Hammad and Zayed *Boundary Value Problems* (2022) 2022:101
https://doi.org/10.1186/s13661-022-01684-0

Boundary Value Problems
a SpringerOpen Journal

RESEARCH

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Solving systems of coupled nonlinear Atangana–Baleanu-type fractional differential equations

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Abstract
In this work, we investigate two types of boundary value problems for a system of coupled Atangana–Baleanu-type fractional differential equations with nonlocal boundary conditions. The fractional derivatives are applied to serve as a nonlocal and nonsingular kernel. The existence and uniqueness of solutions for proposed problems using Krasnoselskii's and Banach's fixed-point approaches are established. Moreover, nonlinear analysis is used to build the Ulam–Hyers stability theory. Subsequently, we discuss two compelling examples to demonstrate the utility of our study.

MSC: 34A08; 34A12; 47H10

Keywords: Atangana–Baleanu-type; Fractional differential equation; Fixed point technique; Boundary value problem; Ulam–Hyers stability

1 Introduction

In many seemingly diverse areas of science and industry, fractional calculus and its possible applications have emerged as an effective modeling tool for many complicated phenomena [1–8]. Some academics have acknowledged the requirement of developing the idea of fractional calculus by searching for novel fractional derivatives (FDs) with various singular or nonsingular kernels to simulate a variety of actual situations in several disciplines of science and engineering. With their contributions to physical phenomena and their success in solving issues in the real world, new fractional operators have emerged from this perspective as the most effective instrument for many professionals and researchers. Before 2015, there were only singular kernels for all fractional derivatives. As a result, simulating physical phenomena with these singularities is challenging.

Caputo and Fabrizio [9] investigated a novel FD in the exponential kernel in 2015. Losada and Nieto covered some of this novel type's characteristics in [10]. Atangana and Baleanu (AB) have looked into a novel and intriguing FD with Mittag-Leffler kernels in [11]. The kind examined by AB was extended by Abdeljawad in [12], which formed their related integral operators for orders ranging from $(0,1)$ to higher arbitrary orders. Additionally, he studied the existence and uniqueness (EaU) theorems for the Caputo-type (ABC) FD and Riemann-type (ABR), as well as some initial value problems of higher arbitrary order.

Also, he established a Lyapunov-type inequality for the ABR fractional boundary value problems of order $\rho \in (2,3]$ in the frame of Mittag-Leffler kernels. In [13, 14], Abdeljawad and Baleanu discussed the discrete forms of those novel operators. We recommend the reader to check out a number of papers [15–17] for some theoretical works on AB fractional differential equations (FDEs).

Recently, four fractional integrodifferential equations (FIDEs) were solved by Baleanu *et al.* [18], while the numerical solutions were presented in the works of [19–22] under suitable conditions. The dimension of the set of solutions for the second FIDE is also shown to be infinite-dimensional under a few different conditions. Aydogan *et al.* [23] used the Caputo–Fabrizio fractional derivative [24] to create two new high-order derivatives known as the CFD and DCF. They also looked into the possibility of solutions for two of these high-order FIDEs.

Guoa *et al.* [25, 26] investigated the Hyers–Ulam (HU) stability of FDEs with impulse and fractional Brownian motion under nonlocal circumstances using the semigroups of operators and the Mönch fixed-point (FP) methods. Numerous researchers examined the UH stability of solutions for particular FDEs of problems with initial or boundary stipulations in the publication series [27–32].

Recently, the ABC-type pantograph FDEs with nonlocal conditions were investigated by Abdo *et al.* [33] as follows:

$$\begin{cases} {}^{ABC}D^{\rho}_{c^+}\theta(\ell) = \hbar(\ell,\theta(\ell),\theta(\gamma\ell)), & \ell \in [c,Q], \rho \in (0,1], \\ \theta(c) = \sum_{j=1}^{n} a_j\theta(y_j), & y_j \in (c,Q]. \end{cases}$$

Abdo *et al.* [34] investigated the ABC-type implicit FDEs as follows:

$${}^{ABC}D^{\rho}_{c^+}\theta(\ell) = \hbar\bigl(\ell,\theta(\ell),{}^{ABC}D^{\rho}_{c^+}\theta(\ell)\bigr), \quad \ell \in [c,Q],$$

with nonlinear integral conditions

$$\theta(c) - \theta'(c) = \int_c^Q k\bigl(\varpi,\theta(\varpi)\bigr)\,d\varpi, \quad \text{when } \rho \in (0,1],$$

$$\theta(c) = 0, \qquad \theta(Q) = \int_c^Q k\bigl(\varpi,\theta(\varpi)\bigr)\,d\varpi, \quad \text{when } \rho \in (1,2].$$

Motivated by the previously mentioned results, in this paper, we plan to investigate and create the required circumstances for the EaU and UH stability results for new systems of coupled ABR-type and ABC-type FDEs defined as follows:

$$\begin{cases} {}^{ABR}D^{\rho}_{c^+}\theta(\ell) = \hbar(\ell,\theta(\ell),\varrho(\ell)), & \ell \in [c,d], \rho \in (2,3], \\ {}^{ABR}D^{\rho}_{c^+}\varrho(\ell) = \hbar(\ell,\varrho(\ell),\theta(\ell)), & \ell \in [c,d], \rho \in (2,3], \\ \theta(c) = 0, \quad \theta(d) = {}^{AB}I^{\varsigma}_{c^+}\theta(\xi), & \xi \in (c,d), \\ \varrho(c) = 0, \quad \varrho(d) = {}^{AB}I^{\varsigma}_{c^+}\varrho(\xi), & \xi \in (c,d), \end{cases} \tag{1.1}$$

$$\begin{cases} {}^{ABC}D^{\upsilon}_{c^+}\theta(\ell) = \hbar(\ell,\theta(\ell),\varrho(\ell)), & \ell \in [c,d], \upsilon \in (1,2], \\ {}^{ABC}D^{\upsilon}_{c^+}\varrho(\ell) = \hbar(\ell,\varrho(\ell),\theta(\ell)), & \ell \in [c,d], \upsilon \in (1,2], \\ \theta(c) = 0, \qquad \theta(d) = {}^{AB}I^{\varsigma}_{c^+}\theta(\xi), & \xi \in (c,d), \\ \varrho(c) = 0, \qquad \varrho(d) = {}^{AB}I^{\varsigma}_{c^+}\varrho(\xi), & \xi \in (c,d), \end{cases} \tag{1.2}$$

where ${}^{ABR}D^{\rho}_{c^+}\theta(\ell)$ and ${}^{ABC}D^{\upsilon}_{c^+}\theta(\ell)$ represent the ABR and ABC FD of order $\rho \in (2,3]$ and $\upsilon \in (1,2]$, respectively, $0 < \varsigma \leq 1$ and $\hbar : [c,d] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function. To the extent that we are aware, our investigation is the first attempt to deal with high-order FDs of ABC-type and ABR-type, particularly when applied to AB integral conditions. Additionally, we looked at EaU without relying on the semigroup property. Therefore, our results will be a valuable addition to the structure of research already done on these interesting operators.

2 Basic concepts

Here, we assume that $C(D,\mathbb{R})$ is the space of all continuous functions $\theta : J \to \mathbb{R}$ endowed with $\|\theta\| = \max\{|\theta(\ell)| : \ell \in D\}$, where $D = [a,b]$. Clearly, $(C(D,\mathbb{R}), \|\cdot\|)$ is a Banach space.

Definition 2.1 ([11]) Assume that $\rho \in (0,1]$. For a function θ the left-sided ABC and ABR FD of order ρ are given by

$$\left({}^{ABC}D^{\rho}_{c^+}\theta\right)(\ell) = \frac{\phi(\rho)}{1-\rho}\int_c^{\ell} \nabla_{\rho}\left(\frac{\rho}{\rho-1}(\ell-\varpi)^{\rho}\right)\theta'(\varpi)\,d\varpi, \quad \ell > c,$$

$$\left({}^{ABR}D^{\rho}_{c^+}\theta\right)(\ell) = \frac{\phi(\rho)}{1-\rho}\frac{d}{d\ell}\int_c^{\ell} \nabla_{\rho}\left(\frac{\rho}{\rho-1}(\ell-\varpi)^{\rho}\right)\theta'(\varpi)\,d\varpi, \quad \ell > c,$$

respectively, where $\phi(\rho)$ refers to the normalization function so that $\phi(0) = 1 = \phi(1)$ and ∇_{ρ} represents the Mittag-Leffler function described as:

$$\nabla_{\rho}(\theta) = \sum_{j=0}^{\infty} \frac{\theta^j}{\Gamma(1+j\rho)}, \quad Re(\rho) > 0, \theta \in C.$$

Moreover, the equivalent fractional integral of AB is given by

$${}^{AB}I^{\rho}_{c^+}\theta(\ell) = \frac{(1-\rho)}{\phi(\rho)}\theta(\ell) + \frac{\rho}{\phi(\rho)\Gamma(\rho)}\int_c^{\ell} (\ell-\varpi)^{\rho-1}\hbar(\varpi)\,d\varpi. \tag{2.1}$$

Lemma 2.2 ([13]) *Let* $\rho \in (0,1]$. *If the ABC FD exists, then we get*

$${}^{AB}I^{\rho}_{c^+}\,{}^{ABC}D^{\rho}_{c^+}\theta(\ell) = \theta(\ell) - \theta(c).$$

Definition 2.3 ([12]) The relation between the ABR and ABC FD is described as

$${}^{ABC}D^{\rho}_{c^+}\theta(\ell) = {}^{ABR}D^{\rho}_{c^+}\theta(\ell) - \frac{\phi(\rho)}{1-\rho}\theta(c)\nabla_{\rho}\left(\frac{\rho}{1-\rho}(\ell-c)^{\rho}\right). \tag{2.2}$$

Remark 2.4 If we take ${}^{AB}I^{\rho}_{c^+}$ on both sides of the equation (2.2) and use Lemma 2.2, we have

$${}^{AB}I^{\rho}_{c^+}\,{}^{ABR}D^{\rho}_{c^+}\theta(\ell) = \theta(\ell).$$

Lemma 2.5 ([11]) *For $\theta > 0$, ${}^{AB}I^{\rho}_{c^+} : C(D,\mathbb{R}) \to C(D,\mathbb{R})$ is bounded.*

Definition 2.6 ([12]) Let $\rho \in (k, k+1]$ and θ be such that $\theta^{(k)} \in H^1(c,d)$. Put $\eta = \rho - k$. Then $\eta \in (0,1]$, and we define

$$\left({}^{ABC}D^{\rho}_{c^+}\theta\right)(\ell) = \left({}^{ABC}D^{\eta}_{c^+}\theta^{(k)}\right)(\ell) \quad \text{and} \quad \left({}^{ABR}D^{\rho}_{c^+}\theta\right)(\ell) = \left({}^{ABR}D^{\eta}_{c^+}\theta^{(k)}\right)(\ell).$$

Further, the AB fractional integral is described as

$$\left({}^{AB}I^{\rho}_{c^+}\theta\right)(\ell) = \left(I^{k}_{c^+}{}^{\eta}I^{\rho}_{c^+}\eta\right)(\ell).$$

Lemma 2.7 ([12]) *For $\theta(\ell)$ defined on D and $\rho \in (n, n+1]$, for some $n \in N_0$, the assertions below are true*:

(1) $({}^{ABC}D^{\rho}_{c^+}{}^{AB}I^{\rho}_{c^+}\theta)(\ell) = \theta(\ell)$;
(2) $({}^{AB}I^{\rho}_{c^+}{}^{ABR}D^{\rho}_{c^+}\theta)(\ell) = \theta(\ell) - \sum_{j=0}^{n-1}\frac{\theta^{(j)}(c)}{j!}(\ell - c)^j$;
(3) $({}^{AB}I^{\rho}_{c^+}{}^{ABC}D^{\rho}_{c^+}\theta)(\ell) = \theta(\ell) - \sum_{j=0}^{n-1}\frac{\theta^{(j)}(c)}{j!}(\ell - c)^j$.

Lemma 2.8 *Consider $\rho \in (n, n+1]$. Then for $k = 1, 2, \dots, n-1$, ${}^{ABR}D^{\rho}_{c^+}(\ell - c)^k = 0$.*

Proof Based on Definition 2.6 after putting $\theta(\ell) = (\ell - c)^k$, we have

$${}^{ABC}D^{\rho}_{c^+}\theta(\ell) = \left({}^{ABC}D^{\eta}_{c^+}\theta^{(k)}\right)(\ell) = {}^{ABC}D^{\eta}_{c^+}\left[(\ell - c)^k\right]^{(n)} = {}^{ABC}D^{\eta}_{c^+}\left[\frac{d}{d\ell}\right]^{(n)}(\ell - c)^k.$$

As $k < n \in \mathbb{N}$, we get $[\frac{d}{d\ell}]^{(n)}(\ell - c)^k = 0$. Consequently, ${}^{ABC}D^{\rho}_{c^+}\theta(\ell) = 0$. □

Theorem 2.9 ([34]) *Let Q be a closed subspace of a Banach space P and M be a mapping such that*

$$\left\|M(\ell) - M(\widehat{\ell})\right\| \leq \nu\|\ell - \widehat{\ell}\|, \quad \textit{for all } \ell, \widehat{\ell} \in Q, \nu \in (0,1),$$

i.e., M is a contraction mapping. Then M has an FP on Q.

Theorem 2.10 ([35]) *Assume that Q is a non-empty, closed, convex, and bounded subset of a Banach space P. Let the operators M^1 and M^2 be such that*

- *for all $\ell, \widehat{\ell} \in P$, $M^1\ell + M^2\widehat{\ell} \in P$;*
- *M^1 is equicontinuous;*
- *M^2 is a contraction.*

Then there is a function $z \in Q$ so that $z = M^1 z + M^2 z$.

Lemma 2.11 ([12]) *Suppose that $\rho \in (2,3]$ and $\hbar \in C(D,\mathbb{R})$. The problem below*

$${}^{ABR}D^{\rho}_{c^+}\theta(\ell) = \hbar(\ell), \quad \ell \in [c,d], \theta(c) = a_1, \theta(d) = a_2,$$

has a solution

$$\theta(\ell) = a_1 + a_2(\ell - c) + {}^{AB}I^{\rho}_{c^+}\hbar(\ell)$$

$$= a_1 + a_2(\ell - c) + I_{c^+}^{2}\,{}^{AB}I_{c^+}^{\eta}\hbar(\ell), \quad (\rho - 2) = \eta \in (0,1],$$

where

$$\begin{aligned} {}^{AB}I_{c^+}^{\eta}\hbar(\ell) &= \frac{1-\eta}{\phi(\eta)}\hbar(\ell) + \frac{\eta}{\phi(\eta)\Gamma(\eta)}\int_c^{\ell}(\ell - \varpi)^{\eta-1}\hbar(\varpi)\,d\varpi \\ &= \frac{3-\rho}{\phi(\rho-2)}\hbar(\ell) + \frac{\rho-2}{\phi(\rho-2)\Gamma(\rho-2)}\int_c^{\ell}(\ell-\varpi)^{\rho-3}\hbar(\varpi)\,d\varpi, \end{aligned}$$

and

$$\begin{aligned} {}^{AB}I_{c^+}^{\rho}\hbar(\ell) &= I_{c^+}^{2}\,{}^{AB}I_{c^+}^{\eta}\hbar(\ell) \\ &= \frac{3-\rho}{\phi(\rho-2)}\int_c^{\ell}(\ell-\varpi)\hbar(\varpi)\,d\varpi \\ &\quad + \frac{\rho-2}{\phi(\rho-2)\Gamma(\rho-2)}\int_c^{\ell}(\ell-\varpi)^{\rho-1}\hbar(\varpi)\,d\varpi. \end{aligned} \tag{2.3}$$

Lemma 2.12 ([12]) *Assume that* $\rho \in (1,2]$ *and* $\hbar \in C(D,\mathbb{R})$. *The linear problem*

$${}^{ABC}D_{c^+}^{\rho}\theta(\ell) = \hbar(\ell), \quad \ell \in [c,d], \theta(c) = a_1, \theta(d) = a_2,$$

has a solution

$$\theta(\ell) = a_1 + a_2(\ell - c) + {}^{AB}I_{c^+}^{\rho}\hbar(\ell),$$

where

$${}^{AB}I_{c^+}^{\rho}\hbar(\ell) = \frac{2-\rho}{\phi(\rho-1)}\int_c^{\ell}\hbar(\varpi)\,d\varpi + \frac{\rho-1}{\phi(\rho-1)\Gamma(\rho)}\int_c^{\ell}(\ell-\varpi)^{\rho-1}\hbar(\varpi)\,d\varpi. \tag{2.4}$$

3 Equivalent integral equations for the proposed problem (1.1)

Lemma 3.1 *Assume that* $\rho \in (2,3]$, $\varsigma \in (0,1]$, $\Xi = (d-c) - {}^{AB}I_{c^+}^{\varsigma}(\xi - c) \neq 0$ *and* $\hbar \in C(D \times \mathbb{R}^2, \mathbb{R})$. *The functions* θ *and* ϱ *are a solution to the coupled ABR problem* (1.1) *if the coupled* (θ,ϱ) *fulfills the fractional integral equations* (*FIEs*) *below*:

$$\begin{cases} \theta(\ell) = \frac{(\ell-c)}{\Xi}\big({}^{AB}I_{c^+}^{\varsigma}\,{}^{AB}I_{c^+}^{\rho}\hbar(\xi,\theta(\xi),\varrho(\xi)) - {}^{AB}I_{c^+}^{\rho}\hbar(d,\theta(d),\varrho(d))\big) \\ \qquad + {}^{AB}I_{c^+}^{\rho}\hbar(\ell,\theta(\ell),\varrho(\ell)), \\ \varrho(\ell) = \frac{(\ell-c)}{\Xi}\big({}^{AB}I_{c^+}^{\varsigma}\,{}^{AB}I_{c^+}^{\rho}\hbar(\xi,\varrho(\xi),\theta(\xi)) - {}^{AB}I_{c^+}^{\rho}\hbar(d,\varrho(d),\theta(d))\big) \\ \qquad + {}^{AB}I_{c^+}^{\rho}\hbar(\ell,\varrho(\ell),\theta(\ell)). \end{cases} \tag{3.1}$$

Proof Let θ be a solution of the first equation of (3.1). Then by Lemma 2.11, we get

$$\theta(\ell) = a_1 + a_2(\ell - c) + {}^{AB}I_{c^+}^{\rho}\hbar\big(\ell,\theta(\ell),\varrho(\ell)\big). \tag{3.2}$$

From the condition $\theta(c) = 0$, we have $a_1 = 0$. Hence the equation (3.2) can be written as

$$\theta(\ell) = a_2(\ell - c) + {}^{AB}I_{c^+}^{\rho}\hbar\big(\ell,\theta(\ell),\varrho(\ell)\big). \tag{3.3}$$

On both sides of (3.3) applying ${}^{AB}I^{\varsigma}_{c^+}$, one can write

$$ {}^{AB}I^{\varsigma}_{c^+}\theta(\ell) = {}^{AB}I^{\varsigma}_{c^+}a_2(\ell-c) + {}^{AB}I^{\varsigma}_{c^+}\,{}^{AB}I^{\rho}_{c^+}\hbar\big(\ell,\theta(\ell),\varrho(\ell)\big). $$

Putting $\ell = \xi$, we have

$$ {}^{AB}I^{\varsigma}_{c^+}\theta(\xi) = {}^{AB}I^{\varsigma}_{c^+}a_2(\xi-c) + {}^{AB}I^{\varsigma}_{c^+}\,{}^{AB}I^{\rho}_{c^+}\hbar\big(\xi,\theta(\xi),\varrho(\xi)\big). \tag{3.4} $$

It follows from the condition $\theta(d) = {}^{AB}I^{\varsigma}_{c^+}\theta(\xi)$, (3.3), and (3.4) that

$$ a_2 = \frac{1}{\Xi}\Big({}^{AB}I^{\varsigma}_{c^+}\,{}^{AB}I^{\rho}_{c^+}\hbar\big(\xi,\theta(\xi),\varrho(\xi)\big) - {}^{AB}I^{\rho}_{c^+}\hbar\big(d,\theta(d),\varrho(d)\big)\Big). $$

Applying the value of a_1 and a_2 in (3.2), we obtain that

$$ \begin{aligned} \theta(\ell) ={}& \frac{(\ell-c)}{\Xi}\Big({}^{AB}I^{\varsigma}_{c^+}\,{}^{AB}I^{\rho}_{c^+}\hbar\big(\xi,\theta(\xi),\varrho(\xi)\big) - {}^{AB}I^{\rho}_{c^+}\hbar\big(d,\theta(d),\varrho(d)\big)\Big) \\ &+ {}^{AB}I^{\rho}_{c^+}\hbar\big(\ell,\theta(\ell),\varrho(\ell)\big). \end{aligned} \tag{3.5} $$

Following the same approach as the previous, one can easily obtain

$$ \begin{aligned} \varrho(\ell) ={}& \frac{(\ell-c)}{\Xi}\Big({}^{AB}I^{\varsigma}_{c^+}\,{}^{AB}I^{\rho}_{c^+}\hbar\big(\xi,\varrho(\xi),\theta(\xi)\big) - {}^{AB}I^{\rho}_{c^+}\hbar\big(d,\varrho(d),\theta(d)\big)\Big) \\ &+ {}^{AB}I^{\rho}_{c^+}\hbar\big(\ell,\varrho(\ell),\theta(\ell)\big). \end{aligned} \tag{3.6} $$

On the contrary, suppose that θ fulfills (3.5). Then taking the operator ${}^{ABR}D^{\rho}_{c^+}$ on both sides of (3.5) and helping Lemma 2.7 and 2.8, we have

$$ \begin{aligned} {}^{ABR}D^{\rho}_{c^+}\theta(\ell) &= {}^{ABR}D^{\rho}_{c^+}\frac{(\ell-c)}{\Xi}\Big({}^{AB}I^{\varsigma}_{c^+}\,{}^{AB}I^{\rho}_{c^+}\hbar\big(\xi,\theta(\xi),\varrho(\xi)\big) - {}^{AB}I^{\rho}_{c^+}\hbar\big(d,\theta(d),\varrho(d)\big)\Big) \\ &\quad + {}^{ABR}D^{\rho}_{c^+}\,{}^{AB}I^{\rho}_{c^+}\hbar\big(\ell,\theta(\ell),\varrho(\ell)\big) \\ &= \hbar\big(\xi,\theta(\xi),\varrho(\xi)\big), \end{aligned} $$

Analogously, one can write

$$ \begin{aligned} {}^{ABR}D^{\rho}_{c^+}\varrho(\ell) &= {}^{ABR}D^{\rho}_{c^+}\frac{(\ell-c)}{\Xi}\Big({}^{AB}I^{\varsigma}_{c^+}\,{}^{AB}I^{\rho}_{c^+}\hbar\big(\xi,\varrho(\xi),\theta(\xi)\big) - {}^{AB}I^{\rho}_{c^+}\hbar\big(d,\varrho(d),\theta(d)\big)\Big) \\ &\quad + {}^{ABR}D^{\rho}_{c^+}\,{}^{AB}I^{\rho}_{c^+}\hbar\big(\ell,\varrho(\ell),\theta(\ell)\big) \\ &= \hbar\big(\xi,\varrho(\xi),\theta(\xi)\big). \end{aligned} $$

Next, letting $\ell \to c$ in (3.5) and (3.6), we have $\theta(c) = 0 = \varrho(c)$. Conversely, taking ${}^{AB}I^{\varsigma}_{c^+}$ on both sides of (3.5) and (3.6), we obtain that

$$ \begin{aligned} {}^{AB}I^{\varsigma}_{c^+}\theta(\ell) ={}& \frac{{}^{AB}I^{\varsigma}_{c^+}(\ell-c)}{\Xi}\Big({}^{AB}I^{\varsigma}_{c^+}\,{}^{AB}I^{\rho}_{c^+}\hbar\big(\xi,\theta(\xi),\varrho(\xi)\big) - {}^{AB}I^{\rho}_{c^+}\hbar\big(d,\theta(d),\varrho(d)\big)\Big) \\ &+ {}^{AB}I^{\varsigma}_{c^+}\,{}^{AB}I^{\rho}_{c^+}\hbar\big(\ell,\theta(\ell),\varrho(\ell)\big). \end{aligned} \tag{3.7} $$

and

$$\begin{aligned}{}^{AB}I_{c^+}^{\varsigma}\varrho(\ell) &= \frac{{}^{AB}I_{c^+}^{\varsigma}(\ell-c)}{\Xi}\bigl({}^{AB}I_{c^+}^{\varsigma}\,{}^{AB}I_{c^+}^{\rho}\hbar\bigl(\xi,\varrho(\xi),\theta(\xi)\bigr)-{}^{AB}I_{c^+}^{\rho}\hbar\bigl(d,\varrho(d),\theta(d)\bigr)\bigr)\\ &\quad+{}^{AB}I_{c^+}^{\varsigma}\,{}^{AB}I_{c^+}^{\rho}\hbar\bigl(\ell,\varrho(\ell),\theta(\ell)\bigr).\end{aligned}\tag{3.8}$$

Putting $\ell=\xi$ in (3.7) and (3.8), we have

$$\begin{aligned}{}^{AB}I_{c^+}^{\varsigma}\theta(\xi) &= \frac{{}^{AB}I_{c^+}^{\varsigma}(\xi-c)}{\Xi}\bigl({}^{AB}I_{c^+}^{\varsigma}\,{}^{AB}I_{c^+}^{\rho}\hbar\bigl(\xi,\theta(\xi),\varrho(\xi)\bigr)-{}^{AB}I_{c^+}^{\rho}\hbar\bigl(d,\theta(d),\varrho(d)\bigr)\bigr)\\ &\quad+{}^{AB}I_{c^+}^{\varsigma}\,{}^{AB}I_{c^+}^{\rho}\hbar\bigl(\xi,\theta(\xi),\varrho(\xi)\bigr).\end{aligned}\tag{3.9}$$

and

$$\begin{aligned}{}^{AB}I_{c^+}^{\varsigma}\varrho(\xi) &= \frac{{}^{AB}I_{c^+}^{\varsigma}(\xi-c)}{\Xi}\bigl({}^{AB}I_{c^+}^{\varsigma}\,{}^{AB}I_{c^+}^{\rho}\hbar\bigl(\xi,\varrho(\xi),\theta(\xi)\bigr)-{}^{AB}I_{c^+}^{\rho}\hbar\bigl(d,\varrho(d),\theta(d)\bigr)\bigr)\\ &\quad+{}^{AB}I_{c^+}^{\varsigma}\,{}^{AB}I_{c^+}^{\rho}\hbar\bigl(\xi,\varrho(\xi),\theta(\xi)\bigr).\end{aligned}\tag{3.10}$$

From the definition of Ξ, equations (3.9) and (3.10) can be written as

$$\begin{aligned}{}^{AB}I_{c^+}^{\varsigma}\theta(\xi) &= \frac{(d-c)-[(d-c)-{}^{AB}I_{c^+}^{\varsigma}(\xi-c)]}{(d-c)-{}^{AB}I_{c^+}^{\varsigma}(\xi-c)}\\ &\quad\times\bigl({}^{AB}I_{c^+}^{\varsigma}\,{}^{AB}I_{c^+}^{\rho}\hbar\bigl(\xi,\theta(\xi),\varrho(\xi)\bigr)-{}^{AB}I_{c^+}^{\rho}\hbar\bigl(d,\theta(d),\varrho(d)\bigr)\bigr)\\ &\quad+{}^{AB}I_{c^+}^{\varsigma}\,{}^{AB}I_{c^+}^{\rho}\hbar\bigl(\xi,\theta(\xi),\varrho(\xi)\bigr)\\ &= \frac{(d-c)}{(d-c)-{}^{AB}I_{c^+}^{\varsigma}(\xi-c)}\bigl({}^{AB}I_{c^+}^{\varsigma}\,{}^{AB}I_{c^+}^{\rho}\hbar\bigl(\xi,\theta(\xi),\varrho(\xi)\bigr)-{}^{AB}I_{c^+}^{\rho}\hbar\bigl(d,\theta(d),\varrho(d)\bigr)\bigr)\\ &\quad+{}^{AB}I_{c^+}^{\rho}\hbar\bigl(d,\theta(d),\varrho(d)\bigr)\\ &= \frac{(d-c)}{\Xi}\bigl({}^{AB}I_{c^+}^{\varsigma}\,{}^{AB}I_{c^+}^{\rho}\hbar\bigl(\xi,\theta(\xi),\varrho(\xi)\bigr)-{}^{AB}I_{c^+}^{\rho}\hbar\bigl(d,\theta(d),\varrho(d)\bigr)\bigr)\\ &\quad+{}^{AB}I_{c^+}^{\rho}\hbar\bigl(d,\theta(d),\varrho(d)\bigr)\\ &= \theta(d).\end{aligned}$$

And

$$\begin{aligned}{}^{AB}I_{c^+}^{\varsigma}\varrho(\xi) &= \frac{(d-c)-[(d-c)-{}^{AB}I_{c^+}^{\varsigma}(\xi-c)]}{(d-c)-{}^{AB}I_{c^+}^{\varsigma}(\xi-c)}\\ &\quad\times\bigl({}^{AB}I_{c^+}^{\varsigma}\,{}^{AB}I_{c^+}^{\rho}\hbar\bigl(\xi,\varrho(\xi),\theta(\xi)\bigr)-{}^{AB}I_{c^+}^{\rho}\hbar\bigl(d,\varrho(d),\theta(d)\bigr)\bigr)\\ &\quad+{}^{AB}I_{c^+}^{\varsigma}\,{}^{AB}I_{c^+}^{\rho}\hbar\bigl(\xi,\varrho(\xi),\theta(\xi)\bigr)\\ &= \frac{(d-c)}{(d-c)-{}^{AB}I_{c^+}^{\varsigma}(\xi-c)}\bigl({}^{AB}I_{c^+}^{\varsigma}\,{}^{AB}I_{c^+}^{\rho}\hbar\bigl(\xi,\varrho(\xi),\theta(\xi)\bigr)-{}^{AB}I_{c^+}^{\rho}\hbar\bigl(d,\varrho(d),\theta(d)\bigr)\bigr)\\ &\quad+{}^{AB}I_{c^+}^{\rho}\hbar\bigl(d,\varrho(d),\theta(d)\bigr)\\ &= \frac{(d-c)}{\Xi}\bigl({}^{AB}I_{c^+}^{\varsigma}\,{}^{AB}I_{c^+}^{\rho}\hbar\bigl(\xi,\varrho(\xi),\theta(\xi)\bigr)-{}^{AB}I_{c^+}^{\rho}\hbar\bigl(d,\varrho(d),\theta(d)\bigr)\bigr)\\ &\quad+{}^{AB}I_{c^+}^{\rho}\hbar\bigl(d,\varrho(d),\theta(d)\bigr)\end{aligned}$$

$$= \varrho(d).$$

Thus, the coupled AB fractional integral conditions are fulfilled. □

The following theorem results from Lemma 3.1:

Theorem 3.2 *Suppose that* $\rho \in (2,3]$, $\varsigma \in (0,1]$, $\Xi = (d-c) - {}^{AB}I_{c^+}^{\varsigma}(\xi - c) \neq 0$ *and* $\hbar : D \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *is a continuous function. Then the supposed problem* (1.1) *are comparable to the FIEs below*:

$$\begin{aligned}
&\theta(\ell) \\
&\quad = \frac{(\ell - c)}{\Xi}\bigg\{\Omega_1 \int_c^{\xi} (\xi - \varpi)\hbar\big(\varpi, \theta(\varpi), \varrho(\varpi)\big)\,d\varpi \\
&\qquad + \frac{\Omega_2}{\Gamma(\rho - 2)} \int_c^{\xi} (\xi - \varpi)^{\rho-1}\hbar\big(\varpi, \theta(\varpi), \varrho(\varpi)\big)\,d\varpi \\
&\qquad + \frac{\Omega_3}{\Gamma(\varsigma)} \int_c^{\xi} (\xi - \varpi)^{\varsigma-1} \int_c^{\varpi} (\varpi - \varkappa)\hbar\big(\varkappa, \theta(\varkappa), \varrho(\varkappa)\big)\,d\varkappa\,d\varpi \\
&\qquad + \frac{\Omega_4}{\Gamma(\varsigma + \rho - 2)} \int_c^{\xi} (\xi - \varpi)^{\varsigma+\rho-1}\hbar\big(\varpi, \theta(\varpi), \varrho(\varpi)\big)\,d\varpi \\
&\qquad - \Omega_5 \int_c^{d} (d - \varpi)\hbar\big(\varpi, \theta(\varpi), \varrho(\varpi)\big)\,d\varpi \\
&\qquad - \frac{\Omega_6}{\Gamma(\rho - 2)} \int_c^{d} (d - \varpi)^{\rho-1}\hbar\big(\varpi, \theta(\varpi), \varrho(\varpi)\big)\,d\varpi\bigg\} \\
&\qquad + \Omega_5 \int_c^{\ell} (\ell - \varpi)\hbar\big(\varpi, \theta(\varpi), \varrho(\varpi)\big)\,d\varpi \\
&\qquad + \frac{\Omega_6}{\Gamma(\rho - 2)} \int_c^{\ell} (\ell - \varpi)^{\rho-1}\hbar\big(\varpi, \theta(\varpi), \varrho(\varpi)\big)\,d\varpi,
\end{aligned}$$

and

$$\begin{aligned}
&\varrho(\ell) \\
&\quad = \frac{(\ell - c)}{\Xi}\bigg\{\Omega_1 \int_c^{\xi} (\xi - \varpi)\hbar\big(\varpi, \varrho(\varpi), \theta(\varpi)\big)\,d\varpi \\
&\qquad + \frac{\Omega_2}{\Gamma(\rho - 2)} \int_c^{\xi} (\xi - \varpi)^{\rho-1}\hbar\big(\varpi, \varrho(\varpi), \theta(\varpi)\big)\,d\varpi \\
&\qquad + \frac{\Omega_3}{\Gamma(\varsigma)} \int_c^{\xi} (\xi - \varpi)^{\varsigma-1} \int_c^{\varpi} (\varpi - \varkappa)\hbar\big(\varkappa, \varrho(\varkappa), \theta(\varkappa)\big)\,d\varkappa\,d\varpi \\
&\qquad + \frac{\Omega_4}{\Gamma(\varsigma + \rho - 2)} \int_c^{\xi} (\xi - \varpi)^{\varsigma+\rho-1}\hbar\big(\varpi, \varrho(\varpi), \theta(\varpi)\big)\,d\varpi \\
&\qquad - \Omega_5 \int_c^{d} (d - \varpi)\hbar\big(\varpi, \varrho(\varpi), \theta(\varpi)\big)\,d\varpi \\
&\qquad - \frac{\Omega_6}{\Gamma(\rho - 2)} \int_c^{d} (d - \varpi)^{\rho-1}\hbar\big(\varpi, \varrho(\varpi), \theta(\varpi)\big)\,d\varpi\bigg\}
\end{aligned}$$

$$+ \Omega_5 \int_c^\ell (\ell - \varpi)\hbar\big(\varpi, \varrho(\varpi), \theta(\varpi)\big)\, d\varpi$$
$$+ \frac{\Omega_6}{\Gamma(\rho - 2)} \int_c^\ell (\ell - \varpi)^{\rho-1}\hbar\big(\varpi, \varrho(\varpi), \theta(\varpi)\big)\, d\varpi,$$

where

$$\Omega_1 = \frac{(1-\varsigma)(3-\rho)}{\phi(\varsigma)\phi(\rho-2)}, \qquad \Omega_2 = \frac{(1-\varsigma)(\rho-2)}{\phi(\varsigma)\phi(\rho-2)}, \qquad \Omega_3 = \frac{\varsigma(3-\rho)}{\phi(\varsigma)\phi(\rho-2)},$$
$$\Omega_4 = \frac{\varsigma(\rho-2)}{\phi(\varsigma)\phi(\rho-2)}, \qquad \Omega_5 = \frac{3-\rho}{\phi(\rho-2)}, \qquad \Omega_6 = \frac{\rho-2}{\phi(\rho-2)}.$$

Proof In light of Lemma 3.1, we find that (3.1) holds. From the definitions of ${}^{AB}I_{c^+}^{\varsigma}$, $\varsigma \in (0,1)$ described in (2.1) and ${}^{AB}I_{c^+}^{\rho}$, $\varsigma \in (2,3]$ described in (2.3), the coupled system (3.1) can be written as

$$\begin{aligned}
&\theta(\ell) \\
&= \frac{(\ell - c)}{\Xi}\Bigg\{\frac{(1-\varsigma)}{\phi(\varsigma)}\bigg(\frac{(3-\rho)}{\phi(\rho-2)}\int_c^\varsigma (\varsigma - \varpi)\hbar\big(\varpi, \theta(\varpi), \varrho(\varpi)\big)\, d\varpi \\
&\quad + \frac{\rho-2}{\phi(\rho-2)\Gamma(\rho-2)}\int_c^\varsigma (\varsigma - \varpi)^{\rho-1}\hbar\big(\varpi, \theta(\varpi), \varrho(\varpi)\big)\, d\varpi\bigg) \\
&\quad + \frac{\varsigma}{\phi(\varsigma)\Gamma(\varsigma)}\int_c^\varsigma (\varsigma - \varpi)^{\varsigma-1}\bigg(\frac{(3-\rho)}{\phi(\rho-2)}\int_c^\varpi (\varpi - \varkappa)\hbar\big(\varkappa, \varrho(\varkappa), \theta(\varkappa)\big)\, d\varkappa \\
&\quad \frac{\rho-2}{\phi(\rho-2)\Gamma(\rho-2)}\int_c^\varpi (\varpi - \varkappa)^{\rho-1}\hbar\big(\varkappa, \varrho(\varkappa), \theta(\varkappa)\big)\, d\varkappa\bigg)\, d\varpi \\
&\quad - \frac{(3-\rho)}{\phi(\rho-2)}\int_c^d (d - \varpi)\hbar\big(\varpi, \theta(\varpi), \varrho(\varpi)\big)\, d\varpi \\
&\quad - \frac{\rho-2}{\phi(\rho-2)\Gamma(\rho-2)}\int_c^d (d - \varpi)^{\rho-1}\hbar\big(\varpi, \theta(\varpi), \varrho(\varpi)\big)\, d\varpi\Bigg\} \\
&\quad + \frac{(3-\rho)}{\phi(\rho-2)}\int_c^\ell (\ell - \varpi)\hbar\big(\varpi, \theta(\varpi), \varrho(\varpi)\big)\, d\varpi \\
&\quad + \frac{\rho-2}{\phi(\rho-2)\Gamma(\rho-2)}\int_c^\ell (\ell - \varpi)^{\rho-1}\hbar\big(\varpi, \theta(\varpi), \varrho(\varpi)\big)\, d\varpi,
\end{aligned}$$

which yields that

$$\begin{aligned}
&\theta(\ell) \\
&= \frac{(\ell - c)}{\Xi}\Bigg\{\frac{(1-\varsigma)}{\phi(\varsigma)}\frac{(3-\rho)}{\phi(\rho-2)}\int_c^\varsigma (\varsigma - \varpi)\hbar\big(\varpi, \theta(\varpi), \varrho(\varpi)\big)\, d\varpi \\
&\quad + \frac{(1-\varsigma)}{\phi(\varsigma)}\frac{\rho-2}{\phi(\rho-2)\Gamma(\rho-2)}\int_c^\varsigma (\varsigma - \varpi)^{\rho-1}\hbar\big(\varpi, \theta(\varpi), \varrho(\varpi)\big)\, d\varpi \\
&\quad + \frac{(3-\rho)}{\phi(\rho-2)}\frac{\varsigma}{\phi(\varsigma)\Gamma(\varsigma)} \\
&\quad \times \int_c^\varsigma (\varsigma - \varpi)^{\varsigma-1}\int_c^\varpi (\varpi - \varkappa)\hbar\big(\varkappa, \varrho(\varkappa), \theta(\varkappa)\big)\, d\varkappa\, d\varpi
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\rho-2}{\phi(\rho-2)\Gamma(\rho-2)}\\
&\times\frac{\varsigma}{\phi(\varsigma)\Gamma(\varsigma)}\int_c^{\varsigma}(\varsigma-\varpi)^{\varsigma-1}\int_c^{\varpi}(\varpi-\varkappa)\hbar\bigl(\varkappa,\varrho(\varkappa),\theta(\varkappa)\bigr)\,d\varkappa\,d\varpi\\
&-\frac{(3-\rho)}{\phi(\rho-2)}\int_c^{d}(d-\varpi)\hbar\bigl(\varpi,\theta(\varpi),\varrho(\varpi)\bigr)\,d\varpi\\
&-\frac{\rho-2}{\phi(\rho-2)\Gamma(\rho-2)}\int_c^{d}(d-\varpi)^{\rho-1}\hbar\bigl(\varpi,\theta(\varpi),\varrho(\varpi)\bigr)\,d\varpi\Bigr\}\\
&+\frac{(3-\rho)}{\phi(\rho-2)}\int_c^{\ell}(\ell-\varpi)\hbar\bigl(\varpi,\theta(\varpi),\varrho(\varpi)\bigr)\,d\varpi\\
&+\frac{\rho-2}{\phi(\rho-2)\Gamma(\rho-2)}\int_c^{\ell}(\ell-\varpi)^{\rho-1}\hbar\bigl(\varpi,\theta(\varpi),\varrho(\varpi)\bigr)\,d\varpi.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\theta(\ell)\\
&\quad=\frac{(\ell-c)}{\Xi}\Bigl\{\Omega_1\int_c^{\xi}(\xi-\varpi)\hbar\bigl(\varpi,\theta(\varpi),\varrho(\varpi)\bigr)\,d\varpi\\
&\qquad+\frac{\Omega_2}{\Gamma(\rho-2)}\int_c^{\xi}(\xi-\varpi)^{\rho-1}\hbar\bigl(\varpi,\theta(\varpi),\varrho(\varpi)\bigr)\,d\varpi\\
&\qquad+\frac{\Omega_3}{\Gamma(\varsigma)}\int_c^{\xi}(\xi-\varpi)^{\varsigma-1}\int_c^{\varpi}(\varpi-\varkappa)\hbar\bigl(\varkappa,\theta(\varkappa),\varrho(\varkappa)\bigr)\,d\varkappa\,d\varpi\\
&\qquad+\frac{\Omega_4}{\Gamma(\varsigma+\rho-2)}\int_c^{\xi}(\xi-\varpi)^{\varsigma+\rho-1}\hbar\bigl(\varpi,\theta(\varpi),\varrho(\varpi)\bigr)\,d\varpi\\
&\qquad-\Omega_5\int_c^{d}(d-\varpi)\hbar\bigl(\varpi,\theta(\varpi),\varrho(\varpi)\bigr)\,d\varpi\\
&\qquad-\frac{\Omega_6}{\Gamma(\rho-2)}\int_c^{d}(d-\varpi)^{\rho-1}\hbar\bigl(\varpi,\theta(\varpi),\varrho(\varpi)\bigr)\,d\varpi\Bigr\}\\
&\qquad+\Omega_5\int_c^{\ell}(\ell-\varpi)\hbar\bigl(\varpi,\theta(\varpi),\varrho(\varpi)\bigr)\,d\varpi\\
&\qquad+\frac{\Omega_6}{\Gamma(\rho-2)}\int_c^{\ell}(\ell-\varpi)^{\rho-1}\hbar\bigl(\varpi,\theta(\varpi),\varrho(\varpi)\bigr)\,d\varpi.
\end{aligned}
$$

By the same scenario, one can obtain

$$
\begin{aligned}
&\varrho(\ell)\\
&\quad=\frac{(\ell-c)}{\Xi}\Bigl\{\Omega_1\int_c^{\xi}(\xi-\varpi)\hbar\bigl(\varpi,\varrho(\varpi),\theta(\varpi)\bigr)\,d\varpi\\
&\qquad+\frac{\Omega_2}{\Gamma(\rho-2)}\int_c^{\xi}(\xi-\varpi)^{\rho-1}\hbar\bigl(\varpi,\varrho(\varpi),\theta(\varpi)\bigr)\,d\varpi\\
&\qquad+\frac{\Omega_3}{\Gamma(\varsigma)}\int_c^{\xi}(\xi-\varpi)^{\varsigma-1}\int_c^{\varpi}(\varpi-\varkappa)\hbar\bigl(\varkappa,\varrho(\varkappa),\theta(\varkappa)\bigr)\,d\varkappa\,d\varpi\\
&\qquad+\frac{\Omega_4}{\Gamma(\varsigma+\rho-2)}\int_c^{\xi}(\xi-\varpi)^{\varsigma+\rho-1}\hbar\bigl(\varpi,\varrho(\varpi),\theta(\varpi)\bigr)\,d\varpi
\end{aligned}
$$

$$- \Omega_5 \int_c^d (d-\varpi)\hbar\big(\varpi,\varrho(\varpi),\theta(\varpi)\big)\,d\varpi$$
$$- \frac{\Omega_6}{\Gamma(\rho-2)} \int_c^d (d-\varpi)^{\rho-1}\hbar\big(\varpi,\varrho(\varpi),\theta(\varpi)\big)\,d\varpi \Big\}$$
$$+ \Omega_5 \int_c^\ell (\ell-\varpi)\hbar\big(\varpi,\varrho(\varpi),\theta(\varpi)\big)\,d\varpi$$
$$+ \frac{\Omega_6}{\Gamma(\rho-2)} \int_c^\ell (\ell-\varpi)^{\rho-1}\hbar\big(\varpi,\varrho(\varpi),\theta(\varpi)\big)\,d\varpi.$$

This completes the proof. □

For short, let us consider

$$\begin{aligned}
\wp_1 &= \frac{\Omega_1(\xi-c)^2}{2} + \frac{\Omega_2(\xi-c)^\rho}{\Gamma(\rho-1)} + \frac{\Omega_3(\xi-c)^{\varsigma+1}}{\Gamma(\varsigma+1)} + \frac{\Omega_4(\xi-c)^{\varsigma+\rho}}{\Gamma(\varsigma+\rho-1)} \\
&\quad + \frac{\Omega_5(d-c)^2}{2} + \frac{\Omega_6(d-c)^\rho}{\Gamma(\rho-1)}, \\
\wp_2 &= \frac{\Omega_5(d-c)^2}{2} + \frac{\Omega_6(d-c)^\rho}{\Gamma(\rho-1)}, \\
\wp_3 &= \Omega_7(\xi-c) + \frac{\Omega_8(\xi-c)^\rho}{\Gamma(\rho+1)} + \frac{\Omega_9(\xi-c)^{\varsigma+1}}{\Gamma(\varsigma+1)} + \frac{\Omega_{10}(\xi-c)^{\varsigma+\rho}}{\Gamma(\varsigma+\rho+1)} \\
&\quad + \Omega_{11}(d-c) + \frac{\Omega_{12}(d-c)^\rho}{\Gamma(\rho+1)}, \\
\wp_4 &= \Omega_{11}(d-c) + \frac{\Omega_{12}(d-c)^\rho}{\Gamma(\rho+1)}.
\end{aligned}$$

4 EaU of solutions for problem (1.1)

This part is devoted to showing the EaU of solutions for the suggested problem (1.1).

Theorem 4.1 *For a continuous function $\hbar : D\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$, assume that*

$$\big|\hbar(\ell,\theta,\varrho)-\hbar(\ell,\widehat{\theta},\widehat{\varrho})\big| \le \frac{A_\hbar}{2}\big(|\theta-\widehat{\theta}|+|\varrho-\widehat{\varrho}|\big), \quad A_\hbar>0, \varrho,\theta,\widehat{\varrho},\widehat{\theta}\in\mathbb{R}. \tag{4.1}$$

Then the coupled ABR-type FDEs (1.1) have a unique solution on D provided that the condition below holds

$$\mho = A_\hbar\left(\frac{\wp_1(d-c)}{\Xi}+\wp_2\right) < 1.$$

Proof Describe an operator $\chi : C(D,\mathbb{R})\times C(D,\mathbb{R})\to C(D,\mathbb{R})$ as

$$\chi(\theta,\varrho)(\ell)$$
$$= \frac{(\ell-c)}{\Xi}\Big\{\Omega_1\int_c^\xi (\xi-\varpi)\hbar\big(\varpi,\theta(\varpi),\varrho(\varpi)\big)\,d\varpi$$
$$+ \frac{\Omega_2}{\Gamma(\rho-2)}\int_c^\xi (\xi-\varpi)^{\rho-1}\hbar\big(\varpi,\theta(\varpi),\varrho(\varpi)\big)\,d\varpi$$

$$
\begin{aligned}
&+ \frac{\Omega_3}{\Gamma(\varsigma)} \int_c^{\xi} (\xi - \varpi)^{\varsigma-1} \int_c^{\varpi} (\varpi - \varkappa)\hbar\big(\varkappa, \theta(\varkappa), \varrho(\varkappa)\big)\, d\varkappa\, d\varpi \\
&+ \frac{\Omega_4}{\Gamma(\varsigma + \rho - 2)} \int_c^{\xi} (\xi - \varpi)^{\varsigma+\rho-1} \hbar\big(\varpi, \theta(\varpi), \varrho(\varpi)\big)\, d\varpi \\
&- \Omega_5 \int_c^{d} (d - \varpi)\hbar\big(\varpi, \theta(\varpi), \varrho(\varpi)\big)\, d\varpi \\
&- \frac{\Omega_6}{\Gamma(\rho - 2)} \int_c^{d} (d - \varpi)^{\rho-1} \hbar\big(\varpi, \theta(\varpi), \varrho(\varpi)\big)\, d\varpi \bigg\} \\
&+ \Omega_5 \int_c^{\ell} (\ell - \varpi)\hbar\big(\varpi, \theta(\varpi), \varrho(\varpi)\big)\, d\varpi \\
&+ \frac{\Omega_6}{\Gamma(\rho - 2)} \int_c^{\ell} (\ell - \varpi)^{\rho-1} \hbar\big(\varpi, \theta(\varpi), \varrho(\varpi)\big)\, d\varpi.
\end{aligned}
$$

Let us consider a closed ball Λ_φ so that

$$
\Lambda_\varphi = \big\{\theta \in C(D, \mathbb{R}) : \|\theta\| \le \varphi\big\},
$$

with radius $\varphi \ge \frac{\mho_1}{1-\mho}$, where $\mho_1 = (\frac{(d-c)}{\Xi}\wp_1 + \wp_2)\pi_\hbar$ and $\pi_\hbar = \sup_{\ell \in D} |\hbar(\ell, 0, 0)|$. Firstly, we prove that $\chi\Lambda_\varphi \subset \Lambda_\varphi$. For each $\theta, \varrho \in \Lambda_\varphi$ and $\ell \in D$, we get

$$
\begin{aligned}
&\big|\chi(\theta, \varrho)(\ell)\big| \\
&\quad \le \frac{(\ell - c)}{\Xi} \bigg\{ \Omega_1 \int_c^{\xi} (\xi - \varpi)\big|\hbar\big(\varpi, \theta(\varpi), \varrho(\varpi)\big)\big|\, d\varpi \\
&\quad + \frac{\Omega_2}{\Gamma(\rho - 2)} \int_c^{\xi} (\xi - \varpi)^{\rho-1} \big|\hbar\big(\varpi, \theta(\varpi), \varrho(\varpi)\big)\big|\, d\varpi \\
&\quad + \frac{\Omega_3}{\Gamma(\varsigma)} \int_c^{\xi} (\xi - \varpi)^{\varsigma-1} \int_c^{\varpi} (\varpi - \varkappa)\big|\hbar\big(\varkappa, \theta(\varkappa), \varrho(\varkappa)\big)\big|\, d\varkappa\, d\varpi \\
&\quad + \frac{\Omega_4}{\Gamma(\varsigma + \rho - 2)} \int_c^{\xi} (\xi - \varpi)^{\varsigma+\rho-1} \big|\hbar\big(\varpi, \theta(\varpi), \varrho(\varpi)\big)\big|\, d\varpi \\
&\quad + \Omega_5 \int_c^{d} (d - \varpi)\big|\hbar\big(\varpi, \theta(\varpi), \varrho(\varpi)\big)\big|\, d\varpi \\
&\quad + \frac{\Omega_6}{\Gamma(\rho - 2)} \int_c^{d} (d - \varpi)^{\rho-1} \big|\hbar\big(\varpi, \theta(\varpi), \varrho(\varpi)\big)\big|\, d\varpi \bigg\} \\
&\quad + \Omega_5 \int_c^{\ell} (\ell - \varpi)\big|\hbar\big(\varpi, \theta(\varpi), \varrho(\varpi)\big)\big|\, d\varpi \\
&\quad + \frac{\Omega_6}{\Gamma(\rho - 2)} \int_c^{\ell} (\ell - \varpi)^{\rho-1} \big|\hbar\big(\varpi, \theta(\varpi), \varrho(\varpi)\big)\big|\, d\varpi.
\end{aligned}
$$

Using (4.1), we have

$$
\begin{aligned}
\big|\hbar\big(\ell, \theta(\ell), \varrho(\ell)\big)\big| &\le \big|\hbar\big(\ell, \theta(\ell), \varrho(\ell)\big) - \hbar(\ell, 0, 0)\big| + \big|\hbar(\ell, 0, 0)\big| \\
&\le \frac{A_\hbar}{2}\big(\big|\theta(\ell)\big| + \big|\varrho(\ell)\big|\big) + \big|\hbar(\ell, 0, 0)\big| \\
&\le \frac{A_\hbar}{2}(2\varphi) + \pi_\hbar = A_\hbar\varphi + \pi_\hbar. \qquad (4.2)
\end{aligned}
$$

Hence

$$\begin{aligned}\|\chi(\theta,\varrho)\| &\leq \frac{(A_\hbar\varphi+\pi_\hbar)(\ell-c)}{\Xi}\wp_1+(A_\hbar\varphi+\pi_\hbar)\wp_2\\ &\leq A_\hbar\left(\frac{\wp_1(d-c)}{\Xi}+\wp_2\right)\varphi+A_\hbar\left(\frac{\wp_1(d-c)}{\Xi}+\wp_2\right)\pi_\hbar\\ &=\mho\varphi+\mho_1<\varphi.\end{aligned}$$

Analogously, we can write $\|\chi(\varrho,\theta)\|<\varphi$. Therefore, $\chi\Lambda_\varphi\subset\Lambda_\varphi$. Next, we shall prove that χ is a contraction. Assume that $\varrho,\theta,\widehat{\varrho},\widehat{\theta}\in C(D,\mathbb{R})$ and $\ell\in D$. Then, we get

$$\begin{aligned}&\big|\chi(\theta,\varrho)(\ell)-\chi(\widehat{\theta},\widehat{\varrho})(\ell)\big|\\ &=\frac{(\ell-c)}{\Xi}\bigg\{\Omega_1\int_c^\xi(\xi-\varpi)\big|\hbar(\varpi,\theta(\varpi),\varrho(\varpi))-\hbar(\varpi,\widehat{\theta}(\varpi),\widehat{\varrho}(\varpi))\big|\,d\varpi\\ &+\frac{\Omega_2}{\Gamma(\rho-2)}\int_c^\xi(\xi-\varpi)^{\rho-1}\big|\hbar(\varpi,\theta(\varpi),\varrho(\varpi))-\hbar(\varpi,\widehat{\theta}(\varpi),\widehat{\varrho}(\varpi))\big|\,d\varpi\\ &+\frac{\Omega_3}{\Gamma(\varsigma)}\int_c^\xi(\xi-\varpi)^{\varsigma-1}\int_c^\varpi(\varpi-\varkappa)\big|\hbar(\varkappa,\theta(\varkappa),\varrho(\varkappa))-\hbar(\varkappa,\widehat{\theta}(\varkappa),\widehat{\varrho}(\varkappa))\big|\,d\varkappa\,d\varpi\\ &+\frac{\Omega_4}{\Gamma(\varsigma+\rho-2)}\int_c^\xi(\xi-\varpi)^{\varsigma+\rho-1}\big|\hbar(\varpi,\theta(\varpi),\varrho(\varpi))-\hbar(\varpi,\widehat{\theta}(\varpi),\widehat{\varrho}(\varpi))\big|\,d\varpi\\ &+\Omega_5\int_c^d(d-\varpi)\big|\hbar(\varpi,\theta(\varpi),\varrho(\varpi))-\hbar(\varpi,\widehat{\theta}(\varpi),\widehat{\varrho}(\varpi))\big|\,d\varpi\\ &+\frac{\Omega_6}{\Gamma(\rho-2)}\int_c^d(d-\varpi)^{\rho-1}\big|\hbar(\varpi,\theta(\varpi),\varrho(\varpi))-\hbar(\varpi,\widehat{\theta}(\varpi),\widehat{\varrho}(\varpi))\big|\,d\varpi\bigg\}\\ &+\Omega_5\int_c^\ell(\ell-\varpi)\big|\hbar(\varpi,\theta(\varpi),\varrho(\varpi))-\hbar(\varpi,\widehat{\theta}(\varpi),\widehat{\varrho}(\varpi))\big|\,d\varpi\\ &+\frac{\Omega_6}{\Gamma(\rho-2)}\int_c^\ell(\ell-\varpi)^{\rho-1}\big|\hbar(\varpi,\theta(\varpi),\varrho(\varpi))-\hbar(\varpi,\widehat{\theta}(\varpi),\widehat{\varrho}(\varpi))\big|\,d\varpi.\end{aligned}$$

Based on our condition (4.1)

$$\big|\hbar(\ell,\varrho,\theta)-\hbar(\ell,\widehat{\varrho},\widehat{\theta})\big|\leq\frac{A_\hbar}{2}\big(|\varrho-\widehat{\varrho}|+|\theta-\widehat{\theta}|\big)\leq\frac{A_\hbar}{2}\big(\|\varrho-\widehat{\varrho}\|+\|\theta-\widehat{\theta}\|\big),$$

we have

$$\begin{aligned}\|\chi(\theta,\varrho)-\chi(\widehat{\theta},\widehat{\varrho})\| &=\frac{A_\hbar(\ell-c)}{2\Xi}\bigg\{\frac{\Omega_1(\xi-c)^2}{2}+\frac{\Omega_2(\xi-c)^\rho}{\Gamma(\rho-1)}+\frac{\Omega_3(\xi-c)^{\varsigma+1}}{\Gamma(\varsigma+1)}\\ &\quad+\frac{\Omega_4(\xi-c)^{\varsigma+\rho}}{\Gamma(\varsigma+\rho-1)}+\frac{\Omega_5(d-c)^2}{2}+\frac{\Omega_6(d-c)^\rho}{\Gamma(\rho-1)}\bigg\}\big(\|\theta-\widehat{\theta}\|+\|\varrho-\widehat{\varrho}\|\big)\\ &\quad+\frac{A_\hbar}{2}\left(\frac{\Omega_5(d-c)^2}{2}+\frac{\Omega_6(d-c)^\rho}{\Gamma(\rho-1)}\right)\big(\|\theta-\widehat{\theta}\|+\|\varrho-\widehat{\varrho}\|\big)\\ &\leq\frac{A_\hbar}{2}\left(\frac{(d-c)}{\Xi}\wp_1+\wp_2\right)\big(\|\theta-\widehat{\theta}\|+\|\varrho-\widehat{\varrho}\|\big)\\ &\leq A_\hbar\left(\frac{(d-c)}{\Xi}\wp_1+\wp_2\right)\big(\|\theta-\widehat{\theta}\|+\|\varrho-\widehat{\varrho}\|\big)\end{aligned}$$

$$= \mho\big(\|\varrho - \widehat{\varrho}\| + \|\theta - \widehat{\theta}\|\big).$$

Similarly, one can obtain that

$$\big\|\chi(\varrho,\theta) - \chi(\widehat{\varrho},\widehat{\theta})\big\| \le \mho\big(\|\varrho - \widehat{\varrho}\| + \|\theta - \widehat{\theta}\|\big).$$

Because of $\mho < 1$, we conclude that the operator χ is a contraction. Hence, Theorem 2.9 tells us there is a FP of χ in D. □

Theorem 4.2 *There is at least one solution to the considered problem* (1.1) *in D provided that all hypotheses of Theorem* 4.1 *are satisfied.*

Proof Divide the operator χ defined in Theorem 4.1 into two operators as

$$\chi(\theta,\varrho)(\ell) = \chi_1(\theta,\varrho)(\ell) + \chi_2(\theta,\varrho)(\ell),$$

where

$$\begin{aligned}
&\chi_1(\theta,\varrho)(\ell)\\
&\quad= \frac{(\ell - c)}{\Xi}\bigg\{\Omega_1\int_c^{\xi}(\xi-\varpi)\hbar\big(\varpi,\theta(\varpi),\varrho(\varpi)\big)\,d\varpi\\
&\qquad+\frac{\Omega_2}{\Gamma(\rho-2)}\int_c^{\xi}(\xi-\varpi)^{\rho-1}\hbar\big(\varpi,\theta(\varpi),\varrho(\varpi)\big)\,d\varpi\\
&\qquad+\frac{\Omega_3}{\Gamma(\varsigma)}\int_c^{\xi}(\xi-\varpi)^{\varsigma-1}\int_c^{\varpi}(\varpi-\varkappa)\hbar\big(\varkappa,\theta(\varkappa),\varrho(\varkappa)\big)\,d\varkappa\,d\varpi\\
&\qquad+\frac{\Omega_4}{\Gamma(\varsigma+\rho-2)}\int_c^{\xi}(\xi-\varpi)^{\varsigma+\rho-1}\hbar\big(\varpi,\theta(\varpi),\varrho(\varpi)\big)\,d\varpi\\
&\qquad-\Omega_5\int_c^{d}(d-\varpi)\hbar\big(\varpi,\theta(\varpi),\varrho(\varpi)\big)\,d\varpi\\
&\qquad-\frac{\Omega_6}{\Gamma(\rho-2)}\int_c^{d}(d-\varpi)^{\rho-1}\hbar\big(\varpi,\theta(\varpi),\varrho(\varpi)\big)\,d\varpi\bigg\},
\end{aligned}$$

and

$$\begin{aligned}
\chi_2(\theta,\varrho)(\ell) = {}&\Omega_5\int_c^{\ell}(\ell-\varpi)\hbar\big(\varpi,\theta(\varpi),\varrho(\varpi)\big)\,d\varpi\\
&+\frac{\Omega_6}{\Gamma(\rho-2)}\int_c^{\ell}(\ell-\varpi)^{\rho-1}\hbar\big(\varpi,\theta(\varpi),\varrho(\varpi)\big)\,d\varpi.
\end{aligned}$$

Consider a closed ball Λ_φ, which is what Theorem 4.1 defines. In order to apply Theorem 2.10, we will break up the proof into the steps below:

(i) We prove that $\chi_1(\theta,\varrho) + \chi_2(\theta,\varrho) \in \Lambda_\varphi$, for all $\theta,\varrho \in \Lambda_\varphi$. For $\chi_1 \in \Lambda_\varphi$, we get

$$\begin{aligned}
&\big|\chi_1(\theta,\varrho)(\ell)\big|\\
&\quad\le \frac{(\ell-c)}{\Xi}\bigg\{\Omega_1\int_c^{\xi}(\xi-\varpi)\big|\hbar\big(\varpi,\theta(\varpi),\varrho(\varpi)\big)\big|\,d\varpi
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{\Omega_2}{\Gamma(\rho-2)} \int_c^{\xi} (\xi-\varpi)^{\rho-1} \left|\hbar\big(\varpi,\theta(\varpi),\varrho(\varpi)\big)\right| d\varpi \\
&+ \frac{\Omega_3}{\Gamma(\varsigma)} \int_c^{\xi} (\xi-\varpi)^{\varsigma-1} \int_c^{\varpi} (\varpi-\varkappa)\left|\hbar\big(\varkappa,\theta(\varkappa),\varrho(\varkappa)\big)\right| d\varkappa\, d\varpi \\
&+ \frac{\Omega_4}{\Gamma(\varsigma+\rho-2)} \int_c^{\xi} (\xi-\varpi)^{\varsigma+\rho-1} \left|\hbar\big(\varpi,\theta(\varpi),\varrho(\varpi)\big)\right| d\varpi \\
&+ \Omega_5 \int_c^{d} (d-\varpi)\left|\hbar\big(\varpi,\theta(\varpi),\varrho(\varpi)\big)\right| d\varpi \\
&+ \frac{\Omega_6}{\Gamma(\rho-2)} \int_c^{d} (d-\varpi)^{\rho-1} \left|\hbar\big(\varpi,\theta(\varpi),\varrho(\varpi)\big)\right| d\varpi \bigg\}.
\end{aligned}
$$

Applying (4.2), we have

$$
\left\|\chi_1(\theta,\varrho)(\ell)\right\| \le \frac{(A_\hbar\varphi+\pi_\hbar)(d-c)}{\Xi}\wp_1. \tag{4.3}
$$

Similarly

$$
\left\|\chi_1(\varrho,\theta)\right\| \le \frac{(A_\hbar\varphi+\pi_\hbar)(d-c)}{\Xi}\wp_1. \tag{4.4}
$$

Again using (4.2), we get

$$
\begin{aligned}
\left|\chi_2(\theta,\varrho)(\ell)\right| &= \Omega_5 \int_c^{\ell} (\ell-\varpi)\left|\hbar\big(\varpi,\theta(\varpi),\varrho(\varpi)\big)\right| d\varpi \\
&\quad + \frac{\Omega_6}{\Gamma(\rho-2)} \int_c^{\ell} (\ell-\varpi)^{\rho-1}\left|\hbar\big(\varpi,\theta(\varpi),\varrho(\varpi)\big)\right| d\varpi \\
&\le (A_\hbar\varphi+\pi_\hbar)\wp_2 = \left|\chi_2(\varrho,\theta)(\ell)\right|,
\end{aligned}
$$

which yields that

$$
\left\|\chi_2(\theta,\varrho)\right\| = (A_\hbar\varphi+\pi_\hbar)\wp_2 = \left\|\chi_2(\varrho,\theta)\right\|. \tag{4.5}
$$

Inequalities (4.3)-(4.5) give

$$
\begin{aligned}
\left\|\chi_1(\theta,\varrho)(\ell)+\chi_2(\theta,\varrho)\right\| &\le \left\|\chi_1(\theta,\varrho)(\ell)\right\| + \left\|\chi_2(\theta,\varrho)\right\| \\
&\le \frac{(A_\hbar\varphi+\pi_\hbar)(d-c)}{\Xi}\wp_1 + (A_\hbar\varphi+\pi_\hbar)\wp_2 \\
&= A_\hbar\left(\frac{(d-c)}{\Xi}\wp_1+\wp_2\right)\varphi + \left(\frac{(d-c)}{\Xi}\wp_1+\wp_2\right)\pi_\hbar \\
&= \mho\varphi + \mho_1\pi_\hbar < \varphi.
\end{aligned}
$$

By the same method one can show that $\|\chi_1(\varrho,\theta)(\ell)+\chi_2(\varrho,\theta)\| < \varphi$. Thus, $\chi_1(\theta,\varrho)+\chi_2(\theta,\varrho) \in \Lambda_\varphi$.

(ii) The mapping χ_1 is a contraction. The contractivity of χ finishes this step.

(iii) χ_2 is continuous and compact. The continuity of $\hbar$ leads to the continuity of χ_2 and by (4.5), χ_2 is uniformly bounded on Λ_φ. Now, we claim that $\chi_2(\Lambda_\varphi)$ is equicontinuous. For this, assume that $\theta, \varrho \in \Lambda_\varphi$ and $c \leq \ell_1 < \ell_2 \leq d$. Then by (4.2), one can write

$$\begin{aligned}
&\left|\chi_2(\theta,\varrho)(\ell_2) - \chi_2(\theta,\varrho)(\ell_1)\right| \\
&= \Omega_5\biggl(\int_c^{\ell_1}\bigl[(\ell_2-\varpi)-(\ell_1-\varpi)\bigr]\bigl|\hbar\bigl(\varpi,\theta(\varpi),\varrho(\varpi)\bigr)\bigr|\,d\varpi \\
&\quad + \int_{\ell_1}^{\ell_2}(\ell_2-\varpi)\bigl|\hbar\bigl(\varpi,\theta(\varpi),\varrho(\varpi)\bigr)\bigr|\,d\varpi\biggr) \\
&\quad + \frac{\Omega_6}{\Gamma(\rho-2)}\int_c^{\ell_1}\bigl[(\ell_2-\varpi)^{\rho-1}-(\ell_1-\varpi)^{\rho-1}\bigr]\bigl|\hbar\bigl(\varpi,\theta(\varpi),\varrho(\varpi)\bigr)\bigr|\,d\varpi \\
&\quad + \frac{\Omega_6}{\Gamma(\rho-2)}\int_{\ell_1}^{\ell_2}(\ell_2-\varpi)^{\rho-1}\bigl|\hbar\bigl(\varpi,\theta(\varpi),\varrho(\varpi)\bigr)\bigr|\,d\varpi \\
&\leq (A_\hbar\varphi+\pi_\hbar)\biggl(\frac{\Omega_5(\ell_2-\ell_1)}{2}+\frac{\Omega_6}{\Gamma(\rho-1)}\bigl[(\ell_2-\varpi)^{\rho}-(\ell_1-\varpi)^{\rho-1}\bigr]\biggr).
\end{aligned}$$

Hence,

$$\bigl\|\chi_2(\theta,\varrho)(\ell_2)-\chi_2(\theta,\varrho)(\ell_1)\bigr\| \to 0 \leftarrow \bigl\|\chi_2(\varrho,\theta)(\ell_2)-\chi_2(\varrho,\theta)(\ell_1)\bigr\|, \quad \text{as } \ell_2 \to \ell_2.$$

Based on the above cases and thanks to Arzela–Ascoli theorem, the relatively compactness of $\chi_2(\Lambda_\varphi)$ is verified, which proves the completely continuous of χ_2. From Theorem 2.10, there is at least one solution in D to the supposed problem (1.1). □

5 EaU of solutions for problem (1.2)

This part is devoted to discussing the EaU of solutions for the coupled ABC-type fractional DEs (1.2).

Lemma 5.1 *Assume that* $\rho \in (1,2]$, $\varsigma \in (0,1]$, $\Xi = (d-c) - {}^{AB}I_{c^+}^{\varsigma}(\xi - c) \neq 0$ *and* $\hbar_1 \in C(D \times \mathbb{R}^2, \mathbb{R})$. *The functions* θ *and* ϱ *are a solution to the coupled ABR problem* (1.2) *if the coupled* (θ,ϱ) *fulfills the following FIEs*:

$$\begin{cases}
\theta(\ell) = \frac{(\ell-c)}{\Xi}\bigl({}^{AB}I_{c^+}^{\varsigma}\,{}^{AB}I_{c^+}^{\rho}\hbar_1(\xi,\theta(\xi),\varrho(\xi)) - {}^{AB}I_{c^+}^{\rho}\hbar(d,\theta(d),\varrho(d))\bigr) \\
\qquad + {}^{AB}I_{c^+}^{\rho}\hbar(\ell,\theta(\ell),\varrho(\ell)), \\
\varrho(\ell) = \frac{(\ell-c)}{\Xi}\bigl({}^{AB}I_{c^+}^{\varsigma}\,{}^{AB}I_{c^+}^{\rho}\hbar_1(\xi,\varrho(\xi),\theta(\xi)) - {}^{AB}I_{c^+}^{\rho}\hbar_1(d,\varrho(d),\theta(d))\bigr) \\
\qquad + {}^{AB}I_{c^+}^{\rho}\hbar_1(\ell,\varrho(\ell),\theta(\ell)).
\end{cases} \tag{5.1}$$

Proof Let θ and ϱ be the solution to the equations of (1.2). Then Lemma 2.12 and Lemma 3.1 can be used to demonstrate that the solution to equations (1.2) is given as (5.1), where ${}^{AB}I_{c^+}^{\rho}\hbar_1$ is defined in (2.4). □

Theorem 5.2 *Assume that* $\rho \in (1,2]$, $\varsigma \in (0,1]$, $\Xi = (d-c) - {}^{AB}I_{c^+}^{\varsigma}(\xi - c) \neq 0$ *and* $\hbar : D \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *is a continuous function. Then according to Lemma* 5.1, *the considered problem*

(1.2) *is comparable to the following FIEs*:

$$\begin{aligned}
&\theta(\ell)\\
&\quad= \frac{(\ell-c)}{\Xi}\biggl\{\Omega_7\int_c^\xi \hbar\bigl(\varpi,\theta(\varpi),\varrho(\varpi)\bigr)\,d\varpi\\
&\qquad+\frac{\Omega_8}{\Gamma(\rho)}\int_c^\xi(\xi-\varpi)^{\rho-1}\hbar\bigl(\varpi,\theta(\varpi),\varrho(\varpi)\bigr)\,d\varpi\\
&\qquad+\frac{\Omega_9}{\Gamma(\varsigma)}\int_c^\xi(\xi-\varpi)^{\varsigma-1}\int_c^\varpi \hbar\bigl(\varkappa,\theta(\varkappa),\varrho(\varkappa)\bigr)\,d\varkappa\,d\varpi\\
&\qquad+\frac{\Omega_{10}}{\Gamma(\varsigma+\rho)}\int_c^\xi(\xi-\varpi)^{\varsigma+\rho-1}\hbar\bigl(\varpi,\theta(\varpi),\varrho(\varpi)\bigr)\,d\varpi\\
&\qquad-\Omega_{11}\int_c^d \hbar\bigl(\varpi,\theta(\varpi),\varrho(\varpi)\bigr)\,d\varpi-\frac{\Omega_{12}}{\Gamma(\rho)}\int_c^d(d-\varpi)^{\rho-1}\hbar\bigl(\varpi,\theta(\varpi),\varrho(\varpi)\bigr)\,d\varpi\biggr\}\\
&\qquad+\Omega_{11}\int_c^\ell \hbar\bigl(\varpi,\theta(\varpi),\varrho(\varpi)\bigr)\,d\varpi+\frac{\Omega_{12}}{\Gamma(\rho)}\int_c^\ell(\ell-\varpi)^{\rho-1}\hbar\bigl(\varpi,\theta(\varpi),\varrho(\varpi)\bigr)\,d\varpi
\end{aligned}$$

and

$$\begin{aligned}
&\varrho(\ell)\\
&\quad= \frac{(\ell-c)}{\Xi}\biggl\{\Omega_7\int_c^\xi \hbar\bigl(\varpi,\varrho(\varpi),\theta(\varpi)\bigr)\,d\varpi\\
&\qquad+\frac{\Omega_8}{\Gamma(\rho)}\int_c^\xi(\xi-\varpi)^{\rho-1}\hbar\bigl(\varpi,\varrho(\varpi),\theta(\varpi)\bigr)\,d\varpi\\
&\qquad+\frac{\Omega_9}{\Gamma(\varsigma)}\int_c^\xi(\xi-\varpi)^{\varsigma-1}\int_c^\varpi \hbar\bigl(\varkappa,\varrho(\varkappa),\theta(\varkappa)\bigr)\,d\varkappa\,d\varpi\\
&\qquad+\frac{\Omega_{10}}{\Gamma(\varsigma+\rho)}\int_c^\xi(\xi-\varpi)^{\varsigma+\rho-1}\hbar\bigl(\varpi,\varrho(\varpi),\theta(\varpi)\bigr)\,d\varpi\\
&\qquad-\Omega_{11}\int_c^d \hbar\bigl(\varpi,\varrho(\varpi),\theta(\varpi)\bigr)\,d\varpi-\frac{\Omega_{12}}{\Gamma(\rho)}\int_c^d(d-\varpi)^{\rho-1}\hbar\bigl(\varpi,\varrho(\varpi),\theta(\varpi)\bigr)\,d\varpi\biggr\}\\
&\qquad+\Omega_{11}\int_c^\ell \hbar\bigl(\varpi,\varrho(\varpi),\theta(\varpi)\bigr)\,d\varpi+\frac{\Omega_{12}}{\Gamma(\rho)}\int_c^\ell(\ell-\varpi)^{\rho-1}\hbar\bigl(\varpi,\varrho(\varpi),\theta(\varpi)\bigr)\,d\varpi
\end{aligned}$$

where

$$\begin{aligned}
&\Omega_7=\frac{(1-\varsigma)(2-\rho)}{\phi(\varsigma)\phi(\rho-1)},\qquad \Omega_8=\frac{(1-\varsigma)(\rho-1)}{\phi(\varsigma)\phi(\rho-1)},\qquad \Omega_9=\frac{\varsigma(2-\rho)}{\phi(\varsigma)\phi(\rho-1)},\\
&\Omega_{10}=\frac{\varsigma(\rho-1)}{\phi(\varsigma)\phi(\rho-1)},\qquad \Omega_{11}=\frac{2-\rho}{\phi(\rho-1)},\qquad \Omega_6=\frac{\rho-1}{\phi(\rho-1)}.
\end{aligned}$$

Proof In the light of Lemma 5.1, we get (5.1). According to definitions ${}^{AB}I_{c^+}^{\varsigma}$ when $\varsigma\in(0,1)$ defined in (2.1) and ${}^{AB}I_{c^+}^{\rho}$ when $\rho\in(1,2]$ defined in (2.4), the coupled system (5.1) can be written as

$$\begin{aligned}
&\theta(\ell)\\
&\quad= \frac{(\ell-c)}{\Xi}\Bigg\{\frac{(1-\varsigma)}{\phi(\varsigma)}\bigg(\frac{(2-\rho)}{\phi(\rho-1)}\int_c^{\varsigma}(\varsigma-\varpi)\hbar\big(\varpi,\theta(\varpi),\varrho(\varpi)\big)\,d\varpi\\
&\qquad+\frac{\rho-2}{\phi(\rho-2)\Gamma(\rho-2)}\int_c^{\varsigma}(\varsigma-\varpi)^{\rho-1}\hbar\big(\varpi,\theta(\varpi),\varrho(\varpi)\big)\,d\varpi\bigg)\\
&\qquad+\frac{\varsigma}{\phi(\varsigma)\Gamma(\varsigma)}\int_c^{\varsigma}(\varsigma-\varpi)^{\varsigma-1}\bigg(\frac{(2-\rho)}{\phi(\rho-1)}\int_c^{\varpi}\hbar\big(\varkappa,\varrho(\varkappa),\theta(\varkappa)\big)\,d\varkappa\\
&\qquad\frac{\rho-1}{\phi(\rho-2)\Gamma(\rho)}\int_c^{\varpi}(\varpi-\varkappa)^{\rho-1}\hbar\big(\varkappa,\varrho(\varkappa),\theta(\varkappa)\big)\,d\varkappa\bigg)\,d\varpi\\
&\qquad-\frac{(2-\rho)}{\phi(\rho-1)}\int_c^{d}\hbar\big(\varpi,\theta(\varpi),\varrho(\varpi)\big)\,d\varpi\\
&\qquad-\frac{\rho-1}{\phi(\rho-2)\Gamma(\rho)}\int_c^{d}(d-\varpi)^{\rho-1}\hbar\big(\varpi,\theta(\varpi),\varrho(\varpi)\big)\,d\varpi\Bigg\}\\
&\qquad+\frac{(2-\rho)}{\phi(\rho-1)}\int_c^{\ell}\hbar\big(\varpi,\theta(\varpi),\varrho(\varpi)\big)\,d\varpi\\
&\qquad+\frac{\rho-1}{\phi(\rho-2)\Gamma(\rho)}\int_c^{\ell}(\ell-\varpi)^{\rho-1}\hbar\big(\varpi,\theta(\varpi),\varrho(\varpi)\big)\,d\varpi,
\end{aligned}$$

which yields that

$$\begin{aligned}
&\theta(\ell)\\
&\quad= \frac{(\ell-c)}{\Xi}\Bigg\{\Omega_7\int_c^{\xi}\hbar\big(\varpi,\theta(\varpi),\varrho(\varpi)\big)\,d\varpi\\
&\qquad+\frac{\Omega_8}{\Gamma(\rho)}\int_c^{\xi}(\xi-\varpi)^{\rho-1}\hbar\big(\varpi,\theta(\varpi),\varrho(\varpi)\big)\,d\varpi\\
&\qquad+\frac{\Omega_9}{\Gamma(\varsigma)}\int_c^{\xi}(\xi-\varpi)^{\varsigma-1}\int_c^{\varpi}\hbar\big(\varkappa,\theta(\varkappa),\varrho(\varkappa)\big)\,d\varkappa\,d\varpi\\
&\qquad+\frac{\Omega_{10}}{\Gamma(\varsigma+\rho)}\int_c^{\xi}(\xi-\varpi)^{\varsigma+\rho-1}\hbar\big(\varpi,\theta(\varpi),\varrho(\varpi)\big)\,d\varpi\\
&\qquad-\Omega_{11}\int_c^{d}\hbar\big(\varpi,\theta(\varpi),\varrho(\varpi)\big)\,d\varpi-\frac{\Omega_{12}}{\Gamma(\rho)}\int_c^{d}(d-\varpi)^{\rho-1}\hbar\big(\varpi,\theta(\varpi),\varrho(\varpi)\big)\,d\varpi\Bigg\}\\
&\qquad+\Omega_{11}\int_c^{\ell}\hbar\big(\varpi,\theta(\varpi),\varrho(\varpi)\big)\,d\varpi+\frac{\Omega_{12}}{\Gamma(\rho)}\int_c^{\ell}(\ell-\varpi)^{\rho-1}\hbar\big(\varpi,\theta(\varpi),\varrho(\varpi)\big)\,d\varpi.
\end{aligned}$$

And

$$\begin{aligned}
&\varrho(\ell)\\
&\quad= \frac{(\ell-c)}{\Xi}\Bigg\{\Omega_7\int_c^{\xi}\hbar\big(\varpi,\varrho(\varpi),\theta(\varpi)\big)\,d\varpi\\
&\qquad+\frac{\Omega_8}{\Gamma(\rho)}\int_c^{\xi}(\xi-\varpi)^{\rho-1}\hbar\big(\varpi,\varrho(\varpi),\theta(\varpi)\big)\,d\varpi
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{\Omega_9}{\Gamma(\varsigma)} \int_c^{\xi} (\xi - \varpi)^{\varsigma-1} \int_c^{\varpi} \hbar\big(\varkappa, \varrho(\varkappa), \theta(\varkappa)\big)\, d\varkappa\, d\varpi \\
&+ \frac{\Omega_{10}}{\Gamma(\varsigma+\rho)} \int_c^{\xi} (\xi - \varpi)^{\varsigma+\rho-1} \hbar\big(\varpi, \varrho(\varpi), \theta(\varpi)\big)\, d\varpi \\
&- \Omega_{11} \int_c^{d} \hbar\big(\varpi, \varrho(\varpi), \theta(\varpi)\big)\, d\varpi - \frac{\Omega_{12}}{\Gamma(\rho)} \int_c^{d} (d - \varpi)^{\rho-1} \hbar\big(\varpi, \varrho(\varpi), \theta(\varpi)\big)\, d\varpi \Big\} \\
&+ \Omega_{11} \int_c^{\ell} \hbar\big(\varpi, \varrho(\varpi), \theta(\varpi)\big)\, d\varpi + \frac{\Omega_{12}}{\Gamma(\rho)} \int_c^{\ell} (\ell - \varpi)^{\rho-1} \hbar\big(\varpi, \varrho(\varpi), \theta(\varpi)\big)\, d\varpi.
\end{aligned}
$$

This finishes the proof. □

Theorem 5.3 *Let $\hbar : D \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a continuous function, and there is a constant number $A_\hbar > 0$ so that*

$$
\big|\hbar(\ell, \theta, \varrho) - \hbar(\ell, \widehat{\theta}, \widehat{\varrho})\big| \le \frac{A_\hbar}{2}\big(|\theta - \widehat{\theta}| + |\varrho - \widehat{\varrho}|\big), \quad \text{for any } \varrho, \theta, \widehat{\varrho}, \widehat{\theta} \in \mathbb{R}, \ell \in D. \tag{5.2}
$$

If

$$
\mho_2 = A_\hbar \left(\frac{\wp_3 (d - c)}{\Xi} + \wp_4 \right) < 1,
$$

then there is a unique solution to the proposed problem (1.2) *in D.*

Proof Describe an operator $\Im : C(D, \mathbb{R}) \times C(D, \mathbb{R}) \to C(D, \mathbb{R})$ as

$$
\begin{aligned}
&\Im(\theta, \varrho)(\ell) \\
&\quad = \frac{(\ell - c)}{\Xi} \Big\{ \Omega_7 \int_c^{\xi} \hbar\big(\varpi, \theta(\varpi), \varrho(\varpi)\big)\, d\varpi \\
&\quad + \frac{\Omega_8}{\Gamma(\rho)} \int_c^{\xi} (\xi - \varpi)^{\rho-1} \hbar\big(\varpi, \theta(\varpi), \varrho(\varpi)\big)\, d\varpi \\
&\quad + \frac{\Omega_9}{\Gamma(\varsigma)} \int_c^{\xi} (\xi - \varpi)^{\varsigma-1} \int_c^{\varpi} \hbar\big(\varkappa, \theta(\varkappa), \varrho(\varkappa)\big)\, d\varkappa\, d\varpi \\
&\quad + \frac{\Omega_{10}}{\Gamma(\varsigma+\rho)} \int_c^{\xi} (\xi - \varpi)^{\varsigma+\rho-1} \hbar\big(\varpi, \theta(\varpi), \varrho(\varpi)\big)\, d\varpi \\
&\quad - \Omega_{11} \int_c^{d} \hbar\big(\varpi, \theta(\varpi), \varrho(\varpi)\big)\, d\varpi - \frac{\Omega_{12}}{\Gamma(\rho)} \int_c^{d} (d - \varpi)^{\rho-1} \hbar\big(\varpi, \theta(\varpi), \varrho(\varpi)\big)\, d\varpi \Big\} \\
&\quad + \Omega_{11} \int_c^{\ell} \hbar\big(\varpi, \theta(\varpi), \varrho(\varpi)\big)\, d\varpi + \frac{\Omega_{12}}{\Gamma(\rho)} \int_c^{\ell} (\ell - \varpi)^{\rho-1} \hbar\big(\varpi, \theta(\varpi), \varrho(\varpi)\big)\, d\varpi.
\end{aligned}
$$

Let $\Phi_\daleth$ be a closed ball defined by

$$
\Phi_\daleth = \big\{ \theta \in C(D, \mathbb{R}) : \|\theta\| \le \daleth \big\},
$$

with radius $\daleth \ge \frac{\mho_3}{1-\mho_2}$, where $\mho_3 = (\frac{(d-c)}{\Xi}\wp_3 + \wp_4)\pi_\hbar$ and $\pi_\hbar = \sup_{\ell \in D} |\hbar(\ell, 0, 0)|$. Now, we show that $\Im \Phi_\daleth \subset \Phi_\daleth$. For each $\theta, \varrho \in \Phi_\daleth$, $\ell \in D$, by (5.2) and helping the proof of Theorem 4.1, we have

$$
\begin{aligned}
\left\|\Im(\theta,\varrho)\right\| &\leq \frac{(A_{\hbar}\beth+\pi_{\hbar})(\ell-c)}{\Xi}\wp_3+(A_{\hbar}\beth+\pi_{\hbar})\wp_4\\
&\leq A_{\hbar}\left(\frac{\wp_3(d-c)}{\Xi}+\wp_4\right)\beth+A_{\hbar}\left(\frac{\wp_3(d-c)}{\Xi}+\wp_4\right)\pi_{\hbar}\\
&=\mho_2\beth+\mho_3<\beth.
\end{aligned}
$$

Similarly, one can write $\|\Im(\varrho,\theta)\|<\beth$. Hence, $\Im\Phi_{\beth}\subset\Phi_{\beth}$. Next, we claim that $\Im$ is a contraction. Let $\varrho,\theta,\widehat{\varrho},\widehat{\theta}\in C(D,\mathbb{R})$ and $\ell\in D$. Then, we have

$$
\begin{aligned}
&\left|\Im(\theta,\varrho)(\ell)-\Im(\widehat{\theta},\widehat{\varrho})(\ell)\right|\\
&=\frac{(\ell-c)}{\Xi}\Bigg\{\Omega_7\int_c^{\xi}\left|\hbar\big(\varpi,\theta(\varpi),\varrho(\varpi)\big)-\hbar\big(\varpi,\widehat{\theta}(\varpi),\widehat{\varrho}(\varpi)\big)\right|d\varpi\\
&\quad+\frac{\Omega_8}{\Gamma(\rho)}\int_c^{\xi}(\xi-\varpi)^{\rho-1}\left|\hbar\big(\varpi,\theta(\varpi),\varrho(\varpi)\big)-\hbar\big(\varpi,\widehat{\theta}(\varpi),\widehat{\varrho}(\varpi)\big)\right|d\varpi\\
&\quad+\frac{\Omega_9}{\Gamma(\varsigma)}\int_c^{\xi}(\xi-\varpi)^{\varsigma-1}\int_c^{\varpi}\left|\hbar\big(\varkappa,\theta(\varkappa),\varrho(\varkappa)\big)-\hbar\big(\varkappa,\widehat{\theta}(\varkappa),\widehat{\varrho}(\varkappa)\big)\right|d\varkappa\,d\varpi\\
&\quad+\frac{\Omega_{10}}{\Gamma(\varsigma+\rho)}\int_c^{\xi}(\xi-\varpi)^{\varsigma+\rho-1}\left|\hbar\big(\varpi,\theta(\varpi),\varrho(\varpi)\big)-\hbar\big(\varpi,\widehat{\theta}(\varpi),\widehat{\varrho}(\varpi)\big)\right|d\varpi\\
&\quad-\Omega_{11}\int_c^{d}\left|\hbar\big(\varpi,\theta(\varpi),\varrho(\varpi)\big)-\hbar\big(\varpi,\widehat{\theta}(\varpi),\widehat{\varrho}(\varpi)\big)\right|d\varpi\\
&\quad-\frac{\Omega_{12}}{\Gamma(\rho)}\int_c^{d}(d-\varpi)^{\rho-1}\left|\hbar\big(\varpi,\theta(\varpi),\varrho(\varpi)\big)-\hbar\big(\varpi,\widehat{\theta}(\varpi),\widehat{\varrho}(\varpi)\big)\right|d\varpi\Bigg\}\\
&\quad+\Omega_{11}\int_c^{\ell}\left|\hbar\big(\varpi,\theta(\varpi),\varrho(\varpi)\big)-\hbar\big(\varpi,\widehat{\theta}(\varpi),\widehat{\varrho}(\varpi)\big)\right|d\varpi\\
&\quad+\frac{\Omega_{12}}{\Gamma(\rho)}\int_c^{\ell}(\ell-\varpi)^{\rho-1}\left|\hbar\big(\varpi,\theta(\varpi),\varrho(\varpi)\big)-\hbar\big(\varpi,\widehat{\theta}(\varpi),\widehat{\varrho}(\varpi)\big)\right|d\varpi.
\end{aligned}
$$

From our assumption (5.2), one sees that

$$
\begin{aligned}
\left\|z(\theta,\varrho)-\Im(\widehat{\theta},\widehat{\varrho})\right\| &= \frac{A_{\hbar}(\ell-c)}{2\Xi}\Bigg\{\frac{\Omega_7(\xi-c)}{2}+\frac{\Omega_8(\xi-c)^{\rho}}{\Gamma(\rho+1)}+\frac{\Omega_9(\xi-c)^{\varsigma+1}}{\Gamma(\varsigma+1)}\\
&\quad+\frac{\Omega_{10}(\xi-c)^{\varsigma+\rho}}{\Gamma(\varsigma+\rho+1)}+\Omega_{11}(d-c)+\frac{\Omega_{12}(d-c)^{\rho}}{\Gamma(\rho+1)}\Bigg\}\big(\|\theta-\widehat{\theta}\|+\|\varrho-\widehat{\varrho}\|\big)\\
&\quad+\frac{A_{\hbar}}{2}\left(\Omega_{11}(\ell-c)+\frac{\Omega_{12}(\ell-c)^{\rho}}{\Gamma(\rho+1)}\right)\big(\|\theta-\widehat{\theta}\|+\|\varrho-\widehat{\varrho}\|\big)\\
&\leq\frac{A_{\hbar}}{2}\left(\frac{(d-c)}{\Xi}\wp_3+\wp_4\right)\big(\|\theta-\widehat{\theta}\|+\|\varrho-\widehat{\varrho}\|\big)\\
&\leq A_{\hbar}\left(\frac{(d-c)}{\Xi}\wp_3+\wp_4\right)\big(\|\theta-\widehat{\theta}\|+\|\varrho-\widehat{\varrho}\|\big)\\
&=\mho_2\big(\|\varrho-\widehat{\varrho}\|+\|\theta-\widehat{\theta}\|\big).
\end{aligned}
$$

In the same approach, we can obtain

$$\left\|\Im(\varrho,\theta)-\Im(\widehat{\varrho},\widehat{\theta})\right\|\leq \mho_2\left(\|\varrho-\widehat{\varrho}\|+\|\theta-\widehat{\theta}\|\right).$$

Since $\mho_2<1$, then the operator $\Im$ is contraction. So by Theorem 2.9, we conclude that there is a unique fixed point of $\Im$ on D. Therefore, the unique solution for the considered problem (1.2) exists in D. □

6 Stability results

This section covers the UH and GUH stabilities for the coupled ABR-type FDEs (1.1) and ABC-type FDEs (1.2). To reach our desired goal, let us consider for $\varepsilon>0$, the following inequalities hold

$$\left|{}^{ABR}D^{\rho}_{c^+}\theta(\ell)-\Im_\theta(\ell)\right|\leq\varepsilon \quad\text{and}\quad \left|{}^{ABR}D^{\rho}_{c^+}\varrho(\ell)-\Im_\varrho(\ell)\right|\leq\varepsilon,\quad \ell\in D. \tag{6.1}$$

Now, we introduce the definition below:

Definition 6.1 The coupled ABR-type FDEs (1.1) are UH stable if there is a real number $S_\hbar>0$ so that for each $\varepsilon>0$ and for a solution $\theta,\varrho\in C(D,\mathbb{R})$ of (6.1), there is a unique solution $\overline{\theta},\overline{\varrho}\in C(D,\mathbb{R})$ of the suggested problem (1.1) so that

$$\left|\theta(\ell)-\overline{\theta}(\ell)\right|+\left|\varrho(\ell)-\overline{\varrho}(\ell)\right|\leq S_\hbar\varepsilon. \tag{6.2}$$

Also, the coupled ABR-type FDEs (1.1) have GUH stable if there is a function $\alpha_\hbar:\mathbb{R}^+\to\mathbb{R}^+$ with $\alpha_\hbar(0)=0$ so that

$$\left|\theta(\ell)-\overline{\theta}(\ell)\right|+\left|\varrho(\ell)-\overline{\varrho}(\ell)\right|\leq \alpha_\hbar\varepsilon.$$

Remark 6.2 The functions $\overline{\theta},\overline{\varrho}\in C(D,\mathbb{R})$ are a solution to the inequalities (6.1) if we can find a continuous functions $U,V:D\to\mathbb{R}$ depending on θ and ϱ, respectively, such that

(i) $|U(\ell)|\leq\frac{\varepsilon}{2}$ and $|V(\ell)|\leq\frac{\varepsilon}{2}$, for all $\ell\in D$;

(ii) ${}^{ABR}D^{\rho}_{c^+}\theta(\ell)=\hbar(\ell,\theta(\ell),\varrho(\ell))+U(\ell)$, $\ell\in D$;

(iii) ${}^{ABR}D^{\rho}_{c^+}\varrho(\ell)=\hbar(\ell,\varrho(\ell),\theta(\ell))+V(\ell)$, $\ell\in D$.

Lemma 6.3 *If $\overline{\theta}$ and $\overline{\varrho}$ are a solution to the inequalities* (6.1), *and $\overline{\theta}$ and $\overline{\varrho}$ satisfy the following inequalities*:

$$\begin{aligned}&\left|\theta(\ell)-\Phi_\theta-\Omega_5\int_c^\ell(\ell-\varpi)\hbar\bigl(\varpi,\theta(\varpi),\varrho(\varpi)\bigr)\,d\varpi\right.\\&\left.\quad-\frac{\Omega_6}{\Gamma(\rho-2)}\int_c^\ell(\ell-\varpi)^{\rho-1}\hbar\bigl(\varpi,\theta(\varpi),\varrho(\varpi)\bigr)\,d\varpi\right|\\&\quad\leq\frac{\varepsilon}{2}\left(\frac{\wp_1(d-c)}{\Xi}+\wp_2\right)\end{aligned}$$

and

$$\begin{aligned}&\bigg|\varrho(\ell) - \Phi_{\varrho} - \Omega_5 \int_c^{\ell} (\ell - \varpi)\hbar\big(\varpi, \varrho(\varpi), \theta(\varpi)\big)\, d\varpi \\ &\quad + \frac{\Omega_6}{\Gamma(\rho - 2)} \int_c^{\ell} (\ell - \varpi)^{\rho-1}\hbar\big(\varpi, \varrho(\varpi), \theta(\varpi)\big)\, d\varpi\bigg| \\ &\quad \leq \frac{\varepsilon}{2}\bigg(\frac{\wp_1(d - c)}{\Xi} + \wp_2\bigg),\end{aligned}$$

where

$$\begin{aligned}\Phi_{\theta} &= \frac{(\ell - c)}{\Xi}\bigg\{\Omega_1 \int_c^{\xi} (\xi - \varpi)\hbar\big(\varpi, \theta(\varpi), \varrho(\varpi)\big)\, d\varpi \\ &\quad + \frac{\Omega_2}{\Gamma(\rho - 2)} \int_c^{\xi} (\xi - \varpi)^{\rho-1}\hbar\big(\varpi, \theta(\varpi), \varrho(\varpi)\big)\, d\varpi \\ &\quad + \frac{\Omega_3}{\Gamma(\varsigma)} \int_c^{\xi} (\xi - \varpi)^{\varsigma-1} \int_c^{\varpi} (\varpi - \varkappa)\hbar\big(\varkappa, \theta(\varkappa), \varrho(\varkappa)\big)\, d\varkappa\, d\varpi \\ &\quad + \frac{\Omega_4}{\Gamma(\varsigma + \rho - 2)} \int_c^{\xi} (\xi - \varpi)^{\varsigma+\rho-1}\hbar\big(\varpi, \theta(\varpi), \varrho(\varpi)\big)\, d\varpi \\ &\quad - \Omega_5 \int_c^{d} (d - \varpi)\hbar\big(\varpi, \theta(\varpi), \varrho(\varpi)\big)\, d\varpi \\ &\quad - \frac{\Omega_6}{\Gamma(\rho - 2)} \int_c^{d} (d - \varpi)^{\rho-1}\hbar\big(\varpi, \theta(\varpi), \varrho(\varpi)\big)\, d\varpi\bigg\}\end{aligned}$$

and

$$\begin{aligned}\Phi_{\varrho} &= \frac{(\ell - c)}{\Xi}\bigg\{\Omega_1 \int_c^{\xi} (\xi - \varpi)\hbar\big(\varpi, \varrho(\varpi), \theta(\varpi)\big)\, d\varpi \\ &\quad + \frac{\Omega_2}{\Gamma(\rho - 2)} \int_c^{\xi} (\xi - \varpi)^{\rho-1}\hbar\big(\varpi, \varrho(\varpi), \theta(\varpi)\big)\, d\varpi \\ &\quad + \frac{\Omega_3}{\Gamma(\varsigma)} \int_c^{\xi} (\xi - \varpi)^{\varsigma-1} \int_c^{\varpi} (\varpi - \varkappa)\hbar\big(\varkappa, \varrho(\varkappa), \theta(\varkappa)\big)\, d\varkappa\, d\varpi \\ &\quad + \frac{\Omega_4}{\Gamma(\varsigma + \rho - 2)} \int_c^{\xi} (\xi - \varpi)^{\varsigma+\rho-1}\hbar\big(\varpi, \varrho(\varpi), \theta(\varpi)\big)\, d\varpi \\ &\quad - \Omega_5 \int_c^{d} (d - \varpi)\hbar\big(\varpi, \varrho(\varpi), \theta(\varpi)\big)\, d\varpi \\ &\quad - \frac{\Omega_6}{\Gamma(\rho - 2)} \int_c^{d} (d - \varpi)^{\rho-1}\hbar\big(\varpi, \varrho(\varpi), \theta(\varpi)\big)\, d\varpi\bigg\}.\end{aligned}$$

Proof In the light of Remark 6.2, we have

(1) ${}^{ABR}D_{c^+}^{\rho}\theta(\ell) = \hbar(\ell, \theta(\ell), \varrho(\ell)) + U(\ell)$,

(2) ${}^{ABR}D_{c^+}^{\rho}\varrho(\ell) = \hbar(\ell, \varrho(\ell), \theta(\ell)) + V(\ell)$,

(3) $\theta(c) = 0 = \varrho(c)$, $\theta(d) = {}^{AB}I_{c^+}^{\varsigma}\theta(\xi)$ and $\varrho(d) = {}^{AB}I_{c^+}^{\varsigma}\varrho(\xi)$.

Then by Theorem 3.2, we have

$$\theta(\ell) = \frac{(\ell - c)}{\Xi}\bigg\{\Omega_1 \int_c^{\xi} (\xi - \varpi)\big[\hbar\big(\varpi, \theta(\varpi), \varrho(\varpi)\big) + U(\varpi)\big]\, d\varpi$$

$$
\begin{aligned}
&+\frac{\Omega_2}{\Gamma(\rho-2)}\int_c^{\xi}(\xi-\varpi)^{\rho-1}\big[\hbar\big(\varpi,\theta(\varpi),\varrho(\varpi)\big)+U(\varpi)\big]\,d\varpi\\
&+\frac{\Omega_3}{\Gamma(\varsigma)}\int_c^{\xi}(\xi-\varpi)^{\varsigma-1}\int_c^{\varpi}(\varpi-\varkappa)\big[\hbar\big(\varkappa,\theta(\varkappa),\varrho(\varkappa)\big)+U(\varkappa)\big]\,d\varkappa\,d\varpi\\
&+\frac{\Omega_4}{\Gamma(\varsigma+\rho-2)}\int_c^{\xi}(\xi-\varpi)^{\varsigma+\rho-1}\big[\hbar\big(\varpi,\theta(\varpi),\varrho(\varpi)\big)+U(\varpi)\big]\,d\varpi\\
&-\Omega_5\int_c^{d}(d-\varpi)\big[\hbar\big(\varpi,\theta(\varpi),\varrho(\varpi)\big)+U(\varpi)\big]\,d\varpi\\
&-\frac{\Omega_6}{\Gamma(\rho-2)}\int_c^{d}(d-\varpi)^{\rho-1}\big[\hbar\big(\varpi,\theta(\varpi),\varrho(\varpi)\big)+U(\varpi)\big]\,d\varpi\Big\}\\
&+\Omega_5\int_c^{\ell}(\ell-\varpi)\big[\hbar\big(\varpi,\theta(\varpi),\varrho(\varpi)\big)+U(\varpi)\big]\,d\varpi\\
&+\frac{\Omega_6}{\Gamma(\rho-2)}\int_c^{\ell}(\ell-\varpi)^{\rho-1}\big[\hbar\big(\varpi,\theta(\varpi),\varrho(\varpi)\big)+U(\varpi)\big]\,d\varpi,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&\bigg|\theta(\ell)-\Phi_\theta-\Omega_5\int_c^{\ell}(\ell-\varpi)\hbar\big(\varpi,\theta(\varpi),\varrho(\varpi)\big)\,d\varpi\\
&\quad-\frac{\Omega_6}{\Gamma(\rho-2)}\int_c^{\ell}(\ell-\varpi)^{\rho-1}\hbar\big(\varpi,\theta(\varpi),\varrho(\varpi)\big)\,d\varpi\bigg|\\
&\quad\le\frac{(\ell-c)}{\Xi}\bigg\{\Omega_1\int_c^{\xi}(\xi-\varpi)\big|U(\varpi)\big|\,d\varpi+\frac{\Omega_2}{\Gamma(\rho-2)}\int_c^{\xi}(\xi-\varpi)^{\rho-1}\big|U(\varpi)\big|\,d\varpi\\
&\quad+\frac{\Omega_3}{\Gamma(\varsigma)}\int_c^{\xi}(\xi-\varpi)^{\varsigma-1}\int_c^{\varpi}(\varpi-\varkappa)\big|U(\varkappa)\big|\,d\varkappa\,d\varpi\\
&\quad+\frac{\Omega_4}{\Gamma(\varsigma+\rho-2)}\int_c^{\xi}(\xi-\varpi)^{\varsigma+\rho-1}\big|U(\varpi)\big|\,d\varpi\\
&\quad+\Omega_5\int_c^{d}(d-\varpi)\big|U(\varpi)\big|\,d\varpi\frac{\Omega_6}{\Gamma(\rho-2)}\int_c^{d}(d-\varpi)^{\rho-1}\big|U(\varpi)\big|\,d\varpi\bigg\}\\
&\quad\Omega_5\int_c^{\ell}(\ell-\varpi)\big|U(\varpi)\big|\,d\varpi+\frac{\Omega_6}{\Gamma(\rho-2)}\int_c^{\ell}(\ell-\varpi)^{\rho-1}\big|U(\varpi)\big|\,d\varpi\\
&\quad\le\frac{\varepsilon}{2}\bigg(\frac{\wp_1(d-c)}{\Xi}+\wp_2\bigg).
\end{aligned}
$$

Similarly, one can write

$$
\begin{aligned}
&\bigg|\varrho(\ell)-\Phi_\varrho-\Omega_5\int_c^{\ell}(\ell-\varpi)\hbar\big(\varpi,\varrho(\varpi),\theta(\varpi)\big)\,d\varpi\\
&\quad-\frac{\Omega_6}{\Gamma(\rho-2)}\int_c^{\ell}(\ell-\varpi)^{\rho-1}\hbar\big(\varpi,\varrho(\varpi),\theta(\varpi)\big)\,d\varpi\bigg|\\
&\quad\le\frac{(\ell-c)}{\Xi}\bigg\{\Omega_1\int_c^{\xi}(\xi-\varpi)\big|V(\varpi)\big|\,d\varpi+\frac{\Omega_2}{\Gamma(\rho-2)}\int_c^{\xi}(\xi-\varpi)^{\rho-1}\big|V(\varpi)\big|\,d\varpi\\
&\quad+\frac{\Omega_3}{\Gamma(\varsigma)}\int_c^{\xi}(\xi-\varpi)^{\varsigma-1}\int_c^{\varpi}(\varpi-\varkappa)\big|V(\varkappa)\big|\,d\varkappa\,d\varpi
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\Omega_4}{\Gamma(\varsigma+\rho-2)}\int_c^{\xi}(\xi-\varpi)^{\varsigma+\rho-1}\left|V(\varpi)\right|d\varpi \\
&+\Omega_5\int_c^d(d-\varpi)\left|V(\varpi)\right|d\varpi\,\frac{\Omega_6}{\Gamma(\rho-2)}\int_c^d(d-\varpi)^{\rho-1}\left|V(\varpi)\right|d\varpi\Big\} \\
&\Omega_5\int_c^{\ell}(\ell-\varpi)\left|V(\varpi)\right|d\varpi+\frac{\Omega_6}{\Gamma(\rho-2)}\int_c^{\ell}(\ell-\varpi)^{\rho-1}\left|V(\varpi)\right|d\varpi \\
&\le\frac{\varepsilon}{2}\left(\frac{\wp_1(d-c)}{\Xi}+\wp_2\right).
\end{aligned}
$$

This finishes the required. □

Theorem 6.4 *Let condition* (5.2) *hold. Then the coupled ABR-type FDEs* (1.1) *are UH stable provided that the assumption below is true*

$$
A_{\hbar}\left(\frac{\Omega_5(d-c)^2}{2}+\frac{\Omega_6(d-c)^{\rho}}{\Gamma(\rho-1)}\right)<1.
$$

Proof Assume that $\varepsilon>0$ and $\theta,\varrho\in C(D,\mathbb{R})$ are functions justifying inequalities (6.1). Let $\overline{\theta},\overline{\varrho}\in C(D,\mathbb{R})$ be a unique solution to the following coupled system:

$$
\begin{cases}
{}^{ABR}D_{c^+}^{\rho}\theta(\ell)=\hbar(\ell,\theta(\ell),\varrho(\ell)), & \ell\in(c,d),\\
{}^{ABR}D_{c^+}^{\rho}\varrho(\ell)=\hbar(\ell,\varrho(\ell),\theta(\ell)), & \rho\in(2,3],\\
\theta(c)=\overline{\theta}(c)=0, \quad \theta(d)=\overline{\theta}(c)={}^{AB}I_{c^+}^{\varsigma}\theta(\xi), & \xi\in(c,d),\\
\varrho(c)=\overline{\varrho}(c)=0, \quad \varrho(d)=\overline{\varrho}(c)={}^{AB}I_{c^+}^{\varsigma}\varrho(\xi).
\end{cases}
\tag{6.3}
$$

Then by Theorem 3.2, we have

$$
\begin{aligned}
\theta(\ell)=\Phi_{\theta}&+\Omega_5\int_c^{\ell}(\ell-\varpi)\hbar\big(\varpi,\theta(\varpi),\varrho(\varpi)\big)\,d\varpi\\
&+\frac{\Omega_6}{\Gamma(\rho-2)}\int_c^{\ell}(\ell-\varpi)^{\rho-1}\hbar\big(\varpi,\theta(\varpi),\varrho(\varpi)\big)\,d\varpi,
\end{aligned}
$$

and

$$
\begin{aligned}
\varrho(\ell)=\Phi_{\varrho}&+\Omega_5\int_c^{\ell}(\ell-\varpi)\hbar\big(\varpi,\varrho(\varpi),\theta(\varpi)\big)\,d\varpi\\
&+\frac{\Omega_6}{\Gamma(\rho-2)}\int_c^{\ell}(\ell-\varpi)^{\rho-1}\hbar\big(\varpi,\varrho(\varpi),\theta(\varpi)\big)\,d\varpi.
\end{aligned}
$$

From our assumptions in (6.3), we have $\Phi_{\theta}=\Phi_{\overline{\theta}}$ and $\Phi_{\varrho}=\Phi_{\overline{\varrho}}$. Hence by Lemma 6.3 and (5.2), we obtain

$$
\begin{aligned}
&\left|\overline{\theta}(\ell)-\theta(\ell)\right|\\
&\quad=\bigg|\overline{\theta}(\ell)-\Phi_{\theta}-\Omega_5\int_c^{\ell}(\ell-\varpi)\hbar\big(\varpi,\theta(\varpi),\varrho(\varpi)\big)\,d\varpi\\
&\qquad-\frac{\Omega_6}{\Gamma(\rho-2)}\int_c^{\ell}(\ell-\varpi)^{\rho-1}\hbar\big(\varpi,\theta(\varpi),\varrho(\varpi)\big)\,d\varpi\bigg|
\end{aligned}
$$

$$
\begin{aligned}
&+ \Omega_5 \int_c^\ell (\ell - \varpi)\left|\hbar\big(\varpi, \overline{\theta}(\varpi), \overline{\varrho}(\varpi)\big) - \hbar\big(\varpi, \theta(\varpi), \varrho(\varpi)\big)\right| d\varpi \\
&+ \frac{\Omega_6}{\Gamma(\rho-2)} \int_c^\ell (\ell - \varpi)^{\rho-1}\left|\hbar\big(\varpi, \overline{\theta}(\varpi), \overline{\varrho}(\varpi)\big) - \hbar\big(\varpi, \theta(\varpi), \varrho(\varpi)\big)\right| d\varpi \\
&\leq \frac{\varepsilon}{2}\left(\frac{\wp_1(d-c)}{\Xi} + \wp_2\right) \\
&+ \frac{A_\hbar}{2}\left(\frac{\Omega_5(d-c)^2}{2} + \frac{\Omega_6(d-c)^\rho}{\Gamma(\rho-1)}\right)\big(\|\overline{\theta} - \theta\| + \left\|\overline{\varrho}(\ell) - \varrho(\ell)\right\|\big).
\end{aligned} \tag{6.4}
$$

Similarly,

$$
\begin{aligned}
&\left|\overline{\varrho}(\ell) - \varrho(\ell)\right| \\
&= \left|\overline{\varrho}(\ell) - \Phi_\theta - \Omega_5 \int_c^\ell (\ell - \varpi)\hbar\big(\varpi, \varrho(\varpi), \theta(\varpi)\big)\, d\varpi \right. \\
&\left. - \frac{\Omega_6}{\Gamma(\rho-2)} \int_c^\ell (\ell - \varpi)^{\rho-1}\hbar\big(\varpi, \varrho(\varpi), \theta(\varpi)\big)\, d\varpi \right| \\
&+ \Omega_5 \int_c^\ell (\ell - \varpi)\left|\hbar\big(\varpi, \overline{\varrho}(\varpi), \overline{\theta}(\varpi)\big) - \hbar\big(\varpi, \varrho(\varpi), \theta(\varpi)\big)\right| d\varpi \\
&+ \frac{\Omega_6}{\Gamma(\rho-2)} \int_c^\ell (\ell - \varpi)^{\rho-1}\left|\hbar\big(\varpi, \overline{\varrho}(\varpi), \overline{\theta}(\varpi)\big) - \hbar\big(\varpi, \varrho(\varpi), \theta(\varpi)\big)\right| d\varpi \\
&\leq \frac{\varepsilon}{2}\left(\frac{\wp_1(d-c)}{\Xi} + \wp_2\right) \\
&+ \frac{A_\hbar}{2}\left(\frac{\Omega_5(d-c)^2}{2} + \frac{\Omega_6(d-c)^\rho}{\Gamma(\rho-2)}\right)\big(\left\|\overline{\varrho}(\ell) - \varrho(\ell)\right\| + \|\overline{\theta} - \theta\|\big).
\end{aligned} \tag{6.5}
$$

Combining (6.4) and (6.5), we have

$$
\begin{aligned}
\|\overline{\theta} - \theta\| + \left\|\overline{\varrho}(\ell) - \varrho(\ell)\right\| &\leq \varepsilon\left(\frac{\wp_1(d-c)}{\Xi} + \wp_2\right) \\
&\quad + A_\hbar\left(\frac{\Omega_5(d-c)^2}{2} + \frac{\Omega_6(d-c)^\rho}{\Gamma(\rho-1)}\right)\big(\|\overline{\theta} - \theta\| + \left\|\overline{\varrho}(\ell) - \varrho(\ell)\right\|\big)
\end{aligned}
$$

or equivalently

$$
\|\overline{\theta} - \theta\| + \left\|\overline{\varrho}(\ell) - \varrho(\ell)\right\| \leq \frac{\varepsilon(\frac{\wp_1(d-c)}{\Xi} + \wp_2)}{1 - A_\hbar(\frac{\Omega_5(d-c)^2}{2} + \frac{\Omega_6(d-c)^\rho}{\Gamma(\rho-2)})} = S_\hbar\varepsilon,
$$

where

$$
S_\hbar = \frac{\frac{\wp_1(d-c)^2}{\Xi} + \wp_2}{1 - A_\hbar(\frac{\Omega_5(d-c)^2}{2} + \frac{\Omega_6(d-c)^\rho}{\Gamma(\rho-1)})}.
$$

Now, if we consider $\alpha_\hbar(\varepsilon) = S_\hbar\varepsilon$ so that $\alpha_\hbar(0) = 0$, then the coupled ABR-type FDEs (1.1) are GUH stable. □

Theorem 6.5 *Assume that hypothesis (5.2) holds. Then the coupled ABC-type FDEs (1.2) are UH stable if the following assumption is satisfied*

$$A_\hbar\left(\frac{\Omega_{11}(d-c)^2}{2}+\frac{\Omega_{12}(d-c)^\rho}{\Gamma(\rho+1)}\right)<1.$$

Proof In the same way as for the proof of Theorem 6.4, we can write

$$\|\overline{\theta}-\theta\|+\left\|\overline{\varrho}(\ell)-\varrho(\ell)\right\|\leq S_\hbar^*\varepsilon,$$

where

$$S_\hbar^*=\frac{\frac{\wp_3(d-c)}{\Xi}+\wp_4}{1-A_\hbar(\frac{\Omega_{11}(d-c)}{2}+\frac{\Omega_{12}(d-c)^\rho}{\Gamma(\rho+1)})}.$$

Also, if we put $\alpha_\hbar(\varepsilon)=S_\hbar^*\varepsilon$ so that $\alpha_\hbar(0)=0$, then the coupled ABC-type FDEs (1.2) are GUH stable. □

7 Supportive examples

In this section, we support Theorems 4.1, 5.3, 6.4, and 6.5 through the following examples:

Example 7.1 For $\rho\in(2,3]$, consider the following system:

$$\begin{cases}{}^{ABR}D_{c^+}^{\frac{5}{2}}\theta(\ell)={}^{ABR}D_{c^+}^{\frac{5}{2}}\varrho(\ell)=\frac{\ell^2}{20e^{\ell-1}}(\frac{|\theta(\ell)|+|\varrho(\ell)|}{1+|\theta(\ell)|+|\varrho(\ell)|}), & \ell\in(0,1),\\ \theta(0)=\varrho(0)=0, \qquad \theta(1)=\varrho(1)={}^{AB}I_{c^+}^{\frac{1}{2}}\theta(\frac{1}{2}).\end{cases} \tag{7.1}$$

Here, $D=[0,1]$, $\rho=\frac{5}{2}\in(2,3]$, $c=0$, $d=1$, $\varsigma=\xi=\frac{1}{2}$ and

$$\hbar\big(\ell,\theta(\ell),\varrho(\ell)\big)=\hbar\big(\ell,\varrho(\ell),\theta(\ell)\big)=\frac{\ell^2}{20e^{\ell-1}}\left(\frac{|\theta(\ell)|+|\varrho(\ell)|}{1+|\theta(\ell)|+|\varrho(\ell)|}\right).$$

If we consider $\ell\in[0,1]$ and $\theta,\varrho,\overline{\theta},\overline{\varrho}\in\mathbb{R}$, we find that

$$\left|\hbar\big(\ell,\theta(\ell),\varrho(\ell)\big)-\hbar\big(\ell,\overline{\theta}(\ell),\overline{\varrho}(\ell)\big)\right|\leq\frac{1}{20}\big(\left|\theta(\ell)-\overline{\theta}(\ell)\right|+\left|\varrho(\ell)-\overline{\varrho}(\ell)\right|\big).$$

Hence, condition (4.1) is satisfied with $A_\hbar=\frac{1}{10}$. Also, $\wp_1\approx0.4975$, $\wp_2\approx0.4071$, $\Xi\approx0.9093$ and $\mho\approx0.09542<1$. Then all hypotheses of Theorem 4.1 are fulfilled, and hence, the coupled ABR fractional problem (7.1) has a unique solution on $[0,1]$. Further, for each $\varepsilon=\max\{\varepsilon_1,\varepsilon_2\}$ and each $\theta,\varrho\in C([0,1],\mathbb{R})$ satisfying

$$\left|{}^{ABR}D_{c^+}^{2.5}\theta(\ell)-\Im_\theta(\ell)\right|\leq\frac{\varepsilon}{2}\quad\text{and}\quad\left|{}^{ABR}D_{c^+}^{\frac{1}{2}}\varrho(\ell)-\Im_\varrho(\ell)\right|\leq\frac{\varepsilon}{2},\quad\ell\in[0,1],$$

there are a solution $\overline{\theta},\overline{\varrho}\in C([0,1],\mathbb{R})$ of the coupled ABR fractional problem (7.1) with

$$\left|\theta(\ell)-\overline{\theta}(\ell)\right|+\left|\varrho(\ell)-\overline{\varrho}(\ell)\right|\leq S_\hbar^*\varepsilon,$$

where $S_{\hbar}$ can be easily calculated from

$$S_{\hbar} = \frac{\frac{\wp_1 (d-c)^2}{\Xi} + \wp_2}{1 - A_{\hbar}(\frac{\Omega_5 (d-c)^2}{2} + \frac{\Omega_6 (d-c)^{\rho}}{\Gamma(\rho-1)})} > 0.$$

Therefore, all requirements of Theorem 6.4 are satisfied. Hence, the coupled ABR fractional problem (7.1) is UH stable.

Example 7.2 Let $\rho \in (1,2]$ and consider the following problem

$$\begin{cases} {}^{ABC}D_{c^+}^{\frac{3}{2}}\theta(\ell) = {}^{ABC}D_{c^+}^{\frac{3}{2}}\varrho(\ell) = \frac{\ell^2}{20e^{\ell-1}}(\frac{|\theta(\ell)|+|\varrho(\ell)|}{1+|\theta(\ell)|+|\varrho(\ell)|}), \quad \ell \in (0,1), \\ \theta(0) = \varrho(0) = 0, \qquad \theta(1) = \varrho(1) = {}^{AB}I_{c^+}^{\frac{1}{2}}\theta(\frac{1}{2}). \end{cases} \tag{7.2}$$

Here, $D = [0,1]$, $\rho = \frac{3}{2} \in (2,3]$, $c = 0$, $d = 1$, $\varsigma = \xi = \frac{1}{2}$ and

$$\hbar\big(\ell,\theta(\ell),\varrho(\ell)\big) = \hbar\big(\ell,\varrho(\ell),\theta(\ell)\big) = \frac{\ell^2}{20e^{\ell-1}}\left(\frac{|\theta(\ell)| + |\varrho(\ell)|}{1 + |\theta(\ell)| + |\varrho(\ell)|}\right).$$

If we consider $\ell \in [0,1]$ and $\theta, \varrho, \overline{\theta}, \overline{\varrho} \in \mathbb{R}$, we find that

$$\begin{aligned} \big|\hbar\big(\ell,\theta(\ell),\varrho(\ell)\big) - \hbar\big(\ell,\overline{\theta}(\ell),\overline{\varrho}(\ell)\big)\big| &= \frac{\ell^2}{20e^{\ell-1}}\left|\frac{|\theta(\ell)| + |\varrho(\ell)|}{1 + |\theta(\ell)| + |\varrho(\ell)|} - \frac{|\overline{\theta}(\ell)| + |\overline{\varrho}(\ell)|}{1 + |\overline{\theta}(\ell)| + |\overline{\varrho}(\ell)|}\right| \\ &\leq \frac{1}{20}\big(\big|\theta(\ell) - \overline{\theta}(\ell)\big| + \big|\varrho(\ell) - \overline{\varrho}(\ell)\big|\big). \end{aligned}$$

Therefore, condition (5.2) is fulfilled with $A_{\hbar} = \frac{1}{10}$. Also, $\wp_3 \approx 0.5104$, $\wp_4 \approx 0.2618$, $\Xi \approx 0.82798$ and $\mho_2 \approx 0.08782 < 1$. Hence, all assumptions of Theorem 5.3 are fulfilled. Thus, the coupled ABC fractional problem (7.2) has a unique solution on $[0,1]$. In addition, for each $\varepsilon = \max\{\varepsilon_1, \varepsilon_2\}$ and each $\theta, \varrho \in C([0,1],\mathbb{R})$ satisfying

$$\big|{}^{ABC}D_{c^+}^{2.5}\theta(\ell) - \Im_{\theta}(\ell)\big| \leq \frac{\varepsilon}{2} \quad \text{and} \quad \big|{}^{ABC}D_{c^+}^{\frac{1}{2}}\varrho(\ell) - \Im_{\varrho}(\ell)\big| \leq \frac{\varepsilon}{2}, \quad \ell \in [0,1],$$

there are a solution $\overline{\theta}, \overline{\varrho} \in C([0,1],\mathbb{R})$ of the coupled ABC fractional problem (7.2) with

$$\big|\theta(\ell) - \overline{\theta}(\ell)\big| + \big|\varrho(\ell) - \overline{\varrho}(\ell)\big| \leq S_{\hbar}\varepsilon,$$

where $S_{\hbar}$ can be easily calculated from

$$S_{\hbar}^{*} = \frac{\frac{\wp_3 (d-c)}{\Xi} + \wp_4}{1 - A_{\hbar}(\frac{\Omega_{11}(d-c)}{2} + \frac{\Omega_{12}(d-c)^{\rho}}{\Gamma(\rho+1)})} > 0.$$

Therefore, all requirements of Theorem 6.5 are satisfied. Hence, the coupled ABC fractional problem (7.2) is UH stable.

8 Conclusion and future works

Researchers who have examined and established some qualitative aspects of solutions to FDEs utilizing such operators have recently become interested in the theory of fractional operators in the framework of Atangana and Baleanu.

The EaU of solutions to the coupled nonlinear AB FDEs subject to integral stipulations have been devised and studied to serve this goal. Additionally, stability analysis from mathematical analytical methods has been examined. To support our findings, two examples are provided. Our method in this study is novel because it relies on a minimal number of hypotheses and allows us to provide the results on existence, uniqueness, and UH stability without using the semigroup characteristic. Our strategy is founded on the transformation of the provided problem into an FIE and the application of several common FP theorems resulting from Banach and Krasnoselskii-type theorems. We further examined the stability findings in the UH sense using mathematical analysis approaches. In future works, we will investigate brand-new numerical discoveries relating to this operator with a higher order. We also look forward to what will happen under the Mittag-Leffler kernels.

Funding

This work is funded through research groups program under grant R.G.P.2/207/43 provided by the Deanship of Scientific Research at King Khalid University, Saudi Arabia.

Abbreviations

FDs, Fractional derivatives; AB, Atangana and Baleanu; EaU, Existence and uniqueness; ABR, Atangana–Baleanu–Riemann; ABC, Atangana–Baleanu–Caputo; FDEs, Fractional differential equations; FIDEs, Fractional integro-differential equations; CFD, Caputo–Fabrizio derivative; FP, Fixed point; HU, Hyers–Ulam; GHU, Generalized Hyers–Ulam; FIEs, Fractional integral equations.

Availability of data and materials

No data is associated with this study.

Declarations

Competing interests

The authors declare no competing interests.

Author contributions

H.A.H. wrote the main manuscript text and M. Z. edited and revised the article.

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Publisher's Note

Received: 29 July 2022 Accepted: 6 December 2022 Published online: 16 December 2022

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