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The Neumann problem for a class of generalized Kirchhoff-type potential systems

Nabil Chems Eddine[1] and Dušan D. Repovš[2,3,4*]

*Correspondence:
dusan.repovs@guest.arnes.si
[2]Faculty of Education, University of Ljubljana, Ljubljana, Slovenia
[3]Faculty of Mathematics and Physics, University of Ljubljana, Ljubljana, Slovenia
Full list of author information is available at the end of the article

Abstract

In this paper, we are concerned with the Neumann problem for a class of quasilinear stationary Kirchhoff-type potential systems, which involves general variable exponents elliptic operators with critical growth and real positive parameter. We show that the problem has at least one solution, which converges to zero in the norm of the space as the real positive parameter tends to infinity, via combining the truncation technique, variational method, and the concentration–compactness principle for variable exponent under suitable assumptions on the nonlinearities.

MSC: 35B33; 35D30; 35J50; 35J60; 46E35

Keywords: Kirchhoff-type problems; Neumann boundary conditions; $p(x)$-Laplacian operator; Generalized capillary operator; Sobolev spaces with variable exponent; Critical Sobolev exponents; Concentration–compactness principle; Critical point theory; Truncation technique

1 Introduction

In this article, we are concerned with the existence and asymptotic behavior of nontrivial solutions for the following class of nonlocal quasilinear elliptic systems:

$$\begin{aligned} &M_i\big(\mathcal{A}_i(u_i)\big)\big(-\operatorname{div}\big(\mathcal{B}_{1_i}(\nabla u_i)\big)+\mathcal{B}_{2_i}(u_i)\big)=|u_i|^{s_i(x)-2}u_i+\lambda F_{u_i}(x,u) \quad \text{in } \Omega,\\ &M_i\big(\mathcal{A}_i(u_i)\big)\mathcal{B}_{1_i}(\nabla u_i)\cdot\mathfrak{N}_i=|u_i|^{\ell_i(x)-2}u_i \quad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

for $i=1,2,\dots,n$ ($n\in\mathbb{N}^*$), where $\Omega\subset\mathbb{R}^N$ ($N\geq 2$) is a bounded domain with smooth boundary $\partial\Omega$, $\mathfrak{N}_i$ is the outward normal vector field on $\partial\Omega$, λ is a positive parameter, $\nabla F=(F_{u_1},\dots,F_{u_2})$ is the gradient of a C^1-function $F:\mathbb{R}^N\times\mathbb{R}^n\to\mathbb{R}$, $p_i,q_i,w_i,s_i\in C(\overline{\Omega})$, and $\ell_i\in C(\partial\Omega)$ are such that

$$1<p_i^-\leq p_i(x)\leq p_i^+<q_i^-\leq q_i(x)\leq q_i^+<N, \tag{1.2}$$

$$h_i^-\leq h_i(x)\leq h_i^+\leq w_i^-\leq w_i(x)\leq w_i^+\leq s_i^-\leq s_i(x)\leq s_i^+\leq h_i^*(x)<\infty, \tag{1.3}$$

and

$$h_i^-\leq h_i(x)\leq h_i^+\leq w_i^-\leq w_i(x)\leq w_i^+\leq \ell_i^-\leq \ell_i(x)\leq \ell_i^+\leq h_i^\partial(x)<\infty, \tag{1.4}$$

for all $x \in \overline{\Omega}$, where functions w_i are given by condition ($\boldsymbol{F}_4$) below, $p_i^- := \inf_{x\in\overline{\Omega}} p_i(x)$, $p_i^+ := \sup_{x\in\overline{\Omega}} p_i(x)$, and analogously to w_i^-, w_i^+, q_i^-, q_i^+, h_i^-, h_i^+, s_i^-, s_i^+, ℓ_i^- and ℓ_i^+, with $h_i(x) = (1 - \mathcal{K}(k_i^3))p_i(x) + \mathcal{K}(k_i^3)q_i(x)$, where k_i^3 is given by condition ($\boldsymbol{A}_2$) below, and

$$h_i^*(x) = \begin{cases} \frac{Nh_i(x)}{N-h_i(x)} & \text{for } h_i(x) < N, \\ +\infty & \text{for } h_i(x) \geq N, \end{cases} \quad \text{and} \quad h_i^\partial(x) = \begin{cases} \frac{(N-1)h_i(x)}{N-h_i(x)} & \text{for } h_i(x) < N, \\ +\infty & \text{for } h_i(x) \geq N, \end{cases}$$

for all $x \in \overline{\Omega}$, and the function $\mathcal{K} : \mathbb{R}_0^+ \to \{0, 1\}$ is defined by

$$\mathcal{K}(k_i) = \begin{cases} 1 & \text{if } k_i > 0, \\ 0 & \text{if } k_i < 0. \end{cases}$$

In addition, we consider both $\mathcal{C}_{h_i}^1$ and $\mathcal{C}_{h_i}^2$ as nonempty disjoint sets, which are respectively defined by

$$\mathcal{C}_{h_i}^1 := \{x \in \partial\Omega, \ell_i(x) = h_i^\partial(x)\} \quad \text{and} \quad \mathcal{C}_{h_i}^2 := \{x \in \overline{\Omega}, s_i(x) = h_i^*(x)\}.$$

The operators $\mathcal{A}_{j_i} : X_i \to \mathbb{R}^n$, for $j = 1$ or 2, and the operators $\mathcal{B}_i : X_i \to \mathbb{R}$ are respectively defined by

$$\begin{aligned} &\mathcal{B}_{j_i}(u_i) = a_{j_i}\left(|u_i|^{p_i(x)}\right)|u_i|^{p_i(x)-2}u_i \quad \text{and} \\ &\mathcal{A}_i(u_i) = \int_\Omega \frac{1}{p_i(x)}\left(A_{1_i}\left(|\nabla u_i|^{p_i(x)}\right) + A_{2_i}\left(|u_i|^{p_i(x)}\right)\right) dx, \end{aligned} \tag{1.5}$$

for all $1 \leq i \leq n$, where $X_i := W^{1,h_i(x)}(\Omega) \cap W^{1,p_i(x)}(\Omega)$ is a Banach space, and functions A_{j_i} are defined by $A_{j_i}(t) = \int_0^t a_{j_i}(k)\, dk$, where functions a_{j_i} are described in condition ($\boldsymbol{A}_1$).

In what follows, we shall consider the functions a_{j_i} satisfying the following assumptions for all $i \in \{1, 2, \ldots, n\}$ and $j \in \{1, 2\}$:

($\boldsymbol{A}_1$) $a_{j_i} : \mathbb{R}^+ \to \mathbb{R}^+$ are of class C^1.

($\boldsymbol{A}_2$) There exist positive constants $k_{j_i}^0$, $k_{j_i}^1$, $k_{j_i}^2$ and k_i^3 for all $i \in \{1, 2, \ldots, n\}$, $j \in \{1, 2\}$ such that

$$k_{j_i}^0 + \mathcal{K}(k_i^3)k_{j_i}^2 \xi^{\frac{q_i(x)-p_i(x)}{p_i(x)}} \leq a_{j_i}(\xi) \leq k_{j_i}^1 + k_i^3 \xi^{\frac{q_i(x)-p_i(x)}{p_i(x)}}, \quad \text{for all } \xi \geq 0 \text{ and a.e. } x \in \overline{\Omega}.$$

($\boldsymbol{A}_3$) There exists $c_i > 0$ for all $1 \leq i \leq n$ such that

$$\min\left\{a_{j_i}\left(\xi^{p_i(x)}\right)\xi^{p_i(x)-2}, a_{j_i}\left(\xi^{p_i(x)}\right)\xi^{p_i(x)-2} + \xi\frac{\partial(a_{j_i}(\xi^{p_i(x)})\xi^{p_i(x)-2})}{\partial\xi}\right\} \geq c_i \xi^{p_i(x)-2},$$

for a.e. $x \in \overline{\Omega}$ and all $\xi \geq 0$.

($\boldsymbol{A}_4$) There exist positive constants β_{j_i} and γ_i for all $i \in \{1, 2, \ldots, n\}$, $j \in \{1, 2\}$ such that

$$A_{j_i}(\xi) \geq \frac{1}{\beta_{j_i}} a_{j_i}(\xi)\xi \quad \text{with } h_i^+ < \gamma_i < s_i^- \text{ and } \frac{q_i^+}{p_i^+} \leq \beta_{j_i} < \frac{\gamma_i}{p_i^+}, \text{ for all } \xi \geq 0.$$

($\boldsymbol{M}$) $M_i : \mathbb{R}^+ \to \mathbb{R}$ are continuous and increasing functions such that $M_i(t) \geq M_i(0) = \mathfrak{M}_i^0 > 0$, for all $t \geq 0$, $i \in \{1, 2, \ldots, n\}$.

As it is well known, there are many examples of functions M_i that satisfy assumption (***M***), for example,

$$M_1(\xi) = \mathfrak{M}_1^0 + \mathfrak{B}_1\xi^{\theta_1}, \quad \text{with } \mathfrak{M}_1^0, \mathfrak{B}_1 \geq 0, \mathfrak{M}_1^0 + \mathfrak{B}_1 > 0 \text{ and } \theta_1 \geq 1.$$

In particular, when $\mathfrak{M}_1^0 = 0$ and $\mathfrak{B}_1 > 0$, the Kirchhoff equation associated with M_1 is said to be degenerate. On the other hand, when $\mathfrak{M}_1^0 > 0$ and $\mathfrak{B}_1 \geq 0$, the Kirchhoff equation associated with M_1 is said to be nondegenerate. In this case, when $\mathfrak{B}_1 = 0$, the Kirchhoff equation associated with M_1 (is a constant) reduces to a local quasilinear elliptic problem.

The study of differential equations and variational problems driven by nonhomogeneous differential operators has received extensive attention and has been extensively investigated, see, e.g., Papageorgiou et al. [42]. This is due to their ability to model many physical phenomena. It should be noted that the $p(x)$-Laplacian operator is a special case of the divergence form operator $\operatorname{div}(\mathcal{B}_{j_i}(\nabla u_i))$. The natural functional framework for this operator is described by the Sobolev space with a variable exponent $W^{1,p(x)}$.

In recent decades, there has been a particular focus on variable exponent Lebesgue and Sobolev spaces, $L^{p(x)}$ and $W^{1,p(x)}$, where p is a real function, e.g., Rădulescu and Repovš [45]. Traditional Lebesgue spaces L^p and Sobolev spaces $W^{1,p}$ with constant exponents have proven insufficient to tackle the complexities of nonlinear problems in the applied sciences and engineering. To address these limitations, the use of variable exponent Lebesgue and Sobolev spaces has been on the rise in recent years.

This area of research reflects a new type of physical phenomena, such as electrorheological fluids, or "smart fluids," which can exhibit dramatic changes in mechanical properties in response to an electromagnetic field. These and other nonhomogeneous materials require the use of variable exponent Lebesgue and Sobolev spaces, where the exponent p is allowed to vary. Moreover, variable exponent Lebesgue and Sobolev spaces have found a wide range of applications, from image restoration and processing to fields such as thermorheological fluids, mathematical biology, flow in porous media, polycrystal plasticity, heterogeneous sand pile growth, and fluid dynamics. For a comprehensive overview of these and other applications, see, e.g., Chen et al. [15], Diening et al. [20, 21], Halsey [28], Rădulescu [44], Ružička [46, 47], and the references therein.

Furthermore, every single equation of the system (1.1) is a generalization of the stationary problem of the first model introduced by Kirchhoff [33] in 1883 and having the following form:

$$\rho\partial_{tt}^2 u - \left(\frac{\rho_0}{h} + \frac{E}{2L}\int_0^L \left|\partial_x u(x)\right|^2 dx\right)\partial_{xx}^2 u = 0, \tag{1.6}$$

where the parameters ρ, h, ρ_0, t, L, E are all constants which respectively have some physical meaning, which is an extension of the classical D'Alembert wave equation, by considering the effect of the change in the length of a vibrating string.

Nearly a century later, in 1978, Jacques-Louis Lions [35] returned to the equation and proposed a general Kirchhoff equation in arbitrary dimension with an external force term which was written as

$$\partial_{tt}^2 u - \left(a + b\int_\Omega |\nabla u|^2\, dx\right)\Delta u = f(x,u) \quad \text{in } \Omega,$$

$$u = 0 \quad \text{on } \partial\Omega.$$

Later on, many interesting results have been obtained by Caristi et al. [9], Dai and Hao [19], Ma [37]; see also the references therein. The main difficulty in studying these equations appears to be due to the fact that they do not satisfy a pointwise identity any longer. It is generated by having the term containing M_i in the equations, and it makes (1.1) a nonlocal problem. The nonlocal problem models arise in the description of biological systems and also various physical phenomena, where u describes a process that depends on the average of itself, such as population density. For more references on this subject, we refer the interested reader to Ambrosio et al. [3], Arosio and Pannizi [5], Cavalcanti et al. [10], Chipot and Lovat [17], He et al. [31], Corrêa and Nascimento [18], Yang and Zhou [50], Wang et al. [48], and the references therein. On the one hand, the differential equations with constant or variable critical exponents in bounded or unbounded domains have attracted increasing attention recently. They were first discussed in the seminal paper by Brezis and Nirenberg [8] in 1983, which treated Laplacian equations. Since then, there have been extensions of [8] in many directions.

One of the main features of elliptic equations involving critical growth is the lack of compactness arising in connection with the variational approach. In order to overcome the lack of compactness, Lions [36] established the method using the so-called concentration compactness principle (CCP, for short) to show that a minimizing sequence or a Palais–Smale ((PS), for short) sequence is precompact. Afterward, the variable exponent version of the Lions concentration–compactness principle for a bounded domain was independently obtained in Bonder et al. [6, 7] and Fu [26], and for an unbounded domain in Fu [27]. Since then, many authors have applied these results to study critical elliptic problems involving variable exponents, see, e.g., Alves et al. [1, 2], Chems Eddine et al. [12–14], Fang and Zhang [24], Hurtado et al. [32], Mingqi et al. [39], Zhang and Fu [51].

When M_i satisfies conditions $a_{1_i} \equiv 1$ (with $k_i^1 = 1$, and $k_i^2 > 0$ and $k_i^3 = 0$), and $a_{2_i} \equiv 0$, Chems Eddine [12] proved the existence of nontrivial weak solutions for the following class of Kirchhoff-type potential systems with Dirichlet boundary conditions:

$$
\begin{aligned}
&-M_i\left(\int_\Omega \frac{1}{p_i(x)}|\nabla u_i|^{p_i(x)}\right)\Delta_{p_i(x)}u_i = |u_i|^{q_i(x)-2}u_i + \lambda F_{u_i}(x,u) \quad \text{in } \Omega,\\
&u_i = 0 \quad \text{on } \partial\Omega,
\end{aligned}
$$

for $1 \leq i \leq n$ ($n \in \mathbb{N}^*$), where $p_i, q_i : \overline{\Omega} \to \mathbb{R}$ are Lipschitz continuous functions such that $1 \leq q_i(x) \leq p_i^*(x)$ for all x in Ω and the potential function F satisfies mixed and subcritical growth conditions. Following that, Chems Eddine [11] established the existence of infinitely many solutions for the following system:

$$
\begin{aligned}
&-M_i\big(\mathcal{A}_i(u_i)\big)\operatorname{div}\big(\mathcal{B}_i(\nabla u_i)\big) = |u_i|^{s_i(x)-2}u_i + \lambda F_{u_i}(x,u) \quad \text{in } \Omega,\\
&u = 0 \quad \text{on } \partial\Omega,
\end{aligned}
$$

for $i = 1, 2, \ldots, n$ ($n \in \mathbb{N}$), where $\{x \in \Omega, s_i(x) = p_i^*(x)\}$ is nonempty and $F \in C^1(\Omega \times \mathbb{R}^n, \mathbb{R})$ satisfies some mixed and subcritical growth conditions. Next, Chems Eddine and Ragusa [14] dealt with cases when the above class of Kirchhoff-type potential systems is considered under Neumann boundary conditions with two critical exponents, and established the existence and multiplicity of solutions.

Our objective in this article is to show the existence of nontrivial solutions for the nonlocal problem (1.1). As we shall see in this paper, there are some substantial difficulties in our situation, which can be summed up in three main problems. First, the assumption (***M***) provides only a positive lower bound for the Kirchhoff functions M_i near zero, creating serious mathematical technical difficulties, to overcome which we need to do a truncation of the Kirchhoff functions M_i to obtain a priori estimates of the boundedness from above, and thus obtain a new auxiliary problem, thereafter, we do another truncation to control the energy functional corresponding to the auxiliary problem. The second difficulty in solving problem (1.1) is the lack of compactness which can be illustrated by the fact that the embeddings $W^{1,h(x)}(\Omega) \hookrightarrow L^{h^*(x)}(\Omega)$ and $W^{1,h(x)}(\Omega) \hookrightarrow L^{h^\partial(x)}(\partial\Omega)$ are no longer compact and, to overcome this difficulty, we use two versions of Lions's principle for the variable exponent, extended by Bonder et al. [6, 7]. Then by combining the variational method and Mountain Pass Theorem, we obtain the existence of at least one nontrivial solution to the auxiliary problem, see Theorem 3.1, and by truncating functions M_i, we obtain the existence of at least one nontrivial solution to problem (1.1), see Theorem 1.2.

Throughout this paper, we shall assume that F satisfies the following conditions:

($\boldsymbol{F}_1$) $F \in C^1(\Omega \times \mathbb{R}^n, \mathbb{R})$ and $F(x, 0_{\mathbb{R}^n}) = 0$.

($\boldsymbol{F}_2$) For all $(i,j) \in \{1,2,\dots,n\}^2$, there exist positive functions b_{i_j} such that

$$\left|F_{\xi_i}(x,\xi_1,\dots,\xi_n)\right| \le \sum_{j=1}^n b_{i_j}(x)|\xi_j|^{r_{i_j}(x)-1},$$

where $1 < r_{i_j}(x) < \inf_{x\in\Omega} h_i(x)$ for all $x \in \Omega$. The weight-functions b_{ii} (resp. b_{i_j} if $i \neq j$) belong to the generalized Lebesgue spaces $L^{\alpha_{ii}}(\Omega)$ (resp. $L^{\alpha_{i_j}}(\Omega)$), with

$$\alpha_{ii}(x) = \frac{h_i(x)}{h_i(x)-1}, \qquad \alpha_{i_j}(x) = \frac{h_i^*(x)h_j^*(x)}{h_i^*(x)h_j^*(x) - h_i^*(x) - h_j^*(x)}.$$

($\boldsymbol{F}_3$) There exist $K > 0$ and $\gamma_i \in (h_i^+, \inf\{s_i^-, \ell_i^-\})$ such that for all $(x,\xi_1,\dots,\xi_n) \in \Omega \times \mathbb{R}^n$ where $|u_i|^{\gamma_i} \ge K$,

$$0 < F(x,\xi_1,\dots,\xi_n) < \sum_{i=1}^n \frac{\xi_i}{\gamma_i} F_{\xi_i}(x,\xi_1,\dots,\xi_n).$$

($\boldsymbol{F}_4$) There exists a positive constant c such that

$$\left|F(x,\xi_1,\dots,\xi_n)\right| \le c\left(\sum_{i=1}^n |\xi_i|^{w_i(x)}\right), \quad \text{for all } (x,\xi_1,\dots,\xi_n) \in \Omega \times \mathbb{R}^n,$$

where $w_i \in C_+(\overline{\Omega})$ and $q_i^+ < w_i^- \le w_i^+ < \inf\{s_i^-, \ell_i^-\} \le \inf\{s_i^+, \ell_i^+\}$, for all $1 \le i \le n$.

Example 1.1 There are many potential functions F satisfying assumptions ($\boldsymbol{F}_1$) and ($\boldsymbol{F}_2$). For example, when $n = 2$, take $F(x,u_1,u_2) = b_{12}(x)|u_1|^{r_1(x)}|u_2|^{r_2(x)}$, where $\frac{r_1(x)}{h_1(x)} + \frac{r_2(x)}{h_2(x)} < 1$, and the positive weight-function $b_{12} \in L^{\alpha(x)}(\Omega)$ with

$$\alpha(x) = \frac{h_2^*(x)h_1^*(x)}{h_2^*(x)h_1^*(x) - h_2^*(x)r_1(x) - h_2^*(x)r_2(x)}, \quad \text{for all } x \in \Omega.$$

By the standard calculus, we can verify that F satisfies the assumption ($\boldsymbol{F}_1$). Moreover, by using Young inequality, we can check the assumption ($\boldsymbol{F}_2$).

The main result of our paper is the following.

Theorem 1.2 *Assume that conditions* ($\boldsymbol{A}_1$)–($\boldsymbol{A}_4$), ($\boldsymbol{M}$), *and* ($\boldsymbol{F}_1$)–($\boldsymbol{F}_4$) *hold. Then there exists* $\lambda^\star > 0$ *such that for all* $\lambda \geq \lambda_\star$, *problem* (1.1) *has at least one nontrivial solution in* X. *Moreover, if* u_λ *is a weak solution of problem* (1.1), *then* $\lim_{\lambda\to+\infty}\|u_\lambda\| = 0$.

The paper is organized as follows: In Sect. 2 we give some preliminary results of the variable exponent spaces. In Sect. 3 we introduce the auxiliary problem and obtain a nontrivial solution for the auxiliary problem. Section 4 is dedicated to proving the main results. Finally, in Sect. 5, we illustrate the degree of generality of the kind of problems we studied in this paper.

2 Preliminaries and basic notations

In this section, we introduce some definitions and results which will be used in the next section. Throughout our work, let Ω be a bounded domain of $\mathbb{R}^N$ ($N \geq 2$) with a Lipschitz boundary $\partial\Omega$, and let us denote by $\mathcal{D}$ either Ω or its boundary $\partial\Omega$. Denote

$$\mathcal{M}_+(\mathcal{D}) := \bigl\{p : \mathcal{D} \to \mathbb{R} \text{ measurable real-valued function} : p(x) > 1 \text{ for a.a. } x \in \mathcal{D}\bigr\},$$
$$C_+(\mathcal{D}) := \bigl\{p \in C(\mathcal{D}) : p(x) > 1 \text{ for a.a. } x \in \mathcal{D}\bigr\}.$$

For all $p \in \mathcal{M}_+(\mathcal{D})$, denote $p^+ := \sup_{x\in\mathcal{D}} p(x)$ and $p^- := \inf_{x\in\mathcal{D}} p(x)$. Also for all $p \in \mathcal{M}_+(\mathcal{D})$ and for a measure η on $\mathcal{D}$, we define the variable exponent Lebesgue space as

$$L^{p(x)}(\mathcal{D}) := L^{p(x)}(\mathcal{D}, d\eta) := \bigl\{u \text{ measurable real-valued function} : \rho_{p,\mathcal{D}}(u) < \infty\bigr\},$$

where the functional $\rho_{p,\mathcal{D}} : L^{p(x)}(\mathcal{D}) \to \mathbb{R}$ is defined as

$$\rho_{p,\mathcal{D}}(u) := \int_\Omega \bigl|u(x)\bigr|^{p(x)}\, d\eta.$$

The functional $\rho_{p,\mathcal{D}}$ is called the $p(x)$-modular of the $L^{p(x)}(\mathcal{D})$ space, it has played an important role in manipulating the generalized Lebesgue–Sobolev spaces. We endow the space $L^{p(x)}(\mathcal{D})$ with the Luxemburg norm

$$\|u\|_{L^{p(x)}(\mathcal{D})} := \inf\left\{\tau > 0 : \rho_{p,\mathcal{D}}\left(\frac{u(x)}{\tau}\right) \leq 1\right\}.$$

Then $(L^{p(x)}(\mathcal{D}), \|u\|_{L^{p(x)}(\mathcal{D})})$ is a separable and reflexive Banach space (see, e.g., Kováčik and Rákosník [34, Theorem 2.5, Corollary 2.7]). In the subsequent sections, the $L^{p(x)}$-spaces under consideration will be $L^{p(x)}(\Omega) := L^{p(x)}(\Omega, dx)$, and $L^{p(x)}(\partial\Omega) := L^{p(x)}(\partial\Omega, d\sigma)$ for an appropriate measure σ supported on $\partial\Omega$. Let us now recall more basic properties concerning the Lebesgue spaces.

Proposition 2.1 (Kováčik and Rákosník [34, Theorem 2.8]) *Let* p *and* q *be variable exponents in* $\mathcal{M}_+(\mathcal{D})$ *such that* $p \leq q$ *in* $\mathcal{D}$, *where* $0 < \operatorname{meas}(\mathcal{D}) < \infty$. *Then the embedding* $L^{q(x)}(\mathcal{D}) \hookrightarrow L^{p(x)}(\mathcal{D})$ *is continuous.*

Furthermore, the following Hölder-type inequality:

$$\left|\int_{\mathcal{D}} u(x)v(x)\,dx\right| \leq \left(\frac{1}{p^-}+\frac{1}{(p')^-}\right)\|u\|_{L^{p(x)}(\mathcal{D})}\|v\|_{L^{p'(x)}(\mathcal{D})} \leq 2\|u\|_{L^{p(x)}(\mathcal{D})}\|v\|_{L^{p'(x)}(\mathcal{D})} \quad (2.1)$$

holds for all $u \in L^{p(x)}(\mathcal{D})$ and $v \in L^{p'(x)}(\mathcal{D})$ (see, e.g., Kováčik and Rákosník [34, Theorem 2.1]), where we denoted by $L^{p'(x)}(\mathcal{D})$ the topological dual space (or the conjugate space) of $L^{p(x)}(\mathcal{D})$, obtained by conjugating the exponent pointwise, that is, $1/p(x) + 1/p'(x) = 1$ (see, e.g., Kováčik and Rákosník [34, Corollary 2.7]). Moreover, if $h_1, h_2, h_3 : \mathcal{D} \to (1,\infty)$ are Lipschitz continuous functions such that

$$\frac{1}{h_1(x)}+\frac{1}{h_2(x)}+\frac{1}{h_3(x)}=1,$$

then for all $u \in L^{h_1(x)}(\mathcal{D})$, $v \in L^{h_2(x)}(\mathcal{D})$, $w \in L^{h_3(x)}(\mathcal{D})$, the following inequality holds:

$$\int_{\mathcal{D}} \left|u(x)v(x)w(x)\right|dx \leq \left(\frac{1}{h_1^-}+\frac{1}{h_2^-}+\frac{1}{h_3^-}\right)\|u\|_{L^{h_1(x)}(\mathcal{D})}\|v\|_{L^{h_2(x)}(\mathcal{D})}\|w\|_{L^{h_3(x)}(\mathcal{D})}.$$

If $u \in L^{p(x)}(\mathcal{D})$ and $p < \infty$, we have the following properties (see, for example, Fan and Zhao [23, Theorems 1.3 and 1.4]):

$$\|u\|_{L^{p(x)}(\mathcal{D})} < 1\ (=1; >1) \quad \text{if and only if} \quad \rho_{p,\mathcal{D}}(u) < 1\ (=1; >1), \quad (2.2)$$

$$\text{if } \|u\|_{L^{p(x)}(\mathcal{D})} > 1 \quad \text{then } \|u\|^{p^-}_{L^{p(x)}(\mathcal{D})} \leq \rho_{p,\mathcal{D}}(u) \leq \|u\|^{p^+}_{L^{p(x)}(\mathcal{D})}, \quad (2.3)$$

$$\text{if } \|u\|_{L^{p(x)}(\mathcal{D})} < 1 \quad \text{then } \|u\|^{p^+}_{L^{p(x)}(\mathcal{D})} \leq \rho_{p,\mathcal{D}}(u) \leq \|u\|^{p^-}_{L^{p(x)}(\mathcal{D})}. \quad (2.4)$$

As a consequence, we have the equivalence of modular and norm convergence

$$\|u\|_{L^{p(x)}(\mathcal{D})} \to 0\ (\to \infty) \quad \text{if and only if} \quad \rho_{p,\mathcal{D}}(u) \to 0\ (\to \infty). \quad (2.5)$$

Proposition 2.2 (Edmunds and Rakosnik [22]) *Let h and ℓ be variable exponents in $\mathcal{M}_+(\mathcal{D})$ with $1 \leq h(x), \ell(x) \leq \infty$ a.e. x in Ω and $p \in L^\infty(\Omega)$. Then if $u \in L^{\ell(x)}(\Omega)$, $u \neq 0$, it follows that*

$$\|u\|_{L^{h(x)\ell(x)}(\Omega)} \leq 1 \quad \Longrightarrow \quad \|u\|^{h^-}_{L^{h(x)\ell(x)}(\Omega)} \leq \||u|^{h(x)}\|_{L^{\ell(x)}(\Omega)} \leq \|u\|^{h^+}_{L^{h(x)\ell(x)}(\Omega)},$$

$$\|u\|_{L^{h(x)\ell(x)}(\Omega)} \geq 1 \quad \Longrightarrow \quad \|u\|^{p^+}_{L^{h(x)\ell(x)}(\Omega)} \leq \||u|^{h(x)}\|_{L^{\ell(x)}(\Omega)} \leq \|u\|^{h^-}_{L^{h(x)\ell(x)}(\Omega)}.$$

When $h(x) = h$ is constant, we obtain $\||u|^h\|_{L^{\ell(x)}(\Omega)} = \|u\|^h_{L^{h\ell(x)}(\Omega)}$.

Now, let us pass to the Sobolev space with variable exponent, that is,

$$W^{1,p(x)}(\Omega) := \left\{u \in L^{p(x)}(\Omega) : \partial_{x_i} u \in L^{p(x)}(\Omega) \text{ for } i = 1, \dots, N\right\},$$

where $\partial_{x_i} u = \frac{\partial u}{\partial x_i}$ represent the partial derivatives of u with respect to x_i in the weak sense. This space has a corresponding modular given by

$$\rho_{1,p(x)}(u) := \int_\Omega |u|^{p(x)} + |\nabla u|^{p(x)}\,dx,$$

which yields the norm

$$\|u\|_{1,p(x)} := \|u\|_{W^{1,p(x)}(\Omega)} := \inf\left\{\tau > 0 : \rho_{1,p}\left(\frac{u(x)}{\tau}\right) \leq 1\right\}.$$

Another possible choice of norm in $W^{1,p(x)}(\Omega)$ is $\|u\|_{L^{p(x)}(\Omega)} + \|\nabla u\|_{L^{p(x)}(\Omega)}$. Both norms turn out to be equivalent, but we use the first one for convenience. It is well known that $W^{1,p(x)}(\Omega)$ is a separable and reflexive Banach spaces (see, e.g., Kováčik and Rákosník [34, Theorem 3.1]). As usual, we define by $p^*(x)$ the critical Sobolev exponent and $p^\partial(x)$ the critical Sobolev trace exponent, respectively, by

$$p^*(x) := \begin{cases} \frac{Np(x)}{N-p(x)} & \text{for } p(x) < N, \\ +\infty & \text{for } p(x) \geq N, \end{cases} \quad \text{and} \quad p^\partial(x) := \begin{cases} \frac{(N-1)p(x)}{N-p(x)} & \text{for } p(x) < N, \\ +\infty & \text{for } p(x) \geq N. \end{cases}$$

We recall the following crucial embeddings on $W^{1,p(x)}(\Omega)$.

Proposition 2.3 (Diening et al. [21], Edmunds and Rakosnik [22]) *Let p be Lipschitz continuous and satisfying $1 < p^- \leq p(x) \leq p^+ < N$, and let $q \in C(\overline{\Omega})$ satisfy $1 \leq q(x) \leq p^*(x)$, for all $x \in \overline{\Omega}$. Then there exists a continuous embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$. If we assume in addition that $1 \leq q(x) < p^*(x)$ for all $x \in \overline{\Omega}$, then this embedding is compact.*

Proposition 2.4 (Diening et al. [21], Edmunds and Rakosnik [22]) *Let $p \in W^{1,h}(\Omega)$ with $1 \leq p_- \leq p_+ < N < h$. Then for all $q \in C(\partial\Omega)$ satisfying $1 \leq q(x) \leq p^\partial(x)$ for $x \in \partial\Omega$, there is a continuous boundary trace embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\partial\Omega)$. If we assume in addition that $1 \leq q(x) < p^*(x)$ for all $x \in \overline{\Omega}$, then this embedding is compact.*

For detailed properties of the variable exponent Lebesgue–Sobolev spaces, we refer the reader to Diening et al. [21], Kováčik and Rákosník [34]. As is well known, the use of critical point theory needs the well-known Palais–Smale condition ($(PS)_c$, for short), which plays a central role.

Definition 2.5 Consider a function $E : X \to \mathbb{R}$ of class C^1, where X is a real Banach space. We say that a sequence $\{u_m\}$ is a Palais–Smale sequence for the functional E if

$$E(u_m) \to c \quad \text{and} \quad E'(u_m) \to 0 \quad \text{in } X'. \tag{2.6}$$

We say that $\{u_m\}$ is a Palais–Smale sequence with energy level c (or (u_m) is $(PS)_c$, for short). Moreover, if every $(PS)_c$ sequence for E has a strongly convergent subsequence in X, then we say that E satisfies the Palais–Smale condition at level c (or E is $(PS)_c$, short).

Our main tool is the following classical Mountain Pass Theorem.

Theorem 2.6 (Rabinowitz [43]) *Let X be a real infinite-dimensional Banach space and let $E : X \to \mathbb{R}$ be of class C^1 and satisfying the $(PS)_c$ such that $E(0_X) = 0$. Assume that*

- (H_1) *there exist positive constants $\rho, \mathcal{R}$ such that $E(u) \geq \mathcal{R}$, for all $u \in \partial B_\rho \cap X$;*
- (H_2) *there exists $z \in X$ with $\|z\|_X > \rho$ such that $E(z) < 0$.*

Then E has a critical value $c \geq \mathcal{R}$, which can be characterized as

$$c := \inf_{\phi\in\Gamma} \max_{\delta\in[0,1]} E\bigl(\phi(\delta)\bigr),$$

where

$$\Gamma = \bigl\{\phi : [0,1] \to X \text{ continuous} : \phi(0) = 0_X, E\bigl(\phi(1)\bigr) < 0\bigr\}.$$

In the sequel, we shall use the product space $X := \prod_{i=1}^{n}(W^{1,h_i(x)}(\Omega) \cap W^{1,p_i(x)}(\Omega))$, equipped with the norm $\|u\| := \max_{1\leq i\leq n}\{\|u_i\|_i\}$, for all $u = (u_1, u_2, \dots, u_n) \in X$, where $\|u_i\|_i := \|u_i\|_{1,p_i(x)} + \mathcal{K}(k_i^3)\|u_i\|_{1,q_i(x)}$ is the norm of $W^{1,h_i(x)}(\Omega) \cap W^{1,p_i(x)}(\Omega)$, for all $i \in \{1, 2, \dots, n\}$.

Definition 2.7 We say that $u = (u_1, u_2, \dots, u_n) \in X$ is a weak solution of the system (1.1) if

$$\begin{aligned}
&\sum_{i=1}^{n} M_i\bigl(\mathcal{A}_i(u_i)\bigr) \int_\Omega \bigl(\mathcal{B}_{1_i}(\nabla u_i)\nabla v_i + \mathcal{B}_{2_i}(u_i)v_i\bigr)\,dx \\
&\quad - \sum_{i=1}^{n}\int_\Omega |u_i|^{s_i(x)-2}u_i v_i\,dx - \sum_{i=1}^{n}\int_{\partial\Omega}|u_i|^{\ell_i(x)-2}u_i v_i\,d\sigma_x \\
&\quad - \sum_{i=1}^{n}\int_\Omega \lambda F_{u_i}(x,u)v_i\,dx = 0,
\end{aligned}$$

for all $v = (v_1, v_2, \dots, v_n) \in X = \prod_{i=1}^{n}(W^{1,h_i(x)}(\Omega) \cap W^{1,p_i(x)}(\Omega))$.

The energy functional $E_\lambda : X \to \mathbb{R}$ associated with problem (1.1) is defined as $E_\lambda(\cdot) := \Phi(\cdot) - \Psi(\cdot) - \Upsilon(\cdot) - \mathcal{F}_\lambda(\cdot)$, where $\Phi, \Psi,$ and $\mathcal{F}_\lambda : X \to \mathbb{R}$ are given by

$$\begin{aligned}
&\Phi(u) = \sum_{i=1}^{n}\widehat{M}_i\bigl(\mathcal{A}_i\bigl(u_i(x)\bigr)\bigr), \qquad \Psi(u) = \sum_{i=1}^{n}\int_\Omega \frac{1}{s_i(x)}|u_i|^{s_i(x)}\,dx, \\
&\Upsilon(u) = \sum_{i=1}^{n}\int_{\partial\Omega}\frac{1}{\ell_i(x)}|u_i|^{\ell_i(x)}\,d\sigma_x, \quad \text{and} \quad \mathcal{F}_\lambda(u) = \int_\Omega \lambda F(x,u)\,dx,
\end{aligned}$$

for all $u = (u_1, \dots, u_n)$ in X, where $\widehat{M}_i(\tau) = \int_0^\tau M_i(s)\,ds$. By standard calculus, one can see that, under the above assumptions, the energy functional $E_\lambda : X \to \mathbb{R}^N$ corresponding to problem (1.1) is well defined and $E_\lambda \in C^1(X, \mathbb{R})$ with

$$\begin{aligned}
\bigl\langle E'_\lambda(u), v\bigr\rangle &= \sum_{i=1}^{n} M_i\bigl(\mathcal{A}_i(u_i)\bigr)\int_\Omega \bigl(\mathcal{B}_{1_i}(\nabla u_i)\nabla v_i + \mathcal{B}_{2_i}(u_i)v_i\bigr)\,dx - \sum_{i=1}^{n}\int_\Omega |u_i|^{s_i(x)-2}u_i v_i\,dx \\
&\quad - \sum_{i=1}^{n}\int_{\partial\Omega}|u_i|^{t_i(x)-2}u_i v_i\,d\sigma_x - \sum_{i=1}^{n}\int_\Omega \lambda F_{u_i}(x,u)v_i\,dx,
\end{aligned}$$

for all $v = (v_1, v_2, \dots, v_n) \in X$. So the critical points of functional E_λ are weak solutions of system (1.1).

To prove our existence result, since we have lost compactness in the inclusions $W^{1,h_i(x)}(\Omega) \hookrightarrow L^{h_i^*(x)}(\Omega)$ and $W^{1,h_i(x)}(\Omega) \hookrightarrow L^{h_i^\partial(x)}(\partial\Omega)$, for all i in $\{1, 2, \dots, n\}$, we can no longer expect the Palais–Smale condition to hold. Nevertheless, we can prove a local Palais–Smale condition that will hold for the energy functional E_λ below a certain value of energy, by using the principle of concentration–compactness for the variable exponent Sobolev space $W^{1,h_i(x)}(\Omega)$. For the reader's convenience, we state this result in order to prove Theorem 1.2, see Bonder et al. [6, 7] for its proof.

Now, let $\mathcal{O}$ be a different subset of $\partial\Omega$, a closed set (possibly empty). Set

$$W^{1,h(x)}_{\mathcal{O}}(\Omega) := \overline{\{\nu \in C^\infty(\overline{\Omega}) : \nu \text{ vanishes on a neighborhood of } \mathcal{O}\}},$$

where closure is taken with respect to $\|\nu\|_{1,h(x)}$. This is the subspace of functions vanishing on $\mathcal{O}$. Evidently, $W^{1,h(x)}_{\emptyset}(\Omega) = W^{1,h(x)}(\Omega)$. In general, $W^{1,h(x)}_{\mathcal{O}}(\Omega) = W^{1,h_i(x)}(\Omega)$ if and only if the $h_i(x)$-capacity of $\mathcal{O}$ equals zero; for more details, we refer the interested readers to Harjulehto et al. [29]. The best Sobolev trace constant $T_i(h_i(x), \ell_i(x), \mathcal{O})$ is defined by

$$0 < T_i\big(h_i(x), \ell_i(x), \mathcal{O}\big) := \inf_{\nu \in W^{1,h_i(x)}_{\mathcal{O}}(\Omega)} \frac{\|\nu\|_{W^{1,h_i(x)}(\Omega)}}{\|\nu\|_{L^{\ell_i(x)}(\partial\Omega)}}.$$

Theorem 2.8 (Bonder et al. [6]) *Let $h_i \in C_+(\overline{\Omega})$, $\ell_i \in C_+(\partial\Omega)$ be such that $\ell_i(x) \le h_i^\partial(x)$, for all $x \in \partial\Omega$, and $\{u_m\}_{m\in\mathbb{N}}$ be a sequence in $W^{1,h_i(x)}(\Omega)$ such that $u_{i_m} \rightharpoonup u_i$ weakly in $W^{1,h_i(x)}(\Omega)$. Then there exist a countable index set J_i^1, positive numbers $\{\mu_{i_j}\}_{j\in J_i^1}$ and $\{\nu_{i_j}\}_{j\in J_i^1}$, and $\{x_j\}_{j\in J_i^1} \subset \mathcal{C}^1_{h_i} = \{x \in \partial\Omega : \ell_i(x) = h_i^\partial(x)\}$ such that*

$$|u_{i_m}|^{\ell_i(x)} \rightharpoonup \nu_i = |u_i|^{\ell_i(x)} + \sum_{j\in J_i^1} \nu_{i_j}\delta_{x_j} \quad \textit{weakly-* in the sense of measures,} \tag{2.7}$$

$$|\nabla u_{i_m}|^{h_i(x)} \rightharpoonup \mu_i \ge |\nabla u_i|^{h_i(x)} + \sum_{j\in I} \mu_{i_j}\delta_{x_j} \quad \textit{weakly-* in the sense of measures,} \tag{2.8}$$

$$\overline{T}_{i x_j} \nu_{i_j}^{1/h_i^\partial(x_j)} \le \mu_{i_j}^{1/h_i(x_j)} \quad \textit{for all } j \in J_i^1, \tag{2.9}$$

where

$$\overline{T}_{i x_j} := \sup_{\epsilon>0} T\big(h_i(x), \ell_i(x), \Omega_{\epsilon,j}, \Lambda_{\epsilon,j}\big), \tag{2.10}$$

is the localized Sobolev trace constant with $\Omega_{\epsilon,j} = \Omega \cap B_\epsilon(x_j)$ and $\Lambda_{\epsilon,j} = \Omega \cap \partial B_\epsilon(x_j)$.

Theorem 2.9 (Bonder and Silva [7]) *Let h_i and s_i be variable exponents $\in C_+(\overline{\Omega})$ such that $s_i(x) \le h_i^*(x)$, for all $x \in \overline{\Omega}$, and $\{u_m\}_{m\in\mathbb{N}}$ be a sequence in $W^{1,h_i(x)}(\Omega)$ such that $u_{i_m} \rightharpoonup u_i$ weakly in $W^{1,h_i(x)}(\Omega)$. Then there exist a countable set J_i^2, positive numbers $\{\mu_{i_j}\}_{j\in J_i^2}$ and $\{\nu_{i_j}\}_{j\in J_i^2}$, and $\{x_j\}_{j\in J_i^2} \subset \mathcal{C}^2_{h_i} = \{x \in \Omega : s_i(x) = h_i^*(x)\}$ such that*

$$|u_{i_m}|^{s_i(x)} \rightharpoonup \nu_i = |u_i|^{s_i(x)} + \sum_{j\in I} \nu_{i_j}\delta_{x_j} \quad \textit{weakly-* in the sense of measures,} \tag{2.11}$$

$$|\nabla u_{i_m}|^{h_i(x)} \rightharpoonup \mu_i \ge |\nabla u_i|^{h_i(x)} + \sum_{j\in I} \mu_{i_j}\delta_{x_j} \quad \textit{weakly-* in the sense of measures,} \tag{2.12}$$

$$S_i \nu_{i_j}^{1/h_i^*(x_j)} \le \mu_{i_j}^{1/h_i(x_j)} \quad \textit{for all } j \in J_i^2, \tag{2.13}$$

where

$$S_i = S_{i_{q_i}}(\Omega) := \inf_{\phi \in C_0^\infty(\Omega)} \frac{\| |\nabla v| \|_{L^{p_i(x)}(\Omega)}}{\|v\|_{L^{s_i(x)}(\Omega)}}, \tag{2.14}$$

is the best constant in the Gagliardo–Nirenberg–Sobolev inequality for variable exponents.

Notations. Weak (resp. strong) convergence will be denoted by $\rightharpoonup$ (resp., $\to$), C_i, C_{i_j}, c_j, and c_{i_j} will denote positive constants which may vary from line to line and can be determined in concrete conditions. Here, X^* denotes the dual space of X, δ_{x_j} is the Dirac mass at x_j, for all $\rho > 0$, $x \in \Omega$, where $B(x,\rho)$ denotes the ball of radius ρ centered at x.

3 The auxiliary problem and variational framework

In order to prove Theorem 1.2, we shall introduce the auxiliary problem by defining the auxiliary functional $E_{\theta,\lambda}$ and showing that the energy functional $E_{\theta,\lambda}$ has the geometry of the Mountain Pass Theorem 2.6.

By assumption (***M***), we see that the functions M_i are bounded only from below and do not give us enough information about the behavior of M_i at infinity, which makes it difficult to prove that the functional E_λ has the geometry of the Mountain Pass Theorem and that the sequence of Palais–Smale is bounded in X. Hence, we truncate functions M_i and study the associated truncated problem.

Take γ_i as in assumption ($\boldsymbol{F}_3$) and $\theta_i \in \mathbb{R}$, for all $i \in \{1,\dots,N\}$ such that $\mathfrak{M}_i^0 < \theta_i < \frac{\gamma_i \mathfrak{M}_i^0}{p_i^+ \max\{\beta_{1_i},\beta_{2_i}\}}$. By assumption (***M***), there exists $\tau_i^0 > 0$ such that $M_i(\tau_i^0) = \theta_i$. Thus, by setting

$$M_{\theta_i}(\tau_i) = \begin{cases} M_i(\tau_i) & \text{for } 0 \le \tau_i \le \tau_i^0, \\ \theta_i & \text{for } \tau_i \ge \tau_i^0, \end{cases}$$

we can introduce the following auxiliary problem:

$$\begin{cases} M_{\theta_i}(\mathcal{A}_i(u_i))(-\operatorname{div}(\mathcal{B}_{1_i}(\nabla u_i)) + \mathcal{B}_{2_i}(u_i)) = |u_i|^{s_i(x)-2}u_i + \lambda F_{u_i}(x,u) & \text{in } \Omega, \\ \mathfrak{N} \cdot M_{\theta_i}(\mathcal{A}_i(u_i))\mathcal{B}_{1_i}(\nabla u_i) = |u_i|^{\ell_i(x)-2}u_i & \text{on } \partial\Omega; \end{cases} \tag{3.1}$$

for $1 \le i \le n$, where $\mathcal{A}_i$, $\mathcal{B}_i$, F_{u_i}, and λ are as in Sect. 1. By assumption (***M***), we also know that

$$\mathfrak{M}_i^0 \le M_{\theta_i}(\tau_i) \le \theta_i < \frac{\gamma_i \mathfrak{M}_i^0}{p_i^+ \max\{\beta_{1_i},\beta_{2_i}\}}, \quad \text{for all } \tau_i \ge 0, 1 \le i \le n. \tag{3.2}$$

Now, the next step is to prove that the auxiliary problem (3.1) has a nontrivial weak solution. We obtain the following result.

Theorem 3.1 *Suppose that conditions* ($\boldsymbol{A}_1$)–($\boldsymbol{A}_4$), (***M***), *and* ($\boldsymbol{F}_1$)–($\boldsymbol{F}_4$) *hold. Then there exists a constant* $\lambda_\star > 0$, *such that if* $\lambda \ge \lambda_\star$, *then problem* (3.1) *has at least one nontrivial solution in* X.

For the proof of Theorem 3.1, we shall need some technical results. We observe that the auxiliary problem (3.1) has a variational structure; indeed, it is the Euler–Lagrange equation of the functional $E_{\theta,\lambda} : X \to \mathbb{R}$ defined as follows:

$$
\begin{aligned}
E_{\theta,\lambda}(u) = {}& \sum_{i=1}^{n} \widehat{M_{\theta_i}}\big(\mathcal{A}_i(u_i)\big) - \sum_{i=1}^{n} \int_{\Omega} \frac{1}{s_i(x)} |u_i|^{s_i(x)}\, dx \\
& - \sum_{i=1}^{n} \int_{\partial\Omega} \frac{1}{\ell_i(x)} |u_i|^{\ell_i(x)}\, d\sigma_x - \int_{\Omega} \lambda F(x,u)\, dx,
\end{aligned}
$$

where $\widehat{M}_{\theta_i}(\tau) = \int_0^\tau M_{\theta_i}(s)\, ds$. Moreover, the functional $E_{\theta,\lambda}$ is Fréchet differentiable in $u \in X$ and, for all $v = (v_1, \dots, v_n) \in X$,

$$
\begin{aligned}
\big\langle E'_{\theta,\lambda}(u), v\big\rangle = {}& \sum_{i=1}^{n} M_{\theta_i}\big(\mathcal{A}_i(u_i)\big) \int_{\Omega} \big(\mathcal{B}_{1_i}(\nabla u_i)\nabla v_i + \mathcal{B}_{2_i}(u_i)v_i\big)\, dx \\
& - \sum_{i=1}^{n} \int_{\Omega} |u_i|^{s_i(x)-2} u_i v_i\, dx \\
& - \sum_{i=1}^{n} \int_{\partial\Omega} |u_i|^{s_i(x)-2} u_i v_i\, d\sigma_x - \sum_{i=1}^{n} \int_{\Omega} \lambda F_{u_i}(x,u) v_i\, dx.
\end{aligned}
\tag{3.3}
$$

Now we prove that the functional $E_{\theta,\lambda}$ has the geometric features required by the Mountain Pass Theorem 2.6.

Lemma 3.2 *Suppose that conditions* ($\boldsymbol{A}_1$)–($\boldsymbol{A}_4$), ($\boldsymbol{M}$) *and* ($\boldsymbol{F}_1$)–($\boldsymbol{F}_4$) *hold. Then there exist positive constants* $\mathcal{R}$ *and* ρ *such that* $E_{\theta,\lambda}(u) \geq \mathcal{R} > 0$, *for all* $u \in X$ *with* $\|u\| = \rho$.

Proof For all $u = (u_1, \dots, u_n) \in X$, we obtain, under the assumptions ($\boldsymbol{A}_2$) and ($\boldsymbol{A}_4$),

$$
\begin{aligned}
& \sum_{i=1}^{n} \widehat{M}_{\theta_i}\big(\mathcal{A}_i(u_i)\big) \\
& \geq \sum_{i=1}^{n} \frac{\mathfrak{M}_i^0}{p_i^+} \int_{\Omega} \big(A_{1_i}\big(|\nabla u_i|^{p_i(x)}\big) + A_{2_i}\big(|u_i|^{p_i(x)}\big)\big)\, dx \\
& \geq \sum_{i=1}^{n} \frac{\mathfrak{M}_i^0}{p_i^+} \int_{\mathbb{R}^N} \bigg[\frac{1}{\beta_{1_i}} a_{1_i}\big(|\nabla u_i|^{p_i(x)}\big)|\nabla u_i|^{p_i(x)} + \frac{1}{\beta_{2_i}} a_{2_i}\big(|u_i|^{p_i(x)}\big)|u_i|^{p_i(x)}\bigg]\, dx \\
& \geq \sum_{i=1}^{n} \frac{\mathfrak{M}_i}{p_i^+ \max\{\beta_{1_i}, \beta_{2_i}\}} \int_{\mathbb{R}^N} \big[a_{1_i}\big(|\nabla u_i|^{p_i(x)}\big)|\nabla u_i|^{p_i(x)} + a_{2_i}\big(|u_i|^{p_i(x)}\big)|u_i|^{p_i(x)}\big]\, dx \\
& \geq \sum_{i=1}^{n} \frac{\mathfrak{M}_i^0 \min\{\min\{k_{1_i}^0, k_{2_i}^0\}, \min\{k_{1_i}^2, k_{2_i}^2\}\}}{p_i^+ \max\{\beta_{1_i}, \beta_{2_i}\}} \big(\rho_{1,p_i(x)}(u_i) + \mathcal{K}\big(k_i^3\big)\rho_{1,q_i(x)}(u_i)\big).
\end{aligned}
$$

Hence, by using assumption ($\boldsymbol{F}_4$), and relations (2.2)–(2.3), we have

$$
\begin{aligned}
& E_{\theta,\lambda}(u) \\
& \geq \sum_{i=1}^{n} \frac{\mathfrak{M}_i^0 \min\{\min\{k_{1_i}^0, k_{2_i}^0\}, \min\{k_{1_i}^2, k_{2_i}^2\}\}}{p_i^+ \max\{\beta_{1_i}, \beta_{2_i}\}} \Big(\min\Big\{\|u_i\|_{1,p_i(x)}^{p_i^-}, \|u_i\|_{1,p_i(x)}^{p_i^+}\Big\}
\end{aligned}
$$

$$+\mathcal{K}(k_i^3)\min\{\|u_i\|_{1,q_i(x)}^{q_i^-},\|u_i\|_{1,q_i(x)}^{q_i^+}\}\big)-\sum_{i=1}^n\frac{1}{s_i^-}\max\{\|u_i\|_{L^{s_i(x)}(\Omega)}^{s_i^-},\|u_i\|_{L^{s_i(x)}(\Omega)}^{s_i^+}\}$$
$$-\sum_{i=1}^n\frac{1}{\ell_i^-}\max\{\|u_i\|_{L^{\ell_i(x)}(\partial\Omega)}^{\ell_i^-},\|u_i\|_{L^{\ell_i(x)}(\partial\Omega)}^{\ell_i^+}\}-\sum_{i=1}^n\lambda\max\{\|u_i\|_{L^{w_i(x)}(\Omega)}^{w_i^-},\|u_i\|_{L^{w_i(x)}(\Omega)}^{w_i^+}\}.$$

By the Sobolev Embedding Theorem and taking $0<\|u\|=\max_{1\le i\le n}\{\|u_i\|_{1,p_i(x)}+\mathcal{K}(k_i^3)\times\|u_i\|_{1,q_i(x)}\}=\rho<1$, there exist positive constants c_{1_i}, c_{2_i}, c_{3i}, c_{4i}, and c_{5i} such that

$$\begin{aligned}E_{\theta,\lambda}(u)&\ge\sum_{i=1}^n c_{1_i}\big(\|u_i\|_{1,p_i(x)}^{q^+}+\mathcal{K}(k_i^3)\|u_i\|_{1,q_i(x)}^{q^+}\big)\\&\quad-\sum_{i=1}^n c_{2_i}\|u_i\|_i^{s_i^-}-\sum_{i=1}^n c_{3i}\|u_i\|_i^{\ell_i^-}-\sum_{i=1}^n c_{4i}\|u_i\|_i^{w_i^-}\\&\ge\sum_{i=1}^n\big(c_{5i}\|u_i\|_i^{q_i^+}-c_{2_i}\|u_i\|_i^{s_i^-}-c_{3i}\|u_i\|_i^{\ell_i^-}-c_{4i}\|u_i\|_i^{w_i^-}\big).\end{aligned}$$

Since $q_i^+<w_i^-<\inf\{s_i^-,\ell_i^-\}$, there exists a positive constant $\mathcal{R}$ such that $E_{\theta,\lambda}(u)\ge\mathcal{R}>0$, with $\|u\|=\rho$. □

Lemma 3.3 *Suppose that conditions* $(\boldsymbol{A}_1)$–$(\boldsymbol{A}_4)$, $(\boldsymbol{M})$, *and* $(\boldsymbol{F}_4)$ *hold. Then for every positive function* λ, *there exists a nonnegative function* $e\in X$, *independent of* λ, *such that* $\|z\|>R$ *and* $E_{\theta,\lambda}(z)<0$.

Proof By the assumptions $(\boldsymbol{A}_1)$ and $(\boldsymbol{A}_4)$, for all $\delta>0$ and $u\in X$, we have

$$\begin{aligned}&\sum_{i=1}^n\widehat{M_{\theta_i}}\big(\mathcal{A}_i(\delta u_i)\big)\\&\quad=\sum_{i=1}^n\int_0^{\mathcal{A}_i(\delta u_i)}M_{\theta,i}(s)\,ds\le\sum_{i=1}^n\theta_i\mathcal{A}_i(\delta u_i)\\&\quad\le\sum_{i=1}^n\theta_i\int_\Omega\frac{1}{p_i(x)}(A_{1_i}\big(|\nabla(\delta u_i)|^{p_i(x)}+A_{2_i}\big(|\delta u_i|^{p_i(x)}\big)\big)\,dx\\&\quad\le\sum_{i=1}^n\theta_i\int_\Omega\bigg(\frac{\max\{k_{1_i}^1,k_{2_i}^1\}}{p_i(x)}\big(|\nabla(\delta u_i)|^{p_i(x)}+|\delta u_i|^{p_i(x)}\big)\\&\qquad+\frac{k_i^3}{q_i(x)}\big(|\nabla(\delta u_i)|^{q_i(x)}+|\delta u_i|^{q_i(x)}\big)\bigg)\,dx\\&\quad\le\sum_{i=1}^n\theta_i\bigg(\frac{\max\{k_{1_i}^1,k_{2_i}^1\}}{p_i^-}\max\{\|u_i\|_{1,p_i(x)}^{p_i^-},\|u_i\|_{1,p_i(x)}^{p_i^+}\}+\frac{k_i^3}{q_i^+}\max\{\|u_i\|_{1,q_i(x)}^{q_i^-},\|u_i\|_{1,q_i(x)}^{q_i^+}\}\bigg).\end{aligned}$$

By this inequality and assumption $(\boldsymbol{F}_4)$, we have, for $e=(e_1,\dots,e_n)\in X\setminus\{(0,\dots,0)\}$ and each $\delta>1$,

$$E_{\theta,\lambda}(\delta e)=\sum_{i=1}^n\widehat{M_i}\big(\mathcal{A}_i(\delta e_i)\big)-\sum_{i=1}^n\int_\Omega\frac{1}{s_i(x)}|\delta e_i|^{s_i(x)}\,dx$$

$$
\begin{aligned}
&-\sum_{i=1}^{n}\int_{\partial\Omega}\frac{1}{\ell_i(x)}|\delta e_i|^{\ell_i(x)}\,d\sigma_x-\lambda\int_{\Omega}F(x,\delta e)\,dx\\
&\le\sum_{i=1}^{n}\biggl[\theta_i\delta^{q_i^+}\biggl(\frac{\max\{k_{1_i}^1,k_{2_i}^1\}}{p_i^-}\max\bigl\{\|e_i\|_{1,p_i(x)}^{p_i^-},\|e_i\|_{1,p_i(x)}^{p_i^+}\bigr\}\\
&\quad+\frac{k_i^3}{q_i^+}\max\bigl\{\|e_i\|_{1,q_i(x)}^{q_i^-},\|e_i\|_{1,q_i(x)}^{q_i^+}\bigr\}\biggr)-\frac{\delta^{s_i^-}}{s_i^-}\min\bigl\{\|e_i\|_{s_i(x)}^{s_i^-},\|e_i\|_{s_i(x)}^{s_i^+}\bigr\}\\
&\quad-\frac{\delta^{\ell_i^-}}{\ell_i^-}\min\bigl\{\|e_i\|_{\ell_i(x)}^{\ell_i^-},\|e_i\|_{\ell_i(x)}^{\ell_i^+}\bigr\}-\delta^{w_i^-}\min\bigl\{\|e_i\|_{w_i(x)}^{w_i^-},\|e_i\|_{w_i(x)}^{w_i^+}\bigr\}\biggr],
\end{aligned}
$$

which tends to $-\infty$ as $\delta\to+\infty$ since $\min\{s_i^-,\ell_i^-\}>w_i^->q_i^+$. So, the lemma is proven by choosing $z=\delta_* e$ with $\delta_*>0$ sufficiently large. □

Now, by the Mountain Pass Theorem 2.6 without the Palais–Smale condition, we get a sequence $\{u_m\}_{m\in\mathbb{N}}\subset X$ such that $E_{\theta,\lambda}(u_m)\to c_{\theta,\lambda}$ and $E'_{\theta,\lambda}(u_m)\to 0$, where $c_{\theta,\lambda}:=\inf_{\phi\in\Gamma}\max_{\delta\in[0,1]}E_\lambda(\phi(\delta))$, and

$$\Gamma=\bigl\{\phi:[0,1]\to X\text{ continuous}:\phi(0)=(0,\dots,0),E_{\theta,\lambda}\bigl(\phi(1)\bigr)<0\bigr\}.$$

Lemma 3.4 *Suppose that conditions* $(\boldsymbol{A}_1)$–$(\boldsymbol{A}_2)$, $(\boldsymbol{M})$, *and* $(\boldsymbol{F}_4)$ *hold. Then* $\lim_{\lambda\to+\infty}c_{\theta,\lambda}=0$.

Proof Let $z=(z_1,\dots,z_n)\in X$ be the function given by Lemma 3.3. Then $\lim_{\delta\to\infty}E_{\theta,\lambda}(\delta z)=-\infty$, for each $\lambda>0$, so it follows that there exists $\delta_\lambda>0$ such that $E_{\theta,\lambda}(\delta_\lambda z)=\max_{\delta\ge 0}E_{\theta,\lambda}(\delta z)$. Hence, $\langle E'_{\theta,\lambda}(\delta_\lambda z),\delta_\lambda z\rangle=0$, so it follows by relation (3.3) that

$$
\begin{aligned}
&\sum_{i=1}^{n}M_{\theta_i}\bigl(\mathcal{A}_i(\delta_\lambda z_i)\bigr)\int_\Omega\bigl(a_{1_i}\bigl(\bigl|\nabla(\delta_\lambda z_i)\bigr|^{p_i(x)}\bigr)\bigl|\nabla(\delta_\lambda z_i)\bigr|^{p_i(x)}+a_{2_i}\bigl(|\delta_\lambda z_i|^{p_i(x)}\bigr)|\delta_\lambda z_i|^{p_i(x)}\bigr)\,dx\\
&\quad=\sum_{i=1}^{n}\int_\Omega|\delta_\lambda z_i|^{s_i(x)}\,dx\\
&\qquad+\sum_{i=1}^{n}\int_{\partial\Omega}|\delta_\lambda z_i|^{\ell_i(x)}\,d\sigma_x+\sum_{i=1}^{n}\lambda\delta_\lambda\int_\Omega F_{u_i}(x,\delta_\lambda z)z_i\,dx.
\end{aligned}
\tag{3.4}
$$

By construction, $z_i\ge 0$ a.e. in Ω, for all $i\in\{1,2,\dots,n\}$. Therefore, by assumption $(\boldsymbol{F}_3)$ and relation (3.4), we get

$$
\begin{aligned}
&\sum_{i=1}^{n}M_{\theta_i}\bigl(\mathcal{A}_i(\delta_\lambda z_i)\bigr)\int_\Omega\bigl(a_{1_i}\bigl(\bigl|\nabla(\delta_\lambda z_i)\bigr|^{p_i(x)}\bigr)\bigl|\nabla(\delta_\lambda z_i)\bigr|^{p_i(x)}+a_{2_i}\bigl(|\delta_\lambda z_i|^{p_i(x)}\bigr)|\delta_\lambda z_i|^{p_i(x)}\bigr)\,dx\\
&\quad\ge\sum_{i=1}^{n}\int_\Omega|\delta_\lambda z_i|^{s_i(x)}\,dx+\sum_{i=1}^{n}\int_{\partial\Omega}|\delta_\lambda z_i|^{\ell_i(x)}\,d\sigma_x.
\end{aligned}
\tag{3.5}
$$

On the other hand, by assumption $(\boldsymbol{A}_2)$ and inequalities (2.2), (2.3), we get

$$
\begin{aligned}
&\sum_{i=1}^{n}M_{\theta_i}\bigl(\mathcal{A}_i(\delta_\lambda z_i)\bigr)\int_\Omega\bigl(a_{1_i}\bigl(\bigl|\nabla(\delta_\lambda z_i)\bigr|^{p_i(x)}\bigr)\bigl|\nabla(\delta_\lambda z_i)\bigr|^{p_i(x)}+a_{2_i}\bigl(|\delta_\lambda z_i|^{p_i(x)}\bigr)|\delta_\lambda z_i|^{p_i(x)}\bigr)\,dx\\
&\quad\le\sum_{i=1}^{n}\theta_i\biggl(\int_\Omega\bigl(a_{1_i}\bigl(\bigl|\nabla(\delta_\lambda z_i)\bigr|^{p_i(x)}\bigr)\bigl|\nabla(\delta_\lambda z_i)\bigr|^{p_i(x)}+a_{2_i}\bigl(|\delta_\lambda z_i|^{p_i(x)}\bigr)|\delta_\lambda z_i|^{p_i(x)}\bigr)\,dx\biggr)
\end{aligned}
$$

$$\begin{aligned}
&\leq \sum_{i=1}^{n} \theta_i \biggl(\max\bigl\{k_{1_i}^1, k_{2_i}^1\bigr\} \int_\Omega \bigl(\bigl|\nabla(\delta_\lambda z_i)\bigr|^{p_i(x)} + |\delta_\lambda z_i|^{p_i(x)}\bigr)\,dx \\
&\quad + k_i^3 \int_\Omega \bigl(\bigl|\nabla(\delta_\lambda z_i)\bigr|^{q_i(x)} + |\delta_\lambda z_i|^{q_i(x)}\bigr)\,dx \biggr) \\
&\leq \sum_{i=1}^{n} \theta_i \bigl(\max\bigl\{k_{1_i}^1, k_{2_i}^1\bigr\} \max\bigl\{\|\delta_\lambda z_i\|_{1,p_i(x)}^{p_i^-}, \|\delta_\lambda z_i\|_{1,p_i(x)}^{p_i^+}\bigr\} \\
&\quad + k_i^3 \max\bigl\{\|\delta_\lambda z_i\|_{1,q_i(x)}^{q_i^-}, \|\delta_\lambda z_i\|_{1,q_i(x)}^{q_i^+}\bigr\}\bigr).
\end{aligned} \tag{3.6}$$

Therefore, from relations (3.5), (3.6), and inequalities (2.2), (2.3), we obtain

$$\begin{aligned}
&\sum_{i=1}^{n} \theta_i \bigl(\max\bigl\{k_{1_i}^1, k_{2_i}^1\bigr\} \max\bigl\{\|\delta_\lambda z_i\|_{1,p_i(x)}^{p_i^-}, \|\delta_\lambda z_i\|_{1,p_i(x)}^{p_i^+}\bigr\} \\
&\qquad + k_i^3 \max\bigl\{\|\delta_\lambda z_i\|_{1,q_i(x)}^{q_i^-}, \|\delta_\lambda z_i\|_{1,q_i(x)}^{q_i^+}\bigr\}\bigr) \\
&\quad \geq \sum_{i=1}^{n} \int_\Omega |\delta_\lambda z_i|^{s_i(x)}\,dx + \sum_{i=1}^{n} \int_{\partial\Omega} |\delta_\lambda z_i|^{s_i(x)}\,d\sigma_x \\
&\quad \geq \sum_{i=1}^{n} \min\bigl\{\|\delta_\lambda z_i\|_{L^{s_i(x)}(\Omega)}^{s_i^-}, \|\delta_\lambda z_i\|_{L^{s_i(x)}(\Omega)}^{s_i^+}\bigr\} \\
&\qquad + \sum_{i=1}^{n} \min\bigl\{\|\delta_\lambda z_i\|_{L^{\ell_i(x)}(\partial\Omega)}^{\ell_i^-}, \|\delta_\lambda z_i\|_{L^{\ell_i(x)}(\partial\Omega)}^{\ell_i^+}\bigr\}.
\end{aligned} \tag{3.7}$$

Next, we shall show that the sequence $\{\delta_\lambda\}$ is bounded in $\mathbb{R}$. Indeed, we suppose by contradiction that $\{\delta_\lambda\}$ is unbounded. Then there is a subsequence denoted by $\{\delta_{\lambda_m}\}$, with $\delta_{\lambda_m} \to \infty$, as $m \to +\infty$. Then by relation (3.7),

$$\begin{aligned}
&\sum_{i=1}^{n} \biggl(\frac{\theta_i \max\{k_{1_i}^1, k_{2_i}^1\} \|z_i\|_{1,p_i(x)}^{p_i^+}}{\delta_{\lambda_m}^{q_M^+ - p_i^+}} + k_i^3 \theta_i \|z_i\|_{1,q_i(x)}^{q_i^+} \biggr) \\
&\quad \geq \sum_{i=1}^{n} \delta_\lambda^{s_i^- - q_M^+} \|z_i\|_{L^{s_i(x)}(\Omega)}^{s_i^-} + \sum_{i=1}^{n} \delta_\lambda^{\ell_i^- - q_M^+} \|z_i\|_{L^{\ell_i(x)}(\partial\Omega)}^{\ell_i^-},
\end{aligned} \tag{3.8}$$

where $q_M^+ = \max_{1\leq i\leq n}\{q_i^+\}$. Therefore, when taking the limit as $m \to +\infty$, we get a contradiction because $p_i^+ < q_M^+ < \inf\{s_i^-, \ell_i^-\}$. Thus, we can conclude that $\{\delta_\lambda\}$ is indeed bounded in $\mathbb{R}$.

Consider a sequence $\{\lambda_m\}_{m\in\mathbb{N}}$ such that $\lambda_m \to +\infty$ and let $\delta_0 \geq 0$ be such that $\delta_{\lambda_m} \to \delta_0$, as $m \to +\infty$. Then by continuity of M_{θ_i}, $\{M_{\theta_i}(\mathcal{A}_i(\delta_{\lambda_m} z_i))\}_{m\in\mathbb{N}}$ is bounded, for all $i \in \{1, 2, \ldots, n\}$. Therefore, there exists $C > 0$ such that

$$\begin{aligned}
&\sum_{i=1}^{n} M_{\theta_i}\bigl(\mathcal{A}_i(\delta_{\lambda_m} z_i)\bigr) \int_\Omega \bigl(a_{1_i}\bigl(|\nabla\delta_{\lambda_m} z_i|^{p_i(x)}\bigr)|\nabla\delta_{\lambda_m} z_i|^{p_i(x)} + a_{2_i}\bigl(|\delta_{\lambda_m} z_i|^{p_i(x)}\bigr)|\delta_{\lambda_m} z_i|^{p_i(x)}\bigr)\,dx \\
&\quad \leq C, \quad \text{for all } m \in \mathbb{N},
\end{aligned}$$

so by inequality (3.5), we have

$$\begin{aligned}&\sum_{i=1}^{n}\int_{\Omega}|\delta_{\lambda_m}z_i|^{s_i(x)}\,dx+\sum_{i=1}^{n}\int_{\partial\Omega}|\delta_{\lambda_m}z_i|^{\ell_i(x)}\,d\sigma_x\\&\quad+\sum_{i=1}^{n}\int_{\Omega}\lambda\delta_{\lambda_m}F_{u_i}(x,\delta_{\lambda_m}z)z_i\,dx\le C,\quad\text{for all } m\in\mathbb{N}.\end{aligned}\tag{3.9}$$

We shall prove that $\delta_0 = 0$. Indeed, if $\delta_0 > 0$, then by assumption ($\boldsymbol{F}_2$), there exist positive functions b_{i_j} $(1 \le i, j \le n)$, such that

$$\left|F_{\xi_i}(x,\xi_1,\dots,\xi_n)\right|\le\sum_{j=1}^{n}b_{i_j}(x)|\xi_j|^{\ell_{i_j}-1},\quad\text{where } 1<\ell_{i_j}<\inf_{x\in\Omega}h_i(x),\ \text{for all } x\in\Omega.$$

Thus, by the Lebesgue Dominated Convergence Theorem, we get

$$\sum_{i=1}^{n}\int_{\Omega}\lambda\delta_{\lambda_m}F_{u_i}(x,\delta_{\lambda_m}z)z_i\,dx\to\sum_{i=1}^{n}\int_{\Omega}\lambda\delta_0F_{u_i}(x,\delta_0z)z_i\,dx,\quad\text{as } m\to+\infty.$$

By remembering that $\lambda_m \to +\infty$, we find

$$\begin{aligned}&\sum_{i=1}^{n}\int_{\Omega}|\delta_{\lambda_m}z_i|^{s_i(x)}\,dx+\sum_{i=1}^{n}\int_{\partial\Omega}|\delta_{\lambda_m}z_i|^{\ell_i(x)}\,d\sigma_x\\&\quad+\sum_{i=1}^{n}\int_{\Omega}\lambda\delta_{\lambda_m}F_{u_i}(x,\delta_{\lambda_m}z)z_i\,dx\to+\infty,\quad\text{as } \lambda_m\to+\infty.\end{aligned}$$

This contradicts the fact (3.9), so we can deduce that $\delta_0 = 0$.

Next, we consider the following path $\phi_*(\delta) = \delta z$ for $\delta \in [0, 1]$ which belongs to Γ. By using assumption ($\boldsymbol{F}_3$), we obtain

$$0<c_{\theta,\lambda_m}\le\max_{\delta\in[0,1]}E_{\theta,\lambda_m}\big(\phi_*(\delta)\big)\le E_{\theta,\lambda}(\delta_{\lambda_m}z)\le\sum_{i=1}^{n}\widehat{M_{\theta_i}}\big(\mathcal{A}_i(\delta_{\lambda_m}z_i)\big).\tag{3.10}$$

On the other hand, since M_{θ_i} are continuous for all $1 \le i \le n$, and $\delta_0 = 0$, we get $\lim_{m\to+\infty}\widehat{M_{\theta_i}}(\mathcal{A}_i(\delta_{\lambda_m}z_i)) = 0$, for all $i \in \{1, 2, \dots, n\}$. Thus, from relation (3.10), we get $\lim_{m\to+\infty} c_{\theta,\lambda_m} = 0$. Moreover, by using also assumption ($\boldsymbol{F}_3$), we verify that the sequence $\{c_{\theta,\lambda}\}_\lambda$ is monotone. Therefore, we have completed the proof. □

Lemma 3.5 *If $\{u_m = (u_{1_m}, u_{2_m}, \dots, u_{n_m})\}_{m\in\mathbb{N}}$ is a Palais–Smale sequence for $E_{\theta,\lambda}$, then $\{u_m\}_m$ is bounded in X.*

Proof Let $\{u_m = (u_{1_m}, u_{2_m}, \dots, u_{n_m})\}_m$ be a $(PS)_c$ for $E_{\theta,\lambda}$. Then we have

$$E_{\theta,\lambda}(u_m)=\sum_{i=1}^{n}\widehat{M_{\theta_i}}\big(\mathcal{A}_i\big(u_{i_m}(x)\big)\big)-\sum_{i=1}^{n}\int_{\Omega}\frac{1}{s_i(x)}\big|u_{i_m}(x)\big|^{s_i(x)}\,dx$$

$$
\begin{aligned}
&- \sum_{i=1}^{n} \int_{\partial\Omega} \frac{1}{\ell_i(x)} \left| u_{i_m}(x) \right|^{\ell_i(x)} \, d\sigma_x - \int_{\Omega} \lambda F(x, u_m) \, dx \\
&= c + o_m(1).
\end{aligned}
$$

On the other hand, for all $v = (v_1, v_2, \ldots, v_n) \in X$, we have

$$
\begin{aligned}
\langle E'_{\theta,\lambda}(u_m), v\rangle &= \sum_{i=1}^{n} M_{\theta_i}\big(\mathcal{A}_i(u_{i_m})\big) \int_{\Omega} \big(\mathcal{B}_{1_i}(\nabla u_{i_m}) \nabla u_{i_m} \nabla v_i + \mathcal{B}_{2_i}(u_{i_m}) u_{i_m} v_i\big) \, dx \\
&\quad - \sum_{i=1}^{n} \int_{\Omega} |u_{i_m}|^{s_i(x)-2} u_{i_m} v_i \, dx \\
&\quad - \sum_{i=1}^{n} \int_{\partial\Omega} |u_{i_m}|^{s_i(x)-2} u_{i_m} v_i \, d\sigma_x - \sum_{i=1}^{n} \int_{\Omega} \lambda F_{u_i}(x, u_m) v_i \, dx \\
&= o_m(1).
\end{aligned}
\tag{3.11}
$$

Thus,

$$
\begin{aligned}
&E_{\theta,\lambda}(u_m) - \left\langle E'_{\theta,\lambda}(u_m), \frac{u_m}{\gamma} \right\rangle \\
&\geq \sum_{i=1}^{n} \bigg(\widehat{M_{\theta_i}}\big(\mathcal{A}_i(u_{i_m})\big) - \frac{1}{\gamma_i} M_{\theta_i}\big(\mathcal{A}_i(u_{i_m})\big) \int_{\Omega} \big(a_{1_i}\big(|\nabla u_{i_m}|^{p_i(x)}\big) |\nabla u_{i_m}|^{p_i(x)} \\
&\quad + a_{2_i}\big(|u_{i_m}|^{p_i(x)}\big) |u_{i_m}|^{p_i(x)}\big) \, dx \bigg) + \sum_{i=1}^{n} \left(\frac{1}{\gamma_i} - \frac{1}{s_i^-} \right) \int_{\Omega} |u_{i_m}|^{s_i(x)} \, dx \\
&\quad + \sum_{i=1}^{n} \left(\frac{1}{\gamma_i} - \frac{1}{\ell_i^-} \right) \int_{\partial\Omega} |u_{i_m}|^{\ell_i(x)} \, d\sigma_x + \lambda \int_{\Omega} \left[\sum_{i=1}^{n} \frac{u_{i_m}}{\gamma_i} F_{u_i}(x, u_m) - F(x, u_m) \right] dx,
\end{aligned}
$$

where $\frac{u_m}{\gamma} = (\frac{u_{1_m}}{\gamma_1}, \frac{u_{2_m}}{\gamma_2}, \ldots, \frac{u_{n_m}}{\gamma_n})$. Therefore, by using assumptions ($\boldsymbol{A}_4$), ($\boldsymbol{M}$), and ($\boldsymbol{F}_3$), we can conclude that

$$
\begin{aligned}
&E_{\theta,\lambda}(u_m) - \left\langle E'_{\theta,\lambda}(u_m), \frac{u_m}{\gamma} \right\rangle \\
&\geq \sum_{i=1}^{n} \mathfrak{M}_i^0 \bigg(\frac{\theta_i}{p_i^+} \int_{\Omega} \big(A_{1_i}\big(|\nabla u_{i_m}|^{p_i(x)}\big) + A_{2_i}\big(|u_{i_m}|^{p_i(x)}\big)\big) \, dx \\
&\quad - \frac{1}{\gamma_i} \int_{\Omega} \big(a_{1_i}\big(|\nabla u_{i_m}|^{p_i(x)}\big) |\nabla u_{i_m}|^{p_i(x)} + a_{2_i}\big(|u_{i_m}|^{p_i(x)}\big) |u_{i_m}|^{p_i(x)} \big) \, dx \bigg) \\
&\geq \sum_{i=1}^{n} \left(\frac{\theta_i \mathfrak{M}_i^0}{p_i^+ \max\{\beta_{1_i}, \beta_{2_i}\}} - \frac{\mathfrak{M}_i^0}{\gamma_i} \right) \\
&\quad \times \int_{\Omega} \big(a_{1_i}\big(|\nabla u_{i_m}|^{p_i(x)}\big) |\nabla u_{i_m}|^{p_i(x)} + a_{2_i}\big(|u_{i_m}|^{p_i(x)}\big) |u_{i_m}|^{p_i(x)} \big) \, dx.
\end{aligned}
$$

By using assumption ($\boldsymbol{A}_4$), we can find positive constants C_{i1} and C_{i2} such that

$$\begin{aligned} &E_{\theta,\lambda}(u_m) - \left\langle E'_{\theta,\lambda}(u_m), \frac{u_m}{\gamma}\right\rangle \\ &\quad\geq \sum_{i=1}^{n}\left[C_{i1}\left(\int_\Omega \left(|\nabla u_{i_m}|^{p_i(x)} + |u_{i_m}|^{p_i(x)}\right) dx\right)\right. \\ &\qquad \left. + C_{i2}\mathcal{K}\left(k_i^3\right)\left(\int_\Omega \left(|\nabla u_{i_m}|^{q_i(x)} + |u_{i_m}|^{q_i(x)}\right) dx\right)\right]. \end{aligned} \tag{3.12}$$

To prove the assertion, we assume by contradiction that $\|u_{i_m}\|_i = \|u_i\|_{1,p_i(x)} + \mathcal{K}(k_i^3) \times \|u_i\|_{1,q_i(x)} \to +\infty$. So, if $k_i^3 = 0$, then by using relation (2.2), we have $E_{\theta,\lambda}(u_m) - \langle E'_{\theta,\lambda}(u_m), \frac{u_m}{\gamma}\rangle \geq \sum_{i=1}^n C_i\|u_{i_m}\|_i^{p_i^-}$. Thus, we can find $c + o_m(1) \geq \sum_{i=1}^n C_i\|u_{i_m}\|_i^{p_i^-}$. Since $p_i^- > 1$, we obtain a contradiction. Hence, we deduce that $\{u_m\}$ is bounded in X.

When $k_i^3 > 0$, we have three cases to analyze:

(i) $\|u_{i_m}\|_{1,p_i(x)} \to +\infty$ and $\|u_{i_m}\|_{1,q_i(x)} \to +\infty$, as $m \to +\infty$,

(ii) $\|u_{i_m}\|_{1,p_i(x)} \to +\infty$ and $\|u_{i_m}\|_{1,q_i(x)}$ is bounded,

(iii) $\|u_{i_m}\|_{1,p_i(x)}$ is bounded and $\|u_{i_m}\|_{1,q_i(x)} \to +\infty$.

In the case (i), for m sufficiently large, we have $\|u_{i_m}\|_{1,q_i(x)}^{q^-} \geq \|u_{i_m}\|_{1,q_i(x)}^{p^-}$. Hence, by inequality (3.12), we get

$$\begin{aligned} c + o_m(1) &\geq \sum_{i=1}^n \left[C_{1_i}\|u_{i_m}\|_{1,p_i(x)}^{p_i^-} + C_{2_i}\mathcal{K}\left(k_i^3\right)\|u_{i_m}\|_{1,q_i(x)}^{q_i^-}\right] \\ &\geq \sum_{i=1}^n \left[C_{1_i}\|u_{i_m}\|_{1,p_i(x)}^{p_i^-} + C_{2_i}\mathcal{K}\left(k_i^3\right)\|u_{i_m}\|_{1,q_i(x)}^{p_i^-}\right] \\ &\geq \sum_{i=1}^n C_{3i}\|u_{i_m}\|_i^{p_i^-}, \end{aligned}$$

and this is a contradiction. In the case (ii), by using inequality (3.12), we conclude that

$$c + o_m(1) \geq \sum_{i=1}^n C_{1_i}\|u_{i_m}\|_{1,p_i(x)}^{p_i^-},$$

Hence, we also get a contradiction when limit as $m \to +\infty$ because $p_i^- > 1$. In the last case (iii), the proof is similar as in the case (ii), so we shall omit it. Finally, we can deduce that $\{u_m\}$ is a bounded sequence in X. □

Next, we shall prove that the auxiliary problem (3.1) possesses at least one nontrivial weak solution.

Proof of Theorem 3.1 By Lemmas 3.2 and 3.3, the functional $E_{\theta,\lambda}$ satisfies the geometric structure required by the Mountain Pass Theorem 2.6. Now, it remains to check the validity of the Palais–Smale condition. Let $\{u_m = (u_{1_m}, u_{2m}, \dots, u_{n_m})\}_{m\in\mathbb{N}}$ be a Palais–Smale sequence at the level $c_{\theta,\lambda}$ in X. Then Lemma 3.4 implies that there exists λ_* such that

$$c_{\theta,\lambda} < \min\left\{ \min_{1\le i\le n}\left\{ \left(\frac{1}{\gamma_i} - \frac{1}{\ell_{i^-_{\mathcal{C}^1_{h_i}}}}\right) \inf_{j\in J^1_i}\left\{ \overline{T}^N_{i_{x_j}}(D_i)^{N/h_i(x_j)}\right\}\right\}, \min_{1\le i\le n}\left\{ \left(\frac{1}{\gamma_i} - \frac{1}{s_{i^-_{\mathcal{C}^2_{h_i}}}}\right) \inf_{j\in J^2_i}\left\{ S^N_i(D_i)^{N/h_i(x_j)}\right\}\right\}\right\},$$

where $\overline{T}_{i_{x_j}}$ and S_i are respectively given by relations (2.10) and (2.14), and

$$D_i = \mathfrak{M}^0_i\left(\min\{k^0_{1_i}, k^0_{2_i}\}\left(1 - \mathcal{K}(k^3_i) + \mathcal{K}(k^3_i)\min\{k^2_{1_i}, k^2_{2_i}\}\right)\right).$$

So, there exists a strongly convergent in X subsequence. Indeed, applying Lemma 3.5, $\{u_m\}_{m\in\mathbb{N}}$ is bounded in X, so, passing to a weakly convergent in X subsequence, still denoted by $\{u_m\}_m$, there exist positive bounded measures μ_i, $\nu_i \in \Omega$ and $\overline{\nu_i} \in \partial\Omega$ such that $|\nabla u_{i_m}|^{h_i(x)} \rightharpoonup \mu_i$, $|u_{i_m}|^{s_i(x)} \rightharpoonup \nu_i$, and $|u_{i_m}|^{\ell_i(x)} \rightharpoonup \overline{\nu_i}$. Hence, by Theorems 2.8 and 2.9, if $\bigcup_{i=1}^n (J^1_i \cup J^2_i) = \emptyset$, then $u_{i_m} \rightharpoonup u_i$ in $L^{s_i(x)}(\Omega)$ and $u_{i_m} \rightharpoonup u_i$ in $L^{\ell_i(x)}(\partial\Omega)$, for all $1 \le i \le n$. Let us prove that if

$$c_{\theta,\lambda} < \min\left\{ \min_{1\le i\le n}\left\{ \left(\frac{1}{\gamma_i} - \frac{1}{\ell_{i^-_{\mathcal{C}^1_{h_i}}}}\right) \inf_{j\in J^1_i}\left\{ \overline{T}^N_{i_{x_j}}(D_i)^{N/h_i(x_j)}\right\}\right\}, \min_{1\le i\le n}\left\{ \left(\frac{1}{\gamma_i} - \frac{1}{s_{i^-_{\mathcal{C}^2_{h_i}}}}\right) \inf_{j\in J^2_i}\left\{ S^N_i(D_i)^{N/h_i(x_j)}\right\}\right\}\right\}$$

and $\{u_m\}_{m\in\mathbb{N}}$ is a Palais–Smale sequence with energy level $c_{\theta,\lambda}$, then $J^1_i \cup J^2_i = \emptyset$, for all $1 \le i \le n$. In fact, suppose there is an $i \in \{1,\dots,n\}$ such that $J^1_i \cup J^2_i$ is nonempty, then $J^1_i \ne \emptyset$ or $J^2_i \ne \emptyset$.

First, we consider the case $J^1_i \ne \emptyset$. Let $x_j \in \mathcal{C}^1_{h_i}$ be a singular point of the measures μ_i and $\overline{\nu_i}$. Consider $\psi \in C^\infty_0(\mathbb{R}^N)$ such that $0 \le \psi(x) \le 1$, $\psi(0) = 1$, and $\operatorname{supp}\psi \subset B(0,1)$. We consider, for each $j \in J^1_i$ and any $\varepsilon > 0$, the functions $\psi_{j,\varepsilon} := \psi(\frac{x-x_j}{\varepsilon})$, for all $x \in \mathbb{R}^N$. Notice that $\psi_{j,\varepsilon} \in C^\infty_0(\mathbb{R}^N, [0,1])$, $|\nabla\psi_{j,\epsilon}|_\infty \le \frac{2}{\varepsilon}$, and

$$\psi_{j,\varepsilon}(x) = \begin{cases} 1, & x \in B(x_j,\varepsilon), \\ 0, & x \in \mathbb{R}^N \setminus B(x_j, 2\varepsilon). \end{cases}$$

Since $\{u_{i_m}\}_m$ is bounded in $W^{1,h_i(x)}(\Omega) \cap W^{1,p_i(x)}(\Omega)$, the sequence $\{u_{i_m}\psi_{j,\varepsilon}\}$ is also bounded in $W^{1,h_i(x)}(\Omega) \cap W^{1,p_i(x)}(\Omega)$. So, by relation (3.11), we obtain $\langle E'_{\theta,\lambda}(u_{1_m},\dots,u_{i_m},\dots,u_{n_m}), (0,\dots,u_{i_m}\psi_{j,\varepsilon},\dots,0)\rangle \to 0$ as $m \to +\infty$. Therefore, we have, as $m \to +\infty$,

$$\begin{aligned}
&\left\langle E'_{\theta,\lambda}(u_m)(0,\dots,u_{i_m}\psi_{j,\varepsilon},\dots,0)\right\rangle \\
&= M_{\theta_i}\left(\mathcal{A}_i(u_{i_m})\right) \int_\Omega \left(a_{1_i}\left(|\nabla u_{i_m}|^{p_i(x)}\right)|\nabla u_{i_m}|^{p_i(x)-2}\nabla u_{i_m}\nabla(u_{i_m}\psi_{j,\varepsilon})\right. \\
&\quad \left. + a_{2_i}\left(|u_{i_m}|^{p_i(x)}\right)|u_{i_m}|^{p_i(x)-2}u_{i_m}(u_{i_m}\psi_{j,\varepsilon})\right) dx - \int_\Omega |u_{i_m}|^{s_i(x)-2}u_{i_m}(u_{i_m}\psi_{j,\epsilon})\, dx
\end{aligned}$$

$$
\begin{aligned}
&-\int_{\partial\Omega} |u_{i_m}|^{\ell_i(x)-2} u_{i_m}(u_{i_m}\psi_{j,\epsilon})\,d\sigma_x - \int_\Omega \lambda F_{u_i}(x,u_m)u_{i_m}\psi_{j,\varepsilon}\,dx \\
&\to 0,
\end{aligned}
$$

and so

$$
\begin{aligned}
&M_{\theta_i}\big(\mathcal{A}_i(u_{i_m})\big)\int_\Omega a_{1_i}\big(|\nabla u_{i_m}|^{p_i(x)}\big)|\nabla u_{i_m}|^{p_i(x)-2}\nabla u_{i_m}\nabla\psi_{j,\varepsilon}u_{i_m}\,dx \\
&= \int_\Omega |u_{i_m}|^{s_i(x)}\psi_{j,\epsilon}\,dx + \int_{\partial\Omega}|u_{i_m}|^{\ell_i(x)}\psi_{j,\epsilon}\,d\sigma_x \\
&\quad - M_{\theta_i}\big(\mathcal{A}_i(u_{i_m})\big)\int_\Omega \big(a_{1_i}\big(|\nabla u_{i_m}|^{p_i(x)}\big)|\nabla u_{i_m}|^{p_i(x)} \\
&\quad + a_{2_i}\big(|u_{i_m}|^{p_i(x)}\big)|u_{i_m}|^{p_i(x)}\big)\psi_{j,\varepsilon}\,dx \\
&\quad + \int_\Omega \lambda F_{u_i}(x,u_m)u_{i_m}\psi_{j,\varepsilon}\,dx + o_m(1).
\end{aligned}
\tag{3.13}
$$

Next, we shall prove that

$$
\lim_{\varepsilon\to 0}\left\{\limsup_{m\to+\infty} M_{\theta_i}\big(\mathcal{A}_i(u_{i_m})\big)\int_\Omega a_i\big(|\nabla u_{i_m}|^{p_i(x)}\big)|\nabla u_{i_m}|^{p_i(x)-2}\nabla u_{i_m}\nabla\psi_{j,\varepsilon}u_{i_m}\,dx\right\} = 0. \tag{3.14}
$$

Notice that, due to assumption ($\boldsymbol{A}_2$), it suffices to show that

$$
\lim_{\varepsilon\to 0}\left\{\limsup_{m\to+\infty} M_{\theta_i}\big(\mathcal{A}_i(u_{i_m})\big)\int_\Omega |\nabla u_{i_m}|^{p_i(x)-2}\nabla u_{i_m}\nabla\psi_{j,\varepsilon}u_{i_m}\,dx\right\} = 0 \tag{3.15}
$$

and

$$
\lim_{\varepsilon\to 0}\left\{\limsup_{m\to+\infty} M_{\theta_i}\big(\mathcal{A}_i(u_{i_m})\big)\int_\Omega |\nabla u_{i_m}|^{q_i(x)-2}\nabla u_{i_m}\nabla\psi_{j,\varepsilon}u_{i_m}\,dx\right\} = 0. \tag{3.16}
$$

First, by applying Hölder inequality, we obtain

$$
\left|\int_\Omega |\nabla u_{i_m}|^{p_i(x)-2}\nabla u_{i_m}\nabla\psi_{j,\varepsilon}u_{i_m}\,dx\right| \le 2\big\||\nabla u_{i_m}|^{p_i(x)-1}\big\|_{L^{\frac{p_i(x)}{p_i(x)-1}}(\Omega)}\|\nabla\psi_{j,\varepsilon}u_{i_m}\|_{L^{p_i(x)}(\Omega)},
$$

where, since $\{u_{i_m}\}$ is bounded, the real-valued sequence $\||\nabla u_{i_m}|^{p_i(x)-1}\|_{L^{\frac{p_i(x)}{p_i(x)-1}}(\Omega)}$ is also bounded, thus there exists a positive constant C_i such that

$$
\left|\int_\Omega |\nabla u_{i_m}|^{p_i(x)-2}\nabla u_{i_m}\nabla\psi_{j,\varepsilon}u_{i_m}\,dx\right| \le C_i\|\nabla\psi_{j,\varepsilon}u_{i_m}\|_{L^{p_i(x)}(\Omega)}.
$$

Moreover, the sequence $\{u_{i_m}\}$ is bounded in $W^{1,p_i(x)}(B(x_j,2\varepsilon))$, so there is a subsequence, again denoted by $\{u_{i_m}\}$, converging weakly to u_i in $L^{p_i(x)}(B(x_j,2\varepsilon))$. Therefore,

$$
\begin{aligned}
&\limsup_{m\to+\infty}\left|\int_\Omega |\nabla u_{i_m}|^{p_i(x)-2}\nabla u_{i_m}\nabla\psi_{j,\varepsilon}u_{i_m}\,dx\right| \\
&\le C_i\|\nabla\psi_{j,\varepsilon}u_i\|_{L^{p_i(x)}(\Omega)}
\end{aligned}
$$

$$\leq 2C_i \limsup_{\varepsilon\to 0} \left\| |\nabla\psi_{j,\varepsilon}|^{p_i(x)} \right\|_{L^{(\frac{p_i^*(x)}{p_i(x)})'}(B(x_j,2\varepsilon))} \left\| |u_i|^{p_i(x)} \right\|_{L^{\frac{p_i^*(x)}{p_i(x)}}(B(x_j,2\varepsilon))}$$

$$\leq 2C_i \limsup_{\varepsilon\to 0} \left\| |\nabla\psi_{j,\varepsilon}|^{p_i(x)} \right\|_{L^{\frac{N}{p_i(x)}}(B(x_j,2\varepsilon))} \left\| |u_i|^{p_i(x)} \right\|_{L^{\frac{N}{N-p_i(x)}}(B(x_j,2\varepsilon))}.$$

Note that

$$\int_{B(x_j,2\varepsilon)} \left(|\nabla\psi_{j,\varepsilon}|^{p_i(x)}\right)^{\frac{N}{p_i(x)}} dx = \int_{B(x_j,2\varepsilon)} |\nabla\psi_{j,\varepsilon}|^N dx \leq \left(\frac{2}{\varepsilon}\right)^N \operatorname{meas}\bigl(B(x_j,2\varepsilon)\bigr) = \frac{4^N}{N}\omega_N,$$

where ω_N is the surface area of the N-dimensional unit sphere. Since

$$\int_{B(x_j,2\varepsilon)} \left(|u_i|^{p_i(x)}\right)^{\frac{N}{N-p_i(x)}} dx \to 0, \quad \text{when } \varepsilon \to 0,$$

we obtain that $\|\nabla\psi_{j,\varepsilon}u_i\|_{L^{p_i(x)}(\Omega)} \to 0$, which implies

$$\lim_{\varepsilon\to 0}\left\{\limsup_{n\to+\infty}\left|\int_\Omega |\nabla u_{i_m}|^{p_i(x)-2}\nabla u_{i_m}\nabla\psi_{j,\epsilon}u_{i_m}\,dx\right|\right\} = 0. \tag{3.17}$$

Since the sequence $\{u_{i_m}\}$ is bounded in $W^{1,h_i(x)}(\Omega) \cap W^{1,p_i(x)}(\Omega)$, we may assume that $\mathcal{A}_i(u_{i_m}) \to \xi_i \geq 0$, as $m \to +\infty$. Note that $M_i(\xi_i)$ is is continuous, so we have $M_i(\mathcal{A}_i(u_{i_m})) \to M_i(\xi_i) \geq \mathfrak{M}_i^0 > 0$, as $m \to +\infty$. Therefore, by relation (3.17), we obtain

$$\lim_{\varepsilon\to 0}\left\{\limsup_{m\to+\infty} M_i\bigl(\mathcal{A}_i(u_{i_m})\bigr)\int_\Omega |\nabla u_{i_m}|^{p_i(x)-2}\nabla u_{i_m}\nabla\psi_{j,\varepsilon}u_{i_m}\,dx\right\} = 0. \tag{3.18}$$

Analogously, we can verify relation (3.16). Hence, we have completed the proof of relation (3.14). Similarly, we can also obtain

$$\lim_{\varepsilon\to 0}\int_\Omega \lambda F_{u_i}(x,u_m)\psi_{j,\epsilon}u_{i_m}\,dx = 0, \quad \text{as } m \to +\infty. \tag{3.19}$$

By applying Hölder inequality, assumption ($\boldsymbol{F}_2$), and the fact that $0 \leq \psi_{j,\varepsilon} \leq 1$, we have

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_\Omega \lambda F_{u_i}(x,u_m)\psi_{j,\varepsilon}u_{i_m}\,dx \\
&\quad\leq \lim_{\varepsilon\to 0}\lambda\int_\Omega\left(\sum_{j=1}^n b_{i_j}(x)|u_jm|^{r_{i_j}-1}\right)\psi_{j,\varepsilon}u_{i_m}\,dx \\
&\quad\leq \lim_{\varepsilon\to 0} c\int_\Omega\left(\sum_{j=1}^n b_{i_j}(x)|u_j|^{r_{i_j}-1}\right)|\psi_{j,\varepsilon}u_{i_m}|dx \\
&\quad\leq \lim_{\varepsilon\to 0} c_1\left(\sum_{j=1}^n |b_{i_j}|_{\alpha_{i_j}(x)}\left\||u_{jm}|^{r_{i_j}-1}\right\|_{L^{q_j^*(x)}(\Omega)}\|\psi_{j,\epsilon}u_{i_m}\|_{L^{q_i^*(x)}(\Omega)}\right).
\end{aligned}$$

This yields

$$\begin{aligned}&\lim_{\varepsilon\to 0}\int_\Omega \lambda F_{u_i}(x,u_m)\psi_{j,\varepsilon}u_{i_m}\,dx\\ &\quad\le \lim_{\varepsilon\to 0}c_1\left(\sum_{j=1}^n \|b_{i_j}\|_{L^{\alpha_{i_j}(x)}(B(x_j,2\varepsilon))}\|u_{jm}\|^{r_{i_j}-1}_{L^{q_j(x)}(B(x_j,2\varepsilon))}\right)\|u_{i_m}\|_{L^{q_i(x)}(B(x_j,2\varepsilon))},\end{aligned}$$

and the last term on the right-hand side goes to zero, because

$$\sum_{j=1}^n \|b_{i_j}\|_{L^{\alpha_{i_j}(x)}(B(x_j,2\varepsilon))}\|u_j\|^{r_{i_j}-1}_{L^{q_j(x)}(B(x_j,2\varepsilon))}<\infty.$$

Therefore, we have completed the proof of relation (3.19). On the other hand, we have

$$\lim_{\varepsilon\to 0}\int_\Omega \psi_{j,\epsilon}\,d\mu_{i_j}=\mu_{i_j}\psi(0)\quad\text{and}\quad \lim_{\varepsilon\to 0}\int_{\partial\Omega}\psi_{j,\epsilon}\,d\overline{\nu}_{i_j}=\overline{\nu}_{i_j}\psi(0),$$

and since $\mathcal{C}^1_{h_i}\cap\mathcal{C}^2_{h_i}=\emptyset$, for $\epsilon>0$ sufficiently small, we have

$$\begin{aligned}&\int_\Omega |u_{i_m}|^{p_i(x)}\psi_{j,\varepsilon}\,dx\to\int_\Omega |u_i|^{p_i(x)}\psi_{j,\varepsilon}\,dx,\\ &\int_\Omega |u_{i_m}|^{q_i(x)}\psi_{j,\varepsilon}\,dx\to\int_\Omega |u_i|^{q_i(x)}\psi_{j,\varepsilon}\,dx,\\ &\int_\Omega |u_{i_m}|^{s_i(x)}\psi_{j,\varepsilon}\,dx\to\int_\Omega |u_i|^{s_i(x)}\psi_{j,\varepsilon}\,dx,\end{aligned}$$

hence, when $\epsilon\to 0$,

$$\int_\Omega |u_i|^{p_i(x)}\psi_{j,\varepsilon}\,dx\to 0,\qquad \int_\Omega |u_i|^{q_i(x)}\psi_{j,\varepsilon}\,dx\to 0,\qquad \int_\Omega |u_i|^{s_i(x)}\psi_{j,\varepsilon}\,dx\to 0.$$

The function $\psi_{j,\varepsilon}$ has compact support, so letting $m\to+\infty$ and $\varepsilon\to 0$ in relation (3.13), we get from relations (3.14)–(3.19),

$$\begin{aligned}0&=-\lim_{\varepsilon\to 0}\left[\limsup_{m\to+\infty}\left(M_{\theta_i}(\mathcal{A}_i(u_{i_m}))\int_\Omega\big(a_{1_i}(|\nabla u_{i_m}|^{p_i(x)})|\nabla u_{i_m}|^{p_i(x)}\right.\right.\\ &\quad\left.\left.+a_{2_i}(|u_{i_m}|^{p_i(x)})|u_{i_m}|^{p_i(x)}\big)\psi_{j,\varepsilon}\,dx\right)\right]+\overline{\nu}_{i_j}\\ &\le -\mathfrak{M}_i^0\lim_{\varepsilon\to 0}\left[\limsup_{m\to+\infty}\left(\int_\Omega\big(a_{1_i}(|\nabla u_{i_m}|^{p_i(x)})|\nabla u_{i_m}|^{p_i(x)}\right.\right.\\ &\quad\left.\left.+a_{2_i}(|u_{i_m}|^{p_i(x)})|u_{i_m}|^{p_i(x)}\big)\psi_{j,\varepsilon}\,dx\right)\right]+\overline{\nu}_{i_j}\\ &\le -\mathfrak{M}_i^0\lim_{\varepsilon\to 0}\left[\limsup_{m\to+\infty}\left(\int_\Omega\big(\min\{k^0_{1_i},k^0_{2_i}\}(|\nabla u_{i_m}|^{p_i(x)}+|u_{i_m}|^{p_i(x)})\right.\right.\\ &\quad\left.\left.+\mathcal{K}(k_i^3)\min\{k^2_{1_i},k^2_{2_i}\}(|\nabla u_{i_m}|^{q_i(x)}+|u_{i_m}|^{q_i(x)})\big)\psi_{j,\varepsilon}\,dx\right)\right]+\overline{\nu}_{i_j}.\end{aligned}\tag{3.20}$$

Note that, when $k_i^3 = 0$, we have $h_i(x) = p_i(x)$. Hence, by using relation (2.8), we have

$$\begin{aligned} 0 &\le -\mathfrak{M}_i^0 \min\{k_{1_i}^0, k_{2_i}^0\} \lim_{\varepsilon\to 0}\int_\Omega \psi_{j,\varepsilon}\, d\mu_i + \overline{\nu}_{i_j} \\ &\le -\mathfrak{M}_i^0 \min\{k_{1_i}^0, k_{2_i}^0\}\mu_{i_j} - \mathfrak{M}_i^0 \min\{k_{1_i}^0, k_{2_i}^0\} \lim_{\varepsilon\to 0}\int_\Omega |\nabla u_i|^{p_i(x)}\psi_{j,\varepsilon}\, dx + \overline{\nu}_{i_j}. \end{aligned}$$

By applying the Lebesgue Dominated Convergence Theorem, we get

$$\lim_{\varepsilon\to 0}\int_\Omega |\nabla u_i|^{p_i(x)}\psi_{j,\varepsilon}\, dx = 0.$$

Therefore,

$$\mathfrak{M}_i^0 \min\{k_{1_i}^0, k_{2_i}^0\}\mu_{i_j} \le \overline{\nu}_{i_j}. \tag{3.21}$$

On the other hand, if $k_i^3 > 0$, then $h_i(x) = q_i(x)$ Therefore, it follows from relations (2.10) and (3.20) that

$$\begin{aligned} 0 &\le -\mathfrak{M}_i^0 \lim_{\varepsilon\to 0}\left[\limsup_{m\to 0}\left(\int_\Omega \mathcal{K}(k_i^3)\min\{k_{1_i}^2, k_{2_i}^2\}|\nabla u_{i_m}|^{q_i(x)}\psi_{j,\varepsilon}\, dx\right)\right] + \overline{\nu}_{i_j} \\ &\le -\mathfrak{M}_i^0\mathcal{K}(k_i^3)\min\{k_{1_i}^2, k_{2_i}^2\}\lim_{\varepsilon\to 0}\int_\Omega \psi_{j,\varepsilon}\, d\mu_i + \overline{\nu}_{i_j} \\ &\le -\mathfrak{M}_i^0\mathcal{K}(k_i^3)\min\{k_{1_i}^2, k_{2_i}^2\}\mu_{i_j} - \mathfrak{M}_i^0\mathcal{K}(k_i^3)\min\{k_{1_i}^2, k_{2_i}^2\}\lim_{\varepsilon\to 0}\int_\Omega |\nabla u_i|^{p_i(x)}\psi_{j,\varepsilon}\, dx + \overline{\nu}_{i_j}. \end{aligned}$$

By applying the Lebesgue Dominated Convergence Theorem again, we get

$$\lim_{\epsilon\to 0}\int_\Omega |\nabla u_i|^{q_i(x)}\psi_{j,\varepsilon}\, dx = 0.$$

Hence,

$$\mathfrak{M}_i^0\mathcal{K}(k_i^3)\min\{k_{1_i}^2, k_{2_i}^2\}\mu_{i_j} \le \overline{\nu}_{i_j}. \tag{3.22}$$

By combining relations (3.21) and (3.22), we have $\mathfrak{M}_i^0((1-\mathcal{K}(k_i^3))\min\{k_{1_i}^0, k_{2_i}^0\} + \mathcal{K}(k_i^3)\times \min\{k_{1_i}^2, k_{2_i}^2\})\mu_{i_j} \le \overline{\nu}_{i_j}$. By using relation (2.10), we obtain

$$\overline{T}_{i x_j}\overline{\nu}_{i_j}^{\frac{1}{h_i^*(x_j)}} \le \mu_{i_j}^{\frac{1}{h_i(x_j)}} \le \left(\frac{\overline{\nu}_{i_j}}{\mathfrak{M}_i^0((1-\mathcal{K}(k_i^3))\min\{k_{1_i}^0, k_{2_i}^0\} + \mathcal{K}(k_i^3)\min\{k_{1_i}^2, k_{2_i}^2\})}\right)^{\frac{1}{h_i(x_j)}}.$$

which implies that $\overline{\nu}_{i_j} = 0$ or $\overline{\nu}_{i_j} \ge \overline{T}_{i x_j}^N(\mathfrak{M}_i^0(\min\{k_{1_i}^0, k_{2_i}^0\}(1 - \mathcal{K}(k_i^3) + \mathcal{K}(k_i^3)\min\{k_{1_i}^2, k_{2_i}^2\})))^{N/h_i(x_j)}$, for all $j \in J_i^1$. On the other hand, by using assumptions (***M***) and (***F***$_3$), we have

$$\begin{aligned} c_{\theta,\lambda} &= E_{\theta,\lambda}(u_m) - \left\langle E'_{\theta,\lambda}(u_m), \frac{u_m}{\gamma}\right\rangle \\ &= \sum_{i=1}^n \widehat{M_{\theta_i}}(\mathcal{A}_i(u_{i_m})) - \sum_{i=1}^n \int_\Omega \frac{1}{s_i(x)}|u_{i_m}|^{s_i(x)}\, dx \end{aligned}$$

$$
\begin{aligned}
&- \sum_{i=1}^{n} \int_{\partial\Omega} \frac{1}{\ell_i(x)} |u_{i_m}|^{\ell_i(x)}\, d\sigma_x - \int_{\Omega} \lambda F(x,u_m)\, dx \\
&- \sum_{i=1}^{n} M_{\theta_i}\big(\mathcal{A}_i(u_{i_m})\big) \int_{\Omega} \frac{1}{\gamma_i}\big(a_{1_i}\big(|\nabla u_{i_m}|^{p_i(x)}\big)|\nabla u_{i_m}|^{p_i(x)} + a_{2_i}\big(|u_{i_m}|^{p_i(x)}\big)|u_{i_m}|^{p_i(x)}\big)\, dx \\
&+ \sum_{i=1}^{n} \int_{\Omega} \frac{1}{\gamma_i}|u_{i_m}|^{s_i(x)}\, dx + \sum_{i=1}^{n} \int_{\partial\Omega} \frac{1}{\gamma_i}|u_{i_m}|^{\ell_i(x)}\, d\sigma_x \\
&+ \sum_{i=1}^{n} \int_{\Omega} \frac{\lambda}{\gamma_i} F_{u_i}(x,u_m) u_{i_m}\, dx + o_m(1) \\
\geq{}& \sum_{i=1}^{n} \frac{\theta_i \mathfrak{M}_i^0}{p_i^+ \max\{\beta_{1_i},\beta_{2_i}\}} \int_{\Omega} \big(a_{1_i}\big(|\nabla u_{i_m}|^{p_i(x)}\big)|\nabla u_{i_m}|^{p_i(x)} + a_{2_i}\big(|u_{i_m}|^{p_i(x)}\big)|u_{i_m}|^{p_i(x)}\big)\, dx \\
&- \sum_{i=1}^{n} \frac{1}{s_i^-} \int_{\Omega} |u_{i_m}|^{s_i(x)}\, dx \\
&- \sum_{i=1}^{n} \frac{1}{\ell_i^-} \int_{\partial\Omega} |u_{i_m}|^{\ell_i(x)}\, d\sigma_x - \int_{\Omega} \lambda F(x,u_m)\, dx \\
&- \sum_{i=1}^{n} \frac{\mathfrak{M}_i^0}{\gamma_i} \int_{\Omega} \big(a_{1_i}\big(|\nabla u_{i_m}|^{p_i(x)}\big)|\nabla u_{i_m}|^{p_i(x)} + a_{2_i}\big(|u_{i_m}|^{p_i(x)}\big)|u_{i_m}|^{p_i(x)}\big)\, dx \\
&+ \sum_{i=1}^{n} \frac{1}{\gamma_i} \int_{\Omega} |u_{i_m}|^{s_i(x)}\, dx + \sum_{i=1}^{n} \frac{1}{\gamma_i} \int_{\partial\Omega} |u_{i_m}|^{\ell_i(x)}\, dx \\
&+ \sum_{i=1}^{n} \int_{\Omega} \frac{\lambda}{\gamma_i} F_{u_i}(x,u_m) u_{i_m}\, dx + o_m(1) \\
\geq{}& \sum_{i=1}^{n} \mathfrak{M}_i^0 \left(\frac{\theta_i}{p_i^+ \max\{\beta_{1_i},\beta_{2_i}\}} - \frac{1}{\gamma_i} \right) \\
&\times \int_{\Omega} \big(a_{1_i}\big(|\nabla u_{i_m}|^{p_i(x)}\big)|\nabla u_{i_m}|^{p_i(x)} + a_{2_i}\big(|u_{i_m}|^{p_i(x)}\big)|u_{i_m}|^{p_i(x)}\big)\, dx \\
&+ \sum_{i=1}^{n} \left(\frac{1}{\gamma_i} - \frac{1}{s_i^-} \right) \int_{\Omega} |u_{i_m}|^{s_i(x)}\, dx + \sum_{i=1}^{n} \left(\frac{1}{\gamma_i} - \frac{1}{\ell_i^-} \right) \int_{\partial\Omega} |u_{i_m}|^{\ell_i(x)}\, dx \\
&+ \lambda \int_{\Omega} \left[\sum_{i=1}^{n} \frac{u_{i_m}}{\gamma_i} F_{u_i}(x,u_m) - F(x,u_m) \right] dx + o_m(1).
\end{aligned}
$$

Hence, we have $c_{\theta,\lambda} \geq \sum_{i=1}^{n} (\frac{1}{\gamma_i} - \frac{1}{\ell_i^-}) \int_{\partial\Omega} |u_{i_m}|^{\ell_i(x)}\, d\sigma_x + o_m(1)$. Setting

$$
\mathcal{C}_{i\kappa}^1 = \bigcup_{x \in \mathcal{C}_{h_i}^1} \big(\mathbf{B}_\kappa(x) \cap \Omega\big) = \big\{ x \in \Omega : \operatorname{dist}\big(x, \mathcal{C}_{h_i}^1\big) < \kappa \big\},
$$

as $m \to +\infty$, we obtain

$$
c_{\theta,\lambda} \geq \sum_{i=1}^{n} \left(\frac{1}{\gamma_i} - \frac{1}{\ell^-_{i_{\mathcal{C}_{i\kappa}^1}}} \right) \left(\int_{\Omega} |u_i|^{\ell_i(x)}\, dx + \sum_{j \in J_i^1} \nu_{ij} \delta_{x_j} \right)
$$

$$\geq \sum_{i=1}^{n}\left(\frac{1}{\gamma_i}-\frac{1}{\ell_{i_{\mathcal{C}^1_{i\kappa}}}^{-}}\right)\left(\int_{\Omega}|u_i|^{s_i(x)}\,dx+\inf_{j\in J_i^1}\left\{\overline{T}_{i_{x_j}}^{N}(D_i)^{N/h_i(x_j)}\right\}\operatorname{Card}J_i^1\right),$$

where $D_i=\mathfrak{M}_i^0(\min\{k_{1_i}^0,k_{2_i}^0\}(1-\mathcal{K}(k_i^3)+\mathcal{K}(k_i^3)\min\{k_{1_i}^2,k_{2_i}^2\}))$. Since $\kappa>0$ is arbitrary and ℓ_i are continuous functions for all $i\in\{1,2,\dots,n\}$, we get

$$c_{\theta,\lambda}\geq \sum_{i=1}^{n}\left(\frac{1}{\gamma_i}-\frac{1}{\ell_{i_{\mathcal{C}^1_{h_i}}}^{-}}\right)\left(\int_{\Omega}|u_i|^{s_i(x)}\,dx+\inf_{j\in J_i^1}\left\{\overline{T}_{i_{x_j}}^{N}(D_i)^{N/h_i(x_j)}\right\}\operatorname{Card}J_i^1\right).$$

Suppose that $\bigcup_{i=1}^n J_i^1\neq\emptyset$, then

$$c_{\theta,\lambda}\geq \min_{1\leq i\leq n}\left\{\left(\frac{1}{\gamma_i}-\frac{1}{\ell_{i_{\mathcal{C}^1_{h_i}}}^{-}}\right)\inf_{j\in J_i^1}\left\{\overline{T}_{i_{x_j}}^{N}(D_i)^{N/h_i(x_j)}\right\}\right\}.$$

Therefore, if $c_{\theta,\lambda}<\min_{1\leq i\leq n}\{(\frac{1}{\gamma_i}-\frac{1}{\ell_{i_{\mathcal{C}^1_{h_i}}}^{-}})\inf_{j\in J_i^1}\{\overline{T}_{i_{x_j}}^{N}(D_i)^{N/h_i(x_j)}\}\}$, the set $\bigcup_{i=1}^n J_i^1$ is empty, which means that for all $1\leq i\leq n$, $\|u_{i_m}\|_{L^{\ell_i(x)}(\partial\Omega)}\to\|u_i\|_{L^{\ell_i(x)}(\partial\Omega)}$. Since $u_m\rightharpoonup u$ in X, we have for all $i\in\{1,2,\dots,n\}$ that $u_{i_m}\to u_i$ strongly in $L^{\ell_i(x)}(\partial\Omega)$. Next, consider $J_i^2\neq\emptyset$, by the same approach for the case J_i^1. We have

$$c_{\theta,\lambda}\geq \min_{1\leq i\leq n}\left\{\left(\frac{1}{\gamma_i}-\frac{1}{s_{i_{\mathcal{C}^2_{h_i}}}^{-}}\right)\inf_{j\in J_i^2}\left\{S_i^{N}(D_i)^{N/h_i(x_j)}\right\}\right\},$$

where $D_i=\mathfrak{M}_i^0(\min\{k_{1_i}^0,k_{2_i}^0\}(1-\mathcal{K}(k_i^3)+\mathcal{K}(k_i^3)\min\{k_{1_i}^2,k_{2_i}^2\}))$. Hence, we deduce that $\bigcup_{i=1}^n J_i^2=\emptyset$, which means that for all $1\leq i\leq n$, $\|u_{i_m}\|_{L^{s_i(x)}(\Omega)}\to\|u_i\|_{L^{s_i(x)}(\Omega)}$. Since $u_m\rightharpoonup u$ in X, we have $u_{i_m}\to u_i$ strongly in $L^{s_i(x)}(\Omega)$, for all $i\in\{1,2,\dots,n\}$. On the other hand, we have

$$\begin{aligned}
&E'_{\theta,\lambda}(u_{1_m},\dots,u_{n_m})-\left\langle E'_{\theta,\lambda}(u_{1_k},\dots,u_{n_k}),(u_{1_m}-u_{1_k},0,\dots,0)\right\rangle\\
&\quad=\left\langle \Phi'_\theta(u_{1_m},\dots,u_{n_m})-\Phi'_\theta(u_{1_k},\dots,u_{n_k}),(u_{1_m}-u_{1_k},0,\dots,0)\right\rangle\\
&\qquad-\left\langle \Psi'(u_{1_m},\dots,u_{n_m})-\Psi'(u_{1_k},\dots,u_{n_k}),(u_{1_m}-u_{1_k},0,\dots,0)\right\rangle\\
&\qquad-\left\langle \Upsilon'(u_{1_m},\dots,u_{n_m})-\Upsilon'(u_{1_k},\dots,u_{n_k}),(u_{1_m}-u_{1_k},0,\dots,0)\right\rangle\\
&\qquad-\left\langle \mathcal{F}'_\lambda(u_{1_m},\dots,u_{n_m})-\mathcal{F}'_\lambda(u_{1_k},\dots,u_{n_k}),(u_{1_m}-u_{1_k},0,\dots,0)\right\rangle,
\end{aligned}$$

thus $E'_{\theta,\lambda}(u_{1_m},\dots,u_{n_m})\to 0$, i.e., $E'_{\theta,\lambda}(u_{1_m},\dots,u_{n_m})$ is a Cauchy sequence in X^*. Furthermore, by using Hölder inequality again, we find

$$\begin{aligned}
&\left\langle \Psi'(u_{1_m},\dots,u_{n_m})-\Psi'(u_{1_k},\dots,u_{n_k}),(u_{1_m}-u_{1_k},0,\dots,0)\right\rangle\\
&\quad=\int_\Omega\left(|u_{1_m}|^{s_1(x)-2}u_{1_m}-|u_{1_k}|^{s_1(x)-2}u_{1_k}\right)(u_{1_m}-u_{1_k})\,dx\\
&\quad\leq\left\||u_{1_m}|^{s_1(x)-2}u_{1_m}-|u_{1_k}|^{s_1(x)-2}u_{1_k}\right\|_{L^{s_1'(x)}(\Omega)}\|u_{1_m}-u_{1_k}\|_{L^{s_1(x)}(\Omega)}.
\end{aligned}$$

Similarly, we also have

$$\begin{aligned}
&\left\langle \Upsilon'(u_{1_m},\dots,u_{n_m}) - \Upsilon'(u_{1_k},\dots,u_{n_k}),(u_{1_m}-u_{1_k},0,\dots,0)\right\rangle \\
&\quad = \int_{\partial\Omega} \left(|u_{1_m}|^{\ell_1(x)-2}u_{1_m} - |u_{1_k}|^{\ell_1(x)-2}u_{1_k}\right)(u_{1_m}-u_{1_k})\,d\sigma_x \\
&\quad \leq \left\| |u_{1_m}|^{\ell_1(x)-2}u_{1_m} - |u_{1_k}|^{\ell_1(x)-2}u_{1_k}\right\|_{L^{\ell_1'(x)}(\partial\Omega)} \|u_{1_m}-u_{1_k}\|_{L^{\ell_1(x)}(\partial\Omega)}.
\end{aligned}$$

Since $\{u_{1_m}\}$ is a Cauchy sequence in $L^{s_1(x)}(\Omega)$ and $L^{\ell_1(x)}(\partial\Omega)$, it follows that $\Psi'(u_{1_m},\dots,u_{n_m})$ and $\Upsilon'(u_{1_m},\dots,u_{n_m})$ are Cauchy sequences in $X^\star$. Moreover, by compactness of $\mathcal{F}_\lambda'$, we have

$$(u_{1_m},\dots,u_{n_m}) \rightharpoonup (u_1,\dots,u_n) \quad \Rightarrow \quad \mathcal{F}_\lambda'(u_{1_m},\dots,u_{n_m}) \to \mathcal{F}_\lambda'(u_1,\dots,u_n),$$

which means that $\mathcal{F}_\lambda'(u_{1_m},\dots,u_{n_m})$ is a Cauchy sequence also in X^*. Therefore, invoking some elementary inequalities (see, e.g., Hurtado et al. [32, Auxiliary Results]), we conclude that for all $\varrho,\zeta \in \mathbb{R}^N$,

$$\begin{cases} |\varrho-\zeta|^{p_i(x)} \leq c_{p_i}(\mathcal{B}_{j_i}(\varrho)-\mathcal{B}_{j_i}(\zeta))\cdot(\varrho-\zeta) & \text{if } p_i(x)\geq 2, \\ |\varrho-\zeta|^2 \leq c(|\varrho|+|\zeta|)^{2-p_i(x)}(\mathcal{B}_{j_i}(\varrho)-\mathcal{B}_{j_i}(\zeta))\cdot(\varrho-\zeta) & \text{if } 1<p_i(x)<2, \end{cases} \tag{3.23}$$

where $\cdot$ denotes the standard inner product in $\mathbb{R}^N$. Define the subsets of Ω dependent on p_i by $U_{p_i} := \{x\in\Omega : p(x)\geq 2\}$ and $V_{p_i} := \{x\in\Omega : 1<p(x)<2\}$. For $i=1$, respectively replacing ϱ and ζ by ∇u_{1_m} and ∇u_{1_k} when $j=1$, and by u_{1_m} and u_{1_k} when $j=2$, in the first line of relation (3.23), and integrating over Ω, we obtain

$$\begin{aligned}
&c_1\int_{U_{p_i}} \left(|\nabla u_{1_m}-\nabla u_{1_k}|^{p_1(x)} + |u_{1_m}-u_{1_k}|^{p_1(x)}\right)dx \\
&\quad \leq \left\langle \Phi_\theta'(u_{1_m},\dots,u_{n_m}) - \Phi_\theta'(u_{1_k},\dots,u_{n_k}),(u_{1_m}-u_{1_k},0,\dots,0)\right\rangle.
\end{aligned}$$

On the other hand, by the second line of relation (3.23), we have

$$\begin{aligned}
&c_2\int_{V_{p_i}} \left(\sigma_1(x)^{p_1(x)-2}|\nabla u_{1_m}-\nabla u_{1_k}|^2 + \sigma_2(x)^{p_1(x)-2}|u_{1_m}-u_{1_k}|^2\right)dx \\
&\quad \leq \left\langle \Phi_\theta'(u_{1_m},\dots,u_{n_m}) - \Phi_\theta'(u_{1_k},\dots,u_{n_k}),(u_{1_m}-u_{1_k},0,\dots,0)\right\rangle,
\end{aligned}$$

where $\sigma_1(x) = (|\nabla u_{1_m}|+|\nabla u_{1_k}|)$ and $\sigma_2(x) = (|u_{1_m}|+|u_{1_k}|)$. Hence, by Hölder's inequality and Lemma 2.2,

$$\begin{aligned}
&\int_{V_{p_i}} \left(|\nabla u_{1_m}-\nabla u_{1_k}|^{p_1(x)} + |u_{1_m}-u_{1_k}|^{p_1(x)}\right)dx \\
&\quad = \int_{V_{p_i}} \sigma_1^{\frac{p_1(x)(p_1(x)-2)}{2}} \left(\sigma_1^{\frac{p_1(x)(p_1(x)-2)}{2}}|\nabla u_{1_m}-\nabla u_{1_k}|^{p_1(x)}\right)dx \\
&\qquad + \int_{V_{p_i}} \sigma_1^{\frac{p_1(x)(p_1(x)-2)}{2}} \left(\sigma_1^{\frac{p_1(x)(p_1(x)-2)}{2}}|u_{1_m}-u_{1_k}|^{p_1(x)}\right)dx
\end{aligned}$$

$$
\begin{aligned}
&\leq C_3 \left\| \sigma_1^{\frac{p_1(x)(2-p_1(x))}{2}} \right\|_{L^{\frac{2}{2-p_1(x)}}(V_{p_i})} \left\| \sigma_1^{\frac{p_1(x)(p_1(x)-2)}{2}} |\nabla u_{1_m} - \nabla u_{1_k}|^{p_1(x)} \right\|_{L^{\frac{2}{p_1(x)}}(V_{p_i})} \\
&\quad + C_4 \left\| \sigma_2^{\frac{p_1(x)(2-p_1(x))}{2}} \right\|_{L^{\frac{2}{2-p_1(x)}}(V_{p_i})} \left\| \sigma_2^{\frac{p_1(x)(p_1(x)-2)}{2}} |u_{1_m} - u_{1_k}|^{p_1(x)} \right\|_{L^{\frac{2}{p_1(x)}}(V_{p_i})} \\
&\leq C_5 \max\Big\{ \|\sigma_1\|_{L^{p_1(x)}(V_{p_i})}^{(\frac{p_1(x)(p_1(x)-2)}{2})^-}, \|\sigma_1\|_{L^{p_1(x)}(V_{p_i})}^{(\frac{p_1(x)(p_1(x)-2)}{2})^+} \Big\} \\
&\quad \times \max\bigg\{ \bigg(\int_{V_{p_i}} \sigma_1^{p_1(x)-2} |\nabla u_{1_m} - \nabla u_{1_k}|^2 \, dx \bigg)^{\frac{p_1^-}{2}}, \\
&\quad \bigg(\int_{V_{p_i}} \sigma_1^{p_1(x)-2} |\nabla u_{1_m} - \nabla u_{1_k}|^2 \, dx \bigg)^{\frac{p_1^+}{2}} \bigg\} \\
&\quad + C_6 \max\Big\{ \|\sigma_2\|_{L^{p_1(x)}(V_{p_i})}^{[\frac{p_1(x)(p_1(x)-2)}{2}]^-}, \|\sigma_2\|_{L^{p_1(x)}(V_{p_i})}^{[\frac{p_1(x)(p_1(x)-2)}{2}]^+} \Big\} \\
&\quad \times \max\bigg\{ \bigg(\int_{V_{p_i}} \sigma_1^{p_1(x)-2} |u_{1_m} - u_{1_k}|^2 \, dx \bigg)^{\frac{p_1^-}{2}}, \bigg(\int_{V_{p_i}} \sigma_1^{p_1(x)-2} |u_{1_m} - u_{1_k}|^2 \, dx \bigg)^{\frac{p_1^+}{2}} \bigg\}.
\end{aligned}
$$

Since $\{u_{1_m}\}$ is a bounded sequence in $W^{1,h_1(x)}(\Omega) \cap W^{1,p_1(x)}$, we have

$$
\big\langle \Phi_\theta'(u_{1_m}, \dots, u_{n_m}) - \Phi_\theta'(u_{1_k}, \dots, u_{n_k}), (u_{1_m} - u_{1_k}, 0, \dots, 0) \big\rangle \to 0, \quad \text{as } m, k \to +\infty,
$$

hence $\{u_{1_m}\}$ is a Cauchy sequence in $W^{1,p_1(x)} \cap W^{1,h_1(x)}(\Omega)$. We argue similarly for $\{u_{i_m}\}$,

$$
\begin{aligned}
&\big\langle \Phi_\theta'(u_{1_m}, \dots, u_{i_m}, \dots, u_{n_m}) - \Phi_\theta'(u_{1_k}, \dots, u_{i_k}, \dots, u_{n_k}), (0, \dots, u_{i_m} - u_{i_k}, 0, \dots, 0) \big\rangle, \\
&\quad \text{for all } i \in \{2, \dots, n\}.
\end{aligned}
$$

Thus, we can conclude that $u_m = (u_{1_m}, \dots, u_{n_m}) \to u = (u_1, \dots, u_n)$ strongly in X as $m \to +\infty$. Therefore, we have that $E_{\theta,\lambda}(u) = c_{\theta,\lambda} > 0$ and $E_{\theta,\lambda}'(u) = 0$ in X', i.e., $u \in X$ is a weak solution of problem (3.1). Since $E_{\theta,\lambda}(u) = c_{\theta,\lambda} > 0 = E_{\theta,\lambda}(0)$, we can conclude that $u \not\equiv 0$. □

4 Proof of the main theorem

Now we are in position to prove Theorem 1.2.

Proof Invoking Theorem 3.1, for all $\lambda \geq \lambda_*$ let $u_\lambda = (u_{1,\lambda}, u_{2,\lambda}, \dots, u_{n,\lambda})$ be a solution of system (3.1). We shall prove that

$$
\text{there exists } \lambda^* \geq \lambda_* \quad \text{such that } \mathcal{A}_i(u_{i,\lambda}) \leq \tau_i^0, \quad \text{for all } \lambda \geq \lambda^*, \tag{4.1}
$$

where τ_i^0 is defined as at the beginning of Sect. 3. We argue by contradiction and suppose that there is a sequence $\{\lambda_m\}_{m\in\mathbb{N}} \subset \mathbb{R}$ such that $\mathcal{A}_i(u_{i,\lambda_m}) \geq \tau_i^0$, for all $i \in \{1, 2, \dots, n\}$. By assumption ($\boldsymbol{A}_1$) and the fact $\mathcal{A}_i(u_{i,\lambda_m}) \geq \tau_i^0$, we get

$$
\begin{aligned}
&\int_\Omega \Big(\max\big\{ k_{1_i}^1, k_{2_i}^1 \big\} \big(|\nabla u_{i,\lambda_m}|^{p_i(x)} + |u_{i,\lambda_m}|^{p_i(x)} \big) + k_i^3 \big(|\nabla u_{i,\lambda_m}|^{q_i(x)} + |u_{i,\lambda_m}|^{q_i(x)} \big) \Big) \, dx \\
&\quad \geq \tau_i^0 \quad \text{for all } i = 1, 2, \dots, n.
\end{aligned} \tag{4.2}
$$

Since $u_{\lambda_m} = (u_{1,\lambda_m}, \dots, u_{n,\lambda_m})$ is a critical point of the functional E_{θ,λ_m}, we can conclude, using assumptions (***M***) and (***F***$_3$), that

$$\begin{aligned}
c_{\theta,\lambda_m} &= E_{\theta,\lambda}(u_{\lambda_m}) - \left\langle E'_{\theta,\lambda}(u_{\lambda_m}), \frac{u_{\lambda_m}}{\gamma} \right\rangle \\
&\geq \sum_{i=1}^{n} \widehat{M_{\theta_i}}\big(\mathcal{A}_i(u_{i,\lambda_m})\big) \\
&\quad - \sum_{i=1}^{n} M_i\big(\mathcal{A}_i(u_{i,\lambda_m})\big) \\
&\quad \times \int_\Omega \frac{1}{\gamma_i}\big(a_{1_i}\big(|\nabla u_{i,\lambda_m}|^{p_i(x)}\big)|\nabla u_{i,\lambda_m}|^{p_i(x)} + a_{2_i}\big(|u_{i,\lambda_m}|^{p_i(x)}\big)|u_{i,\lambda_m}|^{p_i(x)}\big)\,dx \\
&\geq \sum_{i=1}^{n} \frac{\mathfrak{M}_i^0}{p_i^+ \max\{\beta_{1_i}, \beta_{2_i}\}} \\
&\quad \times \int_\Omega \big(a_{1_i}\big(|\nabla u_{i,\lambda_m}|^{p_i(x)}\big)|\nabla u_{i,\lambda_m}|^{p_i(x)} + a_{2_i}\big(|u_{i,\lambda_m}|^{p_i(x)}\big)|u_{i,\lambda_m}|^{p_i(x)}\big)\,dx \\
&\quad - \sum_{i=1}^{n} \frac{\theta_i}{\gamma_i} \int_\Omega \big(a_{1_i}\big(|\nabla u_{i,\lambda_m}|^{p_i(x)}\big)|\nabla u_{i,\lambda_m}|^{p_i(x)} + a_{2_i}\big(|u_{i,\lambda_m}|^{p_i(x)}\big)|u_{i,\lambda_m}|^{p_i(x)}\big)\,dx \\
&\geq \sum_{i=1}^{n} \left(\frac{\mathfrak{M}_i^0}{p_i^+ \max\{\beta_{1_i}, \beta_{2_i}\}} - \frac{\theta_i}{\gamma_i}\right) \\
&\quad \times \int_\Omega \big(a_{1_i}\big(|\nabla u_{i,\lambda_m}|^{p_i(x)}\big)|\nabla u_{i,\lambda_m}|^{p_i(x)} + a_{2_i}\big(|u_{i,\lambda_m}|^{p_i(x)}\big)|u_{i,\lambda_m}|^{p_i(x)}\big)\,dx \\
&\geq \sum_{i=1}^{n} \left(\frac{\mathfrak{M}_i^0}{p_i^+ \max\{\beta_{1_i}, \beta_{2_i}\}} - \frac{\theta_i}{\gamma_i}\right) \\
&\quad \times \left[\int_\Omega \max\big\{k_{1_i}^0, k_{2_i}^0\big\}\big(|\nabla u_{i,\lambda_m}|^{p_i(x)} + |u_{i,\lambda_m}|^{p_i(x)}\big)\,dx \right. \\
&\quad \left. + \mathcal{K}\big(k_i^3\big)\max\big\{k_{1_i}^2, k_{2_i}^2\big\} \int_\Omega \big(|\nabla u_{i,\lambda_m}|^{q_i(x)} + |u_{i,\lambda_m}|^{q_i(x)}\big)\,dx\right].
\end{aligned} \tag{4.3}$$

If $k_i^3 = 0$, then, using relation (4.2), we find $\int_\Omega (|\nabla u_i|^{p_i(x)} + |u_i|^{p_i(x)})\,dx \geq \frac{\tau_i^0}{\max\{k_{1_i}^1, k_{2_i}^1\}}$, thus we have

$$c_{\theta,\lambda} \geq \sum_{i=1}^{n} \left(\frac{\mathfrak{M}_i^0}{p_i^+ \max\{\beta_{1_i}, \beta_{2_i}\}} - \frac{\theta_i}{\gamma_i}\right)\left(\frac{\max\{k_{1_i}^0, k_{2_i}^0\}}{\max\{k_{1_i}^1, k_{2_i}^1\}}\right)\tau_i^0 > 0.$$

This contradicts Lemma 3.4, because $\lim_{m\to+\infty} c_{\theta,\lambda_m} = 0$. On the other hand, if $k_i^3 > 0$, we multiplying relation (4.3) by $\max_{1\leq i\leq n}\{\max\{k_{1_i}^1, k_{2_i}^1\} \times k_i^3\} > 0$, and, by using also relation (4.2), we get

$$\begin{aligned}
&\max_{1\leq i\leq n}\big\{\max\big\{k_{1_i}^1, k_{2_i}^1\big\} \times k_i^3\big\} C_{\theta,\lambda} \\
&\quad \geq \sum_{i=1}^{n} \left(\frac{\mathfrak{M}_i^0}{p_i^+ \max\{\beta_{1_i}, \beta_{2_i}\}} - \frac{\theta_i}{\gamma_i}\right)\kappa_i \int_\Omega \big(\max\big\{k_{1_i}^1, k_{2_i}^1\big\}\big(|\nabla u_{i,\lambda_m}|^{p_i(x)} + |u_{i,\lambda_m}|^{p_i(x)}\big)
\end{aligned}$$

$$+ k_i^3\big(|\nabla u_{i,\lambda_m}|^{q_i(x)} + |u_{i,\lambda_m}|^{q_i(x)}\big)\big)\,dx$$
$$\geq \sum_{i=1}^{n}\left(\frac{\mathfrak{M}_i^0}{p_i^+ \max\{\beta_{1_i},\beta_{2_i}\}} - \frac{\theta_i}{\gamma_i}\right)\kappa_i\tau_i^0,$$

where $\kappa_i = \min\{\max\{k_{1_i}^0, k_{2_i}^0\} \times k_i^3, \max\{k_{1_i}^1, k_{2_i}^1\} \times \max\{k_{1_i}^2, k_{2_i}^2\}\}$. This also contradicts Lemma 3.4 because $\lim_{m\to+\infty} c_{\theta,\lambda} = 0$. Hence, we can conclude in both cases that there exists $\lambda^* \geq \lambda_*$ such that $\mathcal{A}_i(u_{i,\lambda}) \geq \tau_i^0$, for all $\lambda \geq \lambda^*$. So, we can find $M_{\theta_i}(\mathcal{A}_i(u_\lambda)) = M_{\theta_i}(\mathcal{A}_i(u_\lambda))$, for all $\lambda \geq \lambda^*$, which implies that $E_{\theta,\lambda}(u_\lambda) = E_\lambda(u_\lambda)$ and $E'_{\theta,\lambda}(u_\lambda) = E'_\lambda(u_\lambda)$, that is, u_λ is a nontrivial weak solution of the problem (1.1), for each $\lambda \geq \lambda^*$.

It now remains to consider the asymptotic behavior of solutions to problem (1.1). By assumptions $(\boldsymbol{A}_2)$, $(\boldsymbol{A}_4)$, $(\boldsymbol{M})$, $(\boldsymbol{F}_3)$, and inequalities (2.2)–(2.3) and (3.2), arguing as above, we obtain

$$\begin{aligned}
c_\lambda &= c_{\theta,\lambda}\\
&\geq \sum_{i=1}^{n}\left(\frac{\mathfrak{M}_i^0}{p_i^+ \max\{\beta_{1_i},\beta_{2_i}\}} - \frac{\theta_i}{\gamma_i}\right)\Big(\min\{k_{1_i}^0, k_{2_i}^0\}\int_\Omega \big(|\nabla u_{i,\lambda}|^{p_i(x)} + |u_{i,\lambda}|^{p_i(x)}\big)\,dx\\
&\quad + \mathcal{K}(k_i^3)\min\{k_{1_i}^2, k_{2_i}^2\}\int_\Omega \big(|\nabla u_{i,\lambda}|^{q_i(x)} + |u_{i,\lambda}|^{q_i(x)}\big)\,dx\Big)\\
&\geq \sum_{i=1}^{n}\left(\frac{\mathfrak{M}_i^0}{p_i^+ \max\{\beta_{1_i},\beta_{2_i}\}} - \frac{\theta_i}{\gamma_i}\right)\Big[\min\{k_{1_i}^0, k_{2_i}^0\}\min\{\|u_{i,\lambda}\|_{1,p_i(x)}^{p_i^-}, \|u_{i,\lambda}\|_{p_i(x)}^{p_i^+}\}\\
&\quad + \mathcal{K}(k_i^3)\min\{k_{1_i}^2, k_{2_i}^2\}\min\{\|u_{i,\lambda}\|_{1,q_i(x)}^{q_i^-}, \|u_{i,\lambda}\|_{1,q_i(x)}^{q_i^+}\}\Big].
\end{aligned}$$

Hence, by Lemma 3.4, we get $\lim_{\lambda\to+\infty}\|u_\lambda\| = \lim_{\lambda\to+\infty}\max_{1\leq i\leq n}\{\|u_i\|_{1,p_i(x)} + \mathcal{K}(k_i^3)\times \|u_i\|_{1,q_i(x)}\} = 0$. □

5 Some examples

In the last section, we shall exhibit some examples which are interesting from the mathematical point of view and have a wide range of applications in physics and other scientific fields that fall within the general class of systems studied in this paper, under adequate assumptions on functions a_{i_j}.

Example 5.1 Taking $a_{1_i} \equiv 1$ and $a_{2_i} \equiv 1$, we see that a_{1_i} satisfies the assumptions $(\boldsymbol{A}_1)$, $(\boldsymbol{A}_2)$, and $(\boldsymbol{A}_3)$, with $k_{j_i}^0 = k_{j_i}^1 = 1$, $k_{j_i}^2 > 0$, and $k_i^3 = 0$, for all $i \in \{1,2,\ldots,n\}$ and $j = 1$ or 2. Hence, system (1.1) becomes

$$\begin{aligned}
&-M_i\big(\mathcal{A}_i(u_i)\big)\big(\Delta_{p_i(x)}u_i - |u_i|^{p_i(x)-2}u_i\big) = |u_i|^{s_i(x)-2}u_i + \lambda F_{u_i}(x,u) \quad \text{in } \Omega,\\
&M_i\big(\mathcal{A}_i(u_i)\big)|\nabla u_i|^{p(x)-2}\nabla u_i\cdot\mathfrak{N}_i = |u_i|^{\ell_i(x)-2}u_i \quad \text{on } \partial\Omega,
\end{aligned} \tag{5.1}$$

for $1 \leq i \leq n$ $(n \in \mathbb{N}^*)$, where

$$\mathcal{A}_i(u_i) = \int_\Omega \frac{1}{p_i(x)}\big(|\nabla u_i|^{p_i(x)} + |u_i|^{p_i(x)}\big)\,dx.$$

The operator $\Delta_{p_i(x)}u_i := \operatorname{div}(|\nabla u_i|^{p_i(x)-2}\nabla u_i)$ is the so-called $p_i(x)$-Laplacian, which coincides with the usual p_i-Laplacian when $p_i(x) = p_i$, and with the Laplacian when $p_i(x) = 2$.

Example 5.2 Taking $a_{j_i}(\xi) = 1 + \xi^{\frac{q_i(x)-p_i(x)}{p_i(x)}}$, we see that a_{j_i} satisfies the assumptions ($\boldsymbol{A}_1$), ($\boldsymbol{A}_2$), and ($\boldsymbol{A}_3$), with $k^0_{j_i} = k^1_{j_i} = k^2_{j_i} = k^3_i = 1$, for all $i \in \{1,2,\dots,n\}$ and $j = 1$ or 2. Hence, system (1.1) becomes the following (p,q)-Laplacian system

$$\begin{aligned} &-M_i(\mathcal{A}_i(u_i))\big(\Delta_{p_i(x)}u_i + \Delta_{q_i(x)}u_i - \big(|u_i|^{p_i(x)-2}u_i + |u_i|^{q_i(x)-2}u_i\big)\big) \\ &\quad = |u_i|^{s_i(x)-2}u_i + \lambda F_{u_i}(x,u) \quad \text{in } \Omega, \\ &M_i(\mathcal{A}_i(u_i))\big(|\nabla u_i|^{p_i(x)-2}\nabla u_i + |\nabla u_i|^{q_i(x)-2}\nabla u_i\big)\cdot \mathfrak{N}_i = |u_i|^{\ell_i(x)-2}u_i \quad \text{on } \partial\Omega, \end{aligned} \tag{5.2}$$

for $1 \le i \le n$ ($n \in \mathbb{N}^*$), where

$$\mathcal{A}_i(u_i) = \int_\Omega \left(\frac{1}{p_i(x)}\big(|\nabla u_i|^{p_i(x)} + w_i(x)|u_i|^{p_i(x)}\big) + \frac{1}{q_i(x)}\big(|\nabla u_i|^{q_i(x)} + |u_i|^{q_i(x)}\big)\right)dx.$$

As explained in Cherfils and Il'yasov [16], the study of system (5.2) was motivated by the following more general reaction–diffusion system:

$$u_t = \operatorname{div}\big[H(u)\nabla u\big] + d(x,u), \quad \text{where } H(u) = |\nabla u|^{p(x)-2} + |\nabla u|^{q(x)-2},$$

which has applications in biophysics (see, e.g., Fife [25], Murray [40]), plasma physics (see, e.g., Wilhelmsson [49]), and chemical reactions design (see, e.g., Aris [4]). In these applications, u represents concentration, $\operatorname{div}[H(u)\nabla u]$ is the diffusion with a diffusion coefficient, and the reaction term $d(x,u)$ relates to the source and loss processes. For further details, we refer the interested reader to, e.g., Mahshid and Razani [38], He and Li [30], and the references therein.

We continue with other examples which are also interesting from the mathematical point of view.

Example 5.3 Taking $a_{j_i}(\xi) = 1 + \frac{\xi}{\sqrt{1+\xi^2}}$ and $a_{2_i} \equiv 1$, we see that a_{j_i} satisfies the assumptions ($\boldsymbol{A}_1$), ($\boldsymbol{A}_2$), and ($\boldsymbol{A}_3$), with $k^0_{1_i} = k^0_{2_i} = k^1_{2_i} = 1$, $k^1_{1_i} = 2$, and $k^3_i = 0$, $k^2_{1_i} > 0$, and $k^2_{2_i} > 0$, for all $i \in \{1,2,\dots,n\}$. Hence, system (1.1) becomes

$$\begin{aligned} &-M_i(\mathcal{A}_i(u_i))\left(\operatorname{div}\left(\left(1 + \frac{|\nabla u_i|^{p_i(x)}}{\sqrt{1+|\nabla u_i|^{2p_i(x)}}}\right)|\nabla u_i|^{p_i(x)-2}\nabla u_i\right) - w_i(x)|u_i|^{p_i(x)-2}u_i\right) \\ &\quad = |u_i|^{s_i(x)-2}u_i + \lambda F_{u_i}(x,u) \quad \text{in } \Omega, \\ &M_i(\mathcal{A}_i(u_i))\left(1 + \frac{|\nabla u_i|^{p_i(x)}}{\sqrt{1+|\nabla u_i|^{2p_i(x)}}}\right)|\nabla u_i|^{p_i(x)-2}\nabla u_i \cdot \mathfrak{N}_i = |u_i|^{\ell_i(x)-2}u_i \quad \text{on } \partial\Omega, \end{aligned} \tag{5.3}$$

for $1 \le i \le n$ ($n \in \mathbb{N}^*$), where

$$\mathcal{A}_i(u_i) = \int_{\mathbb{R}^N} \frac{1}{p_i(x)}\big(|\nabla u_i|^{p_i(x)} + \sqrt{1+|\nabla u_i|^{2p_i(x)}} + |u_i|^{p_i(x)}\big)\,dx.$$

The operator $\operatorname{div}((1 + \frac{|\nabla u|^{p(x)}}{\sqrt{1+|\nabla u|^{2p(x)}}})|\nabla u|^{p(x)-2}\nabla u)$ is said to be $p_i(x)$-Laplacian-like or is called a generalized capillary operator. The capillarity can be briefly explained by considering the effects of two opposing forces: adhesion, i.e., the attractive (or repulsive) force between the molecules of the liquid and those of the container; and cohesion, i.e., the attractive force between the molecules of the liquid. The study of the capillary phenomenon has gained much attention. This increasing interest is motivated not only by the fascination in naturally occurring phenomena, such as motion of drops, bubbles, and waves, but also by its importance in applied fields, raging from industrial and biomedical and pharmaceutical to microfluidic systems; for further details, we refer the interested reader to, e.g., Ni and Serrin [41], and the references therein.

Example 5.4 Taking $a_{1_i}(\xi) = 1 + \frac{1}{(1+\xi)^{\frac{p_i(x)-2}{p_i(x)}}}$ and $a_{2_i} \equiv 1$, we see that a_{j_i} satisfies the assumptions ($\boldsymbol{A}_1$), ($\boldsymbol{A}_2$), and ($\boldsymbol{A}_3$), with $k^0_{1_i} = k^0_{2_i} = k^1_{2_i} = 1$, $k^1_{1_i} = 2$, $k^3_i = 0$, $k^2_{1_i} > 0$, and $k^2_{2_i} > 0$, for all $i \in \{1, 2, \dots, n\}$. Hence, system (1.1) becomes

$$\begin{aligned} &-M_i\big(\mathcal{A}_i(u_i)\big)\left(\operatorname{div}\left(|\nabla u_i|^{p(x)-2}\nabla u_i + \frac{|\nabla u_i|^{p(x)-2}\nabla u_i}{(1+|\nabla u_i|^{p(x)})^{\frac{p(x)-2}{p(x)}}}\right) - |u_i|^{p_i(x)-2}u_i\right) \\ &\quad = |u_i|^{s_i(x)-2}u_i + \lambda F_{u_i}(x,u) \quad \text{in } \Omega, \\ &M_i\big(\mathcal{A}_i(u_i)\big)\left(|\nabla u_i|^{p(x)-2}\nabla u_i + \frac{|\nabla u_i|^{p(x)-2}\nabla u_i}{(1+|\nabla u_i|^{p(x)})^{\frac{p(x)-2}{p(x)}}}\right)\cdot \mathfrak{N}_i = |u_i|^{\ell_i(x)-2}u_i \quad \text{on } \partial\Omega, \end{aligned} \tag{5.4}$$

for $1 \le i \le n$ ($n \in \mathbb{N}^*$), where

$$\mathcal{A}_i(u_i) = \int_{\mathbb{R}^N} \frac{1}{p_i(x)}\Big(|\nabla u_i|^{p_i(x)} + \sqrt{1+|\nabla u_i|^{2p_i(x)}} + |u_i|^{p_i(x)}\Big)\,dx.$$

Example 5.5 Taking $a_{1_i}(\xi) = 1 + \xi^{\frac{q_i(x)-p_i(x)}{p_i(x)}} + \frac{1}{(1+\xi)^{\frac{p_i(x)-2}{p_i(x)}}}$ and $a_{2_i}(\xi) = 1 + \xi^{\frac{q_i(x)-p_i(x)}{p_i(x)}}$, we see that a_{1_i} satisfies the assumptions ($\boldsymbol{A}_1$), ($\boldsymbol{A}_2$), ($\boldsymbol{A}_3$), and ($\boldsymbol{H}_3$), with $k^0_{1_i} = k^0_{2_i} = k^1_{2_i} = 1$, $k^1_{1_i} = 2$, and $k^3_i = k^2_{1_i} = k^2_{2_i} = 1$, for all $i \in \{1, 2, \dots, n\}$. Hence, system (1.1) becomes

$$\begin{aligned} &-M_i\big(\mathcal{A}_i(u_i)\big)\left(\Delta_{p_i(x)}u_i + \Delta_{q_i(x)}u_i + \operatorname{div}\left(\frac{|\nabla u_i|^{p_i(x)-2}\nabla u_i}{(1+|\nabla u_i|^{p_i(x)})^{\frac{p_i(x)-2}{p_i(x)}}}\right)\right. \\ &\qquad \left. - \big(|u_i|^{p_i(x)-2}u_i + |u_i|^{q_i(x)-2}u_i\big)\right) \\ &\quad = |u_i|^{s_i(x)-2}u_i + \lambda F_{u_i}(x,u) \quad \text{in } \Omega, \\ &M_i\big(\mathcal{A}_i(u_i)\big)\left(|\nabla u_i|^{p_i(x)-2}\nabla u_i + \frac{|\nabla u_i|^{p(x)-2}\nabla u_i}{(1+|\nabla u_i|^{p(x)})^{\frac{p(x)-2}{p(x)}}} + |\nabla u_i|^{q_i(x)-2}\nabla u_i\right)\cdot \mathfrak{N}_i \\ &\quad = |u_i|^{\ell_i(x)-2}u_i \quad \text{on } \partial\Omega, \end{aligned} \tag{5.5}$$

for $1 \le i \le n$ ($n \in \mathbb{N}^*$), where

$$\begin{aligned} \mathcal{A}_i(u_i) = \int_\Omega \bigg(&\frac{1}{p_i(x)}\big(|\nabla u_i|^{p_i(x)} + w_i(x)|u_i|^{p_i(x)}\big) + \frac{1}{q_i(x)}\big(|\nabla u_i|^{q_i(x)} + w_i(x)|u_i|^{q_i(x)}\big) \\ &+ \frac{1}{2}\big(1+|\nabla u_i|^{p_i(x)}\big)^{\frac{2}{p_i(x)}}\bigg)\,dx. \end{aligned}$$

On the other hand, the class of systems (1.1) can contain one model of the above divergence operators, as in Examples 5.1–5.5, or many different models of divergence operators simultaneously, depending on the phenomenon studied. Moreover, each equation in this class can also be degenerate or nondegenerate.

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The authors contributed equally.

Author details
[1]Laboratory of Mathematical Analysis and Applications, Department of Mathematics, Faculty of Sciences, Mohammed V University, P.O. Box 1014, Rabat, Morocco. [2]Faculty of Education, University of Ljubljana, Ljubljana, Slovenia. [3]Faculty of Mathematics and Physics, University of Ljubljana, Ljubljana, Slovenia. [4]Institute of Mathematics, Physics and Mechanics, Ljubljana, Slovenia.

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References
1. Alves, C.O., Barreiro, J.P.: Existence and multiplicity of solutions for a $p(x)$-Laplacian equation with critical growth. J. Math. Anal. Appl. **403**, 143–154 (2013)
2. Alves, C.O., Ferreira, M.C.: Existence of solutions for a class of $p(x)$-Laplacian equations involving a concave–convex nonlinearity with critical growth in $\mathbb{R}^N$. Topol. Methods Nonlinear Anal. **45**(2), 399–422 (2015)
3. Ambrosio, V., Isernia, T., Rădulescu, V.D.: Concentration of positive solutions for a class of fractional p-Kirchhoff type equations. Proc. R. Soc. Edinb. A **151**(2), 601–651 (2021)
4. Aris, R.: Mathematical Modelling Techniques. Research Notes in Mathematics, vol. 24. Pitman, Boston (1979) 191 pp.
5. Arosio, A., Pannizi, S.: On the well-posedness of the Kirchhoff string. Trans. Am. Math. Soc. **348**, 305–330 (1996)
6. Bonder, J.F., Saintier, N., Silva, A.: On the Sobolev trace theorem for variable exponent spaces in the critical range. Ann. Mat. Pura Appl. (4) **193**(6), 1607–1628 (2014)
7. Bonder, J.F., Silva, A.: Concentration-compactness principal for variable exponent space and applications. Electron. J. Differ. Equ. **2010**, 141, 1–18 (2010)
8. Brezis, H., Nirenberg, L.: Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents. Commun. Pure Appl. Math. **36**, 437–477 (1983)
9. Caristi, G., Heidarkhani, S., Salari, A., Tersian, S.A.: Multiple solutions for degenerate nonlocal problems. Appl. Math. Lett. **84**, 26–33 (2018)
10. Cavalcanti, M.M., Cavacanti, V.N., Soriano, J.A.: Global existence and uniform decay rates for the Kirchhoff–Carrier equation with nonlinear dissipation. Adv. Differ. Equ. **6**, 701–730 (2001)
11. Chems Eddine, N.: Existence and multiplicity of solutions for Kirchhoff-type potential systems with variable critical growth exponent. Appl. Anal. (2021). https://doi.org/10.1080/00036811.2021.1979223
12. Chems Eddine, N.: Existence of solutions for a critical $(p_1(x),\dots,p_n(x))$-Kirchhoff-type potential systems. Appl. Anal. **101**, 2239–2253 (2022)
13. Chems Eddine, N., Idrissi, A.A.: Multiple solutions to a $(p_1(x),\dots,p_n(x))$-Laplacian-type systems in unbounded domain. Azerb. J. Math. **10**(1), 3–20 (2020)
14. Chems Eddine, N., Ragusa, M.A.: Generalized critical Kirchhoff-type potential systems with Neumann boundary conditions. Appl. Anal. **101**, 3958–3988 (2022)
15. Chen, Y.M., Levine, S., Rao, M.: Variable exponent, linear growth functionals in image restoration. SIAM J. Appl. Math. **66**, 1383–1406 (2006)
16. Cherfils, L., Il'yasov, Y.: On the stationary solutions of generalized reaction diffusion equations with p&q-Laplacian. Commun. Pure Appl. Anal. **4**(1), 9 (2005)
17. Chipot, M., Lovat, B.: Some remarks on nonlocal elliptic and parabolic problems. Nonlinear Anal. TMA **30**(7), 4619–4627 (1997)

18. Corrêa, F.J.S.A., Nascimento, R.G.: On a nonlocal elliptic system of p-Kirchhoff type under Neumann boundary condition. Math. Comput. Model. (2008). https://doi.org/10.1016/j.mcm.2008.03.013
19. Dai, G., Hao, R.: Existence of solutions for a $p(x)$-Kirchhoff-type equation. J. Math. Anal. Appl. **359**, 704–710 (2009)
20. Diening, L.: Theoretical and numerical results for electrorheological fluids. PhD Thesis, University of Freiburg, Germany (2002)
21. Diening, L., Harjulehto, P., Hästö, P., Ruživčka, M.: Lebesgue and Sobolev Spaces with Variable Exponents. Lecture Notes in Mathematics, vol. 2017. Springer, Heidelberg (2011)
22. Edmunds, D.E., Rakosnik, J.: Sobolev embeddings with variable exponent. Stud. Math. **143**, 267–293 (2000)
23. Fan, X., Zhao, D.: On the spaces $L^{p}(x)(\Omega)$ and $W^{m,p}(\Omega)$. J. Math. Anal. Appl. **263**, 424–446 (2001)
24. Fang, Y., Zhang, J.: Multiplicity of solutions for a class of elliptic systems with critical Sobolev exponent. Nonlinear Anal. TMA **73**(9), 2767–2778 (2010)
25. Fife, P.C.: Mathematical Aspects of Reacting and Diffusing Systems. Lecture Notes in Biomathematics, vol. 28. Springer, Berlin (1979)
26. Fu, Y.Q.: The principle of concentration compactness in $L^{p}(x)$ spaces and its application. Nonlinear Anal. **71**, 1876–1892 (2009)
27. Fu, Y.Q., Zhang, X.: Multiple solutions for a class of $p(x)$-Laplacian equations in $\mathbb{R}^n$ involving the critical exponent. Proc. R. Soc. Lond. Ser. A **466**, 1667–1686 (2010)
28. Halsey, T.C.: Electrorheological fluids. Science **258**, 761–766 (1992)
29. Harjulehto, P., Hästö, P., Koskenoja, M., Varonen, S.: The Dirichlet energy integral and variable exponent Sobolev spaces with zero boundary values. Potential Anal. **25**(3), 205–222 (2006)
30. He, C., Li, G.: The regularity of weak solutions to nonlinear scalar field elliptic equations containing p–q-Laplacians. Ann. Acad. Sci. Fenn., Math. **33**, 337–771 (2008)
31. He, W., Qin, D., Wu, Q.: Existence, multiplicity and nonexistence results for Kirchhoff type equations. Adv. Nonlinear Anal. **10**(1), 616–635 (2021)
32. Hurtado, E.J., Miyagaki, O.H., Rodrigues, R.S.: Existence and asymptotic behaviour for a Kirchhoff type equation with variable critical growth exponent. Milan J. Math. **77**, 127–150 (2010)
33. Kirchhoff, G.: Mechanik. Teubner, Leipzig (1983)
34. Kováčik, O., Rákosník, J.: On spaces $L^{p}(x)(\Omega)$ and $W^{1,p}(\Omega)$. Czechoslov. Math. J. **41**, 592–618 (1991)
35. Lions, J.L.: On some questions in boundary value problems of mathematical physics. In: Contemporary Developments in Continuum Mechanics and Partial Differential Equations. North-Holland Mathematics Studies, vol. 30, pp. 284–346 (1978)
36. Lions, P.L.: The concentration–compactness principle in calculus of variation, the limit case part 1 and 2. Rev. Mat. Iberoam. **1**(1), 145–201 (1985)
37. Ma, T.F.: Remarks on an elliptic equation of Kirchhoff type. Nonlinear Anal. TMA **63**, 1967–1977 (2005)
38. Mahshid, M., Razani, A.: A weak solution for a $(p(x), q(x))$-Laplacian elliptic problem with a singular term. Bound. Value Probl. **2021**, Article 80 (2021)
39. Mingqi, X., Rădulescu, V.D., Zhang, B.: Combined effects for fractional Schrödinger–Kirchhoff systems with critical nonlinearities. ESAIM Control Optim. Calc. Var. **24**(3), 1249–1273 (2018)
40. Myers-Beaghton, A., Vedensky, D.D.: Chapman–Kolmogorov equation for Markov models of epitaxial growth. J. Phys. A **22**(11), 467–475 (1989)
41. Ni, W., Serrin, J.: Existence and nonexistence theorems for ground states of quasilinear partial differential equations. Atti Conv. Lincei **77**, 231–257 (1985)
42. Papageorgiou, N.S., Rădulescu, V.D., Repovš, D.D.: Nonlinear Analysis – Theory and Methods. Springer Monographs in Mathematics. Springer, Cham (2019)
43. Rabinowitz, P.H.: Minimax Methods in Critical Point Theory with Applications to Differential Equations. CBMS Reg. Conf. Ser. Math., vol. 65. Am. Math. Soc., Providence (1986)
44. Rădulescu, V.D.: Nonlinear elliptic equations with variable exponent: old and new. Nonlinear Anal. **121**, 336–369 (2015)
45. Rădulescu, V.D., Repovš, D.D.: Partial Differential Equations with Variable Exponents. Variational Methods and Qualitative Analysis. Monographs and Research Notes in Mathematics. CRC Press, Boca Raton (2015)
46. Ružička, M.: Flow of shear dependent electro-rheological fluids. C. R. Acad. Sci., Sér. 1 Math. **329**, 393–398 (1999)
47. Ružička, M.: Electro-Rheological Fluids: Modeling and Mathematical Theory. Lecture Notes in Mathematics. Springer, Berlin (2000)
48. Wang, F., Hu, D., Xiang, M.: Combined effects of Choquard and singular nonlinearities in fractional Kirchhoff problems. Adv. Nonlinear Anal. **10**(1), 636–658 (2021)
49. Wilhelmsson, H.: Explosive instabilities of reaction–diffusion equations. Phys. Rev. A **36**(2), 965–966 (1987)
50. Yang, W., Zhou, J.: Global attractors of the degenerate fractional Kirchhoff wave equation with structural damping or strong damping. Adv. Nonlinear Anal. **11**(1), 993–1029 (2022)
51. Zhang, X., Fu, Y.: Solutions of $p(x)$-Laplacian equations with critical exponent and perturbations in $\mathbb{R}^N$. Electron. J. Differ. Equ. **2012**, 120, 1–14 (2012)