^2}^2\bigr).
\end{aligned} \tag{4.27}
$$

Choosing a suitably small positive constant δ and using (4.24)–(4.27), we have

$$
\begin{aligned}
&\left(\frac{2\mu+\lambda}{2}\sigma\|\operatorname{div} u_t\|_{L^2}^2 + \frac{\mu}{2}\sigma\|\operatorname{curl} u_t\|_{L^2}^2\right)_t + \sigma\int (\rho+m)|u_{tt}|^2\,dx \\
&\quad \le -\left(\frac{1}{2}\sigma\int (\rho+m)_t|u_t|^2\,dx + \sigma\int (\rho+m)_t u\cdot\nabla u\cdot u_t\,dx - \sigma\int P_t\operatorname{div} u_t\,dx\right)_t
\end{aligned}
$$

$$
\begin{aligned}
&+ C\sigma \|P_{tt}\|_{L^2}^2 + C\sigma \|\nabla u_t\|_{L^2}^2 \big(\|\nabla u_t\|_{L^2}^2 + 1\big) \\
&+ C\sigma \big\|(\rho + m)_{tt}\big\|_{L^2}^2 + C\|\nabla u_t\|_{L^2}^2 + C.
\end{aligned} \tag{4.28}
$$

Integrating (4.28) over (0,T], using $(1.1)_1$, $(1.1)_2$, (4.10), (4.15) and Lemma 2.6 gives

$$
\begin{aligned}
&\sigma \|\nabla u_t\|_{L^2}^2 + \int_0^T \sigma \big\|(\rho + m)^{\frac{1}{2}} u_{tt}\big\|_{L^2}^2 \, dt \\
&\leq -\frac{1}{2}\sigma \int (\rho + m)_t |u_t|^2 \, dx - \sigma \int (\rho + m)_t u \cdot \nabla u \cdot u_t \, dx + \sigma \int P_t \operatorname{div} u_t \, dx \\
&\quad + C \int_0^T \sigma \|\nabla u_t\|_{L^2}^2 \big(\|\nabla u_t\|_{L^2}^2 + 1\big) \, dt + C \\
&\leq \frac{1}{2}\sigma \int \operatorname{div}\big((\rho + m)u\big)|u_t|^2 \, dx + \delta\sigma \|\nabla u_t\|_{L^2}^2 \\
&\quad + C \int_0^T \sigma \|\nabla u_t\|_{L^2}^2 \big(\|\nabla u_t\|_{L^2}^2 + 1\big) \, dt + C \\
&\leq C\sigma \big\|(\rho + m)^{\frac{1}{2}} u_t\big\|_{L^2} \|\nabla u_t\|_{L^2} + \delta\sigma \|\nabla u_t\|_{L^2}^2 \\
&\quad + C \int_0^T \sigma \|\nabla u_t\|_{L^2}^2 \big(\|\nabla u_t\|_{L^2}^2 + 1\big) \, dt + C \\
&\leq \delta\sigma \|\nabla u_t\|_{L^2}^2 + C \int_0^T \sigma \|\nabla u_t\|_{L^2}^2 \big(\|\nabla u_t\|_{L^2}^2 + 1\big) \, dt + C.
\end{aligned} \tag{4.29}
$$

Using (4.29), (4.10) and Gronwall's inequality, we can obtain (4.16). □

Lemma 4.4 *For any $q \in (3, 6)$, there exists a positive constant C such that*

$$
\sup_{0 \leq t \leq T} \big(\|\rho - \rho_\infty\|_{W^{2,q}} + \|m - m_\infty\|_{W^{2,q}} + \|P - P_\infty\|_{W^{2,q}}\big) \leq C, \tag{4.30}
$$

$$
\sup_{0 \leq t \leq T} \sigma \|\nabla u\|_{H^2}^2 + \int_0^T \big(\|\nabla u\|_{H^2}^2 + \big\|\nabla^2 u\big\|_{W^{1,q}}^{p_0} + \sigma \|\nabla u_t\|_{H^1}^2\big) \, dt \leq C, \tag{4.31}
$$

where $p_0 = (1, \frac{9q-6}{10q-12}) \in (1, \frac{7}{6})$.

Proof By (4.13), (4.2), and (4.11), it gives

$$
\begin{aligned}
\big\|\nabla^2 u\big\|_{H^1} &\leq \big\|(\rho + m)\dot{u}\big\|_{H^1} + \|P - P_\infty\|_{H^2} + C \\
&\leq \big\|\nabla\big((\rho + m)\dot{u}\big)\big\|_{L^2} + C \\
&\leq C\|\nabla u_t\|_{L^2} + C,
\end{aligned} \tag{4.32}
$$

where we have used the fact that

$$
\big\|\nabla\big((\rho + m)\dot{u}\big)\big\|_{L^2} \leq \big\|\nabla(\rho + m)\dot{u}\big\|_{L^2} + \big\|(\rho + m)\nabla\dot{u}\big\|_{L^2} \leq C\|\nabla u_t\|_{L^2} + C.
$$

Then, we deduce from (4.2), (4.10), (4.16), and (4.32) that

$$
\sup_{0 \leq t \leq T} \sigma \|\nabla u\|_{H^2}^2 + \int_0^T \|\nabla u\|_{H^2}^2 \, dt \leq C. \tag{4.33}
$$

Utilizing (4.2) and (4.15) implies that

$$\begin{aligned}\|\nabla u_t\|_{H^1} &\le C\big(\big\|\big((\rho+m)\dot{u}\big)_t\big\|_{L^2}+\|\nabla P_t\|_{L^2}+\|\nabla u_t\|_{L^2}\big)\\&\le C\big(\big\|(\rho+m)_t\dot{u}\big\|_{L^2}+\big\|(\rho+m)u_{tt}\big\|_{L^2}\\&\quad+\big\|(\rho+m)(u\cdot\nabla u)_t\big\|_{L^2}+\|\nabla u_t\|_{L^2}\big)+C\\&\le C\big\|(\rho+m)_t\big\|_{L^3}\|\nabla u_t\|_{L^2}+C\big\|(\rho+m)^{\frac12}u_{tt}\big\|_{L^2}+C\|\nabla u_t\|_{L^2}+C\\&\le C\|\nabla u_t\|_{L^2}+C\big\|(\rho+m)^{\frac12}u_{tt}\big\|_{L^2}+C,\end{aligned}\tag{4.34}$$

where in the first inequality, we have used the a priori estimate similar to (4.5) since

$$\begin{cases}\mu\Delta u_t+(\lambda+\mu)\nabla\operatorname{div}u_t=((\rho+m)\dot{u})_t+\nabla P_t, & x\in\Omega,\\ u_t\cdot n=0,\qquad \operatorname{curl}u_t\times n=0, & x\in\partial\Omega.\end{cases}$$

Combining (4.34) with (4.16) implies

$$\int_0^T\sigma\|\nabla u_t\|_{H^1}^2\,dt\le C.\tag{4.35}$$

It follows from (4.13), (4.1), and (4.11) that

$$\begin{aligned}\big\|\nabla^2u\big\|_{W^{1,q}}&\le C\big(\big\|(\rho+m)\dot{u}\big\|_{W^{1,q}}+\|\nabla P\|_{W^{1,q}}+\|\nabla u\|_{L^2}+\|P-P_\infty\|_{L^2}+\|P-P_\infty\|_{L^q}\big)\\&\le C\big(\big\|(\rho+m)\dot{u}\big\|_{L^q}+\big\|\nabla\big((\rho+m)\dot{u}\big)\big\|_{L^q}+\|\nabla P\|_{L^q}+\big\|\nabla^2P\big\|_{L^q}+1\big)\\&\le C\big\|\nabla\big((\rho+m)\dot{u}\big)\big\|_{L^q}+C\big\|\nabla^2P\big\|_{L^q}+C\|\nabla u_t\|_{L^2}+C,\end{aligned}\tag{4.36}$$

which together with $(1.1)_1$ and $(1.1)_2$ gives

$$\begin{aligned}\big(\big\|\nabla^2(\rho+m)\big\|_{L^q}\big)_t&\le C\big(\|\nabla u\|_{L^\infty}+1\big)\big\|\nabla^2(\rho+m)\big\|_{L^q}+C\big\|\nabla^2u\big\|_{W^{1,q}}\\&\le C\big[\big(\|\nabla u\|_{L^\infty}+1\big)\big\|\nabla^2(\rho+m)\big\|_{L^q}\\&\quad+\big\|\nabla\big((\rho+m)\dot{u}\big)\big\|_{L^q}+\|\nabla u_t\|_{L^2}+1\big],\end{aligned}\tag{4.37}$$

and

$$\begin{aligned}\big\|\nabla\big((\rho+m)\dot{u}\big)\big\|_{L^q}&\le C\big\|\nabla(\rho+m)\big\|_{L^q}\|u_t\|_{L^\infty}+C\big\|\nabla(\rho+m)\big\|_{L^q}\|u\|_{L^\infty}\|\nabla u\|_{L^\infty}\\&\quad+C\|\nabla u_t\|_{L^q}+C\big\|\nabla^2u\big\|_{L^q}+C\|\nabla u\|_{H^2}^2\\&\le C\|\nabla u_t\|_{L^2}+C\|\nabla u_t\|_{L^2}^{\frac{6-q}{2q}}\|\nabla u_t\|_{L^6}^{\frac{3q-6}{2q}}+C\|\nabla u\|_{H^2}^2+C\\&\le C\big[\|\nabla u_t\|_{L^2}+\big(\sigma\|\nabla u_t\|_{L^2}^2\big)^{\frac{6-q}{2q}}\big(\sigma\|\nabla u_t\|_{H^1}^2\big)^{\frac{3q-6}{4q}}\sigma^{-\frac12}+\|\nabla u\|_{H^2}^2+1\big]\\&\le C\|\nabla u_t\|_{L^2}+C\big(\sigma\|\nabla u_t\|_{H^1}^2\big)^{\frac{3q-6}{2q}}\sigma^{-\frac12}+C\|\nabla u\|_{H^2}^2+C.\end{aligned}\tag{4.38}$$

Hence, integrating inequality (4.38) over [0,T], by (4.1) and (4.35), we obtain

$$\int_0^T\big\|\nabla\big((\rho+m)\dot{u}\big)\big\|_{L^q}^{p_0}\,dt\le C.\tag{4.39}$$

Applying Gronwall's inequality to (4.37), we deduce from (4.2) and (4.39) that

$$\sup_{0\le t\le T}\left\|\nabla^2(\rho+m)\right\|_{L^q}\le C \tag{4.40}$$

and then

$$\sup_{0\le t\le T}\left(\|P-P_\infty\|_{W^{2,q}}+\|\rho-\rho_\infty\|_{W^{2,q}}+\|m-m_\infty\|_{W^{2,q}}\right)\le C.$$

It follows from (4.36), (4.39), (4.40), and (4.10) that

$$\int_0^T\left\|\nabla^2 u\right\|_{W^{1,q}}^{p_0}\,dt\le C.$$

We finish the proof of Lemma 4.4. □

Lemma 4.5 *There exists a positive constant C such that*

$$\sup_{0\le t\le T}\sigma^2\left(\|\nabla u_t\|_{H^1}^2+\|\nabla u\|_{W^{2,q}}^2\right)+\int_0^T\sigma^2\|\nabla u_{tt}\|_{L^2}^2\,dt\le C, \tag{4.41}$$

for any $q\in(3,6)$.

Proof Differentiating $(1.1)_3$ with respect to t twice gives

$$\begin{aligned}&(\rho+m)_{tt}u_t+2(\rho+m)_t u_{tt}+(\rho+m)u_{ttt}\\&\quad+\left[(\rho+m)_t u\cdot\nabla u+(\rho+m)(u_t\cdot\nabla u+u\cdot\nabla u_t)\right]_t\\&\quad-(2\mu+\lambda)\nabla\operatorname{div}u_{tt}+\mu\nabla\times\operatorname{curl}u_{tt}+\nabla P_{tt}=0.\end{aligned} \tag{4.42}$$

Then, multiplying (4.42) by u_{tt} and integrating over Ω, we conclude that

$$\begin{aligned}&\frac{1}{2}\left(\int(\rho+m)u_{tt}^2\,dx\right)_t+(2\mu+\lambda)\int(\operatorname{div}u_{tt})^2\,dx+\mu\int(\operatorname{curl}u_{tt})^2\,dx\\&\quad=-\int(\rho+m)_{tt}u_t\cdot u_{tt}\,dx-\frac{3}{2}\int(\rho+m)_t u_{tt}^2\,dx-\int\nabla P_{tt}\cdot u_{tt}\,dx\\&\qquad-\int\left((\rho+m)_t u\cdot\nabla u+(\rho+m)u_t\cdot\nabla u+(\rho+m)u\cdot\nabla u_t\right)_t\cdot u_{tt}\,dx\\&\quad=:\sum_{i=1}^4 J_i.\end{aligned} \tag{4.43}$$

Now, we estimate all terms on the right-hand side of (4.43). First, by (4.2), (4.10), and (4.15), it holds

$$\begin{aligned}&J_1+J_2+J_3\\&\quad=-\int(\rho+m)_{tt}u_t\cdot u_{tt}\,dx-\frac{3}{2}\int(\rho+m)_t|u_{tt}|^2\,dx-\int\nabla P_{tt}\cdot u_{tt}\,dx\end{aligned}$$

$$
\begin{aligned}
&= \int \operatorname{div}\big((\rho+m)u\big)_t u_t \cdot u_{tt}\,dx + \frac{3}{2}\int \operatorname{div}\big((\rho+m)u\big)|u_{tt}|^2\,dx + \int P_{tt}\operatorname{div} u_{tt}\,dx \\
&\le -\int \big((\rho+m)u\big)_t \cdot \nabla u_t \cdot u_{tt}\,dx - \int \big((\rho+m)u\big)_t \cdot \nabla u_{tt}\cdot u_t\,dx \\
&\quad - \frac{3}{2}\int \big((\rho+m)u\big)\cdot \nabla u_{tt}\cdot u_{tt}\,dx + C\|P_{tt}\|_{L^2}^2 + \delta\|\nabla u_{tt}\|_{L^2}^2 \\
&\le C\big\|(\rho+m)_t\big\|_{L^6}\|u\|_{L^6}\big(\|\nabla u_t\|_{L^2}\|u_{tt}\|_{L^6} + \|\nabla u_{tt}\|_{L^2}\|u_t\|_{L^6}\big) \\
&\quad + \|\nabla u_{tt}\|_{L^2}\big\|(\rho+m)u_{tt}\big\|_{L^2}\|u\|_{L^\infty} \\
&\quad + C\big(\|\nabla u_t\|_{L^2}\|u_{tt}\|_{L^6} + \|\nabla u_{tt}\|_{L^2}\|u_t\|_{L^6}\big)\big\|(\rho+m)u_t\big\|_{L^3} + C\|P_{tt}\|_{L^2}^2 + \delta\|\nabla u_{tt}\|_{L^2}^2 \\
&\le C\big(\|\nabla u_t\|_{L^2}^2 + \big\|(\rho+m)^{\frac{1}{2}}u_{tt}\big\|_{L^2}^2 + \|\nabla u_t\|_{L^2}^4 + \|P_{tt}\|_{L^2}^2\big) + \delta\|\nabla u_{tt}\|_{L^2}^2, \qquad (4.44)
\end{aligned}
$$

and

$$
\begin{aligned}
J_4 &= -\int \big((\rho+m)_t u\cdot\nabla u + (\rho+m)u_t\cdot\nabla u + (\rho+m)u\cdot\nabla u_t\big)_t \cdot u_{tt}\,dx \\
&= -\int (\rho+m)_{tt} u\cdot\nabla u\cdot u_{tt}\,dx - 2\int (\rho+m)_t u_t\cdot\nabla u\cdot u_{tt}\,dx \\
&\quad - 2\int (\rho+m)_t u\cdot\nabla u_t\cdot u_{tt}\,dx - \int (\rho+m)u_{tt}\cdot\nabla u\cdot u_{tt}\,dx \\
&\quad - 2\int (\rho+m)u_t\cdot\nabla u_t\cdot u_{tt}\,dx - \int (\rho+m)u\cdot\nabla u_{tt}\cdot u_{tt}\,dx \\
&\le C\big\|(\rho+m)_{tt}\big\|_{L^2}\|\nabla u\|_{L^3}\|u_{tt}\|_{L^6}\|u\|_{L^\infty} + C\big\|(\rho+m)_t\big\|_{L^3}\|u_t\|_{L^6}\|\nabla u\|_{L^3}\|u_{tt}\|_{L^6} \\
&\quad + C\big\|(\rho+m)_t\big\|_{L^3}\|\nabla u_t\|_{L^2}\|u_{tt}\|_{L^6}\|u\|_{L^\infty} + C\big\|(\rho+m)^{\frac{1}{2}}u_{tt}\big\|_{L^2}\|\nabla u\|_{L^3}\|u_{tt}\|_{L^6} \\
&\quad + C\big\|(\rho+m)u_t\big\|_{L^3}\|\nabla u_t\|_{L^2}\|u_{tt}\|_{L^6} + C\|\nabla u_{tt}\|_{L^2}\big\|(\rho+m)^{\frac{1}{2}}u_{tt}\big\|_{L^2}\|u\|_{L^\infty} \\
&\le \delta\|\nabla u_{tt}\|_{L^2}^2 + C\big\|(\rho+m)_{tt}\big\|_{L^2}^2 + C\big\|(\rho+m)^{\frac{1}{2}}u_{tt}\big\|_{L^2}^2 \\
&\quad + C\|\nabla u_t\|_{L^2}^2\big(\|\nabla u_t\|_{L^2}^2 + 1\big). \qquad (4.45)
\end{aligned}
$$

Due to the fact that

$$
\|\nabla u_{tt}\|_{L^2} \le C\big(\|\operatorname{div} u_{tt}\|_{L^2} + \|\operatorname{curl} u_{tt}\|_{L^2}\big). \qquad (4.46)
$$

Using (4.43)–(4.46) and choosing enough small δ, we obtain

$$
\begin{aligned}
&\left(\int (\rho+m)|u_{tt}|^2\,dx\right)_t + \|\nabla u_{tt}\|_{L^2}^2 \\
&\quad \le C\big\|(\rho+m)_{tt}\big\|_{L^2}^2 + C\big\|(\rho+m)^{\frac{1}{2}}u_{tt}\big\|_{L^2}^2 \\
&\qquad + C\|P_{tt}\|_{L^2}^2 + C\|\nabla u_t\|_{L^2}^2\big(\|\nabla u_t\|_{L^2}^2 + 1\big), \qquad (4.47)
\end{aligned}
$$

which together with (4.10), (4.15), and (4.16) gives that

$$
\sup_{0\le t\le T}\sigma^2\big\|(\rho+m)^{\frac{1}{2}}u_{tt}\big\|_{L^2}^2 + \int_0^T \sigma^2\|\nabla u_{tt}\|_{L^2}^2\,dt \le C. \qquad (4.48)
$$

Furthermore, it follows from (4.34), (4.16), and (4.48) that

$$\sup_{0\le t\le T}\sigma^2\|\nabla u_t\|_{H^1}^2\le C. \tag{4.49}$$

Finally, combining (4.36) with (4.38), (4.16), (4.30), and (4.33) that

$$\sup_{0\le t\le T}\sigma^2\|\nabla u\|_{W^{2,q}}^2\le C, \tag{4.50}$$

which together with (4.48) and (4.49) gives (4.41), and this completes the proof of Lemma 4.5. □

5 Proofs of Theorems 1.1 and 1.2

With the a priori proof in Sect. 3 and Sect. 4 at hand, we prove the main results of this paper in this section.

Proof of Theorem 1.1 By Lemma 2.1, the problem (1.1)–(1.5) has a unique local classical solution (ρ,m,u) on $\Omega\times(0,T_*]$ for some $T_*>0$. Now, we will extend the classical solution (ρ,m,u) globally in time.

First, by (3.2) and (3.3), it is easy to check that

$$A_1(0)+A_2(0)=0,\qquad 0\le\rho_0+m_0\le\bar\rho+\bar m,\qquad A_3(0)\le M.$$

Then, there exists a $T_1\in(0,T_*]$ such that

$$0\le\rho_0+m_0\le 2(\bar\rho+\bar m),\qquad A_1(T_1)+A_2(T_1)\le 2C_0^{\frac12},\qquad A_3\big(\sigma(T_1)\big)\le 2M. \tag{5.1}$$

Set

$$T^*=\sup\big\{T\mid(5.1)\text{ holds}\big\}. \tag{5.2}$$

Clearly, $0<T_1\le T^*$. And for any $0<\tau<T\le T^*$, one deduces from Lemmas 4.3–4.5 that

$$\begin{cases}\rho-\rho_\infty\in C([0,T];W^{2,q}),\\ m-m_\infty\in C([0,T];W^{2,q}),\\ \nabla u_t\in C([\tau,T];L^q),\\ \nabla u,\nabla^2u\in C([\tau,T];C(\bar\Omega)),\end{cases} \tag{5.3}$$

where one has taken advantage of the standard embedding:

$$L^\infty\big(\tau,T;H^1\big)\cap H^1\big(\tau,T;H^{-1}\big)\hookrightarrow C\big([\tau,T];L^q\big),\quad\text{for any } q\in[2,6).$$

This particularly yields

$$(\rho+m)^{\frac12}u_t,(\rho+m)^{\frac12}\dot u\in C\big([\tau,T];L^2\big). \tag{5.4}$$

Next, we claim that

$$T^* = \infty. \tag{5.5}$$

Otherwise, $T^* < \infty$. By Proposition 3.1, it holds that

$$0 \le \rho + m \le \frac{7}{4}(\bar{\rho} + \bar{m}), \qquad A_1\big(T^*\big) + A_2\big(T^*\big) \le C_0^{\frac{1}{2}}, \qquad A_3\big(\sigma\big(T^*\big)\big) \le M. \tag{5.6}$$

We deduce from Lemma 4.4, Lemma 4.5 and (5.4) that $(\rho(x,T^*), m(x,T^*), u(x,T^*))$ satisfy the initial data condition (1.7)–(1.10), where $g(x) \triangleq (\rho + m)^{\frac{1}{2}} \dot{u}(x,T^*)$, $x \in \Omega$. Hence, Lemma 2.1 shows that there is a $T^{**} > T^*$, such that (5.1) holds for $T = T^{**}$, which contradicts the definition of T^*.

Using Lemma 2.1, Lemma 4.4, Lemma 4.5 and (5.3) indicates that (ρ, m, u) is the unique classical solution defined on $\Omega \times (0,T]$ for any $0 < T < T^* = \infty$.

Finally, to finish the proof of Theorem 1.1, it remains to prove (1.14). It is easy to have

$$(P - P_\infty)_t + u \cdot \nabla P + \gamma \rho^\gamma \operatorname{div} u + \alpha m^\alpha \operatorname{div} u = 0. \tag{5.7}$$

Multiplying (5.7) by $4(P - P_\infty)^3$, one has

$$\big(\|P - P_\infty\|_{L^4}^4\big)_t \le C\|\operatorname{div} u\|_{L^2}^2 + C\|P - P_\infty\|_{L^4}^4,$$

which together with (3.7) and (3.32) yields that

$$\int_1^\infty \big(\|P - P_\infty\|_{L^4}^4\big)_t \, dt \le C. \tag{5.8}$$

Combining (3.32) with (5.8) leads to

$$\lim_{t\to\infty} \|P - P_\infty\|_{L^4}^4 = 0. \tag{5.9}$$

For $2 < q < \infty$, by (5.9), we get

$$\lim_{t\to\infty} \|P - P_\infty\|_{L^q} = 0. \tag{5.10}$$

Notice that (3.7) imply

$$\int (\rho + m)^{\frac{1}{2}} |u|^4 \, dx \le \left(\int (\rho + m)|u|^2 \, dx \right)^{\frac{1}{2}} \|u\|_{L^6}^3 \le C\|\nabla u\|_{L^2}^3. \tag{5.11}$$

Thus, (1.14) follows provided that

$$\lim_{t\to\infty} \|\nabla u\|_{L^2} = 0. \tag{5.12}$$

Choosing $h = 0$ in (3.19), integrating it over $(1, \infty)$ and using (2.16), (3.5), (3.7), and (3.32), we get

$$\begin{aligned}
&\int_1^\infty \left|\phi'(t)\right|^2 dt \\
&\leq C\int_1^\infty \left(\|\nabla u\|_{L^2}^2 + \|\nabla u\|_{L^2}^4 + \|\nabla u\|_{L^3}^3\right) dt + C\|\nabla u\|_{L^2}^2 + C\|P - P_\infty\|_{L^2}^2 \\
&\leq C\int_1^\infty \left(\|\nabla u\|_{L^2}^2 + \|\nabla u\|_{L^2}^4 + \|\nabla u\|_{L^2}^3 + \|\nabla u\|_{L^2}^{\frac{3}{2}}\|P - P_\infty\|_{L^4} + \left\|(\rho + m)\dot{u}\right\|_{L^2}^3\right) dt + C \\
&\leq C,
\end{aligned} \tag{5.13}$$

where $\phi(t) = \frac{\lambda+2\mu}{2}\|\operatorname{div} u\|_{L^2}^2 + \frac{\mu}{2}\|\operatorname{curl} u\|_{L^2}^2$. By (3.7), we obtain that

$$\int_1^\infty \|\nabla u\|_{L^2}^2\, dt \leq \int_0^\infty \|\nabla u\|_{L^2}^2\, dt \leq C,$$

which together with (5.13) yields (5.12). □

Proof of Theorem 1.2 Now, we will prove Theorem 1.2 by contradiction. Suppose that there exists some constant $C_1 > 0$ and a subsequence $\{t_{n_j}\}_{j=1}^\infty$ with $t_{n_j} \to \infty$ as $j \to \infty$, such that $\|\nabla P(\cdot, t_{n_j})\|_{L^r} \leq C_1$. Thanks to (2.2), for $a = 3r/(3r + 4(r - 3)) \in (0, 1)$, it holds that

$$\begin{aligned}
\left\|P(x, t_{n_j}) - P_\infty\right\|_{C(\bar{\Omega})} &\leq C\left\|\nabla P(x, t_{n_j})\right\|_{L^r}^a \left\|P(x, t_{n_j}) - P_\infty\right\|_{L^4}^{1-a} \\
&\leq CC_1^a\left\|P(x, t_{n_j}) - P_\infty\right\|_{L^4}^{1-a},
\end{aligned} \tag{5.14}$$

which together with (1.14) yields that

$$\left\|P(x, t_{n_j}) - P_\infty\right\|_{C(\bar{\Omega})} \to 0 \quad \text{as } t_{n_j} \to \infty. \tag{5.15}$$

On the other hand, since (ρ, m, u) is a classical solution satisfying (1.1), there exists a unique particle path $x_0(t)$ with $x_0(t) = x_0$ such that

$$P\big(x_0(t), t\big) \equiv 0 \quad \text{for all } t > 0.$$

Hence, we have

$$\left\|P(x, t_{n_j}) - P_\infty\right\|_{C(\bar{\Omega})} \geq \left|P\big(x_0(t_{n_j}), t_{n_j}\big) - P_\infty\right| \equiv P_\infty > 0,$$

which contradicts (5.15). Then, we get the desired result (1.15). Hence, we complete the proof of Theorem 1.2. □

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References
1. Aramaki, J.: L^p theory for the div-curl system. Int. J. Math. Anal. **8**, 259–271 (2014)
2. Barrett, J.W., Lu, Y., Süli, E.: Existence of large-data finite-energy global weak solutions to a compressible Oldroyd-B model. Commun. Math. Sci. **15**, 1265–1323 (2017)
3. Beale, J.T., Kato, T., Majda, A.: Remarks on the breakdown of smooth solutions for the 3-D Euler equations. Commun. Math. Phys. **94**, 61–66 (1984)
4. Cai, G., Huang, B., Shi, X.: On compressible Navier–Stokes equations subject to large potential forces with slip boundary conditions in 3D bounded domains (2021). arXiv:2102.12572
5. Cai, G., Li, J.: Existence and exponential growth of global classical solutions to the compressible Navier–Stokes equations with slip boundary conditions in 3D bounded domains (2021). arXiv:2102.06348
6. Cai, G., Li, J., Lü, B.: Global classical solutions to the compressible Navier–Stokes equations with slip boundary conditions in 3D exterior domains (2021). arXiv:2112.05586
7. Carrillo, J.A., Goudon, T.: Stability and asymptotic analysis of a fluid-particle interaction model. Commun. Partial Differ. Equ. **31**, 1349–1379 (2006)
8. Constantin, P., Foias, C.: Navier–Stokes Equations. Chicago Lectures in Mathematics. University of Chicago Press, Chicago (1988)
9. Evje, S.: Weak solutions for a gas-liquid model relevant for describing gas-kick in oil wells. SIAM J. Math. Anal. **43**, 1887–1922 (2011)
10. Evje, S., Karlsen, K.: Global existence of weak solutions for a viscous two-phase model. J. Differ. Equ. **245**, 2660–2703 (2008)
11. Evje, S., Wen, H., Zhu, C.: On global solutions to the viscous liquid-gas model with unconstrained transition to single-phase flow. Math. Models Methods Appl. Sci. **27**, 323–346 (2017)
12. Fan, X., Li, J.: Global classical solutions to 3D compressible Navier–Stokes system with vacuum in bounded domains under non-slip boundary conditions (2021). arXiv:2112.13708
13. Gao, X., Guo, Z., Li, Z.: Global strong solution to the Cauchy problem of 1D viscous two-fluid model without any domination condition. Dyn. Partial Differ. Equ. **19**, 51–70 (2022)
14. Guo, Z., Yang, J., Yao, L.: Global strong solution for a three-dimensional viscous liquid-gas two-phase flow model with vacuum. J. Math. Phys. **52**, 243–275 (2011)
15. Hao, C., Li, H.: Well-posedness for a multi-dimensional viscous liquid-gas two-phase flow model. SIAM J. Math. Anal. **44**, 1304–1332 (2011)
16. Hoff, D.: Global solutions of the Navier–Stokes equations for multidimensional compressible flow with discontinuous initial data. J. Differ. Equ. **120**, 215–254 (1995)
17. Huang, X.: On local strong and classical solutions to the three-dimensional barotropic compressible Navier–Stokes equations with vacuum. Sci. China Math. **64**, 1771–1788 (2021)
18. Huang, X., Li, J., Xin, Z.: Global well-posedness of classical solutions with large oscillations and vacuum to the three-dimensional isentropic compressible Navier–Stokes equations. Commun. Pure Appl. Math. **65**, 549–585 (2012)
19. Ishii, M.: Thermo-fluid dynamic theory of two-phase flow. NASA Sti/Recon Tech. Rep. A **75**, 29657 (1975)
20. Ishii, M.: One-dimensional drift-flux model and constitutive equations for relative motion between phases in various two-phase flow regimes. Argonne National Lab., Ill. (USA) (1977)
21. Itoh, S., Tanaka, N., Tani, A.: The initial value problem for the Navier–Stokes equations with general slip boundary condition in Hölder spaces. J. Math. Fluid Mech. **5**, 275–301 (2003)
22. Kato, T.: Remarks on the Euler and Navier–Stokes equations in $\mathbb{R}^2$. In: Nonlinear Functional Analysis and Its Applications. Part 2 (Berkeley, Calif., 1983). Proc. Sympos. Pure Math., vol. 45, pp. 1–7. Am. Math. Soc., Providence (1986)
23. Li, J., Xin, Z.: Global existence of regular solutions with large oscillations and vacuum. In: Handbook of Mathematical Analysis in Mechanics of Viscous Fluids, pp. 2037–2083. Springer, Cham (2018)
24. Louati, H., Meslameni, M., Razafison, U.: Weighted L^p-theory for vector potential operators in three-dimensional exterior domains. Math. Methods Appl. Sci. **39**, 1990–2010 (2016)
25. Matsumura, A., Nishida, T.: The initial value problem for the equations of motion of viscous and heat-conductive gases. J. Math. Kyoto Univ. **20**, 67–104 (1980)

26. Mellet, A., Vasseur, A.: Asymptotic analysis for a Vlasov–Fokker–Planck/compressible Navier–Stokes system of equations. Commun. Math. Phys. **281**, 573–596 (2008)
27. Navier, C.: Sur les lois de l'équilibre et du mouvement des corps élastiques. Mem. Acad. R. Sci. Inst. Fr. **6**, 369 (1827)
28. Nirenberg, L.: On elliptic partial differential equations. In: Il principio di minimo e sue applicazioni alle equazioni funzionali, pp. 1–48. Springer, Berlin (2011)
29. Novotný, A., Pokorný, M.: Weak solutions for some compressible multicomponent fluid models. Arch. Ration. Mech. Anal. **235**, 355–403 (2020)
30. Novotny, A., Straskraba, I.: Introduction to the Mathematical Theory of Compressible Flow. Oxford Lecture Series in Mathematics and Its Applications, vol. 27. Oxford University Press, Oxford (2004)
31. Serrin, J.: Mathematical principles of classical fluid mechanics. In: Truesdell, C. (ed.) Fluid Dynamics I / Strömungsmechanik I. Encyclopedia of Physics / Handbuch der Physik, vol. 3 / 8 / 1. Springer, Berlin (1959)
32. Vasseur, A., Wen, H., Yu, C.: Global weak solution to the viscous two-fluid model with finite energy. J. Math. Pures Appl. **125**, 247–282 (2019)
33. Von Wahl, W.: Estimating ∇u by $\operatorname{div} u$ and $\operatorname{curl} u$. Math. Methods Appl. Sci. **15**, 123–143 (1992)
34. Wallis, G.B.: One-dimensional two-fluid flow (1979)
35. Wen, H.: On global solutions to a viscous compressible two-fluid model with unconstrained transition to single-phase flow in three dimensions. Calc. Var. Partial Differ. Equ. **60**, 158 (2021)
36. Yao, L., Zhang, T., Zhu, C.: Existence and asymptotic behavior of global weak solutions to a 2D viscous liquid-gas two-phase flow model. SIAM J. Math. Anal. **42**, 1874–1897 (2010)
37. Yao, L., Zhu, C.: Free boundary value problem for a viscous two-phase model with mass-dependent viscosity. J. Differ. Equ. **247**, 2705–2739 (2009)
38. Yao, L., Zhu, C.: Existence and uniqueness of global weak solution to a two-phase flow model with vacuum. Math. Ann. **349**, 903–928 (2011)
39. Yu, H.: Global strong solutions to the 3D viscous liquid-gas two-phase flow model. J. Differ. Equ. **272**, 732–759 (2021)
40. Zhang, Y., Zhu, C.: Global existence and optimal convergence rates for the strong solutions in H^2 to the 3D viscous liquid-gas two-phase flow model. J. Differ. Equ. **258**, 2315–2338 (2015)
41. Zlotnik, A.A.: Uniform estimates and stabilization of symmetric solutions of a system of quasilinear equations. Differ. Equ. **36**, 701–716 (2000)
42. Zuber, N.: On the dispersed two-phase flow in the laminar flow regime. Chem. Eng. Sci. **19**, 897–917 (1964)

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