Elhamza and Hafdallah *Boundary Value Problems* (2023) 2023:23
https://doi.org/10.1186/s13661-023-01708-3

Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

Identification of the bulk modulus coefficient in the acoustic equation from boundary observation: a sentinel method

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Abstract

In this paper, we consider an acoustic equation with incomplete data, where the bulk modulus coefficient and initial conditions are partially known. Our goal is to get information about the bulk modulus coefficient independently of the initial conditions from boundary observations. To achieve this goal, we apply the sentinel method introduced by J.L. Lions, which is a functional that links the solution to the given problem with a control function and a state observation. We prove that the existence of the sentinel functional is equivalent to a boundary-null controllability problem with constraints on the control. We use the Hilbert uniqueness method to study this controllability problem to establish the control of minimal norm.

MSC: 35L05; 35L20; 49J20; 93B05; 93C20

Keywords: Bulk modulus coefficient; Incomplete data; Boundary-null controllability problem; Sentinel method; Hilbert uniqueness method

1 Introduction and preliminaries

Let $\Omega \subset \mathbb{R}^n$ be an open bounded domain with boundary Γ of class C^2. For fixed time $T > 0$, we denote $Q = \Omega \times [0, T]$ and $\Sigma = \Gamma \times [0, T]$. We consider the following acoustic wave equation:

$$\begin{cases} \partial_t^2 y - \sum_{i,j=1}^n \frac{\partial}{\partial x_i}(a_{ij}(x)\frac{\partial y}{\partial x_j}) = 0 & \text{in } \Omega \times (0, T), \\ y(0) = B(x) & \text{in } \Omega, \\ \partial_t y(0) = C(x) & \text{in } \Omega, \\ y = 0 & \text{on } \Gamma \times (0, T), \end{cases} \tag{1}$$

where $\partial_t^2 y = \frac{\partial^2 y}{\partial t^2}$, and $a_{ij}(x) = a_{ij}(x)$, $i, j = 1, \dots, n$, are C^∞ function on R^n satisfying the uniform ellipticity condition

$$\sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \geq \alpha \sum_{i,j=1}^n \xi_i^2, \quad x \in \Omega \text{ for some } \alpha > 0. \tag{2}$$

Given $B \in H_0^1(\Omega)$ and $C \in L^2(\Omega)$, problem (1) admits a unique solution $y \in C([0,T]; H_0^1(\Omega) \times L^2(\Omega))$ [19].

In this paper, we assume that $a_{ij}(x) = a(x)\delta_{ij}$ for all $i,j = 1,\dots,n$, where δ_{ij} is the Kronecker index, so that we can write our equation in the divergence form

$$\begin{cases} \partial_t^2 y - div(a(x)\nabla y) = 0 & \text{in } \Omega \times (0,T), \\ y(0) = B(x) & \text{in } \Omega, \\ \partial_t y(0) = C(x) & \text{in } \Omega, \\ y = 0 & \text{on } \Gamma \times (0,T), \end{cases} \tag{3}$$

where $a \in C^1(\overline{\Omega})$, and we assume that $a(x) > 0$ for all $x \in \overline{\Omega}$.

For equation (3), if the coefficient a and the functions B and C are known, then we can prove that the problem has a unique solution; this is a direct problem.

In other cases, according to the studied phenomena, some quantities in governing equations may be unknown (partially known), i.e., we have problems with missing data (incomplete data); for example, in problems of meteorology the initial conditions are unknown, in problems related to pollution, the term source is unknown, etc.

Moreover, the main purpose in studying these problems with incomplete data is identifying those unknown quantities. This determination proceeds through specific measurements and observations of the available data, which means that we have an inverse problem. When we have to identify the coefficient, we have a coefficient inverse problem, and for the determination of the source term, we have a source inverse problem.

These problems have an important role in mathematical fields such as PDEs, microlocal analysis, probability, etc. Moreover, they have important applications in wide fields of applied science, for example, medical imaging (X-rays, scanners, ...), radar, image processing, petroleum engineering, etc.

In this paper, we deal with the identification of the partially known coefficient a of the form

$$a(x) = a_0(x) + \lambda\widehat{a}_0(x),$$

where a_0 is known, and $\lambda\widehat{a}_0$ is unknown important term.

In addition, we assume that the initial conditions are partially known and are of the forms

$$B(x) = y_0(x) + \tau_0\widehat{y}_0(x),$$

$$C(x) = y_1(x) + \tau_1\widehat{y}_1(x),$$

where the function $(y_0, y_1) \in H_0^1(\Omega) \times L^2(\Omega)$ are known, whereas $\tau_0\widehat{y}_0$ and $\tau_1\widehat{y}_1$ are both unknown with $\|\widehat{y}_0(x)\|_{H_0^1(\Omega)} \leq 1$ and $\|\widehat{y}_1(x)\|_{L^2(\Omega)} \leq 1$.

The parameters λ, τ_0, and τ_1 are sufficiently small real numbers.

We can formulate our inverse problem as follows: To get information on the important term $\lambda\widehat{a}_0$ regardless of computing the unimportant terms $\tau_0\widehat{y}_0$ and $\tau_1\widehat{y}_1$ in the equation with incomplete data

$$\begin{cases} \partial_t^2 y - div((a_0(x) + \lambda\widehat{a}_0(x))\nabla y) = 0 & \text{in } \Omega \times (0,T), \\ y(0) = y_0 + \tau_0\widehat{y}_0 & \text{in } \Omega, \\ \partial_t y(0) = y_1 + \tau_1\widehat{y}_1 & \text{in } \Omega, \\ y = 0 & \text{on } \Gamma \times (0,T), \end{cases} \tag{4}$$

from the knowledge of the conormal derivative

$$\left.\frac{\partial y}{\partial \nu_a}\right|_{O\times(0,T)}, \tag{5}$$

where O (the observatory) is a nonempty open subset of Γ, $\frac{\partial y}{\partial \nu_a} = a(x)\nabla y.\nu(x)$ is the conormal derivative, and $\nu(x)$ is the unit outward normal vector to Γ at x.

Physically, our inverse problem consists of the determination of the bulk modulus in the acoustic equation (4) considered in a nonhomogeneous medium [8]. Coefficient inverse problems for acoustic equations are well studied in the literature, Imanuvilov and Yamamoto [8] considered the identification of the bulk modulus coefficient in acoustic equation (3) from a single measurement. For a similar coefficient inverse problem for an acoustic equation, Bellassoued and Yamamoto [4] studied the global stability in determination of the coefficient $a(x)$ from observation of the data in the boundary. For an inverse problem of the acoustic equation where $a(x) = (a_{ij}(x))_{1\leq i,j\leq n}$, $n \in \mathbb{N}$, we refer to [3] and [9]. Other authors have also dealt with other coefficient inverse problems in [2, 15, 16, 18], and [7], where the authors consider the determination of the coefficient $q(x)$ (the potential coefficient) in the equation $\frac{\partial^2 y}{\partial t^2} - \Delta y + q(x)y$ from specific measurement; most of the previously mentioned papers are based on the Carleman estimate.

The aim of this work is to use a method that has never been applied to inverse problems to identify an unknown coefficient. This method was first used by Lions [11]; it is an upgraded method of the least square method, where all unknown quantities are identified with each other and cannot be separated, and this does not serve us in reaching our goal. However, it has been applied to many problems. The most famous identification problems were presented in the book of Lions [11]. We also mention some recent research [1, 12, 17], and we also refer to [13].

Recently, the authors of [5] used the sentinel method to identify the potential coefficient in the wave equation with incomplete data. In addition, the authors of [6] applied the sentinel method to an identification problem of a fractional thermoelastic deformation system with incomplete data.

This method is called the sentinel method and is applied to problems with missing data to define the important terms independent of the nonimportant terms from the knowledge of specific measurements. It is based on the functional S, depending on the parameters λ, τ_0, and τ_1, defined as follows: for $h_0 \in L^2(O \times [0,T])$,

$$S(\lambda,\tau_0,\tau_1) = \int_0^T \int_\Gamma (h_0\chi_O + u\chi_\omega)\frac{\partial y}{\partial \nu_a}\, d\Gamma\, dt, \tag{6}$$

where χ_O and χ_ω denote the characteristic functions for the open subsets O and ω, respectively, and u is a control function determined as

1) For all $\tau_0\widehat{y}_0$ and $\tau_1\widehat{y}_1$,

$$\frac{\partial S}{\partial_{\tau_0}}(\lambda,\tau_0,\tau_1)\Big|_{\lambda=0,\tau_0=0,\tau_1=0} = \frac{\partial S}{\partial_{\tau_1}}(\lambda,\tau_0,\tau_1)\Big|_{\lambda=0,\tau_0=0,\tau_1=0} = 0. \tag{7}$$

2) The control u is of minimal norm in $L^2(\omega\times(0;T))$ among "the admissible controls", i.e.,

$$\|u\|^2_{L^2(\omega\times(0,T))} = \min_{\widetilde{u}\in U_{ad}} \|\widetilde{u}\|^2_{L^2(\omega\times(0;T))}, \tag{8}$$

where

$$U_{ad} = \left\{\widetilde{u}\in L^2\big(\omega\times(0,T)\big) \text{ such that } \big(\widetilde{u},S(\widetilde{u})\big) \text{ satisfies (7)}\right\}.$$

The existence of a sentinel for the given problem is equivalent to solving a controllability problem with constraints on the control. So we deal with the controllability problem to determine the control function, which is the main objective of this paper.

2 Information given by sentinel

In this section, we present the main result of this paper, which allows us to give information on the pollution term.

Let be $y_\lambda = \frac{\partial y}{\partial\lambda}$ the unique solution of the following system:

$$\begin{cases} \partial_t^2 y_\lambda - div(a_0(x)\nabla y_\lambda) - div(\widehat{a}_0(x)\nabla y_0) = 0 & \text{in } \Omega\times(0,T),\\ y_\lambda(0) = 0 & \text{in } \Omega,\\ \partial_t y_\lambda(0) = 0 & \text{in } \Omega,\\ y_\lambda = 0 & \text{on } \Gamma\times(0,T), \end{cases} \tag{9}$$

where y_0 is the solution of (4) when $\lambda = \tau_0 = \tau_1 = 0$.

On the other hand, we consider the sentinel associated with the measure m_0

$$S_{obs}(\lambda,\tau) = \int_0^T \int_\Gamma (h_0\chi_O + u\chi_\omega) m_0(x,t,\lambda,\tau)\,dx\,dt, \tag{10}$$

where m_0 is the measured state of the system on the observatory O through the interval $[0,T]$.

Theorem 1 *The information given by the sentinel about the important term $\lambda\widehat{a}_0(x)$ is as follows*:

$$\int_0^T \int_\Omega div\big(\lambda\widehat{a}_0(x)\nabla y_0\big) q\,dx\,dt = \int_0^T \int_\Gamma (h_0\chi_O + u\chi_\omega)\left(m_0 - \frac{\partial y_0}{\partial\nu_a}\right) d\Gamma\,dt. \tag{11}$$

Proof According to Taylor's formula, we have

$$S(\lambda,\tau_0,\tau_1) \simeq S(0,0,0) + \tau_0\frac{\partial S}{\partial\tau_0}(0,0,0) + \tau_1\frac{\partial S}{\partial\tau_1}(0,0,0) + \lambda\frac{\partial S}{\partial\lambda}(0,0,0) \tag{12}$$

for small λ, τ_0, and τ_1.

Due to (7), considering that $S(\lambda, \tau_0, \tau_1)$ is observed, we have

$$\lambda \frac{\partial S}{\partial \lambda}(\lambda, \tau_0, \tau_1)\bigg|_{\lambda=0,\tau_0=0,\tau_1=0} \simeq S_{obs}(\lambda, \tau_0, \tau_1) - S(0,0,0), \tag{13}$$

where

$$\frac{\partial S}{\partial \lambda} = \int_0^T \int_\Gamma (h_0 \chi_O + u\chi_\omega) \frac{\partial y_\lambda}{\partial \nu_a} \, d\Gamma \, dt,$$

where y_λ is the unique solution of (9).

Multiplying both sides of the first equation in (9) by the solution $q = q(x,t)$ of (19), we have

$$\int_0^T \int_\Omega \big\{\partial_t^2 y_\lambda - div\big(a_0(x)\nabla y_\lambda\big) - div\big(\widehat{a}_0(x)\nabla y_0\big)\big\} q = 0.$$

Integrating by parts and applying Green's formula, we obtain

$$\int_0^T \int_\Omega div\big(\widehat{a}_0(x)\nabla y_0\big) q \, dx \, dt = \int_0^T \int_\Gamma (h_0 \chi_O + u\chi_\omega) \frac{\partial y_\lambda}{\partial \nu_a} \, d\Gamma \, dt. \tag{14}$$

Combining (14) and (13), we obtain the information about the pollution term given in the previous theorem. □

3 Equivalence of the existence of a sentinel and the boundary null-controllability problem

We denote by $y_{\tau_0} = \frac{\partial y}{\partial \tau_0}(\lambda, \tau_0, \tau_1)|_{\lambda=0,\tau_0=0,\tau_1=0}$ the solution of

$$\begin{cases} \partial_t^2 y_{\tau_0} - div(a_0(x)\nabla y_{\tau_0}) = 0 & \text{in } \Omega \times (0,T), \\ y_{\tau_0}(0) = \widehat{y}_0 & \text{in } \Omega, \\ \partial_t y_{\tau_0}(0) = 0 & \text{in } \Omega, \\ y_{\tau_0} = 0 & \text{on } \Gamma \times (0,T), \end{cases} \tag{15}$$

and by $y_{\tau_1} = \frac{\partial y}{\partial \tau_1}(\lambda, \tau_0, \tau_1)|_{\lambda=0,\tau_0=0,\tau_1=0}$ the solution of

$$\begin{cases} \partial_t^2 y_{\tau_1} - div(a_0(x)\nabla y_{\tau_1}) = 0 & \text{in } \Omega \times (0,T), \\ y_{\tau_1}(0) = 0 & \text{in } \Omega, \\ \partial_t y_{\tau_1}(0) = \widehat{y}_1 & \text{in } \Omega, \\ y_{\tau_1} = 0 & \text{on } \Gamma \times (0,T). \end{cases} \tag{16}$$

Remark 1 Condition 1 (the insensitivity condition) can be written in the form

$$\int_{\Gamma\times(0,T)} (h\chi_O + u\chi_w) \frac{\partial y_{\tau_0}}{\partial \nu_a} \, d\Gamma \, dt = 0 \tag{17}$$

and

$$\int_{\Gamma\times(0,T)} (h_0\chi_O + u\chi_w) \frac{\partial y_{\tau_1}}{\partial \nu_a} \, d\Gamma \, dt = 0. \tag{18}$$

In this section, we will show that the existence of the sentinel for (4) is equivalent to solving an optimal control problem.

Our first result is the following:

Proposition 2 *The existence of the sentinel defined in* (6) *for problem* (4) *is equivalent to solving the following null-controllability problem*:

$$\begin{cases} \partial_t^2 q - div(a_0(x)\nabla q) = 0 & in\ \Omega \times (0,T), \\ q(x,T) = 0 & in\ \Omega, \\ \partial_t q(x,T) = 0 & in\ \Omega, \\ q = h_0\chi_O + u\chi_w & on\ \Gamma \times (0,T). \end{cases} \tag{19}$$

with null-controllability property

$$\begin{cases} q(x,0) = 0, \\ \partial_t q(x,0) = 0 \end{cases} \tag{20}$$

in Ω.

Proof Multiplying both sides of the first equation in (19) by the solution y_{τ_0} of (15), integrating by parts, and applying Green's formula, we get

$$\begin{aligned} 0 &= \int_0^T \int_\Omega \big(\partial_t^2 q - div\big(a_0(x)\nabla q\big)\big) y_{\tau_0}\,dx\,dt \\ &= \int_\Omega [y_{\tau_0}\partial_t q]_0^T\,dx - \int_\Omega [q\partial_t y_{\tau_0}]_0^T\,dx \\ &\quad + \int_0^T \int_\Omega \big(\partial_t^2 y_{\tau_0} - div\big(a_0(x)\nabla y_{\tau_0}\big)\big) q\,dx\,dt \\ &\quad + \int_0^T \int_\Gamma q \frac{\partial y_{\tau_0}}{\partial \nu_a}\,d\Gamma\,dt + \int_0^T \int_\Gamma y_{\tau_0} \frac{\partial q}{\partial \nu_a}\,d\Gamma\,dt. \end{aligned}$$

Since y_{τ_0} is a solution of (15), we have

$$\big\langle \partial_t q(x,0), \widehat{y}_0(x)\big\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} = -\int_0^T \int_\Gamma (h_0\chi_O + u\chi_\omega)\frac{\partial y_{\tau_0}}{\partial \nu}\,d\Gamma\,dt,$$

where $\langle \cdot,\cdot\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}$ is the duality product between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$.

It follows from (17) that

$$\big\langle \partial_t q(x,0), \widehat{y}_0(x)\big\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} = 0 \quad \text{for all } \|\widehat{y}_0\|_{H_0^1(\Omega)} \le 1.$$

Hence we find

$$\partial_t q(0) = 0 \quad \text{a.e. in } \Omega.$$

In the same way, multiplying both sides of the first equation in (19) by the solution y_{τ_1} of (16), integrating by parts, and applying Green's formula, we find

$$\int_\Omega q(0)\widehat{y}_1(x)\,dx = \int_0^T\int_\Gamma (h_0\chi_O + u\chi_\omega)\frac{\partial y_{\tau_1}}{\partial \nu_a}\,d\Gamma\,dt,$$

and from (18) we obtain

$$\int_\Omega q(0)\widehat{y}_1(x)\,dx = 0 \quad \text{for all } \|\widehat{y}_1(x)\|_{L^2(\Omega)} \leq 1.$$

So

$$q(x,0) = 0 \quad \text{in } L^2(\Omega).$$

Therefore

$$q(x,0) = 0 \quad \text{a.e. in } \Omega. \qquad \square$$

4 Study of the boundary null-controllability problem with constraint on the control

In the previous section, we established that the existence of the sentinel functional was related to solving an optimal control problem. For this reason, we are interested in solving this problem, and we state our main result as follows.

Theorem 3 *Let $h_0 \in L^2(O \times [0,T])$. Then there exists a control function u of minimal norm in $L^2(\omega \times [0,T])$ such that the solution q of problem* (19) *satisfies* (20) *and ω satisfies the following geometrical condition*:

$$\omega = \{x \in \Gamma, H(x).\nu(x) > 0\},$$

where $\nu(x)$ is the unit outward normal vector to Γ at x, and $H(x)$ is defined in [19].

Problem (19)–(20) is an optimal control problem. We use the method of HUM introduced by Lions [10] to establish this control with minimal norm in $L^2(\omega \times [0,T])$. It is a constructive method based on the uniqueness results and on the construction of an isomorphism operator.

Note that we can write $q(u) = q_0 + z$, where q_0 and z are solutions of the systems

$$\begin{cases} \partial_t^2 q_0 - div(a_0(x)\nabla q_0) = 0 & \text{in } \Omega \times (0,T), \\ q_0(T) = 0 & \text{in } \Omega, \\ \partial_t q_0(T) = 0 & \text{in } \Omega, \\ q_0 = h_0\chi_O & \text{on } \Gamma \times (0,T), \end{cases} \tag{21}$$

and

$$\begin{cases} \partial_t^2 z - div(a_0(x)\nabla z) = 0 & \text{in } \Omega \times (0,T), \\ z(T) = 0 & \text{in } \Omega, \\ \partial_t z(T) = 0 & \text{in } \Omega, \\ z = u\chi_w & \text{on } \Gamma \times (0,T), \end{cases} \tag{22}$$

respectively.

Obviously, (20) can be written as

$$\begin{cases} q(0) = z(u)(x,0) + q_0(0), \\ \partial q(0) = \partial_t z(u)(x,0) + \partial_t q_0(0), \end{cases} \quad \text{in } \Omega. \tag{23}$$

Note that the solution $z = z(u)$ of (22) is the state satisfying the conditions

$$\begin{cases} z(u)(x,0) = -q_0(0), \\ \partial_t z(u)(x,0) = -\partial_t q_0(0), \end{cases} \quad \text{in } \Omega. \tag{24}$$

Introduce the solution Φ of the following system:

$$\begin{cases} \partial_t^2 \Phi - div(a_0(x)\nabla \Phi) = 0 & \text{in } \Omega \times (0,T), \\ \Phi(0) = \Phi^0 & \text{in } \Omega, \\ \partial_t \Phi(0) = \Phi^1 & \text{in } \Omega, \\ \Phi = 0 & \text{on } \Gamma \times (0,T), \end{cases} \tag{25}$$

where $(\Phi_0, \Phi_1) \in H_0^1(\Omega) \times L^2(\Omega)$. System (25) admits a unique solution $\Phi \in C([0,T]; (H_0^1(\Omega) \times L^2(\Omega)))$. More precisely, we have the following inequality for all $T > 0$ and a constant $C > 0$ [19]:

$$\int_{\Gamma_0 \times (0,T)} \left| \frac{\partial \Phi}{\partial \nu_a} \right|^2 d\Gamma\, dt \leq CT \big(\|\Phi_0\|^2_{H_0^1(\Omega)} + \|\Phi_1\|^2_{L^2(\Omega)} \big). \tag{26}$$

Moreover, there exists a constant $c > 0$ such that [19]

$$\int_{\Gamma_0 \times (0,T)} \left| \frac{\partial \Phi}{\partial \nu_a} \right|^2 d\Gamma\, dt \geq c \big(\|\Phi_0\|^2_{H_0^1(\Omega)} + \|\Phi_1\|^2_{L^2(\Omega)} \big). \tag{27}$$

In addition, let Ψ be the solution of the following problem:

$$\begin{cases} \partial_t^2 \Psi - div(a_0(x)\nabla \Psi) = 0 & \text{in } \Omega \times (0,T), \\ \Psi(T) = 0 & \text{in } \Omega, \\ \partial_t \Psi(T) = 0 & \text{in } \Omega, \\ \Psi = \frac{\partial \Phi}{\partial \nu_a}\chi_w & \text{on } \Gamma \times (0,T). \end{cases} \tag{28}$$

Now, we go back to the proof of the previous theorem.

Proof Define the linear operator

$$\begin{aligned} &\Lambda : H_0^1(\Omega) \times L^2(\Omega) \longrightarrow H^{-1}(\Omega) \times L^2(\Omega), \\ &\Lambda\{\Phi^0, \Phi^1\} = \{-\partial_t z(0), z(0)\}, \end{aligned} \tag{29}$$

where $H^{-1}(\Omega)$ is the dual space of $H_0^1(\Omega)$.

Multiplying both sides of the first equation of (28) by the solution Φ of (25), integrating by parts, and applying Green's formula, we obtain

$$\langle \Lambda\{\Phi^0, \Phi^1\}, \{\Phi^0, \Phi^1\}\rangle = \int_0^T \int_\omega \left|\frac{\partial \Phi}{\partial \nu_a}\right|^2 d\Gamma\, dt,$$

where $\langle\cdot,\cdot\rangle$ denotes the duality product between $H_0^1(\Omega) \times L^2(\Omega)$ and $H^{-1}(\Omega) \times L^2(\Omega)$. Moreover, it is clear that Λ is a positive self-adjoint operator.

This leads to the introduction of the seminorm

$$\left|\{\Phi^0, \Phi^1\}\right|_{H_0^1(\Omega)\times L^2(\Omega)} = \left(\int_0^T \int_\omega \left|\frac{\partial \Phi}{\partial \nu_a}\right|^2 d\Gamma\, dt\right)^{\frac{1}{2}}. \tag{30}$$

To prove that Λ is an isomorphism, we have to show that the previous seminorm (30) is a norm on the set of initial data $\{\Phi^0, \Phi^1\}$ and that if

$$\int_0^T \int_\omega \left|\frac{\partial \Phi}{\partial \nu_a}\right|^2 d\Gamma\, dt = 0, \quad \text{then } \Phi = 0 \text{ in } Q.$$

Take $\omega = \Gamma_0$. From inequality (27) (observability inequality) it is easy to show that the previous seminorm is a norm, denoted by

$$\left\|\{\Phi^0, \Phi^1\}\right\|_{H_0^1(\Omega)\times L^2(\Omega)} = \left(\int_0^T \int_\omega \left|\frac{\partial \Phi}{\partial \nu_a}\right|^2 d\Gamma\, dt\right)^{\frac{1}{2}}. \tag{31}$$

Furthermore, it is clear from (26) and (27) that the norm is equivalent to the usual norm of $H_0^1(\Omega) \times L^2(\Omega)$.

We must show that the operator Λ is an isomorphism from $H_0^1(\Omega) \times L^2(\Omega)$ to $H^{-1}(\Omega) \times L^2(\Omega)$. The norm (31) is defined by the scalar product $\langle \Lambda\{\widetilde{\Phi^0}, \widetilde{\Phi^1}\}, \{\Phi^0, \Phi^1\}\rangle$, which defines a Hilbert space on the set of initial data, which is equivalent to (the Hilbert space) $H_0^1(\Omega) \times L^2(\Omega)$.

By the Riesz representation theorem we conclude that is Λ an isomorphism from $H_0^1(\Omega) \times L^2(\Omega)$ to $H^{-1}(\Omega) \times L^2(\Omega)$ [14].

So (29) has a unique solution given by

$$\{\Phi^0, \Phi^1\} = \Lambda^{-1}\{\partial_t q_0(0), -q_0(0)\}. \tag{32}$$

Thus the control is given by the restriction

$$u = \frac{\partial \Phi}{\partial \nu_a}\chi_\omega, \tag{33}$$

where Φ is the solution of (25). □

So sentinel (6) is given by

$$S(\lambda, \tau_0, \tau_1) = \int_0^T \int_\Gamma \left(h_0 \chi_O + \frac{\partial \Phi}{\partial \nu_a} \chi_\omega \right) \frac{\partial y}{\partial \nu_a} \, d\Gamma \, dt. \tag{34}$$

Acknowledgements
This work was supported by the Directorate-General for Scientific Research and Technological Development (DGRSDT).

Declarations

Competing interests
The authors declare no competing interests.

Author contributions
B.E wrote the main manuscript text. A.H reviewed the manuscript.

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Received: 5 October 2022 Accepted: 1 March 2023 Published online: 09 March 2023

References

1. Amel, B., Imad, R.: Identification of the source term in Navier–Stokes system with incomplete data. AIMS Math. **4**(3), 516–526 (2019)
2. Bellassoued, M.: Global logarithmic stability in inverse hyperbolic problem by arbitrary boundary observation. Inverse Probl. **20**(4), 1033 (2004)
3. Bellassoued, M., Jellali, D., Yamamoto, M.: Lipschitz stability in an inverse problem for a hyperbolic equation with a finite set of boundary data. Appl. Anal. **87**(10–11), 1105–1119 (2008)
4. Bellassoued, M., Yamamoto, M.: Logarithmic stability in determination of a coefficient in an acoustic equation by arbitrary boundary observation. J. Math. Pures Appl. **85**(2), 193–224 (2006)
5. Billal, E., Abdelhak, H.: Identification of the potential coefficient in the wave equation with incomplete data: a sentinel method. Russ. Math. **2022**(12), 113–122 (2022)
6. Chafia, L., Abdelhamid, A., Abdelhak, H.: Identification problem of a fractional thermoelastic deformation system with incomplete data: a sentinel method. Nonlinear Stud. **29**(2), 399–410 (2022)
7. Imanuvilov, O.Y., Yamamoto, M.: Global uniqueness and stability in determining coefficients of wave equations. Commun. Partial Differ. Equ. **26**(7–8), 1409–1425 (2001)
8. Imanuvilov, O.Y., Yamamoto, M.: Determination of a coefficient in an acoustic equation with a single measurement. Inverse Probl. **19**(1), 157 (2003)
9. Jellali, D.: An inverse problem for the acoustic wave equation with finite sets of boundary data. J. Inverse Ill-Posed Probl. **2006**(14), 665–684 (2006)
10. Lions, J.L.: Exact controllability, stabilization and perturbations for distributed systems. SIAM Rev. **30**(1), 1–68 (1988)
11. Lions, J.L.: Sentinelles pour les systèmes distribués à données incomplètes, vol. 21. Elsevier, Masson (1992)
12. Merabet, A., Ayadi, A., Omrane, A.: Detection of pollution terms in nonlinear second order wave systems. Int. J. Parallel Emerg. Distrib. Syst. **34**(1), 13–20 (2019)
13. Mophou, G.M., Puel, J.P.: Boundary sentinels with given sensitivity. Rev. Mat. Complut. **22**(1), 165–185 (2009)
14. Pedersen, M.: Functional Analysis in Applied Mathematics and Engineering. CRC Press, Boca Raton (2018)
15. Puel, J.P., Yamamoto, M.: On a global estimate in a linear inverse hyperbolic problem. Inverse Probl. **12**(6), 995–1002 (1996)
16. Rakesh: Reconstruction for an inverse problem for the wave equation with constant velocity. Inverse Probl. **6**(1), 91–98 (1990)
17. Sandel, S., Ayadi, A.: Boundary sentinels for the resolution of a geometrical problem. Turk. J. Math. **42**(2), 548–556 (2018)
18. Yamamoto, M.: Uniqueness and stability in multidimensional hyperbolic inverse problems. J. Math. Pures Appl. **78**(1), 65–98 (1999)
19. Yao, P.F.: On the observability inequalities for exact controllability of wave equations with variable coefficients. SIAM J. Control Optim. **37**(5), 1568–1599 (1999)