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Arbitrary decay for a von Karman system with memory

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Abstract

In this paper we study the von Karman plate model with long range memory. By using the assumptions on the relaxation function due to Tatar (J. Math. Phys. 52:013502, 2011), we show an arbitrary rate of decay, which is not necessarily of an exponential or polynomial decay. Our result is obtained without imposing the usual relation between the relaxation function h and its derivative.

Keywords: Memory dissipation; Decay rate; von Karman system

1 Introduction

Let Ω be an open bounded set of $\mathbb{R}^2$ with a sufficiently smooth boundary $\Gamma = \Gamma_0 \cup \Gamma_1$, Γ_0 and Γ_1 are closed and disjoint. Denote by $\nu = (\nu_1, \nu_2)$ the external unit normal to Γ, and by $\eta = (-\nu_2, \nu_1)$ the unitary tangent positively oriented on Γ. In this paper we consider the following von Karman system with memory:

$$w_{tt} - k\Delta w_{tt} + \Delta^2 w - \int_0^t h(t-s)\Delta^2 w(s)\,ds = [w, v] \quad \text{in } \Omega \times (0, \infty), \tag{1.1}$$

$$\Delta^2 v = -[w, w] \quad \text{in } \Omega \times (0, \infty), \tag{1.2}$$

$$v = \frac{\partial v}{\partial \nu} = 0 \quad \text{on } \Gamma \times (0, \infty), \tag{1.3}$$

$$w = \frac{\partial w}{\partial \nu} = 0 \quad \text{on } \Gamma_0 \times (0, \infty), \tag{1.4}$$

$$\mathcal{B}_1 w - \mathcal{B}_1 \left\{ \int_0^t h(t-s) w(s)\,ds \right\} = 0 \quad \text{on } \Gamma_1 \times (0, \infty), \tag{1.5}$$

$$\mathcal{B}_2 w - k\frac{\partial w_{tt}}{\partial \nu} - \mathcal{B}_2 \left\{ \int_0^t h(t-s) w(s)\,ds \right\} = 0 \quad \text{on } \Gamma_1 \times (0, \infty), \tag{1.6}$$

$$w(x, y, 0) = w_0(x, y), \qquad w_t(x, y, 0) = w_1(x, y) \quad \text{in } \Omega, \tag{1.7}$$

where the function h satisfies some conditions to be specified later and von Karman bracket is given by

$$[w, v] = w_{xx} v_{yy} - 2 w_{xy} v_{xy} + w_{yy} v_{xx}.$$

Here

$$\mathcal{B}_1 w = \Delta w + (1-\mu) B_1 w \quad \text{and} \quad \mathcal{B}_2 w = \frac{\partial \Delta w}{\partial \nu} + (1-\mu) B_2 w,$$

where constant $\mu (0 < \mu < \frac{1}{2})$ is Poisson's ratio and

$$B_1 w = 2\nu_1 \nu_2 w_{xy} - \nu_1^2 w_{yy} - \nu_2^2 w_{xx}, \qquad B_2 w = \frac{\partial}{\partial \eta} \big[(\nu_1^2 - \nu_2^2) w_{xy} + \nu_1 \nu_2 (w_{yy} - w_{xx}) \big].$$

The equations describe small vibrations of a thin plate of uniform thickness. The second term in (1.1) represents rotational inertia.

Munoz Rivera and Menzala [2] discussed the exponential decay of the energy for problem (1.1)–(1.7) under the usual condition

$$-c_0 h(t) \le h'(t) \le -c_1 h(t), \quad 0 \le h''(t) \le c_2 h(t) \tag{1.8}$$

for some c_i, $i = 0, 1, 2$. Moreover, they showed that when the kernel h decays polynomially, the energy also decays with the same rate. Raposo and Santos [3] generalized the decay result of [2]. They investigated the general decay of the solutions for problem (1.1)–(1.7) under a more general condition on h such as

$$h'(t) \le -\xi(t) h(t), \qquad \xi(t) > 0, \qquad \xi'(t) \le 0, \quad \forall t \ge 0, \tag{1.9}$$

where ξ is a nonincreasing and positive function. Kang [4] proved that the solutions for problem (1.1)–(1.7) decay exponentially to zero as time goes to infinity in case

$$h'(t) + \gamma h(t) \ge 0, \qquad \big[h'(t) + \gamma h(t)\big] e^{\alpha t} \in L^1(0, \infty), \quad \forall t \ge 0,$$

for some $\gamma, \alpha > 0$. Lately, Kang [5] improved the decay result of [3] without imposing any restrictive assumptions on the behavior of the relaxation function at infinity. The author considered the general stability result for problem (1.1)–(1.7) under a relaxation function satisfying

$$h'(t) \le -H\big(h(t)\big), \tag{1.10}$$

where H is a nonnegative function, with $H(0) = 0$, and H is linear or strictly increasing and strictly convex on $(0, r]$ for some $r > 0$. Recently, Balegh et al. [6] studied the general decay rate of the energy for problem (1.1)–(1.7) with nonlinear boundary delay term. The relaxation function h satisfies

$$h'(t) \le -\xi(t) H\big(h(t)\big), \tag{1.11}$$

where ξ is a positive nonincreasing differentiable function and H satisfies the same conditions as (1.10) for some $0 < r < 1$.

For the case $h = 0$ in (1.1) with nonlinear boundary dissipation, Horn and Lasiecka [7] and Bradley and Lasiecka [8] proved the uniform decay rates for the solution when t goes to infinity.

Moreover, Cavalcanti et al. [9] considered the following problem (1.1) with the rotational inertia coefficient $k = 0$:

$$\begin{cases} u_{tt} + \Delta^2 u - \int_0^t h(t-s)\Delta^2 u(s)\,ds = [u,v] & \text{in } \Omega \times (0,\infty), \\ \Delta^2 v = -[u,u] & \text{in } \Omega \times (0,\infty), \\ u = \frac{\partial u}{\partial \nu} = 0, \qquad v = \frac{\partial v}{\partial \nu} = 0 & \text{on } \Gamma \times (0,\infty), \end{cases} \tag{1.12}$$

where the relaxation kernel h satisfies (1.10) and H is a positive, strictly increasing, and convex function with $H(0) = 0$. The rotational inertia ensures the regularity of solutions that is needed in the estimates. They proved the global existence of weak and regular solutions and provided sharp and general decay rate estimates without accounting for regularizing effects of rotational inertia by using the method introduced in [10]. Park [11] established an arbitrary rate of decay for problem (1.12) using the assumptions on the relaxation function due to Tatar [1].

When $k = h = 0$ in (1.1) with nonlinear boundary dissipation, Favini et al. [12] and Horn and Lasiecka [13] proved global existence, uniqueness, and regularity of solutions and uniform decay rates of weak solutions, respectively.

For the case $k = h = 0$ in (1.1) with memory-type boundary condition, Feng and Soufyane [14] obtained an optimal explicit and general energy decay result. For more results on von Karman plate equation with memory-type boundary condition, we refer to [15, 16].

On the other hand, for the viscoelastic wave equation, Cavalcanti et al. [17] proved exponential and polynomial decay under the usual condition (1.8). Later, this assumption was relaxed by several authors [18–20]. Messaoudi [21] considered general stability for the viscoelastic equation

$$u_{tt} - \Delta u + \int_0^t h(t-s)\Delta u(s)\,ds = 0 \quad \text{in } \Omega \times (0,\infty), \tag{1.13}$$

where the relaxation function h satisfies

$$h'(t) \le -\xi(t)h(t), \qquad \frac{|\xi'(t)|}{|\xi(t)|} \le k_0, \qquad \xi(t) > 0, \qquad \xi'(t) \le 0, \quad \forall t \ge 0. \tag{1.14}$$

Tatar [22] investigated polynomial asymptotic stability of solutions for problem (1.13) under the condition

$$h'(t) \le 0 \quad \text{for almost all } t > 0. \tag{1.15}$$

Moreover, Tatar [1] established an arbitrary decay rate for problem (1.13) with assumptions as follows:

$$\int_0^\infty h(s)\gamma(s)\,ds < +\infty, \tag{1.16}$$

where a nondecreasing function $\gamma(t) > 0$ such that $\frac{\gamma'(t)}{\gamma(t)} = \eta(t)$ is a decreasing function. As for problem of decay of the solutions for a viscoelastic system under condition (1.16), we also refer the reader to [11, 23] and the references therein. Later, Mustafa and Messaoudi

[24] showed a general decay rate result for problem (1.13) with condition (1.10) on a relaxation function. The stability of the solutions to a viscoelastic system under condition (1.9) was studied in [25–28] and the references therein.

Motivated by these works, we study an arbitrary decay of solutions for problem (1.1)–(1.7) for relaxation functions satisfying condition (1.16). This result improves earlier ones concerning exponential and polynomial decay for problem (1.1)–(1.7).

The plan of the paper is as follows: in Sect. 2, we prepare some notation and material needed for our work. In Sect. 3, we show an arbitrary decay result of the solutions for problem (1.1)–(1.7).

2 Preliminaries

We define

$$V = \{w \in H^1(\Omega); w = 0 \text{ on } \Gamma_0\}, \qquad W = \left\{w \in H^2(\Omega); w = \frac{\partial w}{\partial \nu} = 0 \text{ on } \Gamma_0\right\}.$$

Integration by parts formula yields

$$\left(\Delta^2 w, v\right) = a(w, v) + (\mathcal{B}_2 w, v)_\Gamma - \left(\mathcal{B}_1 w, \frac{\partial v}{\partial \nu}\right)_\Gamma, \tag{2.1}$$

where the bilinear symmetric form $a(w, v)$ is given by

$$a(w, v) = \int_\Omega \{w_{xx}v_{xx} + w_{yy}v_{yy} + \mu(w_{xx}v_{yy} + w_{yy}v_{xx}) + 2(1-\mu)w_{xy}v_{xy}\}\, d\Omega,$$

where $d\Omega = dx\,dy$. Because $\Gamma_0 \neq \emptyset$, we see that for $c_0 > 0$ and $c_1 > 0$,

$$c_0\|w\|^2_{H^2(\Omega)} \leq a(w, w) \leq c_1\|w\|^2_{H^2(\Omega)}. \tag{2.2}$$

The Sobolev imbedding theorem implies that for positive constants C_p and C_s,

$$\|w\|^2 \leq C_p a(w, w), \qquad \|\nabla w\|^2 \leq C_s a(w, w), \quad \forall w \in W. \tag{2.3}$$

By the symmetry of $a(\cdot, \cdot)$, we get that for any $w \in C^1(0, T; H^2(\Omega))$,

$$\begin{aligned} a(h * w, w_t) = {} & -\frac{1}{2}h(t)a(w, w) + \frac{1}{2}\left(h' \Box \partial^2 w\right)(t) \\ & - \frac{1}{2}\frac{d}{dt}\left\{\left(h \Box \partial^2 w\right)(t) - \left(\int_0^t h(s)\, ds\right) a(w, w)\right\}, \end{aligned} \tag{2.4}$$

where

$$\begin{aligned} (h * w)(t) &:= \int_0^t h(t-s)w(s)\, ds, \\ \left(h \Box \partial^2 w\right)(t) &:= \int_0^t h(t-s)a\big(w(\cdot, t) - w(\cdot, s), w(\cdot, t) - w(\cdot, s)\big)\, ds. \end{aligned}$$

We introduce relative results of the Airy stress function and von Karman bracket.

Lemma 2.1 ([2, 29]) *Let $w, u \in H^2(\Omega)$ and $v \in H_0^2(\Omega)$. Then*

$$\int_\Omega w[v,u]\,d\Omega = \int_\Omega v[w,u]\,d\Omega. \tag{2.5}$$

Lemma 2.2 ([12]) *If $w, v \in H^2(\Omega)$, then $[w,v] \in L^2(\Omega)$ and satisfies*

$$\|v\|_{W^{2,\infty}(\Omega)} \le c\|w\|^2_{H^2(\Omega)} \quad \text{and} \quad \|[w,v]\| \le c\|w\|_{H^2(\Omega)}\|v\|_{W^{2,\infty}(\Omega)}, \tag{2.6}$$

where $c > 0$.

As in [1], we consider the following hypotheses on the relaxation function $h(t)$:
(H1) $h(t) \ge 0$ for all $t \ge 0$ and

$$0 < l := \int_0^\infty h(s)\,ds < 1. \tag{2.7}$$

(H2) $h'(t) \le 0$ for almost all $t > 0$.
(H3) There exists a nondecreasing function $\gamma(t) > 0$ such that

$$\frac{\gamma'(t)}{\gamma(t)} := \eta(t) \quad \text{is a decreasing function and} \quad \int_0^\infty h(s)\gamma(s)\,ds < +\infty. \tag{2.8}$$

By using Galerkin's approximation, we get the following result for the solution (see [2]). For $(w_0, w_1) \in W \times V$, $k > 0$, and $T > 0$, system (1.1)–(1.7) has a unique weak solution. For (w_0, w_1) is 2-regular, the weak solution satisfies

$$w \in C\big([0,T]; W \cap H^4(\Omega)\big), \qquad w_t \in C\big([0,T]; V \cap H^3(\Omega)\big).$$

We define the energy of problem (1.1)–(1.7) by

$$E(t) = \frac{1}{2}\|w_t(t)\|^2 + \frac{k}{2}\|\nabla w_t(t)\|^2 + \frac{1}{2}a\big(w(t), w(t)\big) + \frac{1}{4}\|\Delta v\|^2. \tag{2.9}$$

3 Arbitrary decay of the energy

To obtain the stability of problem (1.1)–(1.7), we introduce the following notations as in [1, 30]. For every measurable set $\mathcal{M} \subset \mathbb{R}^+$, we denote the probability measure $\hat{h}$ by

$$\hat{h}(\mathcal{M}) = \frac{1}{l}\int_{\mathcal{M}} h(s)\,ds. \tag{3.1}$$

The flatness set of h is defined by

$$F_h = \{s \in \mathbb{R}^+ : h(s) > 0 \text{ and } h'(s) = 0\}. \tag{3.2}$$

Let $t_0 > 0$ be a number such that $\int_0^{t_0} h(s)\,ds := h_0 > 0$. We define the modified energy by

$$\mathcal{E}(t) = \frac{1}{2}\|w_t(t)\|^2 + \frac{k}{2}\|\nabla w_t(t)\|^2 + \frac{1}{4}\|\Delta v\|^2 + \frac{1}{2}\left(1 - \int_0^t h(s)\,ds\right)a\big(w(t), w(t)\big) + \frac{1}{2}\big(h\Box\partial^2 w\big)(t).$$

Multiplying (1.1) by $w_t(t)$ and using (2.4), we have

$$\mathcal{E}'(t) = -\frac{1}{2}h(t)a\big(w(t), w(t)\big) + \frac{1}{2}\big(h'\Box\partial^2 w\big)(t). \tag{3.3}$$

From (2.7) one sees that

$$E(t) \le \frac{1}{1-l}\mathcal{E}(t), \quad \forall t \ge 0. \tag{3.4}$$

First, we define the standard functionals

$$\Phi(t) = \int_\Omega w_t(t)w(t)\,d\Omega + k\int_\Omega \nabla w_t(t)\nabla w(t)\,d\Omega,$$

$$\Psi(t) = \int_\Omega \big(k\Delta w_t(t) - w_t(t)\big)\int_0^t h(t-s)\big(w(t)-w(s)\big)\,ds\,d\Omega,$$

and the new one

$$\Xi(t) = \int_0^t G_\gamma(t-s)a\big(w(s), w(s)\big)\,ds,$$

where

$$G_\gamma(t) = \gamma(t)^{-1}\int_t^\infty h(s)\gamma(s)\,ds.$$

Now let us define the perturbed modified energy by

$$\mathcal{F}(t) = M\mathcal{E}(t) + \xi_1\Phi(t) + \xi_2\Psi(t) + \xi_3\Xi(t), \tag{3.5}$$

where M and $\xi_i (i = 1, 2, 3)$ are positive constants to be specified later. Using the methods presented in [1, 4, 5], we get the following lemmas.

Lemma 3.1 *Assume that* (H1) *holds. Then, for* $M > 0$ *large, there exist* $\alpha_0 > 0$ *and* $\alpha_1 > 0$ *such that*

$$\alpha_0\big(\mathcal{E}(t) + \Xi(t)\big) \le \mathcal{F}(t) \le \alpha_1\big(\mathcal{E}(t) + \Xi(t)\big), \quad \forall t \ge 0. \tag{3.6}$$

Proof From Young's inequality, (2.3), and (2.7), we obtain

$$\big|\Phi(t)\big| \le \frac{1}{2}\big\|w_t(t)\big\|^2 + \frac{k}{2}\big\|\nabla w_t(t)\big\|^2 + \frac{C_p + C_s k}{2}a\big(w(t), w(t)\big) \le C_1\mathcal{E}(t) \tag{3.7}$$

and

$$\big|\Psi(t)\big| \le \frac{1}{2}\big\|w_t(t)\big\|^2 + \frac{k}{2}\big\|\nabla w_t(t)\big\|^2 + \frac{(C_p + C_s k)l}{2}\big(h\Box\partial^2 w\big)(t) \le C_2\mathcal{E}(t), \tag{3.8}$$

where $C_1 = \max\{1, \frac{C_p + C_s k}{1-l}\}$ and $C_2 = \max\{1, (C_p + C_s k)l\}$. By (3.7) and (3.8), we find that

$$\big|F(t) - M\mathcal{E}(t) - \xi_3\Xi(t)\big| \le C_3\mathcal{E}(t),$$

where $C_3 = \xi_1 C_1 + \xi_2 C_2$. Setting $\alpha_0 = \min\{M - C_3, \xi_3\}$, $\alpha_1 = \max\{M + C_3, \xi_3\}$ and taking $M > 0$ large, we complete the proof of Lemma 3.1. □

Lemma 3.2 *Assume that* (H1)–(H3) *hold. Then, for each* $t_0 > 0$ *and all measurable sets* $\mathcal{M}$ *and* $\mathcal{N}$ *with* $\mathcal{M} = \mathbb{R}^+ \setminus \mathcal{N}$, *it is satisfied that*

$$\begin{aligned}
\mathcal{F}'(t) \le{}& \{\xi_1 + \xi_2(\delta_2 - h_0)\}\|w_t(t)\|^2 + k\{\xi_1 + \xi_2(\delta_2 - g_0)\}\|\nabla w_t(t)\|^2 \\
&+ \left[\xi_2\left\{(1-h_0)\left(\delta_1 + \frac{3l\hat{h}(\mathcal{N})}{2}\right) + \delta_3 C_* E^2(0)\right\}\right. \\
&\left. - \xi_1\left(1 - \frac{l}{2}\right) + \xi_3 G_\gamma(0)\right] a\big(w(t), w(t)\big) \\
&+ \xi_2 l\left(\frac{1-h_0}{4\delta_1} + 1 + \frac{1}{\delta_1} + \frac{C_p}{2\delta_3}\right)\int_{\mathcal{M}_t} h(t-s) a\big(w(t)-w(s), w(t)-w(s)\big)\,ds \\
&+ \xi_2 l\hat{h}(\mathcal{N})\left(1 + \delta_1 + \frac{C_p}{2\delta_3}\right)\int_{\mathcal{N}_t} h(t-s) a\big(w(t)-w(s), w(t)-w(s)\big)\,ds \\
&- \frac{\xi_1}{2}\big(h\Box\partial^2 w\big)(t) + \frac{\xi_2(1-h_0)}{2}\int_{\mathcal{N}_t} h(t-s) a\big(w(s), w(s)\big)\,ds \\
&+ \left(\frac{M}{2} - \frac{\xi_2 h(0)(C_s h + C_p)}{4\delta_2}\right)\big(h'\Box\partial^2 w\big)(t) \\
&+ \left(\frac{\xi_1}{2} - \xi_3\right)\int_0^t h(t-s) a\big(w(s), w(s)\big)\,ds \\
&- \xi_3 \eta(t)\Xi(t) - \xi_1\|\Delta v\|^2, \quad \forall t \ge t_0,
\end{aligned} \tag{3.9}$$

where C_* *is a positive constant.*

Proof From (1.1)–(1.6), (2.1), (2.5), and (2.7), we have

$$\begin{aligned}
\Phi'(t) ={}& \|w_t(t)\|^2 + k\|\nabla w_t(t)\|^2 - a\big(w(t), w(t)\big) + \frac{1}{2}\left(\int_0^t h(s)\,ds\right) a\big(w(t), w(t)\big) \\
&+ \frac{1}{2}\int_0^t h(t-s) a\big(w(s), w(s)\big)\,ds - \frac{1}{2}\big(h\Box\partial^2 w\big)(t) - \|\Delta v\|^2 \\
\le{}& \|w_t(t)\|^2 + k\|\nabla w_t(t)\|^2 - \left(1 - \frac{l}{2}\right) a\big(w(t), w(t)\big) \\
&+ \frac{1}{2}\int_0^t h(t-s) a\big(w(s), w(s)\big)\,ds - \frac{1}{2}\big(h\Box\partial^2 w\big)(t) - \|\Delta v\|^2.
\end{aligned} \tag{3.10}$$

Similarly, we conclude that

$$\begin{aligned}
\Psi'(t) ={}& \left(1 - \int_0^t h(s)\,ds\right)\int_0^t h(t-s) a\big(w(t)-w(s), w(t)\big)\,ds \\
&+ \int_0^t h(t-s) a\left(w(t)-w(s), \int_0^t h(t-\tau)\big(w(t)-w(\tau)\big)\,d\tau\right) ds \\
&- k\int_0^t h'(t-s)\big(\nabla w(t) - \nabla w(s), \nabla w_t(t)\big)\,ds
\end{aligned}$$

$$\begin{aligned}
&-\int_0^t h'(t-s)\big(w(t)-w(s),w_t(t)\big)\,ds\\
&-\int_0^t h(t-s)\big(w(t)-w(s),[w,v]\big)\,ds-\left(\int_0^t h(s)\,ds\right)\|w_t(t)\|^2\\
&-k\left(\int_0^t h(s)\,ds\right)\|\nabla w_t(t)\|^2\\
:=&\left(1-\int_0^t h(s)\,ds\right)I_1+I_2+\cdots+I_5\\
&-\left(\int_0^t h(s)\,ds\right)\|w_t(t)\|^2-k\left(\int_0^t h(s)\,ds\right)\|\nabla w_t(t)\|^2. \qquad (3.11)
\end{aligned}$$

For all measurable sets $\mathcal{M}$ and $\mathcal{N}$ such that $\mathcal{M}=\mathbb{R}^+\setminus\mathcal{N}$, using Young's inequality, (2.7), and (3.1), we obtain that for $\delta_1>0$,

$$\begin{aligned}
I_1 =& \int_{\mathcal{M}_t} h(t-s)a\big(w(t)-w(s),w(t)\big)\,ds+\left(\int_{\mathcal{N}_t} h(s)\,ds\right)a\big(w(t),w(t)\big)\\
&-\int_{\mathcal{N}_t} h(t-s)a\big(w(s),w(t)\big)\,ds\\
\le&\left(\delta_1+\frac{3l\hat{h}(\mathcal{N})}{2}\right)a\big(w(t),w(t)\big)+\frac{l}{4\delta_1}\int_{\mathcal{M}_t} h(t-s)a\big(w(t)-w(s),w(t)-w(s)\big)\,ds\\
&+\frac{1}{2}\int_{\mathcal{N}_t} h(t-s)a\big(w(s),w(s)\big)\,ds, \qquad (3.12)
\end{aligned}$$

where $\mathcal{M}_t:=\mathcal{M}\cap[0,t]$ and $\mathcal{N}_t:=\mathcal{N}\cap[0,t]$. Similarly, we have that for $\delta_1>0$,

$$\begin{aligned}
I_2 =& a\left(\int_0^t h(t-s)\big(w(t)-w(s)\big)\,ds,\int_0^t h(t-s)\big(w(t)-w(s)\big)\,ds\right)\\
=& a\left(\int_{\mathcal{M}_t} h(t-s)\big(w(t)-w(s)\big)\,ds,\int_{\mathcal{M}_t} h(t-s)\big(w(t)-w(s)\big)\,ds\right)\\
&+2a\left(\int_{\mathcal{M}_t} h(t-s)\big(w(t)-w(s)\big)\,ds,\int_{\mathcal{N}_t} h(t-s)\big(w(t)-w(s)\big)\,ds\right)\\
&+a\left(\int_{\mathcal{N}_t} h(t-s)\big(w(t)-w(s)\big)\,ds,\int_{\mathcal{N}_t} h(t-s)\big(w(t)-w(s)\big)\,ds\right)\\
\le&\left(1+\frac{1}{\delta_1}\right)l\int_{\mathcal{M}_t} h(t-s)a\big(w(t)-w(s),w(t)-w(s)\big)\,ds\\
&+(1+\delta_1)l\hat{h}(\mathcal{N})\int_{\mathcal{N}_t} h(t-s)a\big(w(t)-w(s),w(t)-w(s)\big)\,ds. \qquad (3.13)
\end{aligned}$$

Applying Young's inequality and (2.3), we get that for $\delta_2>0$,

$$\begin{aligned}
|I_3|\le& k\delta_2\|\nabla w_t(t)\|^2+\frac{k}{4\delta_2}\int_\Omega\left(\int_0^t h'(t-s)\big|\nabla w(t)-\nabla w(s)\big|\,ds\right)^2 d\Omega\\
\le& k\delta_2\|\nabla w_t(t)\|^2-\frac{h(0)C_s k}{4\delta_2}\big(h'\Box\partial^2 w\big)(t), \qquad (3.14)\\
|I_4|\le&\delta_2\|w_t(t)\|^2-\frac{h(0)C_p}{4\delta_2}\big(h'\Box\partial^2 w\big)(t). \qquad (3.15)
\end{aligned}$$

By Young's inequality, we find that for $\delta_3 > 0$,

$$|I_5| \le \delta_3 \big\| [w,v] \big\|^2 + \frac{1}{4\delta_3} \left\| \int_0^t h(t-s)\big(w(t)-w(s)\big)\,ds \right\|^2. \tag{3.16}$$

Using (2.2), (2.6), (2.9), (3.4) and the fact $\mathcal{E}(t) \le \mathcal{E}(0) = E(0)$, we see that

$$\begin{aligned}\big\| [w,v] \big\|^2 &\le c^4 \big\| w(t) \big\|^2_{H^2(\Omega)} \big\| w(t) \big\|^4_{H^2(\Omega)} \le \frac{c^4}{c_0} a\big(w(t),w(t)\big) \left(\frac{2}{c_0} E(t) \right)^2 \\ &\le \frac{c^4}{c_0} a\big(w(t),w(t)\big) \left(\frac{2}{c_0(1-l)} \mathcal{E}(t) \right)^2 \le C_* E^2(0) a\big(w(t),w(t)\big),\end{aligned}$$

where $C_* = \frac{4c^4}{c_0^3(1-l)^2}$. From Young's inequality, (2.3), and (3.1), we obtain

$$\begin{aligned}&\left\| \int_0^t h(t-s)\big(w(t)-w(s)\big)\,ds \right\|^2 \\ &\quad= \left\| \int_{\mathcal{M}_t} h(t-s)\big(w(t)-w(s)\big)\,ds + \int_{\mathcal{N}_t} h(t-s)\big(w(t)-w(s)\big)\,ds \right\|^2 \\ &\quad\le 2l \int_{\mathcal{M}_t} h(t-s) \big\| w(t)-w(s) \big\|^2\,ds + 2l\hat{h}(\mathcal{N}) \int_{\mathcal{N}_t} h(t-s) \big\| w(t)-w(s) \big\|^2\,ds \\ &\quad\le 2lC_p \int_{\mathcal{M}_t} h(t-s) a\big(w(t)-w(s),w(t)-w(s)\big)\,ds \\ &\qquad+ 2l\hat{h}(\mathcal{N})C_p \int_{\mathcal{N}_t} h(t-s) a\big(w(t)-w(s),w(t)-w(s)\big)\,ds.\end{aligned}$$

Inserting these estimates into (3.16), we have

$$\begin{aligned}|I_5| &\le \delta_3 C_* E^2(0) a\big(w(t),w(t)\big) + \frac{lC_p}{2\delta_3} \int_{\mathcal{M}_t} h(t-s) a\big(w(t)-w(s),w(t)-w(s)\big)\,ds \\ &\quad+ \frac{l\hat{h}(\mathcal{N})C_p}{2\delta_3} \int_{\mathcal{N}_t} h(t-s) a\big(w(t)-w(s),w(t)-w(s)\big)\,ds.\end{aligned} \tag{3.17}$$

Substituting (3.12)–(3.15) and (3.17) into (3.11), we arrive at

$$\begin{aligned}\Psi'(t) &\le k\left(\delta_2 - \int_0^t h(s)\,ds \right) \big\| \nabla w_t(t) \big\|^2 + \left(\delta_2 - \int_0^t h(s)\,ds \right) \big\| w_t(t) \big\|^2 \\ &\quad+ \left\{ \left(1 - \int_0^t h(s)\,ds \right) \left(\delta_1 + \frac{3l\hat{h}(\mathcal{N})}{2} \right) + \delta_3 C_* E^2(0) \right\} a\big(w(t),w(t)\big) \\ &\quad+ l\left\{ \left(1 - \int_0^t h(s)\,ds \right) \frac{1}{4\delta_1} + 1 + \frac{1}{\delta_1} + \frac{C_p}{2\delta_3} \right\} \\ &\quad\times \int_{\mathcal{M}_t} h(t-s) a\big(w(t)-w(s),w(t)-w(s)\big)\,ds \\ &\quad+ l\hat{h}(\mathcal{N}) \left(1 + \delta_1 + \frac{C_p}{2\delta_3} \right) \int_{\mathcal{N}_t} h(t-s) a\big(w(t)-w(s),w(t)-w(s)\big)\,ds\end{aligned}$$

$$+\frac{1}{2}\left(1-\int_0^t h(s)\,ds\right)\int_{\mathcal{N}_t} h(t-s)a\bigl(w(s),w(s)\bigr)\,ds$$
$$-\frac{h(0)(C_sk+C_p)}{4\delta_2}\bigl(h'\Box\partial^2 w\bigr)(t). \tag{3.18}$$

A differentiation of $\Xi(t)$ yields

$$\begin{aligned}\Xi'(t) &= G_\gamma(0)a\bigl(w(t),w(t)\bigr)-\int_0^t \frac{\gamma'(t-s)}{\gamma(t-s)}G_\gamma(t-s)a\bigl(w(s),w(s)\bigr)\,ds\\ &\quad-\int_0^t h(t-s)a\bigl(w(s),w(s)\bigr)\,ds\\ &\le G_\gamma(0)a\bigl(w(t),w(t)\bigr)-\eta(t)\Xi(t)-\int_0^t h(t-s)a\bigl(w(s),w(s)\bigr)\,ds, \end{aligned}\tag{3.19}$$

where we have used the fact that $\frac{\gamma'(t)}{\gamma(t)}=\eta(t)$ is a nonincreasing function. Since h is positive, we get $\int_0^t h(s)\,ds \ge h_0$ for all $t\ge t_0$, and combining (3.3), (3.5), (3.10), (3.18), and (3.19), we obtain the desired estimate (3.9). □

Now, we are ready to prove the following arbitrary decay result.

Theorem 3.1 *Assume that* (H1)–(H3), $E(0)<\frac{l}{\sqrt{C_*C_p}}$, *and* $\hat{h}(F_h)<\frac{1}{8}$ *hold. If* $h_0>\frac{3l}{8-l}$ *and* $G_\gamma(0)<\frac{(8-l)h_0-3l}{16}$, *then there exist positive constants* t_0, ω, *and* C *such that*

$$E(t)\le \frac{C}{\gamma(t)^\omega}\quad \textit{for } t\ge t_0.$$

Proof As in [1, 30], we introduce the sets

$$\mathcal{M}_n=\bigl\{s\in\mathbb{R}^+ : nh'(s)+h(s)\le 0\bigr\}\quad\text{and}\quad \mathcal{N}_n=\mathbb{R}^+\setminus\mathcal{M}_n,\quad n\in\mathbb{N}.$$

Observe that

$$\bigcup_{n=1}^{\infty}\mathcal{M}_n=\mathbb{R}^+\setminus\{F_h\cup N_h\},$$

where N_h is the null set where h' is not defined and F_h is given in (3.2). Because $\mathcal{N}_{n+1}\subset\mathcal{N}_n$ for all n and $\cap_{n=1}^{\infty}\mathcal{N}_n=F_h\cup N_h$, we have

$$\lim_{n\to\infty}\hat{h}(\mathcal{N}_n)=\hat{h}(F_h). \tag{3.20}$$

Choosing $\mathcal{M}=\mathcal{M}_n$, $\mathcal{N}=\mathcal{N}_n$, and $\delta_3=\frac{(2-l)\xi_1}{4C_*E^2(0)\xi_2}$ in (3.9), we find that

$$\begin{aligned}\mathcal{F}'(t)&\le\bigl\{\xi_1+\xi_2(\delta_2-h_0)\bigr\}\bigl\|w_t(t)\bigr\|^2+k\bigl\{\xi_1+\xi_2(\delta_2-h_0)\bigr\}\bigl\|\nabla w_t(t)\bigr\|^2\\ &\quad+\left\{\xi_2(1-h_0)\left(\delta_1+\frac{3l\hat{h}(\mathcal{N}_n)}{2}\right)-\frac{\xi_1}{2}\left(1-\frac{l}{2}\right)+\xi_3G_\gamma(0)\right\}a\bigl(w(t),w(t)\bigr)\\ &\quad+\left\{\xi_2 l\left(\frac{1-h_0}{4\delta_1}+1+\frac{1}{\delta_1}+\frac{2C_*C_pE^2(0)\xi_2}{(2-l)\xi_1}\right)-\frac{1}{n}\left(\frac{M}{2}-\frac{\xi_2h(0)(C_sk+C_p)}{4\delta_2}\right)\right\}\end{aligned}$$

$$\begin{aligned}
&\times \int_{\mathcal{M}_{nt}} h(t-s)a\bigl(w(t)-w(s),w(t)-w(s)\bigr)\,ds \\
&+\left\{\xi_2 l\hat{h}(\mathcal{N}_n)\left(1+\delta_1+\frac{2C_*C_pE^2(0)\xi_2}{(2-l)\xi_1}\right)-\frac{\xi_1}{2}\right\}\bigl(h\Box\partial^2 w\bigr)(t)-\xi_3\eta(t)\Xi(t) \\
&+\left\{\frac{\xi_2(1-h_0)}{2}+\frac{\xi_1}{2}-\xi_3\right\}\int_0^t h(t-s)a\bigl(w(s),w(s)\bigr)\,ds \\
&-\xi_1\|\Delta v\|^2, \quad \forall t\ge t_0,
\end{aligned} \tag{3.21}$$

where $\mathcal{M}_{nt}=\mathcal{M}_n\cap[0,t]$. For small $0<\varepsilon<h_0$, by taking $\xi_1=(h_0-\varepsilon)\xi_2$, (3.21) yields

$$\begin{aligned}
\mathcal{F}'(t) \le{}& \xi_2(\delta_2-\varepsilon)\bigl\|w_t(t)\bigr\|^2+k\xi_2(\delta_2-\varepsilon)\bigl\|\nabla w_t(t)\bigr\|^2 \\
&+\left\{\xi_2(1-h_0)\left(\delta_1+\frac{3l\hat{h}(\mathcal{N}_n)}{2}\right)-(h_0-\varepsilon)\xi_2\bigl(\beta+(1-\beta)\bigr)\left(\frac{2-l}{4}\right)\right. \\
&\left.+\xi_3G_\gamma(0)\right\}a\bigl(w(t),w(t)\bigr) \\
&+\left\{\xi_2 l\left(\frac{1-h_0}{4\delta_1}+1+\frac{1}{\delta_1}+\frac{2C_*C_pE^2(0)}{(2-l)(h_0-\varepsilon)}\right)-\frac{1}{n}\left(\frac{M}{2}-\frac{\xi_2 h(0)(C_sk+C_p)}{4\delta_2}\right)\right\} \\
&\times \int_{\mathcal{M}_{nt}} h(t-s)a\bigl(w(t)-w(s),w(t)-w(s)\bigr)\,ds \\
&+\xi_2\left\{l\hat{h}(\mathcal{B}_n)\left(1+\delta_1+\frac{2C_*C_pE^2(0)}{(2-l)(h_0-\varepsilon)}\right)-\frac{h_0-\varepsilon}{2}\right\}\bigl(h\Box\partial^2 w\bigr)(t)-\xi_3\eta(t)\Xi(t) \\
&+\left\{\frac{\xi_2(1-\varepsilon)}{2}-\xi_3\right\}\int_0^t h(t-s)a\bigl(w(s),w(s)\bigr)\,ds \\
&-(h_0-\varepsilon)\xi_2\|\Delta v\|^2, \quad \forall t\ge t_0,
\end{aligned} \tag{3.22}$$

where $\beta=\frac{3l(1-h_0)}{4(2-l)h_0}$. From (3.20) and $\hat{h}(F_h)<\frac{1}{8}$, there exists $n_0\in N$ large such that

$$\hat{h}(\mathcal{N}_n)<\frac{1}{8} \tag{3.23}$$

for $n\ge n_0$. By (3.23), we get that for $n\ge n_0$,

$$(1-h_0)\left(\frac{3l\hat{h}(\mathcal{N}_n)}{2}\right)<\beta h_0\left(\frac{2-l}{4}\right).$$

Then we can take a constant $\varepsilon_1>0$ such that

$$(1-h_0)\left(\frac{3l\hat{h}(\mathcal{N}_n)}{2}\right)<\beta(h_0-\varepsilon)\left(\frac{2-l}{4}\right)\quad \text{for } n\ge n_0 \text{ and } 0<\varepsilon\le\varepsilon_1. \tag{3.24}$$

Because $l=\int_0^\infty h(s)\,ds$ and $E(0)<\frac{l}{\sqrt{C_*C_p}}$, there exists $t_1>0$ large such that

$$\frac{l}{2}<h_0 \quad \text{and} \quad \sqrt{C_*C_p}E(0)<h_0<l \quad \text{for } t_0\ge t_1,$$

and then there exists a positive constant $\varepsilon_2 > 0$ with $\varepsilon_2 \leq \varepsilon_1$ small such that

$$\frac{l}{2} < h_0 - \varepsilon \quad \text{and} \quad \sqrt{C_* C_p} E(0) < h_0 - \varepsilon < l \quad \text{for } t_0 \geq t_1 \text{ and } 0 < \varepsilon \leq \varepsilon_2. \tag{3.25}$$

By (3.23) and (3.25), we have that for $t_0 \geq t_1$, $n \geq n_0$, and $0 < \varepsilon \leq \varepsilon_2$,

$$\begin{aligned} l\hat{h}(\mathcal{N}_n)\left(1 + \frac{2C_* C_p E^2(0)}{(2-l)(h_0 - \varepsilon)}\right) - \frac{h_0 - \varepsilon}{2} &< l\hat{h}(\mathcal{N}_n) + \hat{h}(\mathcal{N}_n)\frac{2C_* C_p E^2(0)}{h_0 - \varepsilon} - \frac{h_0 - \varepsilon}{2} \\ &< \frac{l}{8} - \frac{h_0 - \varepsilon}{4} < 0. \end{aligned} \tag{3.26}$$

Then, from (3.24) and (3.26), we can choose $\delta_1 > 0$ small enough such that for $t_0 \geq t_1$, $n \geq n_0$, and $0 < \varepsilon \leq \varepsilon_2$,

$$(1 - h_0)\left(\delta_1 + \frac{3l\hat{h}(\mathcal{N}_n)}{2}\right) - \beta(h_0 - \varepsilon)\left(\frac{2-l}{4}\right) < 0, \tag{3.27}$$

$$l\hat{h}(\mathcal{N}_n)\left(1 + \delta_1 + \frac{2C_* C_p E^2(0)}{(2-l)(h_0 - \varepsilon)}\right) - \frac{h_0 - \varepsilon}{2} < 0. \tag{3.28}$$

From the fact $\frac{3l}{8-l} < h_0 < l$, we see that $1 - \beta = \frac{(8-l)h_0 - 3l}{4(2-l)h_0} > 0$. Once n_0, ε_2, and t_1 are fixed, we choose $n = n_0$, $\varepsilon = \varepsilon_2$, and $t_0 = t_1$. Next we take ξ_2 and ξ_3 satisfying

$$\frac{\xi_2}{2} < \xi_3 < \frac{(8-l)h_0 - 3l}{32 G_\gamma(0)} \xi_2. \tag{3.29}$$

This is possible if $G_\gamma(0) < \frac{(8-l)h_0 - 3l}{16}$. Using (3.25) and (3.29), we obtain

$$\frac{\xi_2(1 - \varepsilon)}{2} - \xi_3 < 0 \tag{3.30}$$

and

$$\xi_3 G_\gamma(0) - \xi_2(1 - \beta)(h_0 - \varepsilon)\left(\frac{2-l}{4}\right) < \frac{(8-l)h_0 - 3l}{16}\left(\frac{1}{2} - \frac{h_0 - \varepsilon}{h_0}\right)\xi_2 < 0. \tag{3.31}$$

Finally, we select $\delta_2 > 0$ small enough and $M > 0$ large enough so that

$$\delta_2 - \varepsilon < 0 \tag{3.32}$$

and

$$\xi_2 l\left(\frac{1 - h_0}{4\delta_1} + 1 + \frac{1}{\delta_1} + \frac{2C_* C_p E^2(0)}{(2-l)(h_0 - \varepsilon)}\right) - \frac{1}{n}\left(\frac{M}{2} - \frac{\xi_2 h(0)(C_s k + C_p)}{4\delta_2}\right) < 0, \tag{3.33}$$

respectively. Combining (3.22), (3.27), (3.28), (3.30)–(3.32), and (3.33), we deduce that

$$\mathcal{F}'(t) \leq -C_4 \mathcal{E}(t) - \xi_3 \eta(t) \Xi(t), \quad t \geq t_0,$$

for some positive constant C_4. Using the fact that $\eta(t)$ is decreasing and Lemma 3.1, we find that

$$\begin{aligned} \mathcal{F}'(t) &\le -C_4 \frac{\eta(t)}{\eta(t_0)} \mathcal{E}(t) - \xi_3 \eta(t)\Xi(t) \le -C_5 \eta(t)\big(\mathcal{E}(t) + \Xi(t)\big) \\ &\le -\omega\eta(t)\mathcal{F}(t), \quad t \ge t_0, \end{aligned} \tag{3.34}$$

where $C_5 = \min\{\frac{C_4}{\eta(t_0)}, \xi_3\}$ and $\omega = \frac{C_5}{\alpha_1}$. From (2.8), (3.6), and (3.34), we conclude that

$$\begin{aligned} \alpha_0\big(\mathcal{E}(t) + \Xi(t)\big) &\le \mathcal{F}(t) \le \mathcal{F}(t_0) e^{-\omega \int_{t_0}^t \eta(s)\,ds} = \mathcal{F}(t_0) e^{-\omega \int_{t_0}^t \frac{\gamma'(s)}{\gamma(s)}\,ds} \\ &= \mathcal{F}(t_0)\gamma(t_0)^{\omega}\gamma(t)^{-\omega}, \quad t \ge t_0. \end{aligned}$$

By the fact $\Xi(t) \ge 0$ and (3.4), we infer that

$$E(t) \le \frac{C}{\gamma(t)^{\omega}}, \quad t \ge t_0,$$

where $C = \frac{\mathcal{F}(t_0)\gamma(t_0)^{\omega}}{\alpha_0(1-l)}$. □

Remark We give some examples to illustrate the decay of energy given by Theorem 3.1 (see [1, 11]).

(1) $\gamma(t) = e^{\alpha t}$, $\alpha > 0$, gives $\eta(t) = \alpha$ and $E(t) \le \frac{C}{e^{\omega\alpha t}}$ for some positive constants C and ω.

(2) $\gamma(t) = (1 + t)^{\alpha}$, $\alpha > 0$, leads to $\eta(t) = \alpha(1 + t)^{-1}$ and $E(t) \le \frac{C}{(1+t)^{\omega}}$ for some positive constants C and ω.

4 Conclusions

In this paper, we study the von Karman plate model with long range memory. Our result is obtained without imposing the usual relation between the relaxation function h and its derivative. Assume that (H1)–(H3), $E(0) < \frac{l}{\sqrt{C_* C_p}}$, and $\hat{h}(F_h) < \frac{1}{8}$ hold. If $h_0 > \frac{3l}{8-l}$ and $G_{\gamma}(0) < \frac{(8-l)h_0 - 3l}{16}$, then there exist positive constants t_0, ω, and C such that

$$E(t) \le \frac{C}{\gamma(t)^{\omega}} \quad \text{for } t \ge t_0.$$

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The author declares that they have no competing interests.

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The authors declare that the study was realized in collaboration with the same responsibility. All authors read and approved the final manuscript.

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