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Existence and uniqueness criterion of a periodic solution for a third-order neutral differential equation with multiple delay

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Abstract

In this paper, we study the existence and uniqueness of a periodic solution for a third-order neutral delay differential equation (NDDE) by applying Mawhin's continuation theorem of coincidence degree and analysis techniques. An illustrative example is given as an application to support our results. To confirm the accuracy of our results, we also present a plot of the behavior of the periodic solution.

MSC: 34C25

Keywords: Existence and uniqueness; Neutral delay differential equation; Mawhin's continuation theorem

1 Introduction

Neutral delay differential equations (NDDEs) are a family of differential equations depending on the past as well as the present state that involve derivatives with delays as well as the function itself. The study of the neutral functional differential equations is essentially based on the questions of the action and estimates of the spectral radii of the operators in the spaces of discontinuous functions, for example, in the spaces of summable or essentially bounded functions.

NDDEs have many interesting applications in various branches of science such as, physics, electrical control and engineering, physical chemistry, and mathematical biology, etc., see [4].

The existence and uniqueness of periodic solutions for NDDE are of great interest in mathematics and its applications to the modeling of various practical problems, see [11, 13, 15]. There have been many papers written on the various aspects of the theory of periodic function differential equations (FDE) and periodic NDDE, see for example [1–3, 5–7, 9, 10, 12, 14, 16–21, 23, 24].

In 2014, Xin and Zhao [24] established sufficient conditions for the existence of a periodic solution to the following neutral equation with variable delay

$$\left(x(t) - c(t)x\left(t - \delta(t)\right)\right)'' + f\left(t, x'(t)\right) + g\left(t, x\left(t - \tau(t)\right)\right) = e(t).$$

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In 2018, Mahmoud and Farghaly [19] studied the sufficient conditions for the existence of a periodic solution for a kind of third-order generalized NDDE with variable parameter

$$\frac{d^3}{dt^3}\big(x(t) - c(t)x\big(t-\delta(t)\big)\big) + f\big(t,\ddot{x}(t)\big) + g\big(t,\dot{x}(t)\big) + h\big(t,x\big(t-\tau(t)\big)\big) = e(t),$$

where $|c(t)| \neq 1, c, \delta \in C^2(\mathbb{R},\mathbb{R})$ and c, δ are ω-periodic functions for some $\omega > 0, \tau, e \in C[0,\omega]$ and $\int_0^\omega e(t)\,dt = 0; f$, g, and h are continuous functions.

In 2022, Taie and Alwaleedy [22] investigated the existence and uniqueness of a periodic solution for the third-order neutral functional differential equation

$$\begin{aligned}&\frac{d^3}{dt^3}\big(x(t) - d(t)x\big(t-\delta(t)\big)\big) + a(t)\ddot{x} + b(t)f\big(t,\dot{x}(t)\big)\\&\quad+\sum_{i=1}^{n} c_i(t)g\big(t,x\big(t-\tau_i(t)\big)\big) = e(t),\end{aligned}$$

where, $|d(t)| \neq 1, d, \delta \in C^3(\mathbb{R},\mathbb{R})$ are ω-periodic functions for some $\omega > 0$, $\dot{\delta}(t) < 1$ for all $t \in [0,\omega]; a, b, c_i, e (i = 1,2,\ldots,n)$ are continuous periodic functions defined on $\mathbb{R}$ with period $\omega > 0$, such that a, b, c_i have the same sign and $\int_0^\omega e(t)\,dt = 0; f, g$ are continuous functions defined on $\mathbb{R}^2$ and periodic in the first argument.

The aim of this paper is to investigate sufficient conditions ensuring the existence and uniqueness of a periodic solution for the following third-order NDDE

$$\begin{aligned}&\frac{d^3}{dt^3}\big(x(t) - \alpha x\big(t-\gamma(t)\big)\big) + af\big(\dot{x}(t)\big)\ddot{x}(t) + bg\big(t,\dot{x}(t)\big)\\&\quad+\sum_{i=1}^{n} c_i h\big(x\big(t-\gamma_i(t)\big)\big) = e(t),\end{aligned}\tag{1.1}$$

where, $\gamma_i, e : \mathbb{R} \to \mathbb{R}$ are T-periodic, $|\alpha| \neq 1$, $\gamma \in C^2(\mathbb{R},\mathbb{R})$, γ are T-periodic functions for some $T > 0, \gamma, e \in C[0,T]$, and $\int_0^T e(t)\,dt = 0; f$, g, and h are continuous functions defined on $\mathbb{R}^2$ and periodic in t with $f(u(t)) = f(u(T)), g(t,u(t)) = g(t+T,u(t+T)), h(x(t)) = h(x(t+T))$, and $g(t,0) = 0$.

2 Preparation

Let $C_T = \{x \in C(\mathbb{R},\mathbb{R}) : x(t+T) = x(t), t \in \mathbb{R}\}$ with the norm $\|x\|_\infty = \max_{t\in[0,T]} |x(t)|$, then $(C_T, \|\cdot\|_\infty)$ is a Banach space. Here, the neutral operator $\mathcal{A}$ is a natural generalization of the familiar operator $\mathcal{A}_1 = x(t) - cx(t-\delta)$, $\mathcal{A}_2 = x(t) - c(t)x(t-\delta)$. However, $\mathcal{A}$ possesses a more complicated nonlinearity than A_1, A_2. Then, for example the neutral operator $\mathcal{A}_1$ is homogeneous in the following estimate $\frac{d}{dt}(A_1x)(t) = (A_1\dot{x})(t)$, but the neutral operator $\mathcal{A}$ is inhomogeneous in general. Hence, many of the new results for differential equations with the neutral operator $\mathcal{A}$, will not be a direct extension of known theorems for NDDEs.

Moreover, define an operator $\mathcal{A} : C_T \to C_T$ as

$$(\mathcal{A}x)(t) = x(t) - \alpha x\big(t-\gamma(t)\big),\tag{2.1}$$

where, $|\alpha| \neq 1$, $\gamma \in C^2(\mathbb{R},\mathbb{R})$ is T-periodic for some $T > 0$.

Lemma 2.1 ([24]) *If* $|\alpha| \neq 1$, *then the operator* $\mathcal{A}$ *has a continuous inverse* $\mathcal{A}^{-1}$ *on* C_T, *satisfying*

(1) $(A^{-1}f)(t) = \begin{cases} f(t) + \sum_{j=1}^{\infty} \alpha^j f(s - \sum_{i=1}^{j-1} \gamma(D_i)), & \text{for } |\alpha| < 1, \forall f \in C_T, \\ -\frac{f(t+\gamma(t))}{\alpha} - \sum_{j=1}^{\infty} \frac{1}{\alpha^{j+1}} f(s + \gamma(t) + \sum_{i=1}^{j-1} \gamma(D_i)), & \text{for } |\alpha| > 1, \forall f \in C_T; \end{cases}$

(2) $|(A^{-1}f)(t)| \leq \frac{\|f\|}{|1-|\alpha||}, \forall f \in C_T$;

(3) $\int_0^T |(A^{-1}f)(t)|\, dt \leq \frac{1}{|1-|\alpha||} \int_0^T |f(t)|\, dt, \forall f \in C_T$;

where $D_1 = t, D_{j+1} = t - \sum_{i=1}^{j} \gamma(D_i), j = 1, 2, \ldots$.

Let X and Y be real Banach spaces and $L : D(L) \subset X \to Y$ be a Fredholm operator with index zero, here $D(L)$ denotes the domain of L. This means that ImL is closed in Y and $\dim \operatorname{Ker} L = \dim(Y/\operatorname{Im} L) < +\infty$. Consider supplementary subspaces X_1, Y_1, of X, Y, respectively, such that $X = \operatorname{Ker} L \oplus X_1$, $Y = \operatorname{Im} L \oplus Y_1$, and let $P_1 : X \to \operatorname{Ker} L$ and $Q_1 : Y \to Y_1$ denote the natural projections. Clearly, $\operatorname{Ker} L \cap (D(L) \cap X_1) = \{0\}$, thus the restriction $L_{P_1} := L|_{D(L)\cap X_1}$ is invertible. Let $L_{P_1}^{-1}$ denote the inverse of L_{P_1}.

Let Ω be an open bounded subset of X with $D(L) \cap \Omega \neq \emptyset$. A map $N : \overline{\Omega} \to Y$ is said to be L-compact in $\overline{\Omega}$ if $Q_1 N(\overline{\Omega})$ is bounded and the operator $L_{P_1}^{-1}(I - Q_1)N : \overline{\Omega} \to X$ is compact.

Lemma 2.2 (Gaines and Mawhin [8]) *Suppose that* X *and* Y *are two Banach spaces, and* $L : D(L) \subset X \to Y$ *is a Fredholm operator with index zero. Furthermore,* $\Omega \subset X$ *is an open bounded set and* $N : \overline{\Omega} \to Y$ *is* L-*compact on* $\overline{\Omega}$. *Assume that the following conditions hold*:

(1) $Lx \neq \lambda Nx$, *for all* $x \in \partial\Omega \cap D(L)$, $\lambda \in (0, 1)$;

(2) $Nx \notin \operatorname{Im} L$, *for all* $x \in \partial\Omega \cap \operatorname{Ker} L$;

(3) $\deg\{JQ_1 N, \Omega \cap \operatorname{Ker} L, 0\} \neq 0$, *where* $J : \operatorname{Im} Q_1 \to \operatorname{Ker} L$ *is an isomorphism.*

Then, the equation $Lx = Nx$ *has a solution in* $\overline{\Omega} \cap D(L)$.

3 Existence result

In this section, we will study the existence of a periodic solution for (1.1).

Now, we rewrite (1.1) in the following form:

$$\begin{cases} \frac{d}{dt}(\mathcal{A}x_1)(t) = x_2(t), \\ \frac{d^2}{dt^2}(\mathcal{A}x_1)(t) = \dot{x}_2(t) = x_3(t), \\ \dot{x}_3(t) = -af(\dot{x}_1(t))\ddot{x}_1(t) - bg(t, \dot{x}_1(t)) - \sum_{i=1}^{n} c_i h(x_1(t - \gamma_i(t))) + e(t). \end{cases} \tag{3.1}$$

Here, if $x(t) = (x_1(t), x_2(t), x_3(t))^\top$ is a T-periodic solution to (3.1), then $x_1(t)$ must be a T-periodic solution to (1.1). Thus, the problem of finding a T-periodic solution for (1.1) reduces to finding one for (3.1).

Recall that $C_T = \{\phi \in C(\mathbb{R}, \mathbb{R}) : \phi(t + T) \equiv \phi(t)\}$ with the norm $\|\phi\| = \max_{t\in[0,T]} |\phi(t)|$. Define $X = Y = C_T \times C_T = \{x = (x_1(\cdot), x_2(\cdot), x_3(\cdot)) \in C(\mathbb{R}, \mathbb{R}^3) : x(t) = x(t + T), t \in \mathbb{R}\}$ with the norm $\|x\| = \max\{\|x_1\|, \|x_2\|, \|x_3\|\}$. Clearly, X and Y are Banach spaces. Moreover, define

$$L : D(L) = \left\{x \in C^1\left(\mathbb{R}, \mathbb{R}^3\right) : x(t + T) = x(t), t \in \mathbb{R}\right\} \subset X \to Y,$$

by

$$(Lx)(t) = \begin{pmatrix} \frac{d}{dt}(\mathcal{A}x_1)(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{pmatrix}.$$

Also, we can define $N : X \to Y$ by

$$(Nx)(t) = \begin{pmatrix} x_2(t) \\ x_3(t) \\ -af(\dot{x}_1(t))\ddot{x}_1(t) - bg(t,\dot{x}_1(t)) - \sum_{i=1}^{n} c_i h(x_1(t-\gamma_i(t))) + e(t) \end{pmatrix}. \tag{3.2}$$

Then, (3.1) can be converted to the abstract equation $Lx = Nx$. From the definition of L, we obtain

$$\operatorname{Ker} L \cong \mathbb{R}^3, \qquad \operatorname{Im} L = \left\{ y \in Y : \int_0^T \begin{pmatrix} y_1(s) \\ y_2(s) \\ y_3(s) \end{pmatrix} ds = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}.$$

Therefore, we find that L is a Fredholm operator with index zero. Let $P_1 : X \to \operatorname{Ker} L$ and $Q_1 : Y \to \operatorname{Im} Q_1 \subset \mathbb{R}^3$ be defined by

$$P_1 x = \begin{pmatrix} (\mathcal{A}x_1)(0) \\ x_2(0) \\ x_3(0) \end{pmatrix}; \qquad Q_1 y = \frac{1}{T}\int_0^T \begin{pmatrix} y_1(s) \\ y_2(s) \\ y_3(s) \end{pmatrix} ds,$$

then $\operatorname{Im} P_1 = \operatorname{Ker} L$ and $\operatorname{Ker} Q_1 = \operatorname{Im} L$. Set $L_{P_1} = L|_{(D(L)\cap \operatorname{Ker} P_1)}$ and $L_{P_1}^{-1} : \operatorname{Im} L \to (D(L) \cap \operatorname{Ker} P_1)$ denotes the inverse of L_{P_1}, it follows that

$$\left[L_{P_1}^{-1} y\right](t) = \begin{pmatrix} (\mathcal{A}^{-1}Fy_1)(t) \\ (Fy_2)(t) \\ (Fy_3)(t) \end{pmatrix}, \tag{3.3}$$

where

$$[Fy_1](t) = \int_0^t y_1(s)\,ds, \qquad [Fy_2](t) = \int_0^t y_2(s)\,ds, \qquad [Fy_3](t) = \int_0^t y_3(s)\,ds.$$

From (3.2), we obtain

$$(Q_1 Nx)(t) = \frac{1}{T}\int_0^T \begin{pmatrix} x_2(t) \\ x_3(t) \\ -af(\dot{x}_1(t))\ddot{x}_1(t) - bg(t,\dot{x}_1(t)) - \sum_{i=1}^{n} c_i h(x_1(t-\gamma_i(t))) + e(t) \end{pmatrix} dt. \tag{3.4}$$

Thus, from (3.3) and (3.4), it is clear that $Q_1 N$ and $L_{P_1}^{-1}(I - Q_1)N$ are continuous, and $Q_1 N(\overline{\Omega})$ is bounded, and then $L_{P_1}^{-1}(I - Q_1)N(\overline{\Omega})$ is compact for any open bounded $\Omega \subset X$, which means N is L-compact on $\bar{\Omega}$.

Now, we will present the following hypotheses that will be used repeatedly during our work:

(H1) There exists a positive constant k_1 such that $|f(u)| \le k_1$, for $u \in \mathbb{R}$;

(H2) There exist positive constants k_2, h_1 such that $|g(t,u)| \le k_2$, $|h(x)| \le h_i$, for $(t,u) \in \mathbb{R} \times \mathbb{R}$ and $(t,x) \in \mathbb{R} \times \mathbb{R}$;

(H3) There exists a positive constant D such that $|h(x)| > \frac{bk_2}{c_i}$ and $x[f(u) + g(t, v) + h(x)] \neq 0$, for $t, u, v, x \in \mathbb{R}$ and $|x| > D$;

(H4) There exist positive constants b_o, c_0 such that $|h(x_1) - h(x_2)| \leq b_o|x_1 - x_2|$, $|g(t, u_1) - g(t, u_2)| \leq c_o|u_1 - u_2|$ for all $t, x_1, x_2, u_1, u_2 \in \mathbb{R}$.

The following theorem is our main result on the existence of a periodic solution for (1.1).

Theorem 3.1 *Suppose that assumptions* (H1)–(H4) *hold. Assume that the following assumption is satisfied*:

If $|\alpha| < 1$ *and*

(i) $1 - |\alpha| - |\alpha|\gamma_1(\gamma_1 - 2) - M_4 > 0$, where

$$M_4 = \frac{1}{2}(\sqrt{M_3} + \alpha\gamma_2 T),$$

$$M_3 = \left(bk_2 + b_0 c \sum_{i=1}^{n} \left\| \frac{1}{(1 - \dot{\gamma}_i)} \right\|_\infty D + nc \max\left\{ \left| h(t, 0) \right| : 0 \leq t \leq T \right\} + \|e\|_\infty \right) M_1 T,$$

$$M_1 = 1 + \alpha(1 + \gamma_1),$$

$$\gamma_1 = \max_{t\in[0,T]} |\dot{\gamma}|, \qquad \gamma_2 = \max_{t\in[0,T]} |\ddot{\gamma}|; \qquad c = \max_{t\in[0,T]} |c_i|,$$

then equation (1.1) *has at least one* T*-periodic solution.*

Proof We know that (3.1) has a T-periodic solution, if and only if, the following operator equation

$$Lx = \lambda Nx, \tag{3.5}$$

has a T-periodic solution. From (3.2), we see that N is L-compact in $\bar{\Omega}$, where Ω is an open bounded subset of X_T. For $\lambda \in (0, 1]$, define $\Omega_1 = \{x \in C_T : Lx = \lambda Nx\}$. Then, $x = (x_1, x_2, x_3)^\top \in \Omega_1$ satisfies:

$$\begin{cases} \frac{d}{dt}(\mathcal{A}x_1)(t) = \lambda x_2(t), \\ \dot{x}_2(t) = \lambda x_3(t), \\ \dot{x}_3(t) = \lambda(-af(\dot{x}_1(t))\ddot{x}_1(t) - bg(t, \dot{x}_1(t)) - \sum_{i=1}^{n} c_i h(x_1(t - \gamma_i(t))) + e(t)). \end{cases} \tag{3.6}$$

Substituting of $x_3(t) = \frac{1}{\lambda^2}\frac{d^2}{dt^2}(\mathcal{A}x_1)(t)$ into the third equation of (3.6), we obtain

$$\frac{d^3}{dt^3}\big(\mathcal{A}x_1(t)\big) = -a\lambda^3 f\big(\dot{x}_1(t)\big)\ddot{x}_1(t) - b\lambda^3 g\big(t, \dot{x}_1(t)\big) - \lambda^3 \sum_{i=1}^{n} c_i h\big(x_1\big(t - \gamma_i(t)\big)\big) + \lambda^3 e(t). \tag{3.7}$$

By integrating both sides of (3.7) over $[0, T]$, we find

$$\int_0^T \left(bg\bigl(t, \dot{x}_1(t)\bigr) + \sum_{i=1}^{n} c_i h\bigl(x_1\bigl(t - \gamma_i(t)\bigr)\bigr) \right) dt = 0, \tag{3.8}$$

which implies that there is at least one point t_1, such that

$$bg\bigl(t_1, \dot{x}_1(t_1)\bigr) + \sum_{i=1}^{n} c_i h\bigl(x_1\bigl(t_1 - \gamma_i(t_1)\bigr)\bigr) = 0.$$

By using (H2), we have

$$bg\bigl(t_1, \dot{x}_1(t_1)\bigr) + \sum_{i=1}^{n} c_i h\bigl(x_1\bigl(t_1 - \gamma_i(t_1)\bigr)\bigr) \le bk_2 + \sum_{i=1}^{n} c_i h_i := K.$$

In view of (H3) we see that $|x_1(t_1 - \gamma(t_1))| \le D$. Since $x_1(t)$ is periodic with period T, $t_1 - \gamma(t_1) = nT + \eta$, $\eta \in [0, T]$ and n is an integer, then $|x_1(\eta)| \le D$.

Thus, for $t \in [\eta, \eta + T]$, we obtain

$$\bigl|x_1(t)\bigr| = \left| x_1(\eta) + \int_\eta^t \dot{x}_1(s)\, ds \right| \le D + \int_\eta^t \bigl|\dot{x}_1(s)\bigr|\, ds$$

and

$$\bigl|x_1(t)\bigr| = \bigl|x_1(t - T)\bigr| = \left| x_1(\eta) - \int_{t-T}^{\eta} \dot{x}_1(s)\, ds \right| \le D + \int_{t-T}^{\eta} \bigl|\dot{x}_1(s)\bigr|\, ds.$$

Combining the above two inequalities, we obtain

$$\begin{aligned} \|x_1\|_\infty &= \max_{t\in[0,T]} \bigl|x_1(t)\bigr| = \max_{t\in[\eta,\eta+T]} \bigl|x_1(t)\bigr| \\ &\le \max_{t\in[\eta,\eta+T]} \left\{ D + \frac{1}{2} \left(\int_\eta^t \bigl|\dot{x}_1(s)\bigr|\, ds + \int_{t-T}^{\eta} \bigl|\dot{x}_1(s)\bigr|\, ds \right) \right\} \\ &\le D + \frac{1}{2} \int_0^T \bigl|\dot{x}_1(s)\bigr|\, ds \le D + \frac{1}{2} T \|\dot{x}_1\|_\infty. \end{aligned} \tag{3.9}$$

Since $x_1(0) = x_1(T)$, there is a constant $\zeta \in [0, T]$ such that $\dot{x}_1(\zeta) = 0$. Thus, we have

$$\begin{aligned} \bigl|\dot{x}_1(t)\bigr| &= \left| \dot{x}_1(\zeta) + \int_\zeta^t \ddot{x}_1(s)\, ds \right| \\ &\le \int_\zeta^t \bigl|\ddot{x}_1(s)\bigr|\, ds, \quad t \in [\zeta, T + \zeta] \end{aligned} \tag{3.10}$$

and

$$\begin{aligned} \bigl|\dot{x}_1(t)\bigr| &= \left| \dot{x}_1(\zeta + T) + \int_{\zeta+T}^{t} \ddot{x}_1(s)\, ds \right| \\ &\le \bigl|\dot{x}_1(\zeta + T)\bigr| + \int_t^{\zeta+T} \bigl|\ddot{x}_1(s)\bigr|\, ds = \int_t^{\zeta+T} \bigl|\ddot{x}_1(s)\bigr|\, ds, \quad t \in [0, T]. \end{aligned} \tag{3.11}$$

Combining the inequalities (3.10) and (3.11), we have

$$\|\dot{x}_1\|_\infty = \max_{t\in[0,T]} \left|\dot{x}_1(t)\right| \le \frac{1}{2}\int_0^T \left|\ddot{x}_1(s)\right| ds, \quad t \in [0,T]. \tag{3.12}$$

Now, by differentiating (2.1) with respect to t, we obtain

$$\begin{aligned}\frac{d}{dt}\big((\mathcal{A}x_1)(t)\big) &= \frac{d}{dt}\big(x_1(t) - \alpha x_1\big(t-\gamma(t)\big)\big)\\ &= \dot{x}_1(t) - \alpha\dot{x}_1\big(t-\gamma(t)\big)\big(1-\dot{\gamma}(t)\big).\end{aligned}$$

Since $\gamma_1 = \max_{t\in[0,T]} |\dot{\gamma}(t)|$ and from (3.9), we find

$$\left|\frac{d}{dt}\big((\mathcal{A}x_1)(t)\big)\right| \le \|\dot{x}_1\|_\infty + \alpha\|\dot{x}_1\|_\infty(1+\gamma_1) \le \big(1+\alpha(1+\gamma_1)\big)\|\dot{x}_1\|_\infty. \tag{3.13}$$

Then,

$$\left|\frac{d}{dt}\big((\mathcal{A}x_1)(t)\big)\right| \le M_1\|\dot{x}_1\|_\infty, \tag{3.14}$$

where

$$M_1 = 1 + \alpha(1+\gamma_1).$$

Also, we find

$$\frac{d^2}{dt^2}\big((\mathcal{A}x_1)(t)\big) = \ddot{x}_1(t) - \alpha\ddot{x}_1\big(t-\gamma(t)\big)\big(1-\dot{\gamma}(t)\big)^2 + \alpha\dot{x}_1\big(t-\gamma(t)\big)\ddot{\gamma}(t).$$

Then, we obtain

$$\begin{aligned}\frac{d^2}{dt^2}\big((\mathcal{A}x_1)(t)\big) &= \big(\ddot{x}_1(t) - \alpha\ddot{x}_1\big(t-\gamma(t)\big)\big)\\ &\quad - \alpha\big(\dot{\gamma}(t)-2\big)\dot{\gamma}(t)\ddot{x}_1\big(t-\gamma(t)\big) + \alpha\dot{x}_1\big(t-\gamma(t)\big)\ddot{\gamma}(t).\end{aligned}$$

Therefore, from the definition of the operator $\mathcal{A}$, we find

$$\begin{aligned}\frac{d^2}{dt^2}\big((\mathcal{A}x_1)(t)\big) &= (\mathcal{A}\ddot{x})(t) - \alpha\big(\dot{\gamma}(t)-2\big)\dot{\gamma}(t)\ddot{x}_1\big(t-\gamma(t)\big)\\ &\quad + \alpha\dot{x}_1\big(t-\gamma(t)\big)\ddot{\gamma}(t).\end{aligned}$$

Then, we can write the above equation as

$$\begin{aligned}(\mathcal{A}\ddot{x})(t) &= \frac{d^2}{dt^2}\big((\mathcal{A}x_1)(t)\big) - \alpha\dot{x}_1\big(t-\gamma(t)\big)\ddot{\gamma}(t)\\ &\quad + \alpha\big(\dot{\gamma}(t)-2\big)\ddot{x}_1\big(t-\gamma(t)\big)\dot{\gamma}(t).\end{aligned} \tag{3.15}$$

Now, by multiplying both sides of (3.7) by $\frac{d}{dt}((Ax_1)(t))$ and integrating it from 0 to T, we obtain

$$\begin{aligned}\int_0^T \frac{d^3}{dt^3}((\mathcal{A}x_1)(t)\frac{d}{dt}\big((\mathcal{A}x_1)(t)\big)\,dt &= -\int_0^T \left|\frac{d^2}{dt^2}\big((\mathcal{A}x_1)(t)\big)\right|^2 dt \\ &= -a\lambda^3\int_0^T f\big(\dot{x}_1(t)\big)\frac{d}{dt}(\mathcal{A}x_1)(t)\ddot{x}_1(t)\,dt \\ &\quad - b\lambda^3\int_0^T g\big(t,\dot{x}_1(t)\big)\frac{d}{dt}\big((\mathcal{A}x_1)(t)\big)\,dt \\ &\quad - \lambda^3\int_0^T\sum_{i=1}^n c_i h\big(x_1\big(t-\gamma_i(t)\big)\big)\frac{d}{dt}\big((\mathcal{A}x_1)(t)\big)\,dt \\ &\quad + \lambda^3\int_0^T e(t)\frac{d}{dt}\big((\mathcal{A}x_1)(t)\big)\,dt.\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}&\int_0^T\left|\frac{d^2}{dt^2}\big((\mathcal{A}x_1)(t)\big)\right|^2 dt \\ &\quad\le ak_1M_1\|\dot{x}_1\|_\infty\big(\dot{x}(T)-\dot{x}(t)\big) \\ &\qquad + b\int_0^T\big|g\big(t,\dot{x}_1(t)\big)\big|\left|\frac{d}{dt}\big((\mathcal{A}x_1)(t)\big)\right|dt \\ &\qquad + \int_0^T\sum_{i=1}^n c_i\big\{\big|h\big(t,x_1\big(t-\gamma_i(t)\big)\big)-h(t,0)+h(t,0)\big|\big\}\left|\frac{d}{dt}\big((\mathcal{A}x_1)(t)\big)\right|dt \\ &\qquad + \int_0^T\big|e(t)\big|\left|\frac{d}{dt}\big((\mathcal{A}x_1)(t)\big)\right|dt.\end{aligned}$$

Then, from the assumption (H4) we obtain

$$\begin{aligned}\int_0^T\left|\frac{d^2}{dt^2}\big((\mathcal{A}x_1)(t)\big)\right|^2 dt &\le b\int_0^T\big|g\big(t,\dot{x}_1(t)\big)\big|\left|\frac{d}{dt}\big((\mathcal{A}x_1)(t)\big)\right|dt \\ &\quad + \int_0^T\sum_{i=1}^n c_i\big(b_0\big|x_1\big(t-\gamma_i(t)\big)\big|+\big|h(t,0)\big|\big)\left|\frac{d}{dt}\big((\mathcal{A}x_1)(t)\big)\right|dt \\ &\quad + \int_0^T\big|e(t)\big|\left|\frac{d}{dt}\big((\mathcal{A}x_1)(t)\big)\right|dt.\end{aligned}$$

Now, by using (3.14), we can see that

$$\begin{aligned}&\int_0^T\sum_{i=1}^n c_ib_0\big|x_1\big(t-\gamma_i(t)\big)\big|\left|\frac{d}{dt}\big((\mathcal{A}x_1)(t)\big)\right|dt \\ &\quad\le M_1\|\dot{x}_1\|_\infty\int_0^T\sum_{i=1}^n c_ib_0\big|x_1\big(t-\gamma_i(t)\big)\big|\,dt \\ &\quad\le b_0\,M_1\|\dot{x}_1\|_\infty\sum_{i=1}^n c_i\int_0^T\left|\frac{1}{(1-\dot{\gamma}_i)}\right|\big|x_1\big(u(t)\big)\big|\,du\end{aligned}$$

$$\leq b_0 M_1 c \sum_{i=1}^{n} \left\| \frac{1}{(1-\dot{\gamma}_i)} \right\|_\infty \|\dot{x}_1\|_\infty \int_0^T \left| x_1\big(u(t)\big)\right| du.$$

By the assumptions (H1) and (H2), we conclude

$$\begin{aligned}\int_0^T \left| \frac{d^2}{dt^2}\big((\mathcal{A}x_1)(t)\big)\right|^2 dt &\leq \left(bk_2 + b_0 c \sum_{i=1}^{n} \left\| \frac{1}{(1-\dot{\gamma}_i)} \right\|_\infty \|x_1\|_\infty \right) M_1 \|\dot{x}_1\|_\infty T \\ &\quad + \big(nc \max\big\{\left|h(t,0)\right| : 0 \leq t \leq T\big\} + \|e\|_\infty\big) M_1 \|\dot{x}_1\|_\infty T.\end{aligned}$$

Thus, by (3.9), we obtain

$$\begin{aligned}\int_0^T \left| \frac{d^2}{dt^2}\big((\mathcal{A}x_1)(t)\big)\right|^2 dt &\leq \frac{1}{2} b_0 c T^2 M_1 \sum_{i=1}^{n} \left\| \frac{1}{(1-\dot{\gamma}_i)} \right\|_\infty \|\dot{x}_1\|_\infty^2 \\ &\quad + \left(bk_2 + b_0 c \sum_{i=1}^{n} \left\| \frac{1}{(1-\dot{\gamma}_i)} \right\|_\infty D \right. \\ &\quad \left. + nc \max\big\{\left|h(t,0)\right| : 0 \leq t \leq T\big\} + \|e\|_\infty \right) M_1 \|\dot{x}_1\|_\infty T.\end{aligned}$$

For positive constants M_2 and M_3, the above inequality becomes

$$\int_0^T \left| \frac{d^2}{dt^2}\big((\mathcal{A}x_1)(t)\big)\right|^2 dt \leq M_2 \|\dot{x}_1\|_\infty + M_3 |\dot{x}_1\|_\infty^2, \tag{3.16}$$

where

$$\begin{aligned} M_2 &= \frac{1}{2} b_0 c T^2 M_1 \sum_{i=1}^{n} \left\| \frac{1}{(1-\dot{\gamma}_i)} \right\|_\infty, \\ M_3 &= \left(bk_2 + b_0 c \sum_{i=1}^{n} \left\| \frac{1}{(1-\dot{\gamma}_i)} \right\|_\infty D + nc \max\big\{\left|h(t,0)\right| : 0 \leq t \leq T\big\} + \|e\|_\infty \right) M_1 T.\end{aligned}$$

By applying Lemma 2.1, we obtain

$$\int_0^T \left|\ddot{x}_1(t)\right| dt = \int_0^T \left|\big(\mathcal{A}^{-1}\mathcal{A}\,\ddot{x}_1\big)(t)\right| dt \leq \frac{\int_0^T |(\mathcal{A}\ddot{x}_1)(t)|\, dt}{1-|\alpha|}.$$

Substituting from (3.15) and by using the conditions of Theorem 3.1, we find

$$\begin{aligned}\int_0^T \left|\ddot{x}_1(t)\right| dt &\leq \frac{1}{1-|\alpha|} \left\{ \int_0^T \left| \frac{d^2}{dt^2}\big((\mathcal{A}x_1)(t)\big)\right| dt \right\} \\ &\quad + \frac{1}{1-|\alpha|} \left\{ \int_0^T \left| \alpha \dot{x}_1\big(t-\gamma(t)\big)\ddot{\gamma}(t)\right| dt \right\} \\ &\quad + \frac{1}{1-|\alpha|} \left\{ \int_0^T \left| \alpha\big(\dot{\gamma}(t)-2\big)\dot{\gamma}(t)\ddot{x}_1\big(t-\gamma(t)\big)\right| dt \right\} \\ &\leq \frac{1}{1-|\alpha|} \left\{ \int_0^T \left| \frac{d^2}{dt^2}\big((\mathcal{A}x_1)(t)\big)\right| dt \right\} + \frac{1}{1-|\alpha|} \left\{ \int_0^T \left| \alpha \dot{x}_1\big(t-\gamma(t)\big)\gamma_2\right| dt \right\}\end{aligned}$$

$$+\frac{1}{1-|\alpha|}\left\{\int_0^T \left|\alpha(\gamma_1-2)\gamma_1\ddot{x}_1\big(t-\gamma(t)\big)\right| dt\right\}.$$

From (3.9) and by using the Schwarz inequality, we conclude

$$\left[1-\alpha\frac{(\gamma_1-2)}{1-|\alpha|}\right]\int_0^T \left|\ddot{x}_1(t)\right| dt \le \frac{1}{1-|\alpha|}\left[T^{\frac{1}{2}}\left(\int_0^T \left|\frac{d^2}{dt^2}\big((\mathcal{A}x_1)(t)\big)\right|^2 dt\right)^{\frac{1}{2}}\right] + \frac{1}{1-|\alpha|}\big(\alpha\gamma_2 T\|\dot{x}_1\|_\infty\big).$$

It follows that

$$\big[|1-|\alpha||-\alpha\gamma_1(\gamma_1-2)\big]\int_0^T \left|\ddot{x}_1(t)\right| dt \le T^{\frac{1}{2}}\left(\int_0^T \left|\frac{d^2}{dt^2}\big((\mathcal{A}x_1)(t)\big)\right|^2 dt\right)^{\frac{1}{2}} + \alpha T\gamma_2\|\dot{x}_1\|_\infty.$$

Applying the inequality $(m+n)^r \le m^r + n^r$ for all $m,n>0$, $0<r<1$, implies from (3.16) that

$$\begin{aligned}
&\big[|1-|\alpha||-\alpha\gamma_1(\gamma_1-2)\big]\int_0^T \left|\ddot{x}_1(t)\right| dt \\
&\quad\le \sqrt{TM_2}\big(\|\dot{x}_1\|_\infty\big)^{\frac{1}{2}} + \sqrt{M_3}\|\dot{x}_1\|_\infty + \alpha T\gamma_2\|\dot{x}_1\|_\infty \\
&\quad\le \sqrt{TM_2}\big(\|\dot{x}_1\|_\infty\big)^{\frac{1}{2}} + (\sqrt{M_3}+\alpha T\gamma_2)\|\dot{x}_1\|_\infty.
\end{aligned}$$

Using (3.12), we find

$$\begin{aligned}
&\big[|1-|\alpha||-\alpha\gamma_1(\gamma_1-2)\big]\int_0^T \left|\ddot{x}_1(t)\right| dt \\
&\quad\le \sqrt{\frac{1}{2}TM_2}\left(\int_0^T \left|\ddot{x}_1(t)\right| dt\right)^{\frac{1}{2}} + M_4\int_0^T \left|\ddot{x}_1(t)\right| dt,
\end{aligned}$$

where

$$M_4 = \frac{1}{2}(\sqrt{M_3}+\alpha T\gamma_2).$$

Then, we conclude

$$\big[|1-|\alpha||-\alpha\gamma_1(\gamma_1-2)-M_4\big]\int_0^T \left|\ddot{x}_1(t)\right| dt \le \sqrt{\frac{1}{2}TM_2}\left(\int_0^T \left|\ddot{x}_1(t)\right| dt\right)^{\frac{1}{2}}. \tag{3.17}$$

Since $|1-|\alpha||-\alpha\gamma_1(\gamma_1-2)-M_4>0$, we can conclude that there exists a positive constant D_1, such that

$$\int_0^T \left|\ddot{x}_1(t)\right| dt \le D_1. \tag{3.18}$$

It follows from (3.12) that

$$\|\dot{x}_1\|_\infty \le \frac{1}{2}D_1.$$

Thus, from (3.9) we obtain

$$\|x_1\|_\infty \le D_2,$$

where

$$D_2 = D + \frac{1}{4} T D_1.$$

Using the first equation of system (3.6), we have

$$\int_0^T x_2(t)\,dt = \int_0^T \frac{d}{dt}\big((\mathcal{A}x_1)(t)\big)\,dt = 0,$$

which mean that there exists a constant $t_1 \in [0, T]$, such that $x_2(t_1) = 0$, then from (3.16) we find

$$\begin{aligned}\|x_2\|_\infty &\le \int_0^T \big|\dot{x}_2(t)\big|\,dt = \int_0^T \left|\frac{d^2}{dt^2}\big((\mathcal{A}x_1)(t)\big)\right|dt \le T^{\frac{1}{2}}\left(\int_0^T \left|\frac{d^2}{dt^2}\big((\mathcal{A}x_1)(t)\big)\right|^2 dt\right)^{\frac{1}{2}} \\ &\le \sqrt{T}\big(\sqrt{M_2}\|\dot{x}_1\|_\infty + M_3\|\dot{x}_1\|_\infty^2\big)^{\frac{1}{2}}.\end{aligned}$$

Therefore, we obtain

$$\|x_2\|_\infty \le D_3, \quad D_3 > 0,$$

where

$$D_3 = \sqrt{T}\big(\sqrt{M_2}\|\dot{x}_1\|_\infty + M_3\|\dot{x}_1\|_\infty^2\big)^{\frac{1}{2}}.$$

From the second equation of system (3.6), we have

$$\int_0^T x_3(t)\,dt = \int_0^T \frac{d^2}{dt^2}\big((\mathcal{A}x_1)(t)\big)\,dt = \int_0^T \dot{x}_2(t)\,dt = 0,$$

then, there is a constant $t_2 \in [0, T]$, such that $x_3(t_2) = 0$, hence

$$\|x_3\|_\infty \le \int_0^T \big|\dot{x}_3(t)\big|\,dt.$$

By the third equation of system (3.6), we have

$$\dot{x}_3(t) = -a\lambda f\big(\dot{x}_1(t)\big)\ddot{x} - b\lambda g\big(t, \dot{x}_1(t)\big) - \lambda \sum_{i=1}^{n} c_i h\big(t, x_1\big(t - \gamma_i(t)\big)\big) + \lambda e(t).$$

Using (H1), (H2), and (H4), we obtain

$$\begin{aligned}
\|x_3\|_\infty &\le \int_0^T |\dot{x}_3(t)|\,dt \\
&\le a\int_0^T |f(\dot{x}_1(t))||\ddot{x}_1(t)|\,dt + b\int_0^T |g(t,\dot{x}_1(t))|\,dt \\
&\quad + \int_0^T \sum_{i=1}^n c_i\big(h(x_1(t-\gamma_i(t))) - h(t,0) + h(t,0)\big)\,dt + \int_0^T |e(t)|\,dt \\
&\le a\int_0^T |f(\dot{x}_1(t))||\ddot{x}_1(t)|\,dt + b\int_0^T |g(t,\dot{x}_1(t))|\,dt \\
&\quad + \int_0^T \sum_{i=1}^n c_i\big(b_o|x_1(t-\gamma_i(t))| + |h(t,0)|\big)\,dt + \int_0^T |e(t)|\,dt \\
&\le \big(bk_2 + b_o\|x_1\|_\infty + nc\max\{|h(t,0)| : 0\le t\le T\} + \|e\|_\infty\big)T := D_4.
\end{aligned}$$

To prove condition (1) of Lemma 2.2, we assume that for any $\lambda \in (0,1)$ and any $x = x(t)$ in the domain of L, which also belongs to $\partial\Omega$, we must have $Lx \neq \lambda Nx$. For otherwise in view of (3.6), we obtain

$$\|x_1\|_\infty \le D_2\|x_2\|_\infty \le D_3, \qquad \|x_3\|_\infty \le D_4.$$

Let $D_5 = \max\{D_2, D_3, D_4\} + 1$, $\Omega = \{x = (x_1, x_2, x_3)^\top : \|x\| < D_5\}$, then we see that x belongs to the interior of Ω, which is contrary to the assumption that $x \in \partial\Omega$. Therefore, condition (1) of Lemma 2.2 is satisfied. Now, for all $x \in \partial\Omega \cap \operatorname{Ker} L$

$$Q_1Nx = \frac{1}{T}\int_0^T \begin{pmatrix} x_2(t) \\ x_3(t) \\ -af(\dot{x}_1(t))\ddot{x}(t) - bg(t,\dot{x}_1(t)) - \sum_{i=1}^n c_ih(x_1(t-\gamma_i(t))) + e(t) \end{pmatrix} dt.$$

If $Q_1Nx = 0$, then $x_2(t) = 0$, $x_3(t) = 0$, $x_1 = D_5$ or $-D_5$. However, if $x_1(t) = D_5$, then by H_3 we obtain

$$0 = \int_0^T h(t, D_5)\,dt,$$

from which there exists a point t_2 such that $h(t_2, D_5) = 0$. From assumption (H3), we have $D_5 \le D$, which yields a contradiction. Similarly if $x_1 = -\mathcal{M}_4$. Therefore, we have $Q_1Nx \neq 0$, hence for all $x \in \partial\Omega \cap \operatorname{Ker} L$, $x \notin \operatorname{Im} L$, so condition (2) of Lemma 2.2 is satisfied.

Define the isomorphism $J : \operatorname{Im} Q_1 \to \operatorname{Ker} L$ as follows:

$$J(x_1, x_2, x_3)^\top = (-x_3, x_1, x_2)^\top.$$

Let $H(\mu, x) = \mu x + (1-\mu)JQ_1Nx$, $(\mu, x) \in [0,1] \times \Omega$, then for all $(\mu, x) \in (0,1) \times (\partial\Omega \cap \operatorname{Ker} L)$,

$$H(\mu, x) = \begin{pmatrix} \mu x_1(t) + \frac{1-\mu}{T}\int_0^T [af(\dot{x}_1(t))\ddot{x}(t) + bg(t,\dot{x}_1(t)) \\ + \sum_{i=1}^n c_ih(x_1(t-\gamma_i(t))) - e(t)]\,dt \\ (\mu + (1-\mu))x_2(t) \\ (\mu + (1-\mu))x_3(t) \end{pmatrix}.$$

Since $\int_0^T e(t)\,dt = 0$, we can obtain

$$H(\mu,x) = \begin{pmatrix} \mu x_1(t) + \frac{1-\mu}{T}\int_0^T [af(\dot{x}_1(t))\ddot{x}(t) + bg(t,\dot{x}_1(t)) \\ + \sum_{i=1}^n c_i h(x_1(t-\gamma_i(t)))]\,dt \\ (\mu + (1-\mu))x_2(t) \\ (\mu + (1-\mu))x_3(t) \end{pmatrix},$$

for all $(\mu,x) \in (0,1) \times (\partial\Omega \cap \operatorname{Ker} L)$.

Using (H3), it is obvious that $x^\top H(\mu,x) \neq 0$, for all $(\mu,x) \in (0,1) \times (\partial\Omega \cap \operatorname{Ker} L)$. Hence,

$$\begin{aligned} \deg\{JQ_1N, \Omega \cap \operatorname{Ker} L, 0\} &= \deg\{H(0,x), \Omega \cap \operatorname{Ker} L, 0\} \\ &= \deg\{H(1,x), \Omega \cap \operatorname{Ker} L, 0\} \\ &= \deg\{I, \Omega \cap \operatorname{Ker} L, 0\} \neq 0. \end{aligned}$$

Hence, condition (3) of Lemma 2.2 is satisfied. By applying Lemma 2.2, we conclude that equation $Lx = Nx$ has a solution $x = (x_1,x_2,x_3)^\top$ on $\bar{\Omega} \cap D(L)$, thus (1.1) has a T-periodic solution $x(t)$. □

4 Uniqueness result

Suppose that

$$|x|_k = \left(\int_0^T |x(t)|^k\,dt\right)^{\frac{1}{k}}, \quad k \geq 1, \qquad |x|_\infty = \max_{t\in[0,T]} |x(t)|,$$

then we have the following uniqueness result.

Theorem 4.1 *Suppose that all conditions of Theorem 3.1 hold and $h(x)$ is a monotone strictly decreasing function in x and $|\alpha| < 1$ and assume that*

(H5) *There exists a positive constant k_3 such that* $f(u(t)) = k_3$, for all $u \in \mathbb{R}$;

(H6) *There exists a positive constant L such that $|g(t,u) - g(t,v)| \leq L|u - v|$; for all $u, v \in \mathbb{R}$.*

such that

$$\frac{1}{(1-|\alpha|)^2}\left(\alpha(1+|\alpha|) + \frac{1}{2}ak_3T + \frac{1}{4}c_0bT^2 + \frac{cb_0}{8}T^{\frac{5}{2}}\sum_{i=0}^n \left\|\frac{1}{(1-\dot{\gamma}_i)}\right\|_\infty\right) < 1.$$

Then, equation (1.1) has at most one T-periodic solution.

Proof Assume that $r_1(t)$ and $r_2(t)$ are two T-periodic solutions of (1.1), then we have $z(t) = r_1(t) - r_2(t)$. Thus, (1.1) takes the form

$$\begin{aligned} &\frac{d^3}{dt^3}\big((r_1(t) - r_2(t)) - \alpha r_1(t-\gamma(t)) - \alpha r_2(t-\gamma(t))\big) \\ &\quad + af(\dot{r}_1(t))\ddot{r}_1(t) - af(\dot{r}_2(t))\ddot{r}_2(t) + bg(t,\dot{r}_1(t)) - bg(t,\dot{r}_2(t)) \\ &\quad + \sum_{i=1}^n c_i\{h(r_1(t-\gamma_i(t))) - h(r_2(t-\gamma_i(t)))\} = 0. \end{aligned}$$

Since $f(u) = k_3$, we obtain

$$\frac{d^3}{dt^3}\big(z(t) - \alpha z\big(t - \gamma(t)\big)\big) + ak_3\ddot{z}(t) + bg\big(t, \dot{r}_1(t)\big) - bg\big(t, \dot{r}_2(t)\big) + \sum_{i=1}^{n} c_i\big\{h\big(r_1\big(t - \gamma_i(t)\big)\big) - h\big(r_2\big(t - \gamma_i(t)\big)\big)\big\} = 0. \tag{4.1}$$

By integrating (4.1) from 0 to T and using the condition $H6$, we obtain

$$\begin{aligned}
&\int_0^T \Big[b\big\{g\big(t, \dot{r}_1(t)\big) - g\big(t, \dot{r}_2(t)\big)\big\} + \sum_{i=1}^{n} c_i\big\{h\big(r_1\big(t - \gamma_i(t)\big)\big) - h\big(r_2\big(t - \gamma_i(t)\big)\big)\big\} \Big]\,dt \\
&\quad\le \int_0^T \Big[bL\big|\dot{r}_1(t) - \dot{r}_2(t)\big| + \sum_{i=1}^{n} c_i\big\{h\big(r_1\big(t - \gamma_i(t)\big)\big) - h\big(r_2\big(t - \gamma_i(t)\big)\big)\big\} \Big]\,dt \\
&\quad\le bL\int_0^T \big|\dot{z}(t)\big|\,dt + \int_0^T \sum_{i=1}^{n} c_i\big\{h\big(r_1\big(t - \gamma_i(t)\big)\big) - h\big(r_2\big(t - \gamma_i(t)\big)\big)\big\}\,dt \\
&\quad\le bL\big|z(T) - z(0)\big| + \int_0^T \sum_{i=1}^{n} c_i\big\{h\big(r_1\big(t - \gamma_i(t)\big)\big) - h\big(r_2\big(t - \gamma_i(t)\big)\big)\big\}\,dt.
\end{aligned}$$

Using the integral mean-value theorem, it follows that there exists a constant $s_1 \in [0, T]$ such that

$$\sum_{i=1}^{n} c_i\big\{h\big(r_1\big(s_1 - \gamma_i(s_1)\big)\big) - h\big(r_2\big(s_1 - \gamma_i(s_1)\big)\big)\big\} = 0. \tag{4.2}$$

Let $\bar{\gamma} = s_1 - \gamma_i(s_1) = nT + \zeta$, where $\zeta \in [0, T]$ and n is an integer. Hence, from equation (4.2) together with condition ($H6$) implies that there exists a constant $\zeta \in [0, T]$ such that

$$z(\zeta) = r_1(\zeta) - r_2(\zeta) = r_1(\bar{\gamma}) - r_2(\bar{\gamma}) = 0.$$

We can write

$$\big|z(t)\big| = \left|z(\zeta) + \int_\zeta^t \dot{z}(s)\,ds\right| \le \int_\zeta^t \big|\dot{z}(s)\big|\,ds.$$

Again, we have

$$\big|z(t)\big| = \left|z(\zeta + T) + \int_{\zeta+T}^t \dot{z}(s)\,ds\right| \le \int_t^{\zeta+T} \big|\dot{z}(s)\big|\,ds.$$

Hence, we have

$$2\big|z(t)\big| \le \int_\zeta^t \big|\dot{z}(s)\big|\,ds + \int_t^{\zeta+T} \big|\dot{z}(s)\big|\,ds = \int_0^T \big|\dot{z}(s)\big|\,ds.$$

By using the Schwartz inequality, we find

$$2\big|z(t)\big| \le \sqrt{T}\left(\int_0^T \big|\dot{z}(s)\big|^2\,ds\right)^{\frac{1}{2}} = \sqrt{T}|\dot{z}|_2.$$

Therefore, we obtain

$$\big|z(t)\big|_\infty \le \frac{1}{2}\sqrt{T}|\dot{z}|_2. \tag{4.3}$$

From the definition of the operator, we have

$$(\mathcal{A}z)(t) = x(t) - \alpha x\big(t - \gamma(t)\big).$$

Multiplying (4.1) by $\dddot{z}(t)$ and integrating it over $[0, T]$, we find

$$\begin{aligned}\int_0^T (\mathcal{A}\dddot{z})(t)\dddot{z}(t)\,dt = -ak_3 &\int_0^T \ddot{z}(t)\dddot{z}(t)\,dt \\ &- b\int_0^T \big[g\big(t,\dot{r}_1(t)\big) - g\big(t,\dot{r}_2(t)\big)\big]\dddot{z}(t)\,dt \\ &- \sum_{i=1}^n c_i \int_0^T h\big(r_1\big(t-\gamma_i(t)\big) - h\big(r_2\big(t-\gamma_i(t)\big)\big)\big)\dddot{z}(t)\,dt.\end{aligned}$$

By using condition H_4, we obtain

$$\begin{aligned}\int_0^T \big|(\mathcal{A}\dddot{z})(t)\big|\big|\dddot{z}(t)\big|\,dt \le ak_3 &\int_0^T \big|\ddot{z}(t)\big|\big|\dddot{z}(t)\big|\,dt \\ &+ bc_0 \int_0^T \big|\dot{z}(t)\big|\big|\dddot{z}(t)\big|\,dt \\ &+ b_0 \sum_{i=1}^n c_i \int_0^T \big|z\big(t-\gamma_i(t)\big)\big|\big|\dddot{z}(t)\big|\,dt.\end{aligned} \tag{4.4}$$

Hence, we have

$$\int_0^T (\mathcal{A}\dddot{z})(t)\dddot{z}(t)\,dt = \int_0^T (\mathcal{A}\dddot{z})(t)\big[\dddot{z}(t) - \alpha\dddot{z}\big(t-\gamma(t)\big) + \alpha\dddot{z}\big(t-\gamma(t)\big)\big]\,dt.$$

From the definition of the operator $\mathcal{A}$, we have

$$\int_0^T \big|(\mathcal{A}\dddot{z})(t)\big|\big|\dddot{z}(t)\big|\,dt = \int_0^T \big|(\mathcal{A}\dddot{z})(t)\big|^2\,dt + \alpha\int_0^T \big|(\mathcal{A}\dddot{z})(t)\big|\big|\dddot{z}\big(t-\gamma(t)\big)\big|\,dt. \tag{4.5}$$

Now, by applying the Schwartz inequality, we obtain

$$\begin{aligned}&\int_0^T \big|\dddot{z}\big(t-\gamma(t)\big)\big|\big|(\mathcal{A}\dddot{z})(t)\big|\,dt \\ &\quad\le \left(\int_0^T \big|\dddot{z}\big(t-\gamma(t)\big)\big|^2\,dt\right)^{\frac{1}{2}}\left(\int_0^T \left|\frac{d^3}{dt^3}\big((\mathcal{A}x_1)(t)\big)\right|^2\,dt\right)^{\frac{1}{2}} \\ &\quad= \left(\int_0^T \big|\dddot{z}\big(t-\gamma(t)\big)\big|^2\,dt\right)^{\frac{1}{2}}\left(\int_0^T \big|\dddot{z}(t) - \alpha\dddot{z}\big(t-\gamma(t)\big)\big|^2\,dt\right)^{\frac{1}{2}}.\end{aligned}$$

Then, we obtain

$$\int_0^T \big|\dddot{z}\big(t-\gamma(t)\big)\big|\big|(\mathcal{A}\dddot{z})(t)\big|\,dt \le |\dddot{z}|_2\big[|\dddot{z}|_2 + |\alpha||\dddot{z}_2|\big] = \big(1+|\alpha|\big)|\dddot{z}|_2^2. \tag{4.6}$$

By substituting from (4.6) into (4.5), we obtain

$$\int_0^T (\mathcal{A}\dddot{z})(t)\dddot{z}(t)\,dt \le \int_0^T \left|(\mathcal{A}\dddot{z})(t)\right|^2 dt + |\alpha|\bigl(1+|\alpha|\bigr)|\dddot{z}|_2^2. \tag{4.7}$$

Substituting from (4.7) into (4.4) and using the Schwarz inequality, we find

$$\begin{aligned}\int_0^T \left|(\mathcal{A}\dddot{z})(t)\right|^2 dt &\le |\alpha|\bigl(1+|\alpha|\bigr)|\dddot{z}|_2^2 ak_3\|\ddot{z}\|_2\|\dddot{z}\|_2 + c_0 b\|\dot{z}\|_2\|\dddot{z}\|_2 \\ &\quad + cb_0\sum_{i=0}^{n}\left\|\frac{1}{(1-\dot{\gamma}_i)}\right\|_\infty \|z\|_\infty\|\dddot{z}\|_2. \end{aligned}\tag{4.8}$$

Since $z(0) = z(T)$, there exists a constant $\xi \in [0, T]$, such that $\dot{z}(\xi) = 0$ and

$$\begin{aligned}\left|\dot{z}(t)\right| &= \left|\dot{z}(\xi) + \int_\xi^t \ddot{z}(s)\,ds\right| \\ &\le \int_\xi^t \left|\ddot{z}(s)\right| ds, \quad t \in [\xi, T+\xi]. \end{aligned}\tag{4.9}$$

Also, for $t \in [0, T]$, we have

$$\begin{aligned}\left|\dot{z}(t)\right| &= \left|\dot{z}(\xi+T) + \int_{\xi+T}^t \ddot{z}(s)\,ds\right| \\ &\le \left|\dot{z}(\xi+T)\right| + \int_t^{\xi+T} \left|\ddot{z}(s)\right| ds \\ &= \int_t^{\xi+T} \left|\ddot{z}(s)\right| ds. \end{aligned}\tag{4.10}$$

By combining (4.9) and (4.10), we obtain

$$\begin{aligned}2\left|\dot{z}(t)\right| &\le \int_\xi^t \left|\ddot{z}(s)\right| ds + \int_t^{\xi+T} \left|\ddot{z}(s)\right| ds \\ &= \int_0^T \left|\ddot{z}(s)\right| ds, \quad t \in [0, T]. \end{aligned}$$

Therefore, by using the Schwartz inequality, we have

$$\left|\dot{z}(t)\right| \le \frac{1}{2}\sqrt{T}\left(\int_0^T \left|\ddot{z}(s)\right|^2 ds\right)^{\frac{1}{2}}, \quad \text{for all} t \in [0, T], \tag{4.11}$$

hence, we obtain

$$|\dot{z}|_\infty \le \frac{1}{2}\sqrt{T}|\ddot{z}|_2, \tag{4.12}$$

therefore, we obtain

$$|\dot{z}|_2 \le \sqrt{T}\max_{t\in[0,T]}\left|\dot{z}(s)\right| \le \frac{1}{2}T\left(\int_0^T \left|\ddot{z}(s)\right|^2 ds\right)^{\frac{1}{2}} = \frac{1}{2}T|\ddot{z}|_2. \tag{4.13}$$

Since $\dot{z}(t)$ is a periodic function for $t \in [0,T]$ by using the above similar technique we obtain

$$\left|\ddot{z}(t)\right| \leq \frac{1}{2}\int_0^T \left|\dddot{z}(t)\right| dt,$$

which, together with the Schwartz inequality, implies

$$|\ddot{z}|_\infty \leq \frac{1}{2}\sqrt{T}\left(\int_0^T \left|\dddot{z}(s)\right|^2 ds\right)^{\frac{1}{2}} = \frac{1}{2}\sqrt{T}|\dddot{z}|_2, \tag{4.14}$$

then, we obtain

$$|\ddot{z}|_2 \leq \sqrt{T}\max_{t\in[0,T]}\left|\ddot{z}(s)\right| \leq \frac{1}{2}\sqrt{T}\int_0^T \left|\dddot{z}(s)\right| ds \leq \frac{1}{2}T|\dddot{z}|_2. \tag{4.15}$$

By substituting (4.15) into (4.13), we obtain

$$|\dot{z}|_2 \leq \frac{1}{4}T^2|\dddot{z}|_2. \tag{4.16}$$

By using (4.13), (4.15), (4.16), and (4.3), (4.8) becomes

$$\begin{aligned}&\int_0^T \left|(\mathcal{A}\dddot{z})(t)\right|^2 dt\\ &\quad\leq \left\{|\alpha|\big(1+|\alpha|\big) + \frac{1}{2}ak_3T + \frac{1}{4}c_0bT^2 + \frac{cb_0}{8}T^{\frac{5}{2}}\sum_{i=0}^{n}\left\|\frac{1}{(1-\dot{\gamma}_i)}\right\|_\infty\right\}\|\dddot{z}\|_2^2.\end{aligned} \tag{4.17}$$

From Lemma 2.1, we have

$$|\dddot{z}|_2^2 = \int_0^T \left|(\mathcal{A}^{-1}\mathcal{A})\dddot{z}(t)\right|^2 dt \leq \frac{1}{(1-|\alpha|)^2}\int_0^T \left|(\mathcal{A}\dddot{z})(t)\right|^2 dt. \tag{4.18}$$

Substituting (4.18) into (4.17), we conclude

$$\begin{aligned}\left|(\mathcal{A}\dddot{z})(t)\right|_2^2 \leq{}& \left\{\alpha\big(1+|\alpha|\big) + \frac{1}{2}ak_3T + \frac{1}{4}c_0bT^2\right.\\ &\left.+ \frac{cb_0}{8}T^{\frac{5}{2}}\sum_{i=0}^{n}\left\|\frac{1}{(1-\dot{\gamma}_i)}\right\|_\infty\right\}\frac{1}{(1-|\alpha|)^2}\left|(\mathcal{A}\dddot{z})(t)\right|_2^2.\end{aligned}$$

Hence, we conclude

$$\begin{aligned}&\left\{1 - \frac{1}{(1-|\alpha|)^2}\left(\alpha\big(1+|\alpha|\big) + \frac{1}{2}ak_3T + \frac{1}{4}c_0bT^2\right.\right.\\ &\quad\left.\left.+ \frac{cb_0}{8}T^{\frac{5}{2}}\sum_{i=0}^{n}\left\|\frac{1}{(1-\dot{\gamma}_i)}\right\|_\infty\right)\right\}\left|(\mathcal{A}\dddot{z})(t)\right|_2^2 \leq 0.\end{aligned}$$

Since

$$\frac{1}{(1-|\alpha|)^2}\left(\alpha\big(1+|\alpha|\big)+\frac{1}{2}ak_3T+\frac{1}{4}c_0bT^2+\frac{cb_0}{8}T^{\frac{5}{2}}\sum_{i=0}^{n}\left\|\frac{1}{(1-\dot{\gamma}_i)}\right\|_\infty\right)<1,$$

we find

$$\big|(\mathcal{A}\dddot{z})(t)\big|_2^2=0.$$

Since $\mathcal{A}z(t)$, $\frac{d}{dt}((\mathcal{A}z)(t))$, $\frac{d^2}{dt^2}((\mathcal{A}z)(t))$, and $\frac{d^3}{dt^3}((\mathcal{A}z)(t))$ are T-periodic and continuous functions, we have

$$\mathcal{A}z(t)\equiv\frac{d}{dt}\big((\mathcal{A}z)(t)\big)\equiv\frac{d^2}{dt^2}\big((\mathcal{A}z)(t)\big)\equiv\frac{d^3}{dt^3}\big(\mathcal{A}z(t)\big)=0,\quad \text{for all } t\in\mathbb{R}.$$

Now, applying Lemma 2.1 in [12], we obtain

$$z(t)\equiv\dot{z}(t)\equiv\ddot{z}(t)\equiv\dddot{z}(t)=0,\quad \forall t\in\mathbb{R}.$$

Hence, we conclude $r_1(t)\equiv r_2(t)$ for all $t\in\mathbb{R}$. □

Hence, (1.1) has a unique T-periodic solution.

5 Example

Consider the following third-order NDDE:

$$\begin{aligned}&\frac{d^3}{dt^3}\left(x(t)-\frac{1}{130}x\left(t-\frac{1}{150}\sin 4t\right)\right)+\frac{1}{6}\cos^2 4t\ddot{x}(t)\\&\quad+\frac{1}{120}\sin 4t\cos\dot{x}(t)+\frac{1}{10}\left(\frac{4}{\pi}x\left(t-\frac{1}{150}\sin 4t\right)\right)=\cos 4t.\end{aligned}\tag{5.1}$$

Comparing (5.1) to (1.1), we find $f(u)=\cos^2 4t$, $a=\frac{1}{6}$, $\alpha=\frac{1}{130}$, $g(t,u)=\sin 4t\cos u$, $b=\frac{1}{120}$, $h(t,x)=\frac{4}{\pi}x(t-\frac{1}{150}\sin 4t)$, $h(t,0)=0$, $b_o=\frac{4}{\pi}$, $c=\frac{1}{10}$, $\gamma(t)=\frac{1}{150}\sin 14t$, $\dot{\gamma}(t)=\frac{4}{150}\cos 4t$, $e(t)=\cos 4t$, and let $T=\frac{\pi}{4}$.

Also, we have

$$\gamma_1=\max_{t\in[0,\frac{\pi}{4}]}\big|\dot{\gamma}(t)\big|=\frac{2}{75},$$

and

$$\gamma_2=\max_{t\in[0,\frac{\pi}{4}]}\big|\ddot{\gamma}(t)\big|=\frac{4}{75},\qquad \left\|\frac{1}{1-\dot{\gamma}}\right\|_\infty=\frac{75}{73}.$$

Therefore, by taking $n=c=k_2=1$, we obtain

$$\begin{aligned}M_1&=1+\alpha(1+\gamma_1)=1.008,\\M_3&=\left\{bk_2+b_0c\left\|\frac{1}{1-\dot{\gamma}}\right\|_\infty D+nc\max\big\{\big|h(t,0)\big|:0\le t\le T\big\}+\|e\|_\infty\right\}M_1T=1.29,\\M_4&=\frac{1}{2}\big(\sqrt{M_3}+|\alpha|\gamma_2T\big)=0.568.\end{aligned}$$

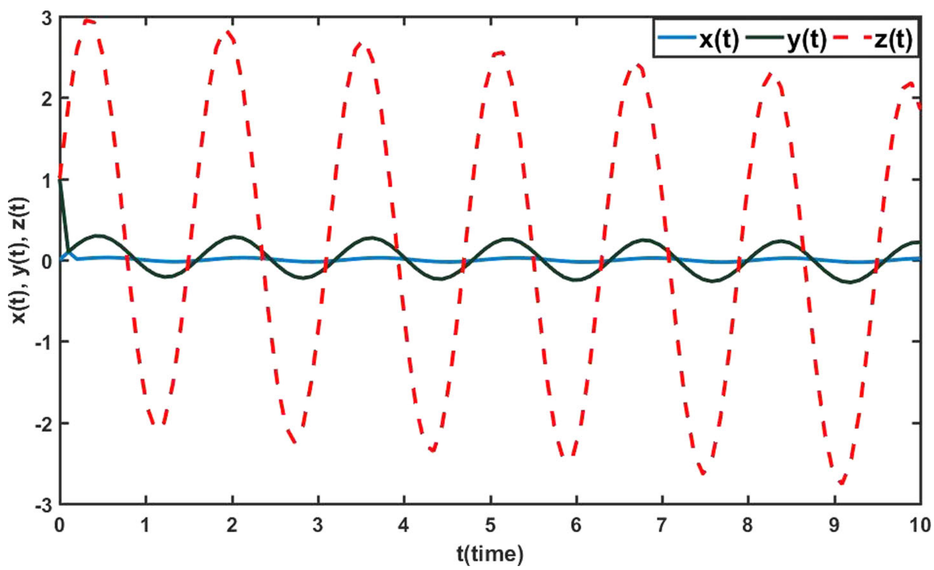

Figure 1 Path of the periodic solution

Hence, we find

$$|1-|\alpha||-|\alpha|\gamma_1(\gamma_1-2)-M_4=0.425>0.$$

To verify how to obtain (3.18) from (3.17), we calculate the following

$$M_2=\frac{1}{2}b_0cT^2M_1\left\|\frac{1}{1-\dot{\gamma}}\right\|_\infty=0.081.$$

Then, (3.17) becomes

$$0.425\times\int_0^T|\ddot{x}_1(t)|\,dt\le\sqrt{\frac{0.081\pi}{2}}\left(\int_0^T|\ddot{x}_1(t)|\,dt\right)^{\frac{1}{2}}.$$

Therefore, we obtain

$$\left(\int_0^T|\ddot{x}_1(t)|\,dt\right)^{\frac{1}{2}}\left\{0.425\left(\int_0^T|\ddot{x}_1(t)|\,dt\right)^{\frac{1}{2}}-\sqrt{\frac{0.081\pi}{2}}\right\}\le 0,$$

which can be considered as a quadratic inequality, its roots are

$$\left(\int_0^T|\ddot{x}_1(t)|\,dt\right)^{\frac{1}{2}}\le 0\quad\text{or}\quad\left(\int_0^T|\ddot{x}_1(t)|\,dt\right)^{\frac{1}{2}}\le 0.839,$$

which implies that

$$\int_0^T|\ddot{x}_1(t)|\,dt\le 0.7044.$$

The rest of the proof is clear. Hence, by Theorem 3.1, (5.1) has at least one $\frac{\pi}{8}$-periodic solution.

Now, by taking $k_3 = 1$ and $c_0 = 1$, we have

$$\frac{1}{(1-|\alpha|)^2}\left(\alpha\left(1+|\alpha|\right)+\frac{1}{2}ak_3T+\frac{1}{4}c_0bT^2+\frac{cb_0}{8}T^{\frac{5}{2}}\left\|\frac{1}{(1-\dot{\gamma})}\right\|_\infty\right)=0.17<1.$$

Thus, (1.1) has a unique periodic solution, see Fig. 1.

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