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Blow-up of solutions to the semilinear wave equation with scale invariant damping on exterior domain

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Abstract

This paper is concerned with the blow-up of solutions to the initial boundary value problem for the wave equation with scale invariant damping term on the exterior domain, where the nonlinear terms are power nonlinearity $|u|^p$, derivative nonlinearity $|u_t|^p$ and combined nonlinearities $|u_t|^p + |u|^q$, respectively. Upper bound lifespan estimates of solutions to the problem are obtained by constructing suitable test functions and utilizing the test function technique. The main novelty is that lifespan estimates of solutions are associated with the well-known Strauss exponent and Glassey exponent. To the best of our knowledge, the results in Theorems 1.1–1.3 are new.

Keywords: Scale invariant damping; Exterior domain; Blow-up; Lifespan estimates; Test function technique

1 Introduction

We consider blow-up dynamics of solutions to the initial boundary value problem for the following damped wave equation on exterior domain

$$\begin{cases} u_{tt} - \Delta u + \frac{\mu}{1+t} u_t = f(u, u_t), & (t,x) \in (0,\infty) \times \Omega^c, \\ u(0,x) = \varepsilon f(x), \quad u_t(0,x) = \varepsilon g(x), & x \in \Omega^c, \\ u(t,x)|_{\partial\Omega} = 0, \quad t \geq 0, \end{cases} \tag{1.1}$$

where $f(u,u_t) = |u|^p, |u_t|^p, |u_t|^p + |u|^q (1 < p, q < \infty)$. μ is a positive constant, $\frac{\mu}{1+t} u_t$ is the scale invariant damping term. $\Omega = B_1(0) = \{x||x| \leq 1\}$ and $\Omega^c = \mathbb{R}^n \setminus B_1(0)$. Let $B_R(0) = \{x||x| \leq R\}, R > 2$. The initial values $f(x)$ and $g(x)$ possess compact supports, which satisfy

$$\big(f(x), g(x)\big) \in H^1\big(\Omega^c\big) \times L^2\big(\Omega^c\big)$$

and

$$\operatorname{supp}\big(f(x), g(x)\big) \subset B_R(0). \tag{1.2}$$

Let us recall several results on the Cauchy problem for nonlinear wave equation

$$\begin{cases} u_{tt} - \Delta u = f(u, u_t), & (t,x) \in (0,\infty) \times \mathbb{R}^n, \\ u(0,x) = \varepsilon f(x), \quad u_t(0,x) = \varepsilon g(x), & x \in \mathbb{R}^n, \end{cases} \tag{1.3}$$

where $f(u,u_t) = |u|^p, |u_t|^p, |u_t|^p + |u|^q$. First, we introduce the related results of problem (1.3) with power-type nonlinear term $f(u,u_t) = |u|^p$. In fact, when $n = 1$, $p_S(1) = \infty$. While $n \geq 2$, $p_S(n)$ is the largest root of the quadratic equation

$$\gamma(p,n) = -(n-1)p^2 + (n+1)p + 2 = 0. \tag{1.4}$$

We say that $p_S(n)$ stands for the Strauss critical exponent, which represents the threshold between the blow-up dynamic of solution and the global existence of solution. Glassey [1] verifies the blow-up of solution to the problem in the dimensions $n = 2$. John [2] proves the non-existence of global solution to the problem for $1 < p < p_S(3) = 1 + \sqrt{2}$ in the case $n = 3$. Zhou [3] investigates the existence of global solution to the Cauchy problem in the dimensions $n = 4$. Blow-up results and lifespan estimates of solution when $n \geq 4$ are considered in [4]. Upper bound lifespan estimate of solution to the small initial value problem can be summarized as

$$T(\varepsilon) \leq \begin{cases} C\varepsilon^{-\frac{2p(p-1)}{\gamma(p,n)}}, & 1 < p < p_S(n), \\ \exp(C\varepsilon^{-p(p-1)}), & p = p_S(n). \end{cases} \tag{1.5}$$

We are in the position to consider problem (1.3) with derivative-type nonlinearity $|u_t|^p$. It has been conjectured that the non-existence of global solution occurs for $p > 1$ when $n = 1$. In addition, there is a critical exponent $p_G(n) = \frac{n+1}{n-1}$ that the solution blows up in finite time if $1 < p < p_G(n)$ $(n \geq 2)$. This is the well-known Glassey conjecture studied by many scholars. John [5] investigates the formation of singularity of solution to the problem when $n = 3$. Masuda [6] proves that the solution to the problem blows up in finite time when $n \leq 3$. Rammaha [7] establishes blow-up results of the problem in the case $n \geq 4$ in the sub-critical case using iteration method. Zhou [8] obtains lifespan estimates of solution for $1 < p < p_G(n) (n \geq 2)$ as well as $1 < p < \infty$ $(n = 1)$. Upper bound lifespan estimate of solution to the problem with small initial values can be summarized as

$$T(\varepsilon) \leq \begin{cases} C\varepsilon^{-(\frac{1}{p-1} - \frac{n-1}{2})^{-1}}, & 1 < p < p_G(n), \\ \exp(C\varepsilon^{-(p-1)}), & p = p_G(n), \\ \infty, & p_G(n) < p < \frac{n}{n-2}. \end{cases} \tag{1.6}$$

The classical wave equation with combined type nonlinear terms $|u_t|^p + |u|^q$ has also been widely discussed. The problem can be regarded as a material combination of power-type nonlinearity $|u|^p$ and derivative-type nonlinearity $|u_t|^p$. When the spatial dimension $n = 1$, Zhou and Han [9] show non-existence of global solution to the problem for $1 < p, q < \infty$. Upper bound lifespan estimate of solution is derived by utilizing test function method. Han and Zhou [10] prove the blow-up result of solution when $(q-1)((n-1)p-2) < 4$. The interested readers may refer to [11–13] for more relevant results.

Recently, the research of semilinear damped wave equations attracts more attention (see detailed illustrations in [14–22]). Let $\tilde{u}$ be a solution for the following linear damped wave equation, namely

$$\begin{cases} \tilde{u}_{tt} - \Delta\tilde{u} + \frac{\mu}{(1+t)^\beta}\tilde{u}_t = 0, & (t,x) \in (0,\infty) \times \mathbb{R}^n, \\ \tilde{u}(0,x) = \varepsilon f(x), \quad \tilde{u}_t(0,x) = \varepsilon g(x), & x \in \mathbb{R}^n, \end{cases} \tag{1.7}$$

where $\mu > 0$, $\beta \in \mathbb{R}, f, g \in C_0^\infty(\mathbb{R}^n)$. We summarize the behaviors of solution as the following four cases.

Scope of β	Corresponding damping	Behavior of solution
$\beta < -1$	Over damping	Solution does not decay to zero
$-1 \le \beta < 1$	Effective	Solution is heat-like
$\beta = 1$	Scaling invariant	Behavior of solution depends on μ
$\beta > 1$	Scattering	Solution is wave-like

The case $\beta < -1$ is corresponding to the over damping. The solution does not decay to zero in this case. Ikeda and Wakasugi [14] verify the existence of global solution for $p > 1$. $-1 \le \beta < 1$ is the effective damping case. The solution behaves like that of heat equation, which indicates that the term $\tilde{u}_{tt}$ has no influence. Lin et al. [23] prove blow-up results of the problem for $1 < p \le p_F(n)$, where the Fujita exponent $p_F(n) = 1 + \frac{2}{n}$. Ikeda and D'Abbicco [24, 25] obtain the precise lifespan estimates of solution to the problem. $\beta = 1$ is corresponding to the scale invariant case. The equation is an intermediate situation between wave- and heat-like. In this case, behavior of solution is determined by the value of μ, which provides a threshold between the effective and non-effective damping. Fujiwara et al. [26] show blow-up result and lifespan estimate of solution in the critical case. $\beta > 1$ is the scattering damping case. The solution behaves like that of wave equation. In this case, the damping term has no influence. Lai and Takamura [27] derive the blow-up of solution when $1 < p < p_S(n)$.

Now let us come back to our problem (1.1). D'Abbicco [28] shows the existence of global solution when

$$\mu \ge \begin{cases} \frac{5}{3}, & n = 1, \\ 3, & n = 2, \\ n+2, & n \ge 3. \end{cases} \tag{1.8}$$

On the other hand, Wakasugi [29] proves the blow-up result when $1 < p < p_F(n)(\mu \ge 1)$ and $1 < p < p_F(n + \mu - 1)(0 < \mu < 1)$. The lifespan estimate of solution satisfies

$$T(\varepsilon) \le \begin{cases} C\varepsilon^{-\frac{p-1}{2-n(p-1)}}, & 1 < p < p_F(n), \mu \ge 1, \\ C\varepsilon^{-\frac{p-1}{2-(n+\mu-1)(p-1)}}, & 1 < p < 1 + \frac{2}{n+\mu-1}, 0 < \mu < 1. \end{cases} \tag{1.9}$$

Wakasugi [30] illustrates that the behavior of solution is wave-like when $\mu > 1$. Takamura [31] obtains the following lifespan estimate of solution

$$T(\varepsilon) \le C\varepsilon^{-\frac{2p(p-1)}{\gamma(p,n+2\mu)}}, \quad p < p_S(n+2\mu), 0 < \mu < \frac{n^2+n+2}{2(n+2)}. \tag{1.10}$$

We observe that the lifespan estimate in (1.10) is better than that in (1.9) when $n \geq 2$. Notice that $\mu = 2$ is a special case. In general, applying the Liouville transform

$$v(t,x) = (1+t)^{\frac{\mu}{2}} u(t,x),$$

we rewrite problem (1.1) as

$$\begin{cases} v_{tt} - \Delta v + \frac{\mu(2-\mu)}{4-(1+t)^2} v = \frac{|v|^p}{(1+t)^{\frac{\mu(p-1)}{2}}}, & (t,x) \in (0,\infty) \times \mathbb{R}^n, \\ v(0,x) = \varepsilon f(x), \qquad v_t(0,x) = \varepsilon\{\frac{\mu}{2} f(x) + g(x)\}, & x \in \mathbb{R}^n. \end{cases} \tag{1.11}$$

In fact, we expect this exponent to have some relationship with $p_S(n)$. D'Abbicco et al. [32] obtain formation of singularity of solution to the problem. The critical exponent is

$$p_c(n) = \max\bigl\{p_F(n), p_S(n+2)\bigr\}, \quad n \leq 3.$$

Meanwhile, the authors prove existence of global solution when $p > p_c(n)$ ($n = 2,3$) and blow-up of solution when $1 < p \leq p_c(n)$ ($n \geq 1$). D'Abbicco and Lucente [33] obtain the existence of global solution in higher dimensions $n \geq 5$ when $p_S(n+2) < p < 1 + 2(\max\{2, \frac{n-3}{2}\})^{-1}$.

For problem (1.1) with $f(u,u_t) = |u_t|^p$, Lai and Takamura [34] investigate the blow-up result of solution when $1 < p \leq p_G(n+2\mu)$. Palmieri and Tu [35] show the formation of singularity of solution when $1 < p \leq p_G(n+\sigma)$, where

$$\sigma = \begin{cases} 2\mu, & \mu \in [0,1), \\ 2, & \mu \in [1,2), \\ \mu, & \mu \in [2,\infty). \end{cases} \tag{1.12}$$

In addition, Hamouda and Hamza [36] obtain the blow-up dynamics of problem (1.1) with $f(u,u_t) = |u_t|^p + |u|^q$ in $\mathbb{R}^n$ in the case $\gamma(p,q,n+\mu) < 4$, where

$$\gamma(p,q,n) = (q-1)\bigl((n-1)p-2\bigr). \tag{1.13}$$

We refer readers to [35, 37] for more details.

Motivated by the previous works in [9, 27, 34, 38–43], our main purpose is to consider lifespan estimates of solutions to problem (1.1) on the exterior domain. We note that there are several results for the wave equation on the exterior domain. Han and Zhou [9, 39, 40] investigate the blow-up results of semilinear wave equations with the variable coefficient on the exterior domain in different dimensions by utilizing the Kato lemma. Employing the test function technique, we generalize the problems studied in [9, 39, 40] to problem (1.1) with the scale invariant damping in the constant coefficient case. We observe that Lai and Takamura [27, 34, 41] derive the lifespan estimates of solutions to the semilinear damped wave equations in the scattering case ($\frac{\mu}{(1+t)^\beta} u_t, \beta > 1$) with the iteration method, where the nonlinear terms are power nonlinearity $|u|^p$, derivative nonlinearity $|u_t|^p$, and combined nonlinearities $|u_t|^p + |u|^q$, respectively. Lai et al. [42] consider the blow-up result of solution to the semilinear wave equation with the scale invariant damping term ($\frac{\mu}{1+t} u_t$)

when $0 < \mu < \mu_0(n)(n \geq 2)$ with the improved Kato lemma. The novelty in this paper is that we employ the cut-off test function technique ($\Psi = \eta_T^{2p'}\phi_0(x), \eta_T^{2p'}\Phi(t,x)$), which is different from the iteration method and improved Kato lemma in [27, 42] to verify the upper bound lifespan estimate of solution to problem (1.1) with power nonlinearity $|u|^p$ when $\mu > 0(n \geq 1)$. It is worth mentioning that Lai and Tu [43] investigate the semilinear wave equations with the scattering space dependent damping ($\frac{\mu}{(1+|x|)^\beta}u_t, \beta > 2$) by making use of the test function method ($\Psi = \eta_T^{2p'}\Phi(t,x), \partial_t\psi(t,x)$). Upper bound lifespan estimates of solutions to the problem with power nonlinearity $|u|^p$ and derivative nonlinearity $|u_t|^p$ are obtained, respectively. However, we consider the problem (1.1) that contains the scaling invariant damping term ($\frac{\mu}{1+t}u_t$). Furthermore, we derive lifespan estimates of solution to problem (1.1) with combined nonlinear terms $|u_t|^p + |u|^q$. Utilizing the test function method, Chen [38] shows the lifespan estimate of solution to the damped wave equation with derivative nonlinearity $|u_t|^p$ and combined nonlinearities $|u_t|^p + |u|^q$, respectively. We extend the problem in $\mathbb{R}^n$ studied in [38] to the exterior domain. In addition, we establish the blow-up result of solution to the initial boundary value problem (1.1) with the power nonlinearity $|u|^p$. To our best knowledge, the results in Theorems 1.1–1.3 are new.

The main results in this paper are presented as follows.

Theorem 1.1 *Let $p > 1$. Assume that the initial values $f(x), g(x)$ are non-negative functions and do not vanish identically. It holds that*

$$\operatorname{supp} u(t,x) \subset \{x \mid |x| \leq t + R\}.$$

Then the solution of problem (1.1) with $f(u,u_t) = |u|^p$ blows up in a finite time. The upper bound lifespan estimate satisfies

$$T(\varepsilon) \leq \begin{cases} C\varepsilon^{-\frac{2p(p-1)}{\gamma(p,n+\mu)}}, & \gamma(p,n+\mu) > 0, n \geq 3, \\ C\varepsilon^{-\frac{2p(p-1)}{\gamma(p,2+\mu)}}, & \gamma(p,2+\mu) > 0, n = 2, \\ C\varepsilon^{-\frac{2p(p-1)}{-\mu p^2+\mu p+4}}, & -\mu p^2 + \mu p + 4 > 0, n = 1. \end{cases} \tag{1.14}$$

Theorem 1.2 *Assume that the initial values $f(x), g(x)$ are non-negative functions and do not vanish identically. It holds that*

$$\operatorname{supp} u(t,x) \subset \{x \mid |x| \leq t + R\}.$$

Then the solution of problem (1.1) with $f(u,u_t) = |u_t|^p$ blows up in a finite time. The upper bound lifespan estimate satisfies

$$T(\varepsilon) \leq \begin{cases} C\varepsilon^{-\frac{p-1}{1-\frac{(n+\mu-1)(p-1)}{2}}}, & 1 < p < \frac{2}{n+\mu-1} + 1, n \geq 3, \\ \exp(C\varepsilon^{-(p-1)}), & p = \frac{2}{n+\mu-1} + 1, n \geq 3, \\ C\varepsilon^{-\frac{p-1}{1-\frac{(3+\mu)(p-1)}{2}}}, & 1 < p < \frac{2}{3+\mu} + 1, n = 2, \\ \exp(C\varepsilon^{-(p-1)}), & p = \frac{2}{3+\mu} + 1, n = 2, \\ C\varepsilon^{-\frac{p-1}{1-\frac{(2+\mu)(p-1)}{2}}}, & 1 < p < \frac{2}{2+\mu} + 1, n = 1, \\ \exp(C\varepsilon^{-(p-1)}), & p = \frac{2}{2+\mu} + 1, n = 1. \end{cases} \tag{1.15}$$

Theorem 1.3 *Let $p, q > 1$. Assume that the initial values $f(x), g(x)$ are non-negative functions and do not vanish identically. It holds that*

$$\operatorname{supp} u(t,x) \subset \{x \mid |x| \le t + R\}.$$

Then the solution of problem (1.1) with $f(u, u_t) = |u_t|^p + |u|^q$ blows up in a finite time. The upper bound lifespan estimate satisfies

$$T(\varepsilon) \le \begin{cases} C\varepsilon^{-\frac{2p(q-1)}{4-\gamma(p,q,n+\mu)}}, & \gamma(p,q,n+\mu) < 4, n \ge 3, \\ C\varepsilon^{-\frac{2p(q-1)}{4-\gamma(p,q,2+\mu)}}, & \gamma(p,q,2+\mu) < 4, n = 2, \\ C\varepsilon^{-\frac{2p(q-1)}{4-\mu pq+\mu p}}, & \mu pq - \mu p < 4, n = 1. \end{cases} \tag{1.16}$$

2 Proof of Theorem 1.1

2.1 The case for $n \ge 3$

We present the definition of energy solution and related lemmas.

Definition 2.1 Suppose that u is an energy solution of problem (1.1) on $[0, T)$ if

$$u \in C([0,T), H^1(\Omega^c)) \cap C^1([0,T), L^2(\Omega^c)) \cap L^p_{\mathrm{loc}}(\Omega^c \times (0,T))$$

and

$$\begin{aligned} &\varepsilon \int_{\Omega^c} g(x)\Psi(0,x)\,dx + \varepsilon \int_{\Omega^c} \mu f(x)\Psi(0,x)\,dx + \int_0^T \int_{\Omega^c} f(u,u_t)\Psi(t,x)\,dx\,dt \\ &\quad = -\int_0^T \int_{\Omega^c} u_t(t,x)\Psi_t(t,x)\,dx\,dt + \int_0^T \int_{\Omega^c} \nabla u \nabla \Psi(t,x)\,dx\,dt \\ &\qquad - \int_0^T \int_{\Omega^c} \frac{\mu}{1+t} u(t,x)\Psi_t(t,x)\,dx\,dt \\ &\qquad + \int_0^T \int_{\Omega^c} \frac{\mu}{(1+t)^2} u(t,x)\Psi(t,x)\,dx\,dt, \end{aligned} \tag{2.1}$$

for all $\Psi(t,x) \in C_0^\infty([0,T) \times \Omega^c)$.

Lemma 2.1 ([44]) *There exists a function $\phi_0(x) \in C^2(\Omega^c)$ ($n \ge 3$) satisfying the following boundary value problem*

$$\begin{cases} \Delta\phi_0(x) = 0, & x \in \Omega^c, n \ge 3, \\ \phi_0(x)|_{\partial\Omega} = 0, \\ |x| \to \infty, & \phi_0(x) \to \infty. \end{cases} \tag{2.2}$$

Moreover, for all $x \in \Omega^c$, it holds that $0 < \phi_0(x) < 1$.

We introduce the following ordinary differential equation

$$\lambda''(t) - \frac{\mu}{1+t}\lambda'(t) - \lambda(t) = 0. \tag{2.3}$$

Lemma 2.2 ([38]) *The ODE* (2.3) *admits one solution*

$$\lambda(t) = (1+t)^{\frac{\mu+1}{2}} K_{\frac{\mu+1}{2}}(1+t),$$

where $K_\nu(z)$ is the second kind modified Bessel function. In particular, $\lambda(t)$ is a real and positive function satisfying

$$\lambda(0) = K_{\frac{\mu+1}{2}}(1) > 0, \qquad \lambda'(0) = -K_{\frac{\mu-1}{2}}(1) < 0, \qquad \lambda'(t) < 0.$$

For large t, it holds that

$$\lambda(t) = \frac{1}{e}\sqrt{\frac{\pi}{2}}(1+t)^{\frac{\mu}{2}} e^{-t} \times \left(1 + O\left(\frac{1}{1+t}\right)\right) = -\lambda'(t). \tag{2.4}$$

Lemma 2.3 ([9]) *There exists a function $\phi_1(x) \in C^2(\Omega^c)$ $(n \geq 1)$ satisfying the following boundary value problem*

$$\begin{cases} \Delta\phi_1(x) = \phi_1(x), & x \in \Omega^c, n \geq 1, \\ \phi_1(x)|_{\partial\Omega} = 0, \\ |x| \to \infty, & \phi_1(x) \to \int_{\mathbb{S}^{n-1}} e^{x\omega}\, d\omega. \end{cases} \tag{2.5}$$

Moreover, there exists a positive constant C, such that $0 < \phi_1(x) \leq C(1+|x|)^{-\frac{n-1}{2}} e^{|x|}$ for all $x \in \Omega^c$.

We define the test function

$$\Phi(t,x) = \lambda(t)\phi_1(x). \tag{2.6}$$

Lemma 2.4 *Let $p > 1$. For all $t \geq 0$, it holds that*

$$\int_{\Omega^c \cap \{|x| \leq t+R\}} \left[\Phi(t,x)\right]^{\frac{p}{p-1}}\, dx \leq C\lambda(t)^{p'}(t+R)^{n-1-\frac{(n-1)p'}{2}} e^{p't},$$

where $p' = \frac{p}{p-1}$, and C is a positive constant.

Proof Using Lemma 2.3, we have

$$\begin{aligned} &\int_{\Omega^c \cap \{|x| \leq t+R\}} \left[\Phi(t,x)\right]^{\frac{p}{p-1}}\, dx \\ &\quad = \int_{\Omega^c \cap \{|x| \leq t+R\}} \left[\lambda(t)\phi_1(x)\right]^{\frac{p}{p-1}}\, dx \\ &\quad \leq C \operatorname{area}\left(\mathbb{S}^{n-1}\right)\lambda(t)^{p'} \int_0^{t+R} (1+r)^{n-1-\frac{(n-1)p'}{2}} e^{p'r}\, dr \\ &\quad \leq C\lambda(t)^{p'}(t+R)^{n-1-\frac{(n-1)p'}{2}} e^{p't}. \end{aligned} \tag{2.7}$$

This completes the proof of Lemma 2.4. □

Lemma 2.5 *Let $p > 1$. For all $t \geq 0$, it holds that*

$$\int_0^T \int_{\Omega^c \cap \{|x| \leq t+R\}} \big[\Phi(t,x)\big]^{\frac{p}{p-1}} \phi_0(x)^{-\frac{1}{p-1}}\,dx\,dt \leq CT^{\frac{\mu p'}{2}+n-\frac{(n-1)p'}{2}},$$

where $p' = \frac{p}{p-1}$, and C is a positive constant.

Proof The direct computation shows

$$\begin{aligned}
&\int_{\Omega^c \cap \{|x| \leq t+R\}} \big[\Phi(t,x)\big]^{\frac{p}{p-1}} \phi_0(x)^{-\frac{1}{p-1}}\,dx \\
&\quad = \int_{\Omega^c \cap B_R(0)} \big[\Phi(t,x)\big]^{\frac{p}{p-1}} \phi_0(x)^{-\frac{1}{p-1}}\,dx \\
&\qquad + \int_{(\Omega^c \setminus B_R(0)) \cap \{|x| \leq t+R\}} \big[\Phi(t,x)\big]^{\frac{p}{p-1}} \phi_0(x)^{-\frac{1}{p-1}}\,dx \\
&\quad = M_1(t) + M_2(t).
\end{aligned} \tag{2.8}$$

We bear in mind $0 < \phi_0(x) < 1$ for all $x \in \Omega^c$. There exists a constant $C \in (0,1)$ such that $\phi_0(x) \geq C$ when $x \in (\Omega^c \setminus B_R(0)) \cap \{|x| \leq t+R\}$. Making use of Lemma 2.4, we acquire

$$\begin{aligned}
M_2(t) &= \int_{(\Omega^c \setminus B_R(0)) \cap \{|x| \leq t+R\}} \big[\Phi(t,x)\big]^{\frac{p}{p-1}} \phi_0(x)^{-\frac{1}{p-1}}\,dx \\
&\leq C\lambda(t)^{p'}(t+R)^{n-1-\frac{(n-1)p'}{2}} e^{p't}.
\end{aligned} \tag{2.9}$$

Taking advantage of Lemma 2.5 in [9], we have

$$M_1(t) \leq C\lambda(t)^{p'} \int_{\Omega^c \cap B_R(0)} \operatorname{dist}(x,\partial\Omega)\,dx \leq C\lambda(t)^{p'}. \tag{2.10}$$

We conclude

$$\begin{aligned}
&\int_{\Omega^c \cap \{|x| \leq t+R\}} \big[\Phi(t,x)\big]^{\frac{p}{p-1}} \phi_0(x)^{-\frac{1}{p-1}}\,dx \\
&\quad \leq C\lambda(t)^{p'}(t+R)^{n-1-\frac{(n-1)p'}{2}} e^{p't}.
\end{aligned} \tag{2.11}$$

Utilizing (2.4), we have

$$\begin{aligned}
\int_0^T \int_{\Omega^c} \big[\Phi(t,x)\big]^{\frac{p}{p-1}} \phi_0(x)^{-\frac{1}{p-1}}\,dx\,dt &\leq C\int_{T/2}^T \lambda(t)^{p'} e^{p't}(t+R)^{n-1-\frac{(n-1)p'}{2}}\,dt \\
&\leq CT^{\frac{\mu p'}{2}+n-\frac{(n-1)p'}{2}}.
\end{aligned} \tag{2.12}$$

This proves Lemma 2.5. □

Proof of Theorem 1.1 Let $\eta(t) \in C^\infty([0,\infty))$ satisfy

$$\eta(t) = \begin{cases} 1, & t \le \frac{1}{2}, \\ \text{decreasing}, & \frac{1}{2} < t < 1, \\ 0, & t \ge 1 \end{cases} \tag{2.13}$$

and

$$|\eta'(t)| \le C, \qquad |\eta''(t)| \le C.$$

Let $\eta_T(t) = \eta(\frac{t}{T})$. Choosing $\Psi(t,x) = \eta_T^{2p'}(t)\phi_0(x)$ in (2.1) with $f(u,u_t) = |u|^p$ and integrating by parts, we obtain

$$\begin{aligned} &\varepsilon \int_{\Omega^c} g(x)\phi_0(x)\,dx + \varepsilon \int_{\Omega^c} \mu f(x)\phi_0(x)\,dx + \int_0^T \int_{\Omega^c} |u|^p \eta_T^{2p'} \phi_0(x)\,dx\,dt \\ &\quad = \int_0^T \int_{\Omega^c} u \partial_t^2 \eta_T^{2p'} \phi_0(x)\,dx\,dt - \int_0^T \int_{\Omega^c} \frac{\mu}{1+t} u \partial_t \eta_T^{2p'} \phi_0(x)\,dx\,dt \\ &\qquad + \int_0^T \int_{\Omega^c} \frac{\mu}{(1+t)^2} u \eta_T^{2p'} \phi_0(x)\,dx\,dt \\ &\quad = I_1 + I_2 + I_3. \end{aligned} \tag{2.14}$$

Noting that

$$\begin{aligned} \partial_t \eta_T^{2p'} &= \frac{2p'}{T} \eta_T^{2p'-1} \eta', \\ \partial_t^2 \eta_T^{2p'} &= \frac{2p'(2p'-1)}{T^2} \eta_T^{2p'-2} |\eta'|^2 + \frac{2p'}{T^2} \eta_T^{2p'-1} \eta'', \end{aligned}$$

we derive

$$\begin{aligned} |I_1| &\le CT^{-2} \int_0^T \int_{\Omega^c} \left| u \eta_T^{2p'-2} \phi_0(x) \right| dx\,dt \\ &\le CT^{-2} \left(\int_0^T \int_{\Omega^c} |u|^p \eta_T^{2p'} \phi_0(x)\,dx\,dt \right)^{\frac{1}{p}} \left(\int_0^T \int_{\Omega^c} \phi_0(x)\,dx\,dt \right)^{\frac{1}{p'}} \\ &\le CT^{n+1-2p'} + \frac{1}{4} \int_0^T \int_{\Omega^c} |u|^p \eta_T^{2p'} \phi_0(x)\,dx\,dt, \end{aligned} \tag{2.15}$$

$$\begin{aligned} |I_2| &\le CT^{-1} \int_0^T \int_{\Omega^c} \left| \frac{\mu}{1+t} u \eta_T^{2p'-1} \phi_0(x) \right| dx\,dt \\ &\le CT^{-1} \left(\int_0^T \int_{\Omega^c} |u|^p \eta_T^{2p'} \phi_0(x)\,dx\,dt \right)^{\frac{1}{p}} \left(\int_0^T \int_{\Omega^c} \left(\frac{\mu}{1+t} \right)^{p'} \phi_0(x)\,dx\,dt \right)^{\frac{1}{p'}} \\ &\le CT^{-1} \left(\int_0^T t^{n-p'}\,dt \right)^{\frac{1}{p'}} \left(\int_0^T \int_{\Omega^c} |u|^p \eta_T^{2p'} \phi_0(x)\,dx\,dt \right)^{\frac{1}{p}} \\ &\le CT^{n+1-2p'} + \frac{1}{4} \int_0^T \int_{\Omega^c} |u|^p \eta_T^{2p'} \phi_0(x)\,dx\,dt \end{aligned} \tag{2.16}$$

and

$$\begin{aligned}|I_3| &\le \int_0^T\int_{\Omega^c}\left|\frac{\mu}{(1+t)^2}u\eta_T^{2p'}\phi_0(x)\right|dx\,dt\\&\le C\left(\int_0^T\int_{\Omega^c}|u|^p\eta_T^{2p'}\phi_0(x)\,dx\,dt\right)^{\frac{1}{p}}\left(\int_0^T\int_{\Omega^c}\left[\frac{\mu}{(1+t)^2}\right]^{p'}\phi_0(x)\,dx\,dt\right)^{\frac{1}{p'}}\\&\le CT^{n+1-2p'}+\frac{1}{4}\int_0^T\int_{\Omega^c}|u|^p\eta_T^{2p'}\phi_0(x)\,dx\,dt.\end{aligned}\tag{2.17}$$

From (2.14), (2.15), (2.16), and (2.17), we obtain

$$C_1(f,g)\varepsilon+\int_0^T\int_{\Omega^c}|u|^p\eta_T^{2p'}\phi_0(x)\,dx\,dt\le CT^{n+1-2p'},\tag{2.18}$$

where

$$C_1(f,g)=C\left(\int_{\Omega^c}g(x)\phi_0(x)\,dx+\int_{\Omega^c}\mu f(x)\phi_0(x)\,dx\right).$$

Setting $\Psi(t,x)=\eta_T^{2p'}(t)\Phi(t,x)$ in (2.1) with $f(u,u_t)=|u|^p$ and integrating by parts, we get

$$\begin{aligned}&C_2(f,g)\varepsilon+\int_0^T\int_{\Omega^c}|u|^p\Psi\,dx\,dt\\&=\int_0^T\int_{\Omega^c}u\partial_t^2\eta_T^{2p'}\Phi\,dx\,dt+2\int_0^T\int_{\Omega^c}u\partial_t\eta_T^{2p'}\partial_t\Phi\,dx\,dt\\&\quad-\int_0^T\int_{\Omega^c}u\frac{\mu}{1+t}\partial_t\eta_T^{2p'}\Phi\,dx\,dt+\int_0^T\int_{\Omega^c}u\frac{\mu}{(1+t)^2}\eta_T^{2p'}\Phi\,dx\,dt\\&\quad+\int_0^T\int_{\Omega^c}u\eta_T^{2p'}\phi_1(x)\left(\lambda''(t)-\lambda(t)-\frac{\mu}{1+t}\lambda'(t)\right)dx\,dt\\&=\int_0^T\int_{\Omega^c}u\partial_t^2\eta_T^{2p'}\Phi\,dx\,dt+2\int_0^T\int_{\Omega^c}u\partial_t\eta_T^{2p'}\partial_t\Phi\,dx\,dt\\&\quad-\int_0^T\int_{\Omega^c}u\frac{\mu}{1+t}\partial_t\eta_T^{2p'}\Phi\,dx\,dt+\int_0^T\int_{\Omega^c}u\frac{\mu}{(1+t)^2}\eta_T^{2p'}\Phi\,dx\,dt\\&=I_4+I_5+I_6+I_7,\end{aligned}\tag{2.19}$$

where

$$C_2(f,g)=C\left(\int_{\Omega^c}g(x)\lambda(0)\phi_1(x)\,dx+\int_{\Omega^c}\big(\mu\lambda(0)-\lambda'(0)\big)f(x)\phi_1(x)\,dx\right).$$

Using Lemma 2.5, we deduce

$$\begin{aligned}|I_4|&\le\int_0^T\int_{\Omega^c}\left|u\partial_t^2\eta_T^{2p'}\Phi\right|dx\,dt\\&\le CT^{-2}\left(\int_0^T\int_{\Omega^c}\eta_T^{2p'}|u|^p\phi_0(x)\,dx\,dt\right)^{\frac{1}{p}}\left(\int_0^T\int_{\Omega^c}\Phi^{p'}\phi_0(x)^{-\frac{1}{p-1}}\,dx\,dt\right)^{\frac{1}{p'}}\end{aligned}$$

$$
\begin{aligned}
&\leq CT^{-2+\frac{n}{p'}+\frac{\mu-n+1}{2}}\left(\int_0^T\int_{\Omega^c}\eta_T^{2p'}|u|^p\phi_0(x)\,dx\,dt\right)^{\frac{1}{p}}, \qquad (2.20)\\
|I_5| &\leq \int_0^T\int_{\Omega^c}\left|u\partial_t\eta_T^{2p'}\partial_t\Phi\right|dx\,dt\\
&\leq CT^{-1}\left(\int_0^T\int_{\Omega^c}\eta_T^{2p'}|u|^p\phi_0(x)\,dx\,dt\right)^{\frac{1}{p}}\left(\int_0^T\int_{\Omega^c}\Phi^{p'}\phi_0(x)^{-\frac{1}{p-1}}\,dx\,dt\right)^{\frac{1}{p'}}\\
&\leq CT^{-1+\frac{n}{p'}+\frac{\mu-n+1}{2}}\left(\int_0^T\int_{\Omega^c}\eta_T^{2p'}|u|^p\phi_0(x)\,dx\,dt\right)^{\frac{1}{p}}. \qquad (2.21)
\end{aligned}
$$

In a similar way, we acquire

$$
|I_6|,|I_7| \leq CT^{-2+\frac{n}{p'}+\frac{\mu-n+1}{2}}\left(\int_0^T\int_{\Omega^c}\eta_T^{2p'}|u|^p\phi_0(x)\,dx\,dt\right)^{\frac{1}{p}}. \qquad (2.22)
$$

We conclude from (2.19), (2.20), (2.21), and (2.22) that

$$
C_2(f,g)\varepsilon \leq CT^{-1+\frac{n}{p'}+\frac{\mu-n+1}{2}}\left(\int_0^T\int_{\Omega^c}\eta_T^{2p'}|u|^p\phi_0(x)\,dx\,dt\right)^{\frac{1}{p}}.
$$

This in turn implies

$$
\big(C_2(f,g)\varepsilon\big)^pT^{n-\frac{(n+\mu-1)p}{2}} \leq \int_0^T\int_{\Omega^c}\eta_T^{2p'}|u|^p\phi_0(x)\,dx\,dt. \qquad (2.23)
$$

From (2.18) and (2.23), we arrive at

$$
T \leq C\varepsilon^{-\frac{2p(p-1)}{\gamma(p,n+\mu)}}. \qquad \square
$$

2.2 The case for $n = 2$

We are in the position to present several lemmas.

Lemma 2.6 ([38]) *There exists a function $\phi_0(x) \in C^2(\Omega^c)$ $(n = 2)$ satisfying the following boundary value problem*

$$
\begin{cases}
\Delta\phi_0(x) = 0, & x \in \Omega^c, n = 2,\\
\phi_0(x)|_{\partial\Omega} = 0, & \\
|x| \to \infty, & \phi_0(x) \to \infty.
\end{cases} \qquad (2.24)
$$

Moreover, for all $x \in \Omega^c$, it holds that $0 < \phi_0(x) \leq C\ln r$, where C is a positive constant, and $r = |x|$.

When we set $n = 2$ in Lemmas 2.4 and 2.5, we obtain the following two lemmas.

Lemma 2.7 *Let $n = 2$ and $p > 1$. Then for all $t \geq 0$, it holds that*

$$
\int_{\Omega^c\cap\{|x|\leq t+R\}}\big[\Phi(t,x)\big]^{p'}\,dx \leq C\lambda(t)^{p'}(t+R)^{1-\frac{p'}{2}}e^{p't},
$$

where $p' = \frac{p}{p-1}$, and C is a positive constant.

Lemma 2.8 *Let $n = 2$ and $p > 1$. Then for all $t \geq 0$, it holds that*

$$\int_0^T \int_{\Omega^c \cap \{|x| \leq t+R\}} \left[\Phi(t,x)\right]^{p'} \phi_0(x)^{-\frac{1}{p-1}} \, dx \, dt \leq CT^{\frac{\mu p'}{2}+2-\frac{p'}{2}} (\ln T)^{-\frac{1}{p-1}}, \tag{2.25}$$

where $p' = \frac{p}{p-1}$, and C is a positive constant.

Proof of Theorem 1.1 Similar to the derivation in (2.15)–(2.17), we acquire

$$\begin{aligned} |I_1| &\leq CT^{-2} \left(\int_0^T \int_{\Omega^c} |u|^p \eta_T^{2p'} \phi_0(x) \, dx \, dt \right)^{\frac{1}{p}} \left(\int_0^T \int_{\Omega^c} \phi_0(x) \, dx \, dt \right)^{\frac{1}{p'}} \\ &\leq CT^{-2} \left(\int_0^T t^2 \ln t \, dt \right)^{\frac{1}{p'}} \left(\int_0^T \int_{\Omega^c} |u|^p \eta_T^{2p'} \phi_0(x) \, dx \, dt \right)^{\frac{1}{p}} \\ &\leq CT^{3-2p'} \ln T + \frac{1}{4} \int_0^T \int_{\Omega^c} |u|^p \eta_T^{2p'} \phi_0(x) \, dx \, dt, \end{aligned} \tag{2.26}$$

and

$$|I_2|, |I_3| \leq CT^{3-2p'} \ln T + \frac{1}{4} \int_0^T \int_{\Omega^c} |u|^p \eta_T^{2p'} \phi_0(x) \, dx \, dt. \tag{2.27}$$

Applying (2.14), (2.26), and (2.27), we obtain

$$C_1(f,g)\varepsilon + \int_0^T \int_{\Omega^c} |u|^p \eta_T^{2p'} \phi_0(x) \, dx \, dt \leq CT^{3-2p'} \ln T. \tag{2.28}$$

Combining Lemma 2.8, we deduce

$$\begin{aligned} |I_4| &\leq CT^{-2} \left(\int_0^T \int_{\Omega^c} \eta_T^{2p'} |u|^p \phi_0(x) \, dx \, dt \right)^{\frac{1}{p}} \left(\int_0^T \int_{\Omega^c} \Phi^{p'} \phi_0(x)^{-\frac{1}{p-1}} \, dx \, dt \right)^{\frac{1}{p'}} \\ &\leq CT^{-2+\frac{2}{p'}+\frac{\mu-1}{2}} (\ln T)^{-\frac{1}{p}} \left(\int_0^T \int_{\Omega^c} \eta_T^{2p'} |u|^p \phi_0(x) \, dx \, dt \right)^{\frac{1}{p}}. \end{aligned} \tag{2.29}$$

Similarly, we obtain

$$|I_5|, |I_6|, |I_7| \leq CT^{-1+\frac{2}{p'}+\frac{\mu-1}{2}} (\ln T)^{-\frac{1}{p}} \left(\int_0^T \int_{\Omega^c} \eta_T^{2p'} |u|^p \phi_0(x) \, dx \, dt \right)^{\frac{1}{p}}. \tag{2.30}$$

We conclude from (2.19), (2.29), and (2.30) that

$$\left(C_2(f,g)\varepsilon\right)^p T^{-p+2-\frac{(\mu-1)p}{2}} \ln T \leq \int_0^T \int_{\Omega^c} \eta_T^{2p'} |u|^p \phi_0(x) \, dx \, dt. \tag{2.31}$$

From (2.28) and (2.31), we have

$$T \leq C\varepsilon^{-\frac{2p(p-1)}{\gamma(p,2+\mu)}}. \qquad \square$$

2.3 The case for $n = 1$

We present several related lemmas.

Lemma 2.9 ([38]) *There exists a function $\phi_0(x) \in C^2([0,\infty))$ $(n = 1)$ satisfying the following boundary value problem*

$$\begin{cases} \Delta\phi_0(x) = 0, & x > 0, n = 1, \\ \phi_0(x)|_{x=0} = 0, \\ |x| \to \infty, & \phi_0(x) \to \infty. \end{cases} \tag{2.32}$$

Moreover, there exist two positive constants C_1 and C_2, such that $C_1(x) \le \phi_0(x) \le C_2(x)$ for all $x \ge 0$.

Lemma 2.10 *Let $n = 1$ and $p > 1$. Then, for all $t \ge 0$, it holds that*

$$\int_0^{t+R} \left[\Phi(t,x)\right]^{p'} dx \le C\lambda(t)^{p'} e^{p't},$$

where $p' = \frac{p}{p-1}$, and C is a positive constant.

Proof Making use of Lemma 2.3, we obtain

$$\begin{aligned} \int_0^{t+R} \left[\Phi(t,x)\right]^{p'} dx &= \int_0^{t+R} \left[\lambda(t)\phi_1(x)\right]^{p'} dx \\ &\le \lambda(t)^{p'} \int_0^{t+R} \left(Ce^{|x|}\right)^{p'} dx \\ &\le C\lambda(t)^{p'} e^{p't}. \end{aligned} \tag{2.33}$$

This finishes the proof of Lemma 2.10. □

Lemma 2.11 *Let $n = 1$ and $p > 1$. Then, for all $t \ge 0$, it holds that*

$$\int_0^T \int_0^{t+R} \left[\Phi(t,x)\right]^{p'} \phi_0(x)^{-\frac{1}{p-1}} \,dx\,dt \le CT^{\frac{\mu p'}{2}+1},$$

where $p' = \frac{p}{p-1}$, and C is a positive constant.

Proof The direct calculation shows

$$\begin{aligned} &\int_0^{t+R} \left[\Phi(t,x)\right]^{p'} \phi_0(x)^{-\frac{1}{p-1}} \,dx \\ &\quad= \int_0^{R} \left[\Phi(t,x)\right]^{p'} \phi_0(x)^{-\frac{1}{p-1}} \,dx \\ &\qquad+ \int_R^{t+R} \left[\Phi(t,x)\right]^{p'} \phi_0(x)^{-\frac{1}{p-1}} \,dx \\ &\quad= M_3(t) + M_4(t). \end{aligned} \tag{2.34}$$

We observe that $0 < \phi_0(x) < \infty$ for all $x > 0$. There exists a positive constant C, such that $\phi_0(x) \geq C$ when $R \leq x \leq t + R$. According to Lemma 2.10, we have

$$M_4(t) = \int_R^{t+R} \bigl[\Phi(t,x)\bigr]^{p'} \phi_0(x)^{-\frac{1}{p-1}}\, dx \leq C\lambda(t)^{p'} e^{p't}. \tag{2.35}$$

Taking advantage of Lemma 2.5 in [39], we acquire

$$\begin{aligned} M_3(t) &= \int_0^R \bigl[\Phi(t,x)\bigr]^{p'} \phi_0(x)^{-\frac{1}{p-1}}\, dx \\ &\leq \int_0^R (Cx)^{-\frac{1}{p-1}} \lambda(t)^{p'} \phi_1(x)^{p'}\, dx \\ &\leq C\lambda(t)^{p'} \int_0^R x^{-\frac{1}{p-1}} x^{p'}\, dx \\ &\leq C\lambda(t)^{p'}. \end{aligned} \tag{2.36}$$

We conclude that

$$\int_0^{t+R} \bigl[\Phi(t,x)\bigr]^{p'} \phi_0(x)^{-\frac{1}{p-1}}\, dx \leq C\lambda(t)^{p'} e^{p't}. \tag{2.37}$$

Therefore, we have

$$\int_0^T \int_0^{t+R} \bigl[\Phi(t,x)\bigr]^{p'} \phi_0(x)^{-\frac{1}{p-1}}\, dx\, dt \leq C \int_{T/2}^T \lambda(t)^{p'} e^{p't}\, dt \leq CT^{\mu p'/2+1}. \tag{2.38}$$

This finishes the proof of Lemma 2.11. □

Proof of Theorem 1.1 Similar to (2.15)–(2.17), we derive

$$\begin{aligned} |I_1| &\leq CT^{-2} \biggl(\int_0^T \int_0^{t+R} |u|^p \eta_T^{2p'} \phi_0(x)\, dx\, dt \biggr)^{\frac{1}{p}} \biggl(\int_0^T \int_0^{t+R} \phi_0(x)\, dx\, dt \biggr)^{\frac{1}{p'}} \\ &\leq \frac{1}{2} CT^{-2} \biggl(\int_0^T t^2\, dt \biggr)^{\frac{1}{p'}} \biggl(\int_0^T \int_0^{t+R} |u|^p \eta_T^{2p'} \phi_0(x)\, dx\, dt \biggr)^{\frac{1}{p}} \\ &\leq CT^{3-2p'} + \frac{1}{4} \int_0^T \int_0^{t+R} |u|^p \eta_T^{2p'} \phi_0(x)\, dx\, dt, \end{aligned} \tag{2.39}$$

and

$$|I_2|, |I_3| \leq CT^{3-2p'} + \frac{1}{4} \int_0^T \int_0^{t+R} |u|^p \eta_T^{2p'} \phi_0(x)\, dx\, dt. \tag{2.40}$$

Applying (2.14), (2.39), and (2.40) gives rise to

$$C_1(f,g)\varepsilon + \int_0^T \int_0^{t+R} |u|^p \eta_T^{2p'} \phi_0(x)\, dx\, dt \leq CT^{3-2p'}. \tag{2.41}$$

Making use of Lemma 2.11, we acquire

$$\begin{aligned}
|I_4| &\le CT^{-2}\int_0^T\int_0^{t+R}\bigl|u\eta_T^{2p'}\Phi\bigr|\,dx\,dt\\
&\le CT^{-2}\biggl(\int_0^T\int_0^{t+R}\eta_T^{2p'}|u|^p\phi_0(x)\,dx\,dt\biggr)^{\frac{1}{p}}\\
&\quad\times\biggl(\int_0^T\int_0^{t+R}\Phi^{\frac{p}{p-1}}\phi_0(x)^{-\frac{1}{p-1}}\,dx\,dt\biggr)^{\frac{1}{p'}}\\
&\le CT^{-2+\frac{1}{p'}+\frac{\mu}{2}}\biggl(\int_0^T\int_0^{t+R}\eta_T^{2p'}|u|^p\phi_0(x)\,dx\,dt\biggr)^{\frac{1}{p}}.
\end{aligned}\tag{2.42}$$

In a similar way, we arrive at

$$|I_5|,|I_6|,|I_7|\le CT^{-1+\frac{1}{p'}+\frac{\mu}{2}}\biggl(\int_0^T\int_0^{t+R}\eta_T^{2p'}|u|^p\phi_0(x)\,dx\,dt\biggr)^{\frac{1}{p}}.\tag{2.43}$$

We conclude from (2.19), (2.42), and (2.43) that

$$\bigl(C_2(f,g)\varepsilon\bigr)^pT^{1-\frac{\mu p}{2}}\le\int_0^T\int_0^{t+R}\eta_T^{2p'}|u|^p\phi_0(x)\,dx\,dt.\tag{2.44}$$

Applying (2.41) and (2.44), we obtain

$$T\le C\varepsilon^{-\frac{2p(p-1)}{-\mu p^2+\mu p+4}}. \qquad\square$$

3 Proof of Theorem 1.2

3.1 The case for $n\ge 3$

We introduce

$$\theta(t)=\begin{cases}0, & t<\frac{1}{2},\\ \eta(t), & t\ge\frac{1}{2},\end{cases}\qquad \theta_M(t)=\theta\Bigl(\frac{t}{M}\Bigr).\tag{3.1}$$

Let $M\in(1,T)$. Set $\psi=-\eta_M^{2p'}(t)\lambda(t)\phi_0(x)\phi_1(x)$. Choosing $\Psi(t,x)=\partial_t\psi$ in (2.1) with $f(u,u_t)=|u_t|^p$ and applying Lemma 2.1 lead to

$$\begin{aligned}
&C_3(f,g)\varepsilon+\int_0^T\int_{\Omega^c}|u_t|^p\eta_M^{2p'}(t)\bigl|\lambda'(t)\bigr|\phi_0(x)\phi_1(x)\,dx\,dt\\
&\quad=\int_0^T\int_{\Omega^c}u_t\eta_M^{2p'}(t)\phi_0(x)\phi_1(x)\Bigl(\lambda''(t)-\lambda(t)-\frac{\mu}{1+t}\lambda'(t)\Bigr)\,dx\,dt\\
&\qquad+\int_0^T\int_{\Omega^c}u_t\partial_t^2\eta_M^{2p'}(t)\lambda(t)\phi_0(x)\phi_1(x)\,dx\,dt\\
&\qquad+2\int_0^T\int_{\Omega^c}u_t\partial_t\eta_M^{2p'}(t)\lambda'(t)\phi_0(x)\phi_1(x)\,dx\,dt\\
&\qquad+\int_0^T\int_{\Omega^c}u_t\eta_M^{2p'}(t)\lambda(t)\nabla\phi_0(x)\nabla\phi_1(x)\,dx\,dt
\end{aligned}$$

$$
\begin{aligned}
&-\int_0^T\int_{\Omega^c}\frac{\mu}{1+t}u_t\partial_t\eta_M^{2p'}\lambda(t)\phi_0(x)\phi_1(x)\,dx\,dt\\
=&\int_0^T\int_{\Omega^c}u_t\partial_t^2\eta_M^{2p'}(t)\lambda(t)\phi_0(x)\phi_1(x)\,dx\,dt\\
&+2\int_0^T\int_{\Omega^c}u_t\partial_t\eta_M^{2p'}(t)\lambda'(t)\phi_0(x)\phi_1(x)\,dx\,dt\\
&+\int_0^T\int_{\Omega^c}u_t\eta_M^{2p'}(t)\lambda(t)\nabla\phi_0(x)\nabla\phi_1(x)\,dx\,dt\\
&-\int_0^T\int_{\Omega^c}\frac{\mu}{1+t}u_t\partial_t\eta_M^{2p'}\lambda(t)\phi_0(x)\phi_1(x)\,dx\,dt\\
=&I_8+I_9+I_{10}+I_{11},
\end{aligned}
\tag{3.2}
$$

where $C_3(f,g)=-\int_{\Omega^c}\lambda'(0)g(x)\phi_0(x)\phi_1(x)\,dx+\int_{\Omega^c}\lambda(0)f(x)\phi_0(x)\phi_1(x)\,dx$.

The direct calculation gives rise to

$$
\begin{aligned}
|I_8|\le{}&CM^{-2}\left(\int_0^T\int_{\Omega^c}|u_t|^p\theta_M^{2p'}\left|\lambda'(t)\right|\phi_0(x)\phi_1(x)\,dx\,dt\right)^{\frac{1}{p}}\\
&\times\left(\int_{\frac{M}{2}}^M\int_{\{|x|\le t+R\}}\left|\lambda'(t)\right|^{-\frac{1}{p-1}}\left|\lambda(t)\right|^{\frac{p}{p-1}}\phi_0(x)\phi_1(x)\,dx\,dt\right)^{\frac{1}{p'}}\\
\le{}&CM^{-2+\frac{n+\mu+1}{2}\frac{1}{p'}}\left(\int_0^T\int_{\Omega^c}|u_t|^p\theta_M^{2p'}\left|\lambda'(t)\right|\phi_0(x)\phi_1(x)\,dx\,dt\right)^{\frac{1}{p}},
\end{aligned}
\tag{3.3}
$$

$$
\begin{aligned}
|I_9|\le{}&CM^{-1}\left(\int_0^T\int_{\Omega^c}|u_t|^p\theta_M^{2p'}\left|\lambda'(t)\right|\phi_0(x)\phi_1(x)\,dx\,dt\right)^{\frac{1}{p}}\\
&\times\left(\int_0^T\int_{\Omega^c}\left|\lambda'(t)\right|\phi_0(x)\phi_1(x)\,dx\,dt\right)^{\frac{1}{p'}}\\
\le{}&CM^{-1+\frac{n+\mu+1}{2}\frac{1}{p'}}\left(\int_0^T\int_{\Omega^c}|u_t|^p\theta_M^{2p'}\left|\lambda'(t)\right|\phi_0(x)\phi_1(x)\,dx\,dt\right)^{\frac{1}{p}},
\end{aligned}
\tag{3.4}
$$

$$
\begin{aligned}
|I_{10}|\le{}&C\left(\int_0^T\int_{\Omega^c}|u_t|^p\theta_M^{2p'}\left|\lambda'(t)\right|\phi_0(x)\phi_1(x)\,dx\,dt\right)^{\frac{1}{p}}\\
&\times\left(\int_0^T\int_{\Omega^c}\left|\lambda'(t)\right|^{-\frac{1}{p-1}}\left|\lambda(t)\right|^{\frac{p}{p-1}}\left(\phi_0(x)\right)^{-\frac{1}{p-1}}\left|\nabla\phi_0(x)\right|^{\frac{p}{p-1}}\phi_1(x)\,dx\,dt\right)^{\frac{1}{p'}}\\
\le{}&CM^{-2+\frac{n+\mu+1}{2}\frac{1}{p'}}\left(\int_0^T\int_{\Omega^c}|u_t|^p\theta_M^{2p'}\left|\lambda'(t)\right|\phi_0(x)\phi_1(x)\,dx\,dt\right)^{\frac{1}{p}}
\end{aligned}
\tag{3.5}
$$

and

$$
\begin{aligned}
|I_{11}|\le{}&CM^{-2}\left(\int_0^T\int_{\Omega^c}|u_t|^p\theta_M^{2p'}\left|\lambda'(t)\right|\phi_0(x)\phi_1(x)\,dx\,dt\right)^{\frac{1}{p}}\\
&\times\left(\int_{\frac{M}{2}}^M\int_{\{|x|\le t+R\}}\left|\lambda'(t)\right|^{-\frac{1}{p-1}}\left|\lambda(t)\right|^{\frac{p}{p-1}}\phi_0(x)\phi_1(x)\,dx\,dt\right)^{\frac{1}{p'}}\\
\le{}&CM^{-2+\frac{n+\mu+1}{2}\frac{1}{p'}}\left(\int_0^T\int_{\Omega^c}|u_t|^p\theta_M^{2p'}\left|\lambda'(t)\right|\phi_0(x)\phi_1(x)\,dx\,dt\right)^{\frac{1}{p}}.
\end{aligned}
\tag{3.6}
$$

It is deduced from (3.2)–(3.6) that

$$\begin{aligned} &C_3(f,g)\varepsilon + \int_0^T \int_{\Omega^c} |u_t|^p \eta_M^{2p'} \left|\lambda'(t)\right| \phi_0(x)\phi_1(x)\,dx\,dt \\ &\quad \le CM^{-1+\frac{n+\mu+1}{2}\frac{1}{p'}} \left(\int_0^T \int_{\Omega^c} |u_t|^p \theta_M^{2p'} \left|\lambda'(t)\right| \phi_0(x)\phi_1(x)\,dx\,dt \right)^{\frac{1}{p}}. \end{aligned} \tag{3.7}$$

We introduce the following function

$$\begin{aligned} F(M) &= \int_1^M \left(\int_0^T \int_{\Omega^c} |u_t|^p \left|\lambda'(t)\right| \phi_0(x)\phi_1(x)\theta_\sigma^{2p'}\,dx\,dt \right) \frac{1}{\sigma}\,d\sigma \\ &\le C\ln 2 \int_0^T \int_{\Omega^c} |u_t|^p \left|\lambda'(t)\right| \phi_0(x)\phi_1(x)\eta_M^{2p'}\,dx\,dt. \end{aligned} \tag{3.8}$$

Differentiating (3.8) with respect to M gives rise to

$$F'(M) = \frac{1}{M} \int_0^T \int_{\Omega^c} |u_t|^p \left|\lambda'(t)\right| \phi_0(x)\phi_1(x)\theta_M^{2p'}\,dx\,dt. \tag{3.9}$$

Combining (3.7), (3.8), and (3.9), we derive

$$M^{\frac{(n+\mu-1)(p-1)}{2}} F'(M) \ge C\bigl(C_3(f,g)\varepsilon + F(M)\bigr)^p. \tag{3.10}$$

Thus, we obtain

$$T(\varepsilon) \le \begin{cases} C\varepsilon^{-\frac{p-1}{1-\frac{(n+\mu-1)(p-1)}{2}}}, & 1 < p < \frac{2}{n+\mu-1} + 1, \\ \exp(C\varepsilon^{-(p-1)}), & p = \frac{2}{n+\mu-1} + 1. \end{cases} \tag{3.11}$$

3.2 The case for $n = 2$

In this case, we achieve

$$|I_8| \le CM^{-2+\frac{3+\mu}{2}\frac{1}{p'}} (\ln M)^{\frac{1}{p'}} \left(\int_0^T \int_{\Omega^c} |u_t|^p \theta_M^{2p'} \left|\lambda'(t)\right| \phi_0(x)\phi_1(x)\,dx\,dt \right)^{\frac{1}{p}}, \tag{3.12}$$

$$|I_9| \le CM^{-1+\frac{3+\mu}{2}\frac{1}{p'}} (\ln M)^{\frac{1}{p'}} \left(\int_0^T \int_{\Omega^c} |u_t|^p \theta_M^{2p'} \left|\lambda'(t)\right| \phi_0(x)\phi_1(x)\,dx\,dt \right)^{\frac{1}{p}}, \tag{3.13}$$

$$|I_{10}| \le CM^{-2+\frac{3+\mu}{2}\frac{1}{p'}} (\ln M)^{-\frac{1}{p}} \left(\int_0^T \int_{\Omega^c} |u_t|^p \theta_M^{2p'} \left|\lambda'(t)\right| \phi_0(x)\phi_1(x)\,dx\,dt \right)^{\frac{1}{p}} \tag{3.14}$$

and

$$|I_{11}| \le CM^{-2+\frac{3+\mu}{2}\frac{1}{p'}} (\ln M)^{\frac{1}{p'}} \left(\int_0^T \int_{\Omega^c} |u_t|^p \theta_M^{2p'} \left|\lambda'(t)\right| \phi_0(x)\phi_1(x)\,dx\,dt \right)^{\frac{1}{p}}. \tag{3.15}$$

Making use of (3.2) and (3.12)–(3.15) leads to

$$C_3(f,g)\varepsilon + \int_0^T \int_{\Omega^c} |u_t|^p \eta_M^{2p'} \left|\lambda'(t)\right| \phi_0(x)\phi_1(x)\,dx\,dt$$

$$\le CM^{-1+\frac{3+\mu}{2}\frac{1}{p'}}(\ln M)^{\frac{1}{p'}}\left(\int_0^T\int_{\Omega^c}|u_t|^p\theta_M^{2p'}\left|\lambda'(t)\right|\phi_0(x)\phi_1(x)\,dx\,dt\right)^{\frac{1}{p}}. \tag{3.16}$$

Combining (3.8), (3.9), and (3.16), we derive

$$M^{\frac{(\mu+1)(p-1)}{2}}(\ln M)^{p-1}F'(M)\ge C\big(C_3(f,g)\varepsilon+F(M)\big)^p. \tag{3.17}$$

As a result, we obtain

$$T(\varepsilon)\le\begin{cases}C\varepsilon^{-\frac{p-1}{1-\frac{(3+\mu)(p-1)}{2}}}, & 1<p<\frac{2}{3+\mu}+1,\\ \exp(C\varepsilon^{-(p-1)}), & p=\frac{2}{3+\mu}+1.\end{cases} \tag{3.18}$$

3.3 The case for $n = 1$

In this case, we acquire

$$|I_8|\le CM^{-2+\frac{2+\mu}{2}\frac{1}{p'}}M^{\frac{1}{p'}}\left(\int_0^T\int_{\Omega^c}|u_t|^p\theta_M^{2p'}\left|\lambda'(t)\right|\phi_0(x)\phi_1(x)\,dx\,dt\right)^{\frac{1}{p}}, \tag{3.19}$$

$$|I_9|\le CM^{-1+\frac{2+\mu}{2}\frac{1}{p'}}M^{\frac{1}{p'}}\left(\int_0^T\int_{\Omega^c}|u_t|^p\theta_M^{2p'}\left|\lambda'(t)\right|\phi_0(x)\phi_1(x)\,dx\,dt\right)^{\frac{1}{p}}, \tag{3.20}$$

$$|I_{10}|\le CM^{-2+\frac{2+\mu}{2}\frac{1}{p'}}M^{-\frac{1}{p}}\left(\int_0^T\int_{\Omega^c}|u_t|^p\theta_M^{2p'}\left|\lambda'(t)\right|\phi_0(x)\phi_1(x)\,dx\,dt\right)^{\frac{1}{p}} \tag{3.21}$$

and

$$|I_{11}|\le CM^{-2+\frac{2+\mu}{2}\frac{1}{p'}}M^{\frac{1}{p'}}\left(\int_0^T\int_{\Omega^c}|u_t|^p\theta_M^{2p'}\left|\lambda'(t)\right|\phi_0(x)\phi_1(x)\,dx\,dt\right)^{\frac{1}{p}}. \tag{3.22}$$

Taking into account (3.2) and (3.19)–(3.22) yields

$$\begin{aligned}&C_3(f,g)\varepsilon+\int_0^T\int_{\Omega^c}|u_t|^p\eta_M^{2p'}\left|\lambda'(t)\right|\phi_0(x)\phi_1(x)\,dx\,dt\\&\quad\le CM^{-1+\frac{2+\mu}{2}\frac{1}{p'}}\left(\int_0^T\int_{\Omega^c}|u_t|^p\theta_M^{2p'}\left|\lambda'(t)\right|\phi_0(x)\phi_1(x)\,dx\,dt\right)^{\frac{1}{p}}.\end{aligned} \tag{3.23}$$

Combining (3.8), (3.9), and (3.23), we derive

$$M^{\frac{\mu(p-1)}{2}}F'(M)\ge C\big(C_3(f,g)\varepsilon+F(M)\big)^p. \tag{3.24}$$

Consequently, we conclude that

$$T(\varepsilon)\le\begin{cases}C\varepsilon^{-\frac{p-1}{1-\frac{(2+\mu)(p-1)}{2}}}, & 1<p<\frac{2}{2+\mu}+1,\\ \exp(C\varepsilon^{-(p-1)}), & p=\frac{2}{2+\mu}+1.\end{cases} \tag{3.25}$$

4 Proof of Theorem 1.3

4.1 The case for $n \geq 3$

We introduce the following function

$$\zeta(t) = \begin{cases} 0, & t \leq \frac{1}{4}, \\ \text{increasing}, & \frac{1}{4} < t < \frac{1}{2}, \\ \theta(t), & t \geq \frac{1}{2}. \end{cases}$$

We set

$$\zeta_M(t) = \zeta\left(\frac{t}{M}\right), \qquad \psi_M(t) = \zeta_M^k(t),$$

where $M \in (1, T), k$ is a positive constant. We obtain

$$\begin{aligned} &\left|\partial_t^2 \psi_M(t)\right| \leq CM^{-2} \psi_M^{1-\frac{2}{k}}(t), \\ &\left|\partial_t \psi_M(t)\right| = CM^{-1} \psi_M^{1-\frac{1}{k}}(t). \end{aligned}$$

Choosing $\Psi(t,x) = \psi_M(t)\phi_0(x)$ in (2.1) with $f(u,u_t) = |u_t|^p + |u|^q$ and integrating over $[0,T) \times \Omega^c$, we have

$$\begin{aligned} &\int_0^T \int_{\Omega^c} \left(|u_t|^p + |u|^q\right) \psi_M \phi_0(x)\, dx\, dt \\ &\quad = \int_0^T \int_{\Omega^c} u \partial_t^2 \psi_M \phi_0(x)\, dx\, dt + \int_0^T \int_{\Omega^c} \frac{\mu}{(1+t)^2} u \psi_M \phi_0(x)\, dx\, dt \\ &\qquad - \int_0^T \int_{\Omega^c} \frac{\mu}{1+t} u \partial_t \psi_M \phi_0(x)\, dx\, dt \\ &\quad = I_{12} + I_{13} + I_{14}. \end{aligned} \tag{4.1}$$

We notice that $\operatorname{supp} \psi_M \subset [\frac{R}{4}, R]$. We acquire

$$\begin{aligned} |I_{12}| &\leq \int_0^T \int_{\Omega^c} \left|u \partial_t^2 \psi_M \phi_0(x)\right| dx\, dt \\ &\leq CM^{-2} \left(\int_0^T \int_{\Omega^c} |u|^q \psi_M^{q(1-\frac{2}{k}-\frac{1}{2q'})} \phi_0(x)\, dx\, dt\right)^{\frac{1}{q}} \\ &\qquad \times \left(\int_0^T \int_{|x| \leq t+R} \psi_M^{\frac{1}{2}} \phi_0(x)\, dx\, dt\right)^{\frac{1}{q'}} \\ &\leq CR^{-2+\frac{n+1}{q'}} \left(\int_0^T \int_{\Omega^c} |u|^q \psi_M \phi_0(x)\, dx\, dt\right)^{\frac{1}{q}}, \end{aligned} \tag{4.2}$$

where $q(1 - \frac{2}{k} - \frac{1}{2q'}) \geq 1$ for some sufficiently large k. It holds that

$$|I_{13}| \leq C \int_0^T \int_{\Omega^c} \left|u \psi_M \phi_0(x) \frac{\mu}{(1+t)^2}\right| dx\, dt$$

$$\le C\left(\int_0^T\int_{\Omega^c}|u|^q\psi_M\phi_0(x)\,dx\,dt\right)^{\frac{1}{q}}\left(\int_0^T t^{-2q'}\,dt\int_{|x|\le t+R}\phi_0(x)\,dx\right)^{\frac{1}{q'}}$$

$$\le CR^{-2+\frac{n+1}{q'}}\left(\int_0^T\int_{\Omega^c}|u|^q\psi_M\phi_0(x)\,dx\,dt\right)^{\frac{1}{q}}. \tag{4.3}$$

In a similar way, we derive

$$|I_{14}|\le CR^{-2+\frac{n+1}{q'}}\left(\int_0^T\int_{\Omega^c}|u|^q\psi_M\phi_0(x)\,dx\,dt\right)^{\frac{1}{q}}. \tag{4.4}$$

Combining (4.1), (4.2), (4.3), and (4.4), we have

$$\begin{aligned}
&\int_0^T\int_{\Omega^c}\left(|u_t|^p+|u|^q\right)\psi_M\phi_0(x)\,dx\,dt\\
&\quad\le CR^{-2+\frac{n+1}{q'}}\left(\int_0^T\int_{\Omega^c}|u|^q\psi_M\phi_0(x)\,dx\,dt\right)^{\frac{1}{q}}\\
&\quad\le CM^{n-\frac{q+1}{q-1}}+\frac{1}{2}\int_0^T\int_{\Omega^c}|u|^q\psi_M\phi_0(x)\,dx\,dt,
\end{aligned} \tag{4.5}$$

which yields

$$\int_0^T\int_{\Omega^c}|u_t|^p\psi_M\phi_0(x)\,dx\,dt\le CM^{n-\frac{q+1}{q-1}}. \tag{4.6}$$

Choosing $\Psi(t,x)=-\partial_t(\eta_M^k(t)\lambda(t)\phi_1(x))$ in (2.1) with $f(u,u_t)=|u_t|^p+|u|^q$ and integrating over $[0,T)\times\Omega^c$, we obtain

$$\begin{aligned}
&C_4(f,g)\varepsilon+\int_0^T\int_{\Omega^c}\left(|u_t|^p+|u|^q\right)\eta_M^k(t)\left|\lambda'(t)\right|\phi_1(x)\,dx\,dt\\
&\quad=\int_0^T\int_{\Omega^c}u_t\eta_M^k(t)\phi_1(x)\left(\lambda''(t)-\frac{\mu}{1+t}\lambda'(t)-\lambda(t)\right)dx\,dt\\
&\qquad+\int_0^T\int_{\Omega^c}u_t\partial_t^2\eta_M^k(t)\lambda(t)\phi_1(x)\,dx\,dt+2\int_0^T\int_{\Omega^c}u_t\partial_t\eta_M^k(t)\lambda'(t)\phi_1(x)\,dx\,dt\\
&\qquad-\int_0^T\int_{\Omega^c}u_t\frac{\mu}{1+t}\eta_M^k(t)\lambda(t)\phi_1(x)\,dx\,dt\\
&\quad=I_{15}+I_{16}+I_{17},
\end{aligned} \tag{4.7}$$

where $C_4(f,g)=C(-\int_{\Omega^c}g(x)\lambda'(0)\phi_1(x)\,dx+\int_{\Omega^c}\lambda(0)f(x)\lambda(0)\phi_1(x)\,dx)$.

Furthermore, the direct calculation shows

$$\begin{aligned}
&\left|\partial_t^2\eta_M^k(t)\right|\le CM^{-2}\theta_M^k(t),\\
&\left|\partial_t\eta_M(t)\right|=CM^{-1}\theta_M^{k-1}(t).
\end{aligned}$$

Applying Lemma 2.5 gives rise to

$$|I_{15}|\le\int_0^T\int_{\Omega^c}|u_t|\partial_t^2\eta_M^k(t)\lambda(t)\phi_1(x)\,dx\,dt$$

$$
\begin{aligned}
&\leq CM^{-2}\left(\int_0^T\int_{\Omega^c}|u_t|^p\theta_M^k(t)\phi_0(x)\,dx\,dt\right)^{\frac{1}{p}}\\
&\quad\times\left(\int_{\frac{M}{2}}^M\int_{|x|\leq t+R}\theta_M^{k-2p'}(t)\left|\lambda(t)\phi_1(x)\right|^{p'}\phi_0(x)^{-\frac{1}{p-1}}\,dx\,dt\right)^{\frac{1}{p'}}\\
&\leq CM^{-2+\frac{\mu}{2}+\frac{n}{p'}-\frac{n-1}{2}}\left(\int_0^T\int_{\Omega^c}|u_t|^p\theta_M^k(t)\phi_0(x)\,dx\,dt\right)^{\frac{1}{p}}. \qquad (4.8)
\end{aligned}
$$

Similarly, we obtain

$$
|I_{16}|\leq CM^{-1+\frac{\mu}{2}+\frac{n}{p'}-\frac{n-1}{2}}\left(\int_0^T\int_{\Omega^c}|u_t|^p\theta_M^k(t)\phi_0(x)\,dx\,dt\right)^{\frac{1}{p}}, \qquad (4.9)
$$

$$
|I_{17}|\leq CM^{-2+\frac{\mu}{2}+\frac{n}{p'}-\frac{n-1}{2}}\left(\int_0^T\int_{\Omega^c}|u_t|^p\theta_M^k(t)\phi_0(x)\,dx\,dt\right)^{\frac{1}{p}}. \qquad (4.10)
$$

Using (4.7), (4.8), (4.9), and (4.10), we have

$$
C_4(f,g)\varepsilon\leq CM^{-1+\frac{\mu}{2}+\frac{n}{p'}-\frac{n-1}{2}}\left(\int_0^T\int_{\Omega^c}|u_t|^p\theta_M^k(t)\phi_0(x)\,dx\,dt\right)^{\frac{1}{p}},
$$

which in turn implies

$$
\bigl(C_4(f,g)\varepsilon\bigr)^pM^{n-\frac{(n+\mu-1)p}{2}}\leq\int_0^T\int_{\Omega^c}|u_t|^p\theta_M^k(t)\phi_0(x)\,dx\,dt. \qquad (4.11)
$$

Since

$$
\theta_M^k(t)\leq\psi_M(t)=\zeta_M^k(t),
$$

combining (4.6) and (4.11), we obtain

$$
T\leq C\varepsilon^{-\frac{2p(q-1)}{4-\gamma(p,q,n+\mu)}}.
$$

4.2 The case for $n=2$

In this case, we derive

$$
\begin{aligned}
|I_{12}|&\leq CM^{-2}\int_0^T\int_{\Omega^c}\left|u\psi_M\phi_0(x)\right|dx\,dt\\
&\leq CM^{-2}\left(\int_0^T t^2\ln t\,dt\right)^{\frac{1}{q'}}\left(\int_0^T\int_{\Omega^c}|u|^q\psi_M^{q(1-\frac{2}{k}-\frac{1}{2q'})}\phi_0(x)\,dx\,dt\right)^{\frac{1}{q}}\\
&\leq CR^{-2+\frac{3}{q'}}(\ln R)^{\frac{1}{q'}}\left(\int_0^T\int_{\Omega^c}|u|^q\psi_M\phi_0(x)\,dx\,dt\right)^{\frac{1}{q}}, \qquad (4.12)
\end{aligned}
$$

$$
|I_{13}|,|I_{14}|\leq CR^{-2+\frac{3}{q'}}(\ln R)^{\frac{1}{q'}}\left(\int_0^T\int_{\Omega^c}|u|^q\psi_M\phi_0(x)\,dx\,dt\right)^{\frac{1}{q}}. \qquad (4.13)
$$

Combining (4.1), (4.12), and (4.13), we obtain

$$\int_0^T \int_{\Omega^c} \left(|u_t|^p + |u|^q\right)\psi_M\phi_0(x)\,dx\,dt \\ \leq CM^{2-\frac{q+1}{q-1}}\ln M + \frac{1}{2}\int_0^T \int_{\Omega^c} |u|^q\psi_M\phi_0(x)\,dx\,dt, \tag{4.14}$$

which results in

$$\int_0^T \int_{\Omega^c} |u_t|^p\psi_M\phi_0(x)\,dx\,dt \leq CM^{2-\frac{q+1}{q-1}}\ln M. \tag{4.15}$$

Similar to the derivation in (4.8), applying Lemma 2.8, we deduce

$$\begin{aligned} |I_{15}| &\leq CM^{-2}\left(\int_0^T \int_{\Omega^c} |u_t|^p\theta_M^k(t)\phi_0(x)\,dx\,dt\right)^{\frac{1}{p}} \\ &\quad\times \left(\int_{\frac{M}{2}}^M \int_{|x|\leq t+R} \theta_M^{k-2p'}(t)\left|\lambda(t)\phi_1(x)\right|^{p'}\phi_0(x)^{-\frac{1}{p-1}}\,dx\,dt\right)^{\frac{1}{p'}} \\ &\leq CM^{-2+\frac{\mu-1}{2}+\frac{2}{p'}}(\ln M)^{-\frac{1}{p}}\left(\int_0^T \int_{\Omega^c} |u_t|^p\theta_M^k(t)\phi_0(x)\,dx\,dt\right)^{\frac{1}{p}}. \end{aligned} \tag{4.16}$$

In a similar way, we acquire

$$|I_{16}| \leq CM^{-1+\frac{\mu-1}{2}+\frac{2}{p'}}(\ln M)^{-\frac{1}{p}}\left(\int_0^T \int_{\Omega^c} |u_t|^p\theta_M^k(t)\phi_0(x)\,dx\,dt\right)^{\frac{1}{p}}, \tag{4.17}$$

$$|I_{17}| \leq CM^{-2+\frac{\mu-1}{2}+\frac{2}{p'}}(\ln M)^{-\frac{1}{p}}\left(\int_0^T \int_{\Omega^c} |u_t|^p\theta_M^k(t)\phi_0(x)\,dx\,dt\right)^{\frac{1}{p}}. \tag{4.18}$$

Using (4.7), (4.16), (4.17), and (4.18), we obtain

$$\left(C_4(f,g)\varepsilon\right)^p M^{2-\frac{(\mu+1)p}{2}}\ln M \leq \int_0^T \int_{\Omega^c} |u_t|^p\theta_M^k(t)\phi_0(x)\,dx\,dt. \tag{4.19}$$

Combining (4.15) and (4.19), we conclude that

$$T \leq C\varepsilon^{-\frac{2p(q-1)}{4-\gamma(p,q,2+\mu)}}.$$

4.3 The case for $n = 1$

In this case, using Lemma 2.9, we obtain

$$\begin{aligned} |I_{12}| &\leq CM^{-2}\left(\int_0^T \int_{\Omega^c} |u|^q\psi_M^{q(1-\frac{2}{k}-\frac{1}{2q'})}\phi_0(x)\,dx\,dt\right)^{\frac{1}{q}} \\ &\quad\times \left(\int_0^T \int_{|x|\leq t+R} \psi_M^{\frac{1}{2}}\phi_0(x)\,dx\,dt\right)^{\frac{1}{q'}} \\ &\leq CM^{-2}\left(\int_0^T t^2\,dt\right)^{\frac{1}{q'}}\left(\int_0^T \int_{\Omega^c} |u|^q\psi_M^{q(1-\frac{2}{k}-\frac{1}{2q'})}\phi_0(x)\,dx\,dt\right)^{\frac{1}{q}} \end{aligned}$$

$$\leq CR^{-2+\frac{3}{q'}}\left(\int_0^T\int_{\Omega^c}|u|^q\psi_M\phi_0(x)\,dx\,dt\right)^{\frac{1}{q}}, \tag{4.20}$$

and

$$|I_{13}|,|I_{14}|\leq CR^{-2+\frac{3}{q'}}\left(\int_0^T\int_{\Omega^c}|u|^q\psi_M\phi_0(x)\,dx\,dt\right)^{\frac{1}{q}}. \tag{4.21}$$

Making use of (4.1), (4.20), and (4.21), we derive

$$\int_0^T\int_{\Omega^c}|u_t|^p\psi_M\phi_0(x)\,dx\,dt\leq CM^{2-\frac{q+1}{q-1}}. \tag{4.22}$$

Similar to (4.8), utilizing Lemma 2.11 gives rise to

$$\begin{aligned}|I_{15}|&\leq CM^{-2}\left(\int_0^T\int_{\Omega^c}|u_t|^p\theta_M^k(t)\phi_0(x)\,dx\,dt\right)^{\frac{1}{p}}\\&\quad\times\left(\int_{\frac{M}{2}}^M\int_{|x|\leq t+R}\theta_M^{k-2p'}(t)\big|\lambda(t)\phi_1(x)\big|^{p'}\phi_0(x)^{-\frac{1}{p-1}}\,dx\,dt\right)^{\frac{1}{p'}}\\&\leq CM^{-2+\frac{\mu}{2}+\frac{1}{p'}}\left(\int_0^T\int_{\Omega^c}|u_t|^p\theta_M^k(t)\phi_0(x)\,dx\,dt\right)^{\frac{1}{p}}.\end{aligned} \tag{4.23}$$

In a similar way, we acquire

$$|I_{16}|\leq CM^{-1+\frac{\mu}{2}+\frac{1}{p'}}\left(\int_0^T\int_{\Omega^c}|u_t|^p\theta_M^k(t)\phi_0(x)\,dx\,dt\right)^{\frac{1}{p}}, \tag{4.24}$$

$$|I_{17}|\leq CM^{-2+\frac{\mu}{2}+\frac{1}{p'}}\left(\int_0^T\int_{\Omega^c}|u_t|^p\theta_M^k(t)\phi_0(x)\,dx\,dt\right)^{\frac{1}{p}}. \tag{4.25}$$

Applying (4.7), (4.23), (4.24), and (4.25), we observe

$$\big(C_4(f,g)\varepsilon\big)^pM^{1-\frac{\mu p}{2}}\leq\int_0^T\int_{\Omega^c}|u_t|^p\theta_M^k(t)\phi_0(x)\,dx\,dt. \tag{4.26}$$

Making use of (4.22) and (4.26), we have

$$T\leq C\varepsilon^{-\frac{2p(q-1)}{4-\mu pq+\mu p}}.$$

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