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Fractional evolution equation with Cauchy data in L^p spaces

Nguyen Duc Phuong[1], Dumitru Baleanu[2,3], Ravi P. Agarwal[4] and Le Dinh Long[5,6*]

*Correspondence:
ledinhlong@vlu.edu.vn
[5]Division of Applied Mathematics, Science and Technology Advanced Institute, Van Lang University, Ho Chi Minh City, Vietnam
[6]Faculty of Applied Technology, School of Engineering and Technology, Van Lang University, Ho Chi Minh City, Vietnam
Full list of author information is available at the end of the article

Abstract

In this paper, we consider the Cauchy problem for fractional evolution equations with the Caputo derivative. This problem is not well posed in the sense of Hadamard. There have been many results on this problem when data is noisy in L^2 and H^s. However, there have not been any papers dealing with this problem with observed data in L^p with $p \neq 2$. We study three cases of source functions: homogeneous case, inhomogeneous case, and nonlinear case. For all of them, we use a truncation method to give an approximate solution to the problem. Under different assumptions on the smoothness of the exact solution, we get error estimates between the regularized solution and the exact solution in L^p. To our knowledge, L^p evaluations for the inverse problem are very limited. This work generalizes some recent results on this problem.

MSC: 60G15; 60G22; 60G52; 60G57

Keywords: Fractional evolution equation; Caputo derivative; Cauchy problem; Fourier truncation regularization; Sobolev embeddings

1 Introduction

Today, fractional calculus involves the investigation of so-called integral operators and fractional derivatives over real or complex domains and their applications. There are many types of mathematical models that require the use of noninteger order derivatives. Recently, it has been widely used in modeling practical models because of its ability to provide approximation and ignore the influence of external forces such as in physics, engineering, mechanics science, biology, and some other areas [1–5]. There are several versions of noninteger derivatives, but perhaps the two types of derivatives, Caputo and Riemann–Liouville, are of most interest to mathematicians [6–15].

Let Ω be a bounded domain in $\mathbb{R}^d$ $(d \geq 1)$ with sufficiently smooth boundary $\partial\Omega$. In this paper, we are interested in the following evolution equation with a time-fractional derivative:

$$\begin{cases} {}_C D_t^\alpha u + \Delta u = F(t,x;u), & \text{in } \Omega \times (0,T], \\ u|_{\partial\Omega} = 0, & \text{in } \Omega, \end{cases} \tag{1.1}$$

with the initial value data

$$u(x,0) = g(x), \qquad u_t(x,0) = \varphi(x), \tag{1.2}$$

where F is the source function and u describes the distribution of the temperature at position x and time t. In (1.1), $\alpha \in (1,2)$ is the fractional order and ${}_C D_t^\alpha$ denotes the Caputo fractional derivative with respect to t (time-fractional) and is defined by (see [16, 17])

$$\begin{cases} {}_C D_t^\alpha u(t,x) = \frac{1}{\Gamma(2-\alpha)} \int_0^t (t-z)^{1-\alpha} \frac{\partial^2 u}{\partial z^2}(z,x)\,\mathrm{d}z, & \text{for } 1 < \alpha < 2, \\ {}_C D_t^\alpha u(t,x) = \frac{\partial^\alpha u}{\partial t^\alpha}(t,x), & \text{for } \alpha = 1, 2, \end{cases} \tag{1.3}$$

and Γ is the gamma function. Note that if $\alpha = 2$, then Eq. (1.1) represents a Cauchy problem for elliptic equation. There are some main reasons why we are interested in studying Problem (1.1)–(1.2)

1.1 Our motivation

• The first reason is that Problem (1.1)–(1.2) tends to be the elliptic equation when $\alpha \to 2$. The Cauchy problem for elliptic equations has many applications in many physical systems such as plasma physics. There are many interesting papers that have investigated the Cauchy elliptic equation with classical derivative, for example, [15, 18, 19] and the references therein. During the simulation, there are several applied models that are described by partial differential equations with memory terms attached. For example, the problem of electrical conduction in biological tissues in the radio frequency range is governed by an elliptic equation with memory [20]. In many physical phenomena, we need to use non-integer order derivatives for elliptic equations, to simulate problems related to viscous models. In [9], the authors studied an elliptic equation with a condition on the boundary associated with a generalized Riemann–Liouville derivative of fractional order. Some other articles for an elliptic equation with fractional order have been studied in [21, 22].

• We have another detailed explanation of the application of Problem (1.1)–(1.2) when $\alpha \to 1$. Indeed, if $\alpha = 1$, we have the following problem:

$$\begin{cases} u_t + \Delta u = F(t,x;u), & \text{in } \Omega \times (0,T], \\ u|_{\partial\Omega} = 0, & \text{in } \Omega, \\ u(x,0) = g(x), & \text{in } \Omega. \end{cases} \tag{1.4}$$

By changing the variable $v(x,t) = u(x, T-t)$, we turn this problem into a final value problem

$$\begin{cases} v_t - \Delta u = F(t,x;v), & \text{in } \Omega \times (0,T], \\ v|_{\partial\Omega} = 0, & \text{in } \Omega, \\ v(x,T) = g(x), & \text{in } \Omega. \end{cases} \tag{1.5}$$

The backward in time problem for parabolic equation introduced as above is one of the classic inverse problems involving many applications and has been studied extensively over the past 50 years. Interesting work on Problem (1.5) can be consulted by (1.5).

Let us continue to add justification to our interest in elliptic equations with the Caputo derivative. We observe that the elliptic problem with the Caputo derivative is a subbranch of the elliptic equation with a nonlocal condition. According to the work of Bitsadze and Samarskii [23], the nonlocal elliptic equations have many applications in the theory of plasma. Furthermore, it is difficult for elliptic equations with classical derivatives to describe the past process. When the phenomenon needs to involve factors and past information, the Caputo derivative plays an important role. In addition, many problems in dynamical processes, electrochemistry, and signal processing lead to elliptic differential equations of fractional order.

• In the interesting paper, Jin and Rundell [24] considered the ill-posedness of Problem (1.1)–(1.2) in the simple case $F = 0$; however, the authors have not provided the approximation and error estimate. The first results investigating the above equation seem to be of [25] and [26]. Our present paper has generalized the previous results by [25] and [26] in the sense that the observed data belongs to L^p. It is a strong claim that our paper is part of a series of investigations on the ill-posedness of fractional diffusion equations, which have been published in a recent series of works by Baleanu and colleagues [27–29].

1.2 Our novelty and contribution

We note that research results on the inverse and ill-posed problem with observed data on L^p are very rare. This is a difficult topic attracting the interest of many mathematicians. The problem will be difficult when we get the observed data in L^p space with $p \neq 2$. Note that Parseval's equality cannot be applied to our problem with observed data in L^p space with $p \neq 2$. To overcome these challenges, we learn techniques from the article [30] and in the references [31–43]. This idea can be summed up as the importance of the technique of using embedding between L^p and $\mathbb{H}^s(\Omega)$. In this paper, we used the Fourier truncation method to regularize our problem.

The main contribution of our paper is as follows:

- The first contribution is to investigate the regularized problem for our problem in the cases: homogeneous case, inhomogeneous case, and nonlinear case.
- We showed the existence of a regularized problem with a nonlinear source by the Banach fixed point theorem. Complexity occurs when we encounter components that involve the Mittag-Leffler functions. We need to provide some background and knowledge about the bound of these functions.
- The last main contribution is to provide the error estimate in L^p space when we observe the noisy data in L^p space. As we know, problems involving L^p are always complicated. So we need some new techniques to handle it.

This paper is organized as follows. In Sect. 2, we give some knowledge about some functional spaces and some properties of the bounds of Mittag-Leffler terms. Section 3 considers the homogeneous problem. In Sect. 4, we study the Cauchy problem for the inhomogeneous case. Finally, in Sect. 5, we treat the nonlinear case. In each case, we introduce a regularized solution of the Fourier truncation form. Then, we provide the error between the regularized solution and the exact solution.

2 Preliminary results

This section provides some notation and the functional spaces which will be used throughout this article. Recall that the spectral problem

$$\begin{cases} \Delta e_j(x) = -\lambda_j e_j(x), & x \in \Omega, \\ e_j(x) = 0, & x \in \partial\Omega, \end{cases}$$

admits the eigenvalues $0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_j \le \cdots$ with $\lambda_j \to \infty$ as $j \to \infty$ and corresponding eigenfunctions $e_j \in H_0^1(\Omega)$.

Definition 2.1 (Hilbert scale space) We recall the Hilbert scale space $\mathbb{H}^s(\Omega)$ given as follows:

$$\mathbb{H}^s(\Omega) = \left\{ f \in L^2(\Omega) \,\Big|\, \sum_{j=1}^{\infty} \lambda_j^{2s} \left(\int_\Omega f(x) e_j(x)\,\mathrm{d}x \right)^2 < \infty \right\}$$

for any $s \ge 0$. It is well known that $\mathbb{H}^s(\Omega)$ is a Hilbert space corresponding to the norm

$$\|f\|_{\mathbb{H}^s(\Omega)} = \left(\sum_{j=1}^{\infty} \lambda_j^{2s} \left(\int_\Omega f(x) e_j(x)\,\mathrm{d}x \right)^2 \right)^{1/2}, \quad f \in \mathbb{H}^s(\Omega).$$

Lemma 2.1 *The following statements are true*:

$$\begin{aligned} &L^p(\Omega) \hookrightarrow \mathbb{H}^\mu(\Omega) \quad \text{if } \frac{d}{4} < \mu \le 0 \text{ and } p \ge \frac{2d}{d-4\mu}, \\ &\mathbb{H}^s(\Omega) \hookrightarrow L^p(\Omega) \quad \text{if } 0 \le s < \frac{d}{4} \text{ and } p \le \frac{2d}{d-4s}. \end{aligned} \tag{2.1}$$

Also, for $M, n > 0$, we introduce the n-order Gevrey class $\mathbb{G}_M^n(\Omega)$ of L^2-functions, see e.g. [44], defined by the spectrum of the Laplacian as follows:

$$\mathbb{G}_M^n(\Omega) := \left\{ \psi \in L^2(\Omega) : \sum_{j=1}^{\infty} \lambda_j^n \exp\bigl(2M\lambda_j^{\frac{1}{\alpha}}\bigr) \left| \int_\Omega \psi(x) e_j(x)\,\mathrm{d}x \right|^2 < \infty \right\},$$

equipped with the following norm:

$$\|\psi\|_{\mathbb{G}_M^n(\Omega)} = \left(\sum_{j=1}^{\infty} \lambda_j^n \exp\bigl(2M\lambda_j^{\frac{1}{\alpha}}\bigr) \left| \int_\Omega \psi(x) e_j(x)\,\mathrm{d}x \right|^2 \right)^{1/2}.$$

Definition 2.2 The Mittag-Leffler function is defined by

$$E_{\alpha,\theta}(z) = \sum_{m=0}^{\infty} \frac{z^m}{\Gamma(\alpha m + \theta)}, \quad z \in \mathbb{C}, \tag{2.2}$$

where $\alpha > 0$ and $\theta \in \mathbb{R}$ are arbitrary constants.

The following lemmas provide upper and lower bounds of the Mittag-Leffler functions $E_{\alpha,1}(z)$, $E_{\alpha,2}(z)$, $E_{\alpha,\alpha}(z)$ by the exponential functions.

Lemma 2.2 (see [45]) *Let* $0 < \alpha_0 < \alpha_1 < 2$ *and* $\alpha \in [\alpha_0, \alpha_1]$. *Then there exist constants* $M_1, M_2 > 0$ *and* $z > 0$ *such that*

$$\frac{M_1}{\alpha}\exp\bigl(z^{\frac{1}{\alpha}}\bigr) \le E_{\alpha,1}(z) \le \frac{M_2}{\alpha}\exp\bigl(z^{\frac{1}{\alpha}}\bigr),$$
$$\frac{M_1}{\alpha}\frac{\exp(z^{\frac{1}{\alpha}})}{z^{\frac{1}{\alpha}}} \le E_{\alpha,2}(z) \le \frac{M_2}{\alpha}\frac{\exp(z^{\frac{1}{\alpha}})}{z^{\frac{1}{\alpha}}},$$
$$\frac{M_1}{\alpha}\frac{\exp(z^{\frac{1}{\alpha}})}{z^{1-\frac{1}{\alpha}}} \le E_{\alpha,\alpha}(z) \le \frac{M_2}{\alpha}\frac{\exp(z^{\frac{1}{\alpha}})}{z^{1-\frac{1}{\alpha}}}.$$

Lemma 2.3 (see [26]) *Let* $\alpha \in [\alpha_0, \alpha_1]$ *with* $1 < \alpha_0 < \alpha_1 < 2$ *and* $z \in [0, T]$. *Then there exists a positive constant* $\overline{M} > 0$ *independent of* z *such that*

$$E_{\alpha,1}\bigl(\lambda_j z^\alpha\bigr) \le \frac{\overline{M}}{\alpha}\exp\bigl(\lambda_j^{\frac{1}{\alpha}} z\bigr), \tag{2.3}$$
$$zE_{\alpha,2}\bigl(\lambda_j z^\alpha\bigr) \le \frac{\overline{M}}{\alpha}\lambda_j^{-\frac{1}{\alpha}}\exp\bigl(\lambda_j^{\frac{1}{\alpha}} z\bigr), \tag{2.4}$$
$$z^{\alpha-1}E_{\alpha,\alpha}\bigl(\lambda_j z^\alpha\bigr) \le \frac{\overline{M}}{\alpha}\lambda_j^{\frac{1-\alpha}{\alpha}}\exp\bigl(\lambda_j^{\frac{1}{\alpha}} z\bigr). \tag{2.5}$$

Next, let us give the explicit fomula of the mild solution to Problem (1.1)–(1.2). Suppose that Problem (1.1)–(1.2) has a solution $u(t,x) = \sum_{j=1}^{\infty}[\int_\Omega u(t,x)e_j(x)\,\mathrm{d}x]e_j(x)$. Then the function $u_j(t) = \int_\Omega u(t,x)e_j(x)\,\mathrm{d}x$ solves the following ordinary differential equation:

$$\begin{cases} {}_C D_t^\alpha u_j(t) - \lambda_j u_j(t) = \int_\Omega G(t,x,u(t,x))e_j(x)\,\mathrm{d}x, \\ u_j(0) = \int_\Omega \psi(x)e_j(x)\,\mathrm{d}x, \\ \frac{\mathrm{d}}{\mathrm{d}t}u_j(0) = \int_\Omega \varphi(x)e_j(x)\,\mathrm{d}x. \end{cases} \tag{2.6}$$

By applying the method in [17, Sect. 2], we obtain the solution of (2.6) as follows:

$$\begin{aligned} u_j(t) &= E_{\alpha,1}\bigl(\lambda_j t^\alpha\bigr)\biggl[\int_\Omega \psi(x)e_j(x)\,\mathrm{d}x\biggr] + tE_{\alpha,2}\bigl(\lambda_j t^\alpha\bigr)\biggl[\int_\Omega \varphi(x)e_j(x)\,\mathrm{d}x\biggr] \\ &\quad + \int_0^t (t-\nu)^{\alpha-1}E_{\alpha,\alpha}\bigl(\lambda_j(t-\nu)^\alpha\bigr)\biggl[\int_\Omega G\bigl(\nu,x,u(\nu,x)\bigr)e_j(x)\,\mathrm{d}x\biggr]\,\mathrm{d}\nu. \end{aligned}$$

Consequently, the Fourier series $u \in L^2(\Omega)$ is given as follows:

$$\begin{aligned} u(t,x) &= \sum_{j=1}^{\infty} E_{\alpha,1}\bigl(\lambda_j t^\alpha\bigr)\biggl[\int_\Omega \psi(x)e_j(x)\,\mathrm{d}x\biggr]e_j(x) \\ &\quad + \sum_{j=1}^{\infty} tE_{\alpha,2}\bigl(\lambda_j t^\alpha\bigr)\biggl[\int_\Omega \varphi(x)e_j(x)\,\mathrm{d}x\biggr]e_j(x) \\ &\quad + \sum_{j=1}^{\infty}\biggl[\int_0^t (t-\nu)^{\alpha-1}E_{\alpha,\alpha}\bigl(\lambda_j(t-\nu)^\alpha\bigr)\biggl(\int_\Omega G\bigl(\nu,x,u(\nu,x)\bigr)e_j(x)\,\mathrm{d}x\biggr)\,\mathrm{d}\nu\biggr]e_j(x). \end{aligned} \tag{2.7}$$

3 Cauchy problem with the homogeneous case

In this section, we consider the following problem:

$$\begin{cases} {}_C D_t^\alpha u + \Delta u = 0, & \text{in } \Omega \times (0,T], \\ u|_{\partial\Omega} = 0, & \text{in } \Omega, \\ u(x,0) = \psi(x), \quad u_t(x,0) = \varphi(x), & \text{in } \Omega. \end{cases} \tag{3.1}$$

The solution of the homogeneous problem has the following series representation:

$$\begin{aligned} u(t,x) = {} & \sum_{j=1}^{\infty} E_{\alpha,1}\big(\lambda_j t^\alpha\big)\bigg[\int_\Omega \psi(x)e_j(x)\,\mathrm{d}x\bigg]e_j(x) \\ & + \sum_{j=1}^{\infty} tE_{\alpha,2}\big(\lambda_j t^\alpha\big)\bigg[\int_\Omega \varphi(x)e_j(x)\,\mathrm{d}x\bigg]e_j(x). \end{aligned} \tag{3.2}$$

For $\beta > 0$, we construct a regularized solution as follows:

$$\begin{aligned} W_\beta(t,x) = {} & \sum_{j=1}^{\mathcal{D}(\beta)} E_{\alpha,1}\big(\lambda_j t^\alpha\big)\bigg[\int_\Omega \psi_\beta(x)e_j(x)\,\mathrm{d}x\bigg]e_j(x) \\ & + \sum_{j=1}^{\mathcal{D}(\beta)} tE_{\alpha,2}\big(\lambda_j t^\alpha\big)\bigg[\int_\Omega \varphi_\beta(x)e_j(x)\,\mathrm{d}x\bigg]e_j(x), \end{aligned} \tag{3.3}$$

here $\mathcal{D}(\beta)$ depends only on β.

Theorem 3.1 *Let $\beta,\delta > 0$, $0 < p < \frac{d}{4}$, and $1 < m < 2$. We presume that the Cauchy data (ψ,φ) is disturbed by the noisy data $(\psi_\beta,\varphi_\beta) \in L^m(\Omega) \times L^m(\Omega)$ such that*

$$\|\psi - \psi_\beta\|_{L^m(\Omega)} + \|\varphi - \varphi_\beta\|_{L^m(\Omega)} \le \beta. \tag{3.4}$$

Then, if $u \in L^\infty(0,T;\mathbb{H}^{p+\delta}(\Omega))$, we obtain the following estimate:

$$\begin{aligned} \big\|u(t,\cdot) - W_\beta(t,\cdot)\big\|_{L^{\frac{2d}{d-4p}}(\Omega)} \le {} & (T+1)C_1 C(d,m,p)\exp\big(|\overline{C}|^{\frac{1}{\alpha}} T\big|\mathcal{D}(\beta)\big|^{\frac{1}{\alpha d}}\big)\big|\mathcal{D}(\beta)\big|^{\frac{2p}{d}-\frac{m-2}{2m}}\beta \\ & + C(d,p)(\widetilde{C})^{-\delta}\big|\mathcal{D}(\beta)\big|^{-\frac{2\delta}{d}}\|u\|_{L^\infty(0,T;\mathbb{H}^{p+\delta}(\Omega))}, \end{aligned}$$

where the constants are defined in the proof.

Proof Since $1 < m < 2$, we find that $L^m(\Omega) \hookrightarrow \mathbb{H}^{\frac{dm-2d}{4m}}(\Omega)$. This embedding allows us to give that

$$\begin{aligned} & \|\psi - \psi_\beta\|_{\mathbb{H}^{\frac{dm-2d}{4m}}(\Omega)} + \|\varphi - \varphi_\beta\|_{\mathbb{H}^{\frac{dm-2d}{4m}}(\Omega)} \\ & \quad \le C(d,m)\|\psi - \psi_\beta\|_{L^m(\Omega)} + C(d,m)\|\varphi - \varphi_\beta\|_{L^m(\Omega)} \\ & \quad \le \beta C(d,m). \end{aligned} \tag{3.5}$$

It is obvious that

$$\left\|u(t,\cdot)-W_\beta(t,\cdot)\right\|_{\mathbb{H}^p(\Omega)} \le \left\|u(t,\cdot)-v_\beta(t,\cdot)\right\|_{\mathbb{H}^p(\Omega)} + \left\|v_\beta(t,\cdot)-W_\beta(t,\cdot)\right\|_{\mathbb{H}^p(\Omega)}, \tag{3.6}$$

where the function v_β is defined by

$$\begin{aligned} v_\beta(t,x) &= \sum_{j=1}^{\mathcal{D}(\beta)} E_{\alpha,1}\big(\lambda_j t^\alpha\big)\left[\int_\Omega \psi(x)e_j(x)\,\mathrm{d}x\right]e_j(x) \\ &\quad + \sum_{j=1}^{\mathcal{D}(\beta)} tE_{\alpha,2}\big(\lambda_j t^\alpha\big)\left[\int_\Omega \varphi(x)e_j(x)\,\mathrm{d}x\right]e_j(x). \end{aligned} \tag{3.7}$$

In view of Parseval's equality, the second quantity on the right-hand side of (3.6) is bounded by

$$\left\|v_\beta(t,\cdot)-W_\beta(t,\cdot)\right\|_{\mathbb{H}^p(\Omega)} \le \mathcal{J}_1(t)+\mathcal{J}_2(t), \tag{3.8}$$

where

$$\mathcal{J}_1(t) = \left\|\sum_{j=1}^{\mathcal{D}(\beta)} E_{\alpha,1}\big(\lambda_j t^\alpha\big)\left[\int_\Omega \big(\psi_\beta(x)-\psi(x)\big)e_j(x)\,\mathrm{d}x\right]e_j(x)\right\|_{\mathbb{H}^p(\Omega)} \tag{3.9}$$

and

$$\mathcal{J}_2(t) = \left\|\sum_{j=1}^{\mathcal{D}(\beta)} tE_{\alpha,2}\big(\lambda_j t^\alpha\big)\left[\int_\Omega \big(\varphi_\beta(x)-\varphi(x)\big)e_j(x)\,\mathrm{d}x\right]e_j(x)\right\|_{\mathbb{H}^p(\Omega)}. \tag{3.10}$$

Our upcoming task is to provide the upper bounds for the two components $\mathcal{J}_1(t)$ and $\mathcal{J}_2(t)$. By looking at Parseval's equality for the term $\mathcal{J}_1(t)$, we have the following equality:

$$\begin{aligned} \big|\mathcal{J}_1(t)\big|^2 &= \sum_{j=1}^{\mathcal{D}(\beta)} \lambda_j^{2p}\big|E_{\alpha,1}\big(\lambda_j t^\alpha\big)\big|^2\left[\int_\Omega \big(\psi_\beta(x)-\psi(x)\big)e_j(x)\,\mathrm{d}x\right]^2 \\ &= \sum_{j=1}^{\mathcal{D}(\beta)} \lambda_j^{2p-\frac{dm-2d}{2m}}\big|E_{\alpha,1}\big(\lambda_j t^\alpha\big)\big|^2 \lambda_j^{\frac{dm-2d}{2m}}\left[\int_\Omega \big(\psi_\beta(x)-\psi(x)\big)e_j(x)\,\mathrm{d}x\right]^2. \end{aligned} \tag{3.11}$$

In view of inequality (2.3) as in Lemma (2.3), we find that

$$\lambda_j^{2p-\frac{dm-2d}{2m}}\big|E_{\alpha,1}\big(\lambda_j t^\alpha\big)\big|^2 \le \left|\frac{\overline{M}}{\alpha}\right|^2 \exp\big(2\lambda_j^{\frac{1}{\alpha}}t\big)\lambda_j^{2p-\frac{dm-2d}{2m}}. \tag{3.12}$$

It is obvious to see that

$$\lambda_j \le \overline{C}j^{2/d} \le \overline{C}\big|\mathcal{D}(\beta)\big|^{\frac{2}{d}} \quad \text{if } j\le \mathcal{D}(\beta). \tag{3.13}$$

Thus, we get immediately that

$$\lambda_j^{2p-\frac{dm-2d}{2m}}\big|E_{\alpha,1}\big(\lambda_j t^\alpha\big)\big|^2 \le \left|\frac{\overline{M}}{\alpha}\right|^2 |\overline{C}|^{2p-\frac{dm-2d}{2m}} \exp\big(2|\overline{C}|^{\frac{1}{\alpha}}T\big|\mathcal{D}(\beta)\big|^{\frac{2}{\alpha d}}\big)\big|\mathcal{D}(\beta)\big|^{\frac{4p}{d}-\frac{m-2}{m}}. \tag{3.14}$$

It follows from (3.11) that

$$\begin{aligned}\big|\mathcal{J}_1(t)\big|^2 &\le |C_1|^2 \exp\big(2|\overline{C}|^{\frac{1}{\alpha}} T\big|\mathcal{D}(\beta)\big|^{\frac{2}{\alpha d}}\big)\big|\mathcal{D}(\beta)\big|^{\frac{4p}{d}-\frac{m-2}{m}} \\ &\quad\times \sum_{j=1}^{\mathcal{D}(\beta)} \lambda_j^{\frac{dm-2d}{2m}} \left[\int_\Omega \big(\psi_\beta(x)-\psi(x)\big)e_j(x)\,\mathrm{d}x\right]^2 \\ &\le |C_1|^2 \exp\big(2|\overline{C}|^{\frac{1}{\alpha}} T\big|\mathcal{D}(\beta)\big|^{\frac{2}{\alpha d}}\big)\big|\mathcal{D}(\beta)\big|^{\frac{4p}{d}-\frac{m-2}{m}} \|\psi-\psi_\beta\|^2_{\mathbb{H}^{\frac{dm-2d}{4m}}(\Omega)},\end{aligned} \tag{3.15}$$

where $|C_1|^2 = |\frac{\overline{M}}{\alpha}|^2|\overline{C}|^{2p-\frac{dm-2d}{2m}}$. Using (3.5), we find that

$$\begin{aligned}\mathcal{J}_1(t) &\le C_1 \exp\big(|\overline{C}|^{\frac{1}{\alpha}} T\big|\mathcal{D}(\beta)\big|^{\frac{1}{\alpha d}}\big)\big|\mathcal{D}(\beta)\big|^{\frac{2p}{d}-\frac{m-2}{2m}} \|\psi-\psi_\beta\|_{\mathbb{H}^{\frac{dm-2d}{4m}}(\Omega)} \\ &\le C_1 C(d,m) \exp\big(|\overline{C}|^{\frac{1}{\alpha}} T\big|\mathcal{D}(\beta)\big|^{\frac{1}{\alpha d}}\big)\big|\mathcal{D}(\beta)\big|^{\frac{2p}{d}-\frac{m-2}{2m}} \beta.\end{aligned} \tag{3.16}$$

We continue to estimate the second term $\mathcal{J}_2(t)$. Indeed, we get that

$$\begin{aligned}\big|\mathcal{J}_2(t)\big|^2 &= \sum_{j=1}^{\mathcal{D}(\beta)} \lambda_j^{2p} t^2 \big|E_{\alpha,2}\big(\lambda_j t^\alpha\big)\big|^2 \left[\int_\Omega \big(\varphi_\beta(x)-\varphi(x)\big)e_j(x)\,\mathrm{d}x\right]^2 \\ &= t^2 \sum_{j=1}^{\mathcal{D}(\beta)} \lambda_j^{2p-\frac{dm-2d}{2m}} \big|E_{\alpha,2}\big(\lambda_j t^\alpha\big)\big|^2 \lambda_j^{\frac{dm-2d}{2m}} \left[\int_\Omega \big(\varphi_\beta(x)-\varphi(x)\big)e_j(x)\,\mathrm{d}x\right]^2 \\ &\le T^2 |C_1|^2 \exp\big(2|\overline{C}|^{\frac{1}{\alpha}} T\big|\mathcal{D}(\beta)\big|^{\frac{2}{\alpha d}}\big)\big|\mathcal{D}(\beta)\big|^{\frac{4p}{d}-\frac{m-2}{m}} \|\varphi-\varphi_\beta\|^2_{\mathbb{H}^{\frac{dm-2d}{4m}}(\Omega)}.\end{aligned} \tag{3.17}$$

By the same explanations as above, we claim that

$$\mathcal{J}_2(t) \le T C_1 C(d,m) \exp\big(|\overline{C}|^{\frac{1}{\alpha}} T\big|\mathcal{D}(\beta)\big|^{\frac{1}{\alpha d}}\big)\big|\mathcal{D}(\beta)\big|^{\frac{2p}{d}-\frac{m-2}{2m}} \beta. \tag{3.18}$$

Combining (3.8), (3.15), and (3.18), we derive that

$$\begin{aligned}&\big\|v_\beta(t,\cdot) - W_\beta(t,\cdot)\big\|_{\mathbb{H}^p(\Omega)} \\ &\quad\le (T+1) C_1 C(d,m) \exp\big(|\overline{C}|^{\frac{1}{\alpha}} T\big|\mathcal{D}(\beta)\big|^{\frac{1}{\alpha d}}\big)\big|\mathcal{D}(\beta)\big|^{\frac{2p}{d}-\frac{m-2}{2m}} \beta.\end{aligned} \tag{3.19}$$

We continue by dealing with the first term on the right-hand side of (3.6). It is clear to see that

$$\big\|u(t,\cdot) - v_\beta(t,\cdot)\big\|^2_{\mathbb{H}^p(\Omega)} = \sum_{j>\mathcal{D}(\beta)} \lambda_j^{-2\delta} \lambda_j^{2p+2\delta} \left[\int_\Omega u(t,x) e_j(x)\,\mathrm{d}x\right]^2. \tag{3.20}$$

Because of the fact that

$$\lambda_j \ge \widetilde{C} j^{2/d} > \widetilde{C}\big|\mathcal{D}(\beta)\big|^{2/d},$$

we find that $\lambda_j^{-2\delta} \le (\widetilde{C})^{-2\delta}|\mathcal{D}(\beta)|^{-\frac{4\delta}{d}}$, which allows us to provide that

$$\begin{aligned} &\|u(t,\cdot)-v_\beta(t,\cdot)\|^2_{\mathbb{H}^p(\Omega)} \\ &\quad\le (\widetilde{C})^{-2\delta}|\mathcal{D}(\beta)|^{-\frac{4\delta}{d}} \sum_{j>\mathcal{D}(\beta)} \lambda_j^{2p+2\delta}\left[\int_\Omega u(t,x)e_j(x)\,\mathrm{d}x\right]^2 \\ &\quad\le (\widetilde{C})^{-2\delta}|\mathcal{D}(\beta)|^{-\frac{4\delta}{d}}\|u(t,\cdot)\|^2_{\mathbb{H}^{p+\delta}(\Omega)} \le (\widetilde{C})^{-2\delta}|\mathcal{D}(\beta)|^{-\frac{4\delta}{d}}\|u\|^2_{L^\infty(0,T;\mathbb{H}^{p+\delta}(\Omega))}. \end{aligned} \tag{3.21}$$

Therefore, we derive that

$$\|u(t,\cdot)-v_\beta(t,\cdot)\|_{\mathbb{H}^p(\Omega)} \le (\widetilde{C})^{-\delta}|\mathcal{D}(\beta)|^{-\frac{2\delta}{d}}\|u\|_{L^\infty(0,T;\mathbb{H}^{p+\delta}(\Omega))}. \tag{3.22}$$

Combining (3.6), (3.19), and (3.22), we get that

$$\begin{aligned} \|u(t,\cdot)-W_\beta(t,\cdot)\|_{\mathbb{H}^p(\Omega)} &\le \|v_\beta(t,\cdot)-W_\beta(t,\cdot)\|_{\mathbb{H}^p(\Omega)} + \|u(t,\cdot)-v_\beta(t,\cdot)\|_{\mathbb{H}^p(\Omega)} \\ &\le (T+1)C_1C(d,m)\exp\left(|\overline{C}|^{\frac{1}{\alpha}}T|\mathcal{D}(\beta)|^{\frac{1}{\alpha d}}\right)|\mathcal{D}(\beta)|^{\frac{2p}{d}-\frac{m-2}{2m}}\beta \\ &\quad + (\widetilde{C})^{-\delta}|\mathcal{D}(\beta)|^{-\frac{2\delta}{d}}\|u\|_{L^\infty(0,T;\mathbb{H}^{p+\delta}(\Omega))}. \end{aligned} \tag{3.23}$$

For $0 < p < d/4$, we recall the Sobolev embedding

$$\mathbb{H}^p(\Omega) \hookrightarrow L^{\frac{2d}{d-4p}}(\Omega).$$

Therefore, we obtain the following estimate:

$$\begin{aligned} \|u(t,\cdot)-W_\beta(t,\cdot)\|_{L^{\frac{2d}{d-4p}}(\Omega)} &\le C(d,p)\|u(t,\cdot)-W_\beta(t,\cdot)\|_{\mathbb{H}^p(\Omega)} \\ &\le (T+1)C_1C(d,m,p)\exp\left(|\overline{C}|^{\frac{1}{\alpha}}T|\mathcal{D}(\beta)|^{\frac{1}{\alpha d}}\right)|\mathcal{D}(\beta)|^{\frac{2p}{d}-\frac{m-2}{2m}}\beta \\ &\quad + C(d,p)(\widetilde{C})^{-\delta}|\mathcal{D}(\beta)|^{-\frac{2\delta}{d}}\|u\|_{L^\infty(0,T;\mathbb{H}^{p+\delta}(\Omega))}. \end{aligned} \qquad \square$$

4 Regularization in L^p space under inhomogeneous source term

In this section, we consider the following problem:

$$\begin{cases} {}_CD_t^\alpha u + \Delta u = G(x,t), & \text{in } \Omega\times(0,T], \\ u|_{\partial\Omega} = 0, & \text{in } \Omega, \\ u(x,0) = \psi(x), \quad u_t(x,0) = \varphi(x), & \text{in } \Omega. \end{cases} \tag{4.1}$$

The solution to Problem (4.1) possesses the following series representation:

$$\begin{aligned} u(t,x) &= \sum_{j=1}^\infty E_{\alpha,1}(\lambda_j t^\alpha)\left[\int_\Omega \psi(x)e_j(x)\,\mathrm{d}x\right]e_j(x) \\ &\quad + \sum_{j=1}^\infty tE_{\alpha,2}(\lambda_j t^\alpha)\left[\int_\Omega \varphi(x)e_j(x)\,\mathrm{d}x\right]e_j(x) \end{aligned}$$

$$+\sum_{j=1}^{\infty}\left[\int_0^t (t-\nu)^{\alpha-1}E_{\alpha,\alpha}\big(\lambda_j(t-\nu)^\alpha\big)\left(\int_\Omega G(\nu,x)e_j(x)\,\mathrm{d}x\right)\mathrm{d}\nu\right]e_j(x).$$

In addition, for $\beta > 0$, we construct the Fourier regularized solution as follows:

$$\begin{aligned}
V_\beta(t,x) &= \sum_{j=1}^{\mathcal{D}(\beta)} E_{\alpha,1}\big(\lambda_j t^\alpha\big)\left[\int_\Omega \psi_\beta(x)e_j(x)\,\mathrm{d}x\right]e_j(x)\\
&\quad+\sum_{j=1}^{\mathcal{D}(\beta)} tE_{\alpha,2}\big(\lambda_j t^\alpha\big)\left[\int_\Omega \varphi_\beta(x)e_j(x)\,\mathrm{d}x\right]e_j(x)\\
&\quad+\sum_{j=1}^{\mathcal{D}(\beta)}\left[\int_0^t (t-\nu)^{\alpha-1}E_{\alpha,\alpha}\big(\lambda_j(t-\nu)^\alpha\big)\left(\int_\Omega G_\beta(\nu,x)e_j(x)\,\mathrm{d}x\right)\mathrm{d}\nu\right]e_j(x).
\end{aligned}$$

Here, $\mathcal{D}(\beta)$ depends only on β.

Theorem 4.1 *Let $\beta,\delta > 0$, $0 < p < \frac{d}{4}$, and $1 < m < 2$. We presume that the Cauchy data (ψ,φ,G) is disturbed by the noisy data $(\psi_\beta,\varphi_\beta,G_\beta) \in L^m(\Omega)\times L^m(\Omega)\times L^2(0,T;L^m(\Omega))$ such that*

$$\|\psi-\psi_\beta\|_{L^m(\Omega)} + \|\varphi-\varphi_\beta\|_{L^m(\Omega)} + \|G_\beta - G\|_{L^2(0,T;L^m(\Omega))} \le \beta.$$

Then the following error estimate holds:

$$\begin{aligned}
\big\|u(t,\cdot)-V_\beta(t,\cdot)\big\|_{L^{\frac{2d}{d-4p}}(\Omega)} &\lesssim \exp\big(|\overline{C}|^{\frac{1}{\alpha}}T\big|\mathcal{D}(\beta)\big|^{\frac{1}{\alpha d}}\big)\big|\mathcal{D}(\beta)\big|^{\frac{2p}{d}-\frac{m-2}{2m}}\beta\\
&\quad+\frac{C\overline{M}}{\alpha}\exp\big(|\overline{C}|^{\frac{1}{\alpha}}T\big|\mathcal{D}(\beta)\big|^{\frac{2}{\alpha d}}\big)\big|\mathcal{D}(\beta)\big|^{\frac{2p}{d}+\frac{2-2\alpha}{d\alpha}-\frac{m-2}{2m}}\beta\\
&\quad+C(d,p)(\widetilde{C})^{-\delta}\big|\mathcal{D}(\beta)\big|^{-\frac{2\delta}{d}}\|u\|_{L^\infty(0,T;\mathbb{H}^{p+\delta}(\Omega))}.
\end{aligned}$$

Proof Set the following function:

$$\begin{aligned}
v_\beta(t,x) &= \sum_{j=1}^{\mathcal{D}(\beta)} E_{\alpha,1}\big(\lambda_j t^\alpha\big)\left[\int_\Omega \psi(x)e_j(x)\,\mathrm{d}x\right]e_j(x)\\
&\quad+\sum_{j=1}^{\mathcal{D}(\beta)} tE_{\alpha,2}\big(\lambda_j t^\alpha\big)\left[\int_\Omega \varphi(x)e_j(x)\,\mathrm{d}x\right]e_j(x)\\
&\quad+\sum_{j=1}^{\mathcal{D}(\beta)}\left[\int_0^t (t-\nu)^{\alpha-1}E_{\alpha,\alpha}\big(\lambda_j(t-\nu)^\alpha\big)\left(\int_\Omega G(\nu,x)e_j(x)\,\mathrm{d}x\right)\mathrm{d}\nu\right]e_j(x).
\end{aligned}$$

It is clear to see that

$$\big\|v_\beta(t,\cdot)-V_\beta(t,\cdot)\big\|_{\mathbb{H}^p(\Omega)} \le \mathcal{J}_1(t)+\mathcal{J}_2(t)+\mathcal{J}_3(t), \tag{4.2}$$

where $\mathcal{J}_1$, $\mathcal{J}_2$ are defined respectively in (3.9), (3.10), and the third term $\mathcal{J}_3$ is defined by

$$\begin{aligned}\mathcal{J}_3(t) = \Bigg\| \sum_{j=1}^{\mathcal{D}(\beta)} \bigg[\int_0^t (t-\nu)^{\alpha-1} E_{\alpha,\alpha}\big(\lambda_j (t-\nu)^\alpha\big) \\ \times \left(\int_\Omega \big(G_\beta(\nu,x) - G(\nu,x)\big) e_j(x)\,\mathrm{d}x \right) \mathrm{d}\nu \bigg] e_j(x) \Bigg\|_{\mathbb{H}^p(\Omega)}.\end{aligned} \tag{4.3}$$

Using (2.5) of Lemma (2.3), we derive

$$(t-\nu)^{\alpha-1} E_{\alpha,\alpha}\big(\lambda_j (t-\nu)^\alpha\big) \le \frac{\overline{M}}{\alpha} \lambda_j^{\frac{1-\alpha}{\alpha}} \exp\big(\lambda_j^{\frac{1}{\alpha}} (t-\nu)\big). \tag{4.4}$$

This inequality together with Parseval's equality leads to

$$\begin{aligned}\big|\mathcal{J}_3(t)\big|^2 &= \sum_{j=1}^{\mathcal{D}(\beta)} \lambda_j^{2p} \bigg[\int_0^t (t-\nu)^{\alpha-1} E_{\alpha,\alpha}\big(\lambda_j (t-\nu)^\alpha\big) \left(\int_\Omega \big(G_\beta(\nu,x) - G(\nu,x)\big) e_j(x)\,\mathrm{d}x \right) \mathrm{d}\nu \bigg]^2 \\ &\le \left| \frac{\overline{M}}{\alpha} \right|^2 \sum_{j=1}^{\mathcal{D}(\beta)} \lambda_j^{2p + \frac{2-2\alpha}{\alpha}} \\ &\quad \times \bigg[\int_0^t \exp\big(2\lambda_j^{\frac{1}{\alpha}} (t-\nu)\big) \left(\int_\Omega \big(G_\beta(\nu,x) - G(\nu,x)\big) e_j(x)\,\mathrm{d}x \right) \mathrm{d}\nu \bigg]^2.\end{aligned} \tag{4.5}$$

Since (3.13), we find that

$$\exp\big(2\lambda_j^{\frac{1}{\alpha}} (t-\nu)\big) \le \exp\big(2|\overline{C}|^{\frac{1}{\alpha}} T \big|\mathcal{D}(\beta)\big|^{\frac{2}{\alpha d}}\big). \tag{4.6}$$

Thus, using Hölder inequality, we find that

$$\begin{aligned}\big|\mathcal{J}_3(t)\big|^2 &\le \left| \frac{\overline{M}}{\alpha} \right|^2 \exp\big(2|\overline{C}|^{\frac{1}{\alpha}} T \big|\mathcal{D}(\beta)\big|^{\frac{2}{\alpha d}}\big) \\ &\quad \times \sum_{j=1}^{\mathcal{D}(\beta)} \lambda_j^{2p + \frac{2-2\alpha}{\alpha} - \frac{dm-2d}{2m}} \int_0^T \lambda_j^{\frac{dm-2d}{2m}} \left(\int_\Omega \big(G_\beta(\nu,x) - G(\nu,x)\big) e_j(x)\,\mathrm{d}x \right)^2 d\nu.\end{aligned} \tag{4.7}$$

Since (3.13), we know that

$$\lambda_j^{2p + \frac{2-2\alpha}{\alpha} - \frac{dm-2d}{2m}} \le \big(\overline{C}\big|\mathcal{D}(\beta)\big|^{\frac{2}{d}}\big)^{2p + \frac{2-2\alpha}{\alpha} - \frac{dm-2d}{2m}} = C(p,\alpha,m,d)\big|\mathcal{D}(\beta)\big|^{\frac{4p}{d} + \frac{4-4\alpha}{d\alpha} - \frac{m-2}{m}},$$

which leads to

$$\begin{aligned}\big|\mathcal{J}_3(t)\big|^2 &\le \left| \frac{C\overline{M}}{\alpha} \right|^2 \exp\big(2|\overline{C}|^{\frac{1}{\alpha}} T \big|\mathcal{D}(\beta)\big|^{\frac{2}{\alpha d}}\big) \big|\mathcal{D}(\beta)\big|^{\frac{4p}{d} + \frac{4-4\alpha}{d\alpha} - \frac{m-2}{m}} \\ &\quad \times \int_0^T \big\| G_\beta(\nu,\cdot) - G(\nu,\cdot) \big\|^2_{\mathbb{H}^{\frac{dm-2d}{4m}}(\Omega)} d\nu \\ &\le \left| \frac{C\overline{M}}{\alpha} \right|^2 \exp\big(2|\overline{C}|^{\frac{1}{\alpha}} T \big|\mathcal{D}(\beta)\big|^{\frac{2}{\alpha d}}\big) \big|\mathcal{D}(\beta)\big|^{\frac{4p}{d} + \frac{4-4\alpha}{d\alpha} - \frac{m-2}{m}} \\ &\quad \times \|G_\beta - G\|^2_{L^2(0,T;\mathbb{H}^{\frac{dm-2d}{4m}}(\Omega))}.\end{aligned} \tag{4.8}$$

Noting that Sobolev embedding

$$L^2\big(0,T;L^m(\Omega)\big) \hookrightarrow L^2\big(0,T;\mathbb{H}^{\frac{dm-2d}{4m}}(\Omega)\big),$$

we derive that

$$\|G_\beta - G\|^2_{L^2(0,T;\mathbb{H}^{\frac{dm-2d}{4m}}(\Omega))} \le C(d,m)\|G_\beta - G\|^2_{L^2(0,T;L^m(\Omega))} \le \beta^2 C(d,m). \tag{4.9}$$

Combining (4.8) and (4.9), we obtain that

$$\big|\mathcal{J}_3(t)\big| \lesssim \frac{C\overline{M}}{\alpha}\exp\big(|\overline{C}|^{\frac{1}{\alpha}}T\big|\mathcal{D}(\beta)\big|^{\frac{2}{\alpha d}}\big)\big|\mathcal{D}(\beta)\big|^{\frac{2p}{d}+\frac{2-2\alpha}{d\alpha}-\frac{m-2}{2m}}\beta. \tag{4.10}$$

Here, the hidden constant depends on d, m. Combining (3.16), (3.18), (4.10) and looking at (4.2), we arrive at

$$\begin{aligned}\big\|v_\beta(t,\cdot) - V_\beta(t,\cdot)\big\|_{\mathbb{H}^p(\Omega)} &\lesssim \exp\big(|\overline{C}|^{\frac{1}{\alpha}}T\big|\mathcal{D}(\beta)\big|^{\frac{1}{\alpha d}}\big)\big|\mathcal{D}(\beta)\big|^{\frac{2p}{d}-\frac{m-2}{2m}}\beta \\ &\quad + \frac{C\overline{M}}{\alpha}\exp\big(|\overline{C}|^{\frac{1}{\alpha}}T\big|\mathcal{D}(\beta)\big|^{\frac{2}{\alpha d}}\big)\big|\mathcal{D}(\beta)\big|^{\frac{2p}{d}+\frac{2-2\alpha}{d\alpha}-\frac{m-2}{2m}}\beta.\end{aligned} \tag{4.11}$$

Here, the hidden constant depends on d, m, T, C_1.

On the other hand, similar to (3.22), for any $u \in L^\infty(0,T;\mathbb{H}^{r+\delta}(\Omega))$, we also get the following estimate:

$$\big\|u(t,\cdot) - v_\beta(t,\cdot)\big\|_{\mathbb{H}^p(\Omega)} \le (\widetilde{C})^{-\delta}\big|\mathcal{D}(\beta)\big|^{-\frac{2\delta}{d}}\|u\|_{L^\infty(0,T;\mathbb{H}^{p+\delta}(\Omega))}. \tag{4.12}$$

In conclusion, we obtain the desired estimate

$$\begin{aligned}\big\|u(t,\cdot) - V_\beta(t,\cdot)\big\|_{L^{\frac{2d}{d-4p}}(\Omega)} &\lesssim \exp\big(|\overline{C}|^{\frac{1}{\alpha}}T\big|\mathcal{D}(\beta)\big|^{\frac{1}{\alpha d}}\big)\big|\mathcal{D}(\beta)\big|^{\frac{2p}{d}-\frac{m-2}{2m}}\beta \\ &\quad + \frac{C\overline{M}}{\alpha}\exp\big(|\overline{C}|^{\frac{1}{\alpha}}T\big|\mathcal{D}(\beta)\big|^{\frac{2}{\alpha d}}\big)\big|\mathcal{D}(\beta)\big|^{\frac{2p}{d}+\frac{2-2\alpha}{d\alpha}-\frac{m-2}{2m}}\beta \\ &\quad + C(d,p)(\widetilde{C})^{-\delta}\big|\mathcal{D}(\beta)\big|^{-\frac{2\delta}{d}}\|u\|_{L^\infty(0,T;\mathbb{H}^{p+\delta}(\Omega))}.\end{aligned}$$

The proof is completed. □

5 Regularization in L^p space under nonlinear source term

In this section, we consider the Cauchy problem for nonlinear problem. Indeed, we will focus on Problem (2.7). For the sake of brevity, we denote the operators

$$\mathcal{P}_1(t)v = \sum_{j=1}^{\infty} E_{\alpha,1}\big(\lambda_j t^\alpha\big)\bigg[\int_\Omega v(x)e_j(x)\,\mathrm{d}x\bigg]e_j(x) \tag{5.1}$$

and

$$\mathcal{P}_2(t)v = \sum_{j=1}^{\infty} tE_{\alpha,2}\big(\lambda_j t^\alpha\big)\bigg[\int_\Omega v(x)e_j(x)\,\mathrm{d}x\bigg]e_j(x), \tag{5.2}$$

and

$$\mathcal{P}_3(t)v = \sum_{j=1}^{\infty} t^{\alpha-1}E_{\alpha,\alpha}\big(\lambda_j t^{\alpha}\big)\left[\int_{\Omega} v(x)e_j(x)\,\mathrm{d}x\right]e_j(x) \tag{5.3}$$

for any $v \in L^2(\Omega)$. We rewrite the mild solution as the following form:

$$u(t,x) = \mathcal{P}_1(t)\psi + \mathcal{P}_2(t)\varphi + \int_0^t (t-\nu)^{\alpha-1}\mathcal{P}_3(t-\nu)G\big(\nu,x,u(\nu,x)\big)\,d\nu. \tag{5.4}$$

By using the Fourier method, we approximate Problem (2.7) by a new integral equation. Before providing this integral equation, we need to introduce the operator $\mathbb{S}_R$, which is the orthogonal projection onto the eigenspace span $\{e_j, \lambda_j \le R\}$ for any $R > 0$. Let any function $\theta \in L^2(\Omega)$, then we provide the truncation operator $\mathbb{S}_R\theta$ as follows:

$$\mathbb{S}_R\theta = \sum_{j=1}^{\lambda_j \le R}\left(\int_{\Omega}\theta(x)e_j(x)\,\mathrm{d}x\right)e_j(x). \tag{5.5}$$

Let us assume that the Cauchy data (ψ,φ) is noisy by the observation data $(\psi_\beta,\varphi_\beta) \in L^p(\Omega) \times L^p(\Omega)$ such that

$$\|\psi - \psi_\beta\|_{L^p(\Omega)} + \|\varphi - \varphi_\beta\|_{L^p(\Omega)} \le \beta. \tag{5.6}$$

Since this observation data $(\psi_\beta,\varphi_\beta)$, we can construct a regularized solution as follows:

$$Z_\beta(t,x) = \mathcal{P}_1(t)\mathbb{S}_{R_\beta}\psi_\beta + \mathcal{P}_2(t)\mathbb{S}_{R_\beta}\varphi_\beta + \int_0^t \mathcal{P}_3(t-\nu)\mathbb{S}_{R_\beta}G\big(\nu,x,Z_\beta(\nu,x)\big)\,d\nu. \tag{5.7}$$

Integral equation as above is called "regularized problem". Our main purpose in this section is to

- Show the existence and uniqueness of the mild solution to regularized problem (5.7);
- Estimate the upper bound of the regularized solution Z_β and the sought solution u on L^m space.

Theorem 5.1 *Let the observed data $(\psi_\beta,\varphi_\beta) \in L^p(\Omega) \times L^p(\Omega)$. Then Problem (5.7) has a unique solution $Z_\beta \in L^\infty_a(0,T;L^{\frac{2d}{d-4r}}(\Omega))$. In addition, we get the following bound:*

$$\begin{aligned}\|Z_\beta\|_{L^\infty_a(0,T;L^{\frac{2d}{d-4r}}(\Omega))} &\le 2\left|\frac{\overline{M}}{\alpha}\right|(R_\beta)^{r-\frac{dp-2d}{4p}}\exp\big(R_\beta^{\frac{1}{\alpha}}T\big)\|\psi_\beta\|_{L^p(\Omega)} \\ &\quad + 2\lambda_1^{-1/\alpha}\left|\frac{\overline{M}}{\alpha}\right|(R_\beta)^{r-\frac{dp-2d}{4p}}\exp\big(R_\beta^{\frac{1}{\alpha}}T\big)\|\varphi_\beta\|_{L^p(\Omega)}\end{aligned} \tag{5.8}$$

for any $a > R_\beta^{\frac{1}{\alpha}}$ and $0 \le r < \frac{d}{4}$.

Proof Let $w \in \mathbb{H}^q(\Omega)$. Then, for any $q' \ge q$, we have the following estimate by a simple computation:

$$\big\|\mathcal{P}_1(t)\mathbb{S}_{R_\beta}w\big\|_{\mathbb{H}^{q'}(\Omega)} = \left(\sum_{\lambda_j \le R_\beta}\lambda_j^{2q'-2q}\big|E_{\alpha,1}\big(\lambda_j t^{\alpha}\big)\big|^2\lambda_j^{2q}\left(\int_{\Omega}w(x)e_j(x)\,\mathrm{d}x\right)^2\right)^{\frac{1}{2}}. \tag{5.9}$$

In view of inequality (2.3) of Lemma (2.3), if $\lambda_j \le R_\beta$ then we find that

$$\lambda_j^{2q'-2q}\left|E_{\alpha,1}\left(\lambda_j t^\alpha\right)\right|^2 \le \left|\frac{\overline{M}}{\alpha}\right|^2 \lambda_j^{2q'-2q}\exp\left(2\lambda_j^{\frac{1}{\alpha}}t\right) \le \left|\frac{\overline{M}}{\alpha}\right|^2 (R_\beta)^{2q'-2q}\exp\left(2R_\beta^{\frac{1}{\alpha}}t\right). \tag{5.10}$$

From the two observations above, we deduce that

$$\left\|\mathcal{P}_1(t)\mathbb{S}_{R_\beta}w\right\|_{\mathbb{H}^{q'}(\Omega)} \le \left|\frac{\overline{M}}{\alpha}\right|(R_\beta)^{q'-q}\exp\left(R_\beta^{\frac{1}{\alpha}}t\right)\|w\|_{\mathbb{H}^q(\Omega)}. \tag{5.11}$$

By a similar techniques as above, for any $w \in \mathbb{H}^q(\Omega)$, we also find that

$$\left\|\mathcal{P}_2(t)\mathbb{S}_{R_\beta}w\right\|_{\mathbb{H}^{q'}(\Omega)} \le \lambda_1^{-1/\alpha}\left|\frac{\overline{M}}{\alpha}\right|(R_\beta)^{q'-q}\exp\left(R_\beta^{\frac{1}{\alpha}}t\right)\|w\|_{\mathbb{H}^q(\Omega)} \tag{5.12}$$

and

$$\left\|\mathcal{P}_3(t)\mathbb{S}_{R_\beta}w\right\|_{\mathbb{H}^{q'}(\Omega)} \le \lambda_1^{\frac{1-\alpha}{\alpha}}\left|\frac{\overline{M}}{\alpha}\right|(R_\beta)^{q'-q}\exp\left(R_\beta^{\frac{1}{\alpha}}t\right)\|w\|_{\mathbb{H}^q(\Omega)}. \tag{5.13}$$

Let $a > 0$ be a real constant. Denote by $L_a^\infty(0,T;L^{\frac{2d}{d-4r}}(\Omega))$ the subspace of $L^\infty(0,T;L^{\frac{2d}{d-4r}}(\Omega))$ associated with the following norm:

$$\|f\|_a := \max_{0\le t\le T}\left\|\exp\left(-a(T-t)\right)f(\cdot,t)\right\|_{L^{\frac{2d}{d-4r}}(\Omega)}, \quad \forall f \in L^\infty\left(0,T;L^{\frac{2d}{d-4r}}(\Omega)\right).$$

Let us define a nonlinear map $\mathcal{M} : L_a^\infty(0,T;L^{\frac{2d}{d-4r}}(\Omega)) \to L_a^\infty(0,T;L^{\frac{2d}{d-4r}}(\Omega))$ by

$$\mathcal{M}z(t,x) = \mathcal{P}_1(t)\mathbb{S}_{R_\beta}\psi_\beta + \mathcal{P}_2(t)\mathbb{S}_{R_\beta}\varphi_\beta + \int_0^t \mathcal{P}_3(t-\nu)\mathbb{S}_{R_\beta}G\left(\nu,x,z(\nu,x)\right)d\nu. \tag{5.14}$$

By inserting $z = 0$ into the above expression, we get immediately that

$$\mathcal{M}(z=0) = \mathcal{P}_1(t)\mathbb{S}_{R_\beta}\psi_\beta + \mathcal{P}_2(t)\mathbb{S}_{R_\beta}\varphi_\beta.$$

We aim to show that $\mathcal{M}(z=0) \in L_a^\infty(0,T;L^{\frac{2d}{d-4r}}(\Omega))$ for $0 \le r < \frac{d}{4}$. In view of the embedding $\mathbb{H}^r(\Omega) \hookrightarrow L^{\frac{2d}{d-4r}}(\Omega)$ and (5.11) with $q' = r \ge 0$ and $q = \frac{dp-2d}{4p} < 0$, we find that

$$\begin{aligned}\left\|\mathcal{P}_1(t)\mathbb{S}_{R_\beta}\psi_\beta\right\|_{L^{\frac{2d}{d-4r}}(\Omega)} &\lesssim \left\|\mathcal{P}_1(t)\mathbb{S}_{R_\beta}\psi_\beta\right\|_{\mathbb{H}^r(\Omega)} \\ &\le \left|\frac{\overline{M}}{\alpha}\right|(R_\beta)^{r-\frac{dp-2d}{4p}}\exp\left(R_\beta^{\frac{1}{\alpha}}t\right)\|\psi_\beta\|_{\mathbb{H}^{\frac{dp-2d}{4p}}(\Omega)}.\end{aligned}$$

It is obvious to see that the Sobolev embedding

$$L^p(\Omega) \hookrightarrow \mathbb{H}^{\frac{dp-2d}{4p}}(\Omega) \tag{5.15}$$

since $1 < p < 2$. Thus, we find that

$$\left\|\mathcal{P}_1(t)\mathbb{S}_{R_\beta}\psi_\beta\right\|_{L^{\frac{2d}{d-4r}}(\Omega)} \lesssim \left|\frac{\overline{M}}{\alpha}\right|(R_\beta)^{r-\frac{dp-2d}{4p}}\exp\left(R_\beta^{\frac{1}{\alpha}}t\right)\|\psi_\beta\|_{L^p(\Omega)}. \tag{5.16}$$

By a similar argument as above, we also show that

$$\left\|\mathcal{P}_2(t)\mathbb{S}_{R_\beta}\psi_\beta\right\|_{L^{\frac{2d}{d-4r}}(\Omega)} \lesssim \lambda_1^{-1/\alpha}\left|\frac{\overline{M}}{\alpha}\right|(R_\beta)^{r-\frac{dp-2d}{4p}}\exp\bigl(R_\beta^{\frac{1}{\alpha}}t\bigr)\|\varphi_\beta\|_{L^p(\Omega)}. \tag{5.17}$$

From the two observations above, we can deduce that $\mathcal{M}(z=0)\in L_a^\infty(0,T;L^{\frac{2d}{d-4r}}(\Omega))$.

Let any two functions $z_1,z_2\in L_a^\infty(0,T;L^{\frac{2d}{d-4r}}(\Omega))$ for any $a>0$. Then we have the following identity:

$$\mathcal{M}z_1(t,x)-\mathcal{M}z_2(t,x)=\int_0^t\mathcal{P}_3(t-\nu)\mathbb{S}_{R_\beta}\bigl(G\bigl(\nu,x,z_1(\nu,x)\bigr)-G\bigl(\nu,x,z_2(\nu,x)\bigr)\bigr)\,d\nu. \tag{5.18}$$

By looking at estimate (5) with $q'=r$ and $q=0$, we get the following evaluation:

$$\begin{aligned}
&\left\|\mathcal{P}_3(t-\nu)\mathbb{S}_{R_\beta}\bigl(G\bigl(\nu,x,z_1(\nu,x)\bigr)-G\bigl(\nu,x,z_2(\nu,x)\bigr)\bigr)\right\|_{\mathbb{H}^r(\Omega)}\\
&\quad\le\lambda_1^{\frac{1-\alpha}{\alpha}}\left|\frac{\overline{M}}{\alpha}\right|(R_\beta)^{q'-q}\exp\bigl(R_\beta^{\frac{1}{\alpha}}(t-\nu)\bigr)\left\|G\bigl(\nu,\cdot,z_1(\nu,\cdot)\bigr)-G\bigl(\nu,\cdot,z_2(\nu,\cdot)\bigr)\right\|_{L^2(\Omega)}\\
&\quad\le L_g\lambda_1^{\frac{1-\alpha}{\alpha}}\left|\frac{\overline{M}}{\alpha}\right|(R_\beta)^{q'-q}\exp\bigl(R_\beta^{\frac{1}{\alpha}}(t-\nu)\bigr)\left\|z_1(\nu,\cdot)-z_2(\nu,\cdot)\right\|_{L^2(\Omega)}\\
&\quad\le L_g\lambda_1^{\frac{1-\alpha}{\alpha}}\left|\frac{\overline{M}}{\alpha}\right|(R_\beta)^{q'-q}\exp\bigl(R_\beta^{\frac{1}{\alpha}}(t-\nu)\bigr)\left\|z_1(\nu,\cdot)-z_2(\nu,\cdot)\right\|_{L^{\frac{2d}{d-4r}}(\Omega)},
\end{aligned}\tag{5.19}$$

where we have used the global Lipschitz function of G and the Sobolev embedding $L^2(\Omega)\hookrightarrow L^{\frac{2d}{d-4r}}(\Omega)$. It follows from (5.18) that

$$\begin{aligned}
&\exp(-at)\left\|\mathcal{M}z_1(t,\cdot)-\mathcal{M}z_2(t,\cdot)\right\|_{\mathbb{H}^r(\Omega)}\\
&\quad\le\frac{\overline{M}L_g\lambda_1^{\frac{1-\alpha}{\alpha}}(R_\beta)^{q'-q}}{\alpha}\\
&\qquad\times\int_0^t\exp\bigl[\bigl(R_\beta^{\frac{1}{\alpha}}-a\bigr)(t-\nu)\bigr]\exp(-a\nu)\left\|z_1(\nu,\cdot)-z_2(\nu,\cdot)\right\|_{L^{\frac{2d}{d-4r}}(\Omega)}\,d\nu.
\end{aligned}\tag{5.20}$$

Recall that

$$\|z_1-z_2\|_{L_a^\infty(0,T;L^{\frac{2d}{d-4r}}(\Omega))}:=\operatorname*{ess\,sup}_{0<\nu<T}\exp(-a\nu)\left\|z_1(\nu,\cdot)-z_2(\nu,\cdot)\right\|_{L^{\frac{2d}{d-4r}}(\Omega)},$$

and we get

$$\begin{aligned}
&\exp(-at)\left\|\mathcal{M}z_1(t,\cdot)-\mathcal{M}z_2(t,\cdot)\right\|_{L^{\frac{2d}{d-4r}}(\Omega)}\\
&\quad\lesssim\exp(-at)\left\|\mathcal{M}z_1(t,\cdot)-\mathcal{M}z_2(t,\cdot)\right\|_{\mathbb{H}^r(\Omega)}\\
&\quad\le\frac{\overline{M}L_g\lambda_1^{\frac{1-\alpha}{\alpha}}(R_\beta)^{q'-q}}{\alpha}\left(\int_0^t\exp\bigl[\bigl(R_\beta^{\frac{1}{\alpha}}-a\bigr)(t-\nu)\bigr]\,d\nu\right)\|z_1-z_2\|_{L_a^\infty(0,T;L^{\frac{2d}{d-4r}}(\Omega))}\\
&\quad\le\frac{\overline{M}L_g\lambda_1^{\frac{1-\alpha}{\alpha}}(R_\beta)^{q'-q}}{a\alpha}\|z_1-z_2\|_{L_a^\infty(0,T;L^{\frac{2d}{d-4r}}(\Omega))},
\end{aligned}$$

where we can verify the following inequality:

$$\int_0^t \exp\bigl[-a(t-\nu)\bigr]\exp\bigl[R_\beta^{\frac{1}{\alpha}}(t-\nu)\bigr]\,d\nu < \frac{1}{a}$$

for any $a > R_\beta^{\frac{1}{\alpha}}$. Also we apply the embedding $\mathbb{H}^r(\Omega) \hookrightarrow L^{\frac{2d}{d-4r}}(\Omega)$ to find that

$$\|\mathcal{M}z_1 - \mathcal{M}z_2\|_{L_a^\infty(0,T;L^{\frac{2d}{d-4r}}(\Omega))} \le \frac{\overline{M}L_g\lambda_1^{\frac{1-\alpha}{\alpha}}(R_\beta)^{q'-q}}{a\alpha}\|z_1 - z_2\|_{L_a^\infty(0,T;L^{\frac{2d}{d-4r}}(\Omega))}. \tag{5.21}$$

By choosing a such that

$$a \ge \frac{2\overline{M}L_g\lambda_1^{\frac{1-\alpha}{\alpha}}(R_\beta)^{q'-q}}{\alpha},$$

we know that $\frac{\overline{M}L_g\lambda_1^{\frac{1-\alpha}{\alpha}}(R_\beta)^{q'-q}}{a\alpha} \le 1/2$. Hence, together with the fact that

$$\mathcal{M}(z=0) \in L_a^\infty\bigl(0,T;L^{\frac{2d}{d-4r}}(\Omega)\bigr),$$

we can say that $\mathcal{M}$ is a contraction in $L_a^\infty(0,T;L^{\frac{2d}{d-4r}}(\Omega))$. By applying the Banach fixed point theorem, we can deduce that $\mathcal{M}$ has a fixed point $Z_\beta \in L_a^\infty(0,T;L^{\frac{2d}{d-4r}}(\Omega))$. In the following, we continue to estimate the upper bound of Z_β. By letting $z_1 = Z_\beta$ and $z_2 = 0$ into (5.21), we find that

$$\bigl\|\mathcal{M}Z_\beta - \mathcal{P}_1(t)\mathbb{S}_{R_\beta}\psi_\beta - \mathcal{P}_2(t)\mathbb{S}_{R_\beta}\varphi_\beta\bigr\|_{L_a^\infty(0,T;L^{\frac{2d}{d-4r}}(\Omega))} \le \frac{1}{2}\|Z_\beta\|_{L_a^\infty(0,T;L^{\frac{2d}{d-4r}}(\Omega))}. \tag{5.22}$$

Due to the fact that $\mathcal{M}Z_\beta = Z_\beta$ and combining (5.16) and (5.17), we find that

$$\begin{aligned}
&\|Z_\beta\|_{L_a^\infty(0,T;L^{\frac{2d}{d-4r}}(\Omega))} \\
&\quad = \|\mathcal{M}Z_\beta\|_{L_a^\infty(0,T;L^{\frac{2d}{d-4r}}(\Omega))} \\
&\quad \le \bigl\|\mathcal{P}_1(t)\mathbb{S}_{R_\beta}\psi_\beta + \mathcal{P}_2(t)\mathbb{S}_{R_\beta}\varphi_\beta\bigr\|_{L_a^\infty(0,T;L^{\frac{2d}{d-4r}}(\Omega))} + \frac{1}{2}\|Z_\beta\|_{L_a^\infty(0,T;L^{\frac{2d}{d-4r}}(\Omega))} \\
&\quad \le \left|\frac{\overline{M}}{\alpha}\right|(R_\beta)^{r-\frac{dp-2d}{4p}}\exp\bigl(R_\beta^{\frac{1}{\alpha}}T\bigr)\|\psi_\beta\|_{L^p(\Omega)} \\
&\qquad + \lambda_1^{-1/\alpha}\left|\frac{\overline{M}}{\alpha}\right|(R_\beta)^{r-\frac{dp-2d}{4p}}\exp\bigl(R_\beta^{\frac{1}{\alpha}}T\bigr)\|\varphi_\beta\|_{L^p(\Omega)} + \frac{1}{2}\|Z_\beta\|_{L_a^\infty(0,T;L^{\frac{2d}{d-4r}}(\Omega))}.
\end{aligned}$$

Thus, we arrive at the desired result. □

The following theorem provides the error between the regularized solution and the exact solution in the space of L^p type when the noisy data in L^p.

Theorem 5.2 *Let us assume that Problem* (1.1)–(1.2) *has a unique solution* $u \in L^\infty(0,T;\mathbb{H}^{r+\delta}(\Omega)) \cap L^\infty(0,T;\mathbb{G}_{MT}^s(\Omega))$ *for any* $\delta > 0$ *and* $0 \le r < \frac{d}{4}$ *and* $s > r$. *Let the observed data*

$(\psi_\beta, \varphi_\beta) \in L^p(\Omega) \times L^p(\Omega)$ *satisfy* (5.6) *and* $1 < p < 2$. *Let us choose* R_β *such that*

$$\lim_{\beta\to 0} R_\beta = \infty, \lim_{\beta\to 0} (R_\beta)^{r-\frac{dp-2d}{4p}} \exp\bigl(R_\beta^{\frac{1}{\alpha}} T\bigr)\beta = 0. \tag{5.23}$$

Then the following estimate holds:

$$\bigl\| Z_\beta(t,\cdot) - u(t,\cdot)\bigr\|_{L^{\frac{2d}{d-4r}}(\Omega)} \lesssim \max\bigl((R_\beta)^{r-\frac{dp-2d}{4p}} \exp\bigl(R_\beta^{\frac{1}{\alpha}} T\bigr)\beta, (R_\beta)^{r-s}, (R_\beta)^{-\delta}\bigr). \tag{5.24}$$

Remark 5.1 Let us choose $R_\beta = (\frac{1-\mu}{T})^\alpha \log^\alpha(\frac{1}{\beta})$ for any $0 < \mu < 1$. Then the error between the regularized solution Z_β and the exact solution u in $L^{\frac{2d}{d-4r}}(\Omega)$ is of order

$$\max\left(\left[\log\left(\frac{1}{\beta}\right)\right]^{-\delta}, \left[\log\left(\frac{1}{\beta}\right)\right]^{r-s}, \left[\log\left(\frac{1}{\beta}\right)\right]^{\alpha(r-\frac{dp-2d}{4p})} \beta^\mu\right).$$

Proof It follows from (5.4) that

$$\mathbb{S}_R u(t,x) = \mathcal{P}_1(t)\mathbb{S}_R\psi + \mathcal{P}_2(t)\mathbb{S}_R\varphi + \int_0^t (t-\nu)^{\alpha-1}\mathcal{P}_3(t-\nu)\mathbb{S}_R G\bigl(\nu,x,u(\nu,x)\bigr)\,d\nu. \tag{5.25}$$

This implies that

$$\begin{aligned}\bigl\|Z_\beta(t,\cdot) - u(t,\cdot)\bigr\|_{L^2(\Omega)} &\le \bigl\|Z_\beta(t,\cdot) - \mathbb{S}_{R_\beta} u(t,\cdot)\bigr\|_{L^2(\Omega)} + \bigl\|\mathbb{S}_{R_\beta} u(t,\cdot) - u(t,\cdot)\bigr\|_{L^2(\Omega)} \\ &= (I) + (II). \end{aligned} \tag{5.26}$$

Let us first treat the term (I). Indeed, by the definition of Z_β as in (5.7) and (5.25), we arrive at

$$\begin{aligned} &\bigl\|Z_\beta(t,\cdot) - \mathbb{S}_{R_\beta} u(t,\cdot)\bigr\|_{L^2(\Omega)} \\ &\quad\le \bigl\|\mathcal{P}_1(t)\mathbb{S}_{R_\beta}\psi_\beta - \mathcal{P}_1(t)\mathbb{S}_{R_\beta}\psi\bigr\|_{L^2(\Omega)} + \bigl\|\mathcal{P}_2(t)\mathbb{S}_R\varphi_\beta - \mathcal{P}_2(t)\mathbb{S}_R\varphi\bigr\|_{L^2(\Omega)} \\ &\qquad + \int_0^t (t-\nu)^{\alpha-1}\mathcal{P}_3(t-\nu)\mathbb{S}_{R_\beta}\bigl(G\bigl(\nu,x,Z_\beta(\nu,x)\bigr) - G\bigl(\nu,x,u(\nu,x)\bigr)\bigr)\,d\nu. \end{aligned} \tag{5.27}$$

In view of the estimate (5.11) with $q' = 0$ and $q = \frac{dp-2d}{4p} < 0$, we find that

$$\begin{aligned} &\bigl\|\mathcal{P}_1(t)\mathbb{S}_{R_\beta}\psi_\beta - \mathcal{P}_1(t)\mathbb{S}_{R_\beta}\psi\bigr\|_{L^2(\Omega)} \\ &\quad\le \bigl\|\mathcal{P}_1(t)\mathbb{S}_{R_\beta}\psi_\beta - \mathcal{P}_1(t)\mathbb{S}_{R_\beta}\psi\bigr\|_{L^{\frac{2d}{d-4r}}(\Omega)} \\ &\quad\lesssim \bigl\|\mathcal{P}_1(t)\mathbb{S}_{R_\beta}(\psi_\beta - \psi)\bigr\|_{\mathbb{H}^r(\Omega)} \le \left|\frac{\overline{M}}{\alpha}\right| (R_\beta)^{-\frac{dp-2d}{4p}} \exp\bigl(R_\beta^{\frac{1}{\alpha}} t\bigr) \|\psi_\beta - \psi\|_{\mathbb{H}^{\frac{dp-2d}{4p}}(\Omega)} \\ &\quad\lesssim \left|\frac{\overline{M}}{\alpha}\right| (R_\beta)^{-\frac{dp-2d}{4p}} \exp\bigl(R_\beta^{\frac{1}{\alpha}} t\bigr) \|\psi_\beta - \psi\|_{L^p(\Omega)} \lesssim \left|\frac{\overline{M}}{\alpha}\right| (R_\beta)^{-\frac{dp-2d}{4p}} \exp\bigl(R_\beta^{\frac{1}{\alpha}} t\bigr)\beta, \end{aligned}$$

where we have used (5.15) to get the last estimate. By a similar explanation, we obtain the estimate for the second term on the right-hand side of (5.27) as follows:

$$\begin{aligned}
&\big\|\mathcal{P}_2(t)\mathbb{S}_{R_\beta}\varphi_\beta-\mathcal{P}_2(t)\mathbb{S}_{R_\beta}\varphi\big\|_{L^2(\Omega)}\\
&\quad\le\big\|\mathcal{P}_2(t)\mathbb{S}_{R_\beta}\varphi_\beta-\mathcal{P}_2(t)\mathbb{S}_{R_\beta}\varphi\big\|_{L^{\frac{2d}{d-4r}}(\Omega)}\\
&\quad\lesssim\big\|\mathcal{P}_2(t)\mathbb{S}_{R_\beta}\varphi_\beta-\mathcal{P}_2(t)\mathbb{S}_{R_\beta}\varphi\big\|_{\mathbb{H}^r(\Omega)}\\
&\quad\le\lambda_1^{-1/\alpha}\left|\frac{\overline{M}}{\alpha}\right|(R_\beta)^{-\frac{dp-2d}{4p}}\exp\big(R_\beta^{\frac{1}{\alpha}}t\big)\|\varphi_\beta-\varphi\|_{\mathbb{H}^{\frac{dp-2d}{4p}}(\Omega)}\\
&\quad\lesssim\lambda_1^{-1/\alpha}\left|\frac{\overline{M}}{\alpha}\right|(R_\beta)^{-\frac{dp-2d}{4p}}\exp\big(R_\beta^{\frac{1}{\alpha}}t\big)\|\varphi_\beta-\varphi\|_{L^p(\Omega)}\lesssim\lambda_1^{-1/\alpha}\left|\frac{\overline{M}}{\alpha}\right|(R_\beta)^{-\frac{dp-2d}{4p}}\exp\big(R_\beta^{\frac{1}{\alpha}}t\big)\beta.
\end{aligned}$$

We continue to consider the third term on the right-hand side of (5.27). Indeed, we get

$$\begin{aligned}
&\left\|\int_0^t(t-\nu)^{\alpha-1}\mathcal{P}_3(t-\nu)\mathbb{S}_R\big(G\big(\nu,x,Z_\beta(\nu,x)\big)-G\big(\nu,x,u(\nu,x)\big)\big)\,d\nu\right\|_{L^2(\Omega)}\\
&\quad\le\lambda_1^{\frac{1-\alpha}{\alpha}}\left|\frac{\overline{M}}{\alpha}\int_0^t\right|\exp\big(R_\beta^{\frac{1}{\alpha}}(t-\nu)\big)\big\|G\big(\nu,\cdot,Z_\beta(\nu,\cdot)\big)-G\big(\nu,\cdot,u(\nu,\cdot)\big)\big\|_{L^2(\Omega)}\,d\nu\\
&\quad\le L_g\lambda_1^{\frac{1-\alpha}{\alpha}}\left|\frac{\overline{M}}{\alpha}\right|\int_0^t\exp\big(R_\beta^{\frac{1}{\alpha}}(t-\nu)\big)\big\|Z_\beta(\nu,\cdot)-u(\nu,\cdot)\big\|_{L^2(\Omega)}\,d\nu.
\end{aligned}\tag{5.28}$$

From two above observations and (5.27), we find that

$$\begin{aligned}
&\big\|Z_\beta(t,\cdot)-\mathbb{S}_Ru(t,\cdot)\big\|_{L^2(\Omega)}\\
&\quad\le\big(\lambda_1^{-1/\alpha}+1\big)\left|\frac{\overline{M}}{\alpha}\right|(R_\beta)^{-\frac{dp-2d}{4p}}\exp\big(R_\beta^{\frac{1}{\alpha}}t\big)\beta\\
&\qquad+L_g\lambda_1^{\frac{1-\alpha}{\alpha}}\left|\frac{\overline{M}}{\alpha}\right|\int_0^t\exp\big(R_\beta^{\frac{1}{\alpha}}(t-\nu)\big)\big\|Z_\beta(\nu,\cdot)-u(\nu,\cdot)\big\|_{L^2(\Omega)}\,d\nu.
\end{aligned}\tag{5.29}$$

Next, we continue to provide the following estimate:

$$\begin{aligned}
\big\|u(t)-\mathbb{S}_{R_\beta}u(t,\cdot)\big\|_{L^2(\Omega)}&=\left(\sum_{\lambda_j>R_\beta}\exp\big(-2(T-t)\lambda_j^{\frac{1}{\alpha}}\big)\lambda_j^{-2s}\exp\big(2(T-t)\lambda_j^{\frac{1}{\alpha}}\big)\lambda_j^{2s}\big\langle u(t),e_j\big\rangle^2\right)^{\frac{1}{2}}\\
&\le(R_\beta)^{-s}\exp\big(-(T-t)R_\beta^{\frac{1}{\alpha}}\big)\overline{B},
\end{aligned}\tag{5.30}$$

where we remind that

$$\overline{B}=\operatorname*{ess\,sup}_{0\le t\le T}\left(\sum_{j=1}^\infty\lambda_j^{2s}\exp\big(2(T-t)\lambda_j^{\frac{1}{\alpha}}\big)\big\langle u(t),e_j\big\rangle^2\right)^{\frac{1}{2}}\le\|u\|_{L^\infty(0,T;\mathbb{G}^s_{MT}(\Omega))}.\tag{5.31}$$

Combining (5.29) and (5.30), we arrive at the following estimate:

$$\begin{aligned}
&\big\|Z_\beta(t,\cdot)-u(t,\cdot)\big\|_{L^2(\Omega)}\\
&\quad\le\big\|Z_\beta(t,\cdot)-\mathbb{S}_{R_\beta}u(t,\cdot)\big\|_{L^2(\Omega)}+\big\|u(t)-\mathbb{S}_{R_\beta}u(t,\cdot)\big\|_{L^2(\Omega)}
\end{aligned}$$

$$
\begin{aligned}
&\leq \left(\lambda_1^{-1/\alpha}+1\right)\left|\frac{\overline{M}}{\alpha}\right|(R_\beta)^{-\frac{dp-2d}{4p}}\exp\left(R_\beta^{\frac{1}{\alpha}}t\right)\beta+(R_\beta)^{-s}\exp\left(-(T-t)R_\beta^{\frac{1}{\alpha}}\right)\overline{B} \\
&\quad + L_g\lambda_1^{\frac{1-\alpha}{\alpha}}\left|\frac{\overline{M}}{\alpha}\right|\int_0^t \exp\left(R_\beta^{\frac{1}{\alpha}}(t-\nu)\right)\left\|Z_\beta(\nu,\cdot)-u(\nu,\cdot)\right\|_{L^2(\Omega)}\,d\nu. \qquad (5.32)
\end{aligned}
$$

Multiplying both sides of the above expression by $\exp(R_\beta^{\frac{1}{\alpha}}(T-t))$, we obtain that

$$
\begin{aligned}
&\exp\left(R_\beta^{\frac{1}{\alpha}}(T-t)\right)\left\|Z_\beta(t,\cdot)-u(t,\cdot)\right\|_{L^2(\Omega)} \\
&\quad\leq \frac{\overline{M}(\lambda_1^{-1/\alpha}+1)}{\alpha}(R_\beta)^{-\frac{dp-2d}{4p}}\exp\left(R_\beta^{\frac{1}{\alpha}}T\right)\beta+(R_\beta)^{-s}\overline{B} \\
&\qquad + \frac{\overline{M}L_g\lambda_1^{\frac{1-\alpha}{\alpha}}}{\alpha}\int_0^t \exp\left(R_\beta^{\frac{1}{\alpha}}(T-\nu)\right)\left\|Z_\beta(\nu,\cdot)-u(\nu,\cdot)\right\|_{L^2(\Omega)}\,d\nu. \qquad (5.33)
\end{aligned}
$$

By using Gronwall's inequality, we get that

$$
\begin{aligned}
&\exp\left(R_\beta^{\frac{1}{\alpha}}(T-t)\right)\left\|Z_\beta(t,\cdot)-u(t,\cdot)\right\|_{L^2(\Omega)} \\
&\quad\leq \left[\frac{\overline{M}(\lambda_1^{-1/\alpha}+1)}{\alpha}(R_\beta)^{-\frac{dp-2d}{4p}}\exp\left(R_\beta^{\frac{1}{\alpha}}T\right)\beta+(R_\beta)^{-s}\overline{B}\right]\exp\left(t\frac{\overline{M}L_g\lambda_1^{\frac{1-\alpha}{\alpha}}}{\alpha}\right). \qquad (5.34)
\end{aligned}
$$

This implies that

$$
\begin{aligned}
&\left\|Z_\beta(t,\cdot)-u(t,\cdot)\right\|_{L^2(\Omega)} \\
&\quad\lesssim \exp\left(R_\beta^{\frac{1}{\alpha}}(t-T)\right)\left[\frac{\overline{M}(\lambda_1^{-1/\alpha}+1)}{\alpha}(R_\beta)^{-\frac{dp-2d}{4p}}\exp\left(R_\beta^{\frac{1}{\alpha}}T\right)\beta+(R_\beta)^{-s}\overline{B}\right]. \qquad (5.35)
\end{aligned}
$$

Our next aim is to considering the quantity $\|Z_\beta(t,\cdot)-u(t,\cdot)\|_{\mathbb{H}^r(\Omega)}$. Using the triangle inequality, we arrive at

$$
\left\|Z_\beta(t,\cdot)-u(t,\cdot)\right\|_{\mathbb{H}^r(\Omega)} \leq \left\|Z_\beta(t,\cdot)-\mathbb{S}_{R_\beta}u(t,\cdot)\right\|_{\mathbb{H}^r(\Omega)}+\left\|\mathbb{S}_{R_\beta}u(t,\cdot)-u(t,\cdot)\right\|_{\mathbb{H}^r(\Omega)}. \qquad (5.36)
$$

We control the first term on the right above as follows:

$$
\begin{aligned}
\left\|Z_\beta(t,\cdot)-\mathbb{S}_{R_\beta}u(t,\cdot)\right\|_{\mathbb{H}^r(\Omega)} &\leq \sqrt{\sum_{j=1}^{\lambda_j\leq R_\beta}\lambda_j^{2r}\left(\int_\Omega\left(Z_\beta(x,t)-u(x,t)\right)e_j(x)\,\mathrm{d}x\right)^2} \\
&\leq (R_\beta)^r\left\|Z_\beta(t,\cdot)-\mathbb{S}_{R_\beta}u(t,\cdot)\right\|_{L^2(\Omega)}. \qquad (5.37)
\end{aligned}
$$

The second quantity on the right-hand side of (5.36) is bounded by

$$
\begin{aligned}
\left\|\mathbb{S}_{R_\beta}u(t,\cdot)-u(t,\cdot)\right\|_{\mathbb{H}^r(\Omega)} &= \sqrt{\sum_{\lambda_j>R_\beta}\lambda_j^{2r}\left(\int_\Omega u(x,t)e_j(x)\,\mathrm{d}x\right)^2} \\
&= \sqrt{\sum_{\lambda_j>R_\beta}\lambda_j^{-2\delta}\lambda_j^{2r+2\delta}\left(\int_\Omega u(x,t)e_j(x)\,\mathrm{d}x\right)^2}
\end{aligned}
$$

$$\leq (R_\beta)^{-\delta}\|u\|_{L^\infty(0,T;\mathbb{H}^{r+\delta}(\Omega))}, \tag{5.38}$$

where

$$\|u\|_{L^\infty(0,T;\mathbb{H}^{r+\delta}(\Omega))} \geq \sqrt{\sum_{j=1}^{\infty} \lambda_j^{2r+2\delta}\left(\int_\Omega u(x,t)e_j(x)\,\mathrm{d}x\right)^2}.$$

Combining (5.36), (5.37), and (5.38) together with (5.35), we get that

$$\begin{aligned}
&\|Z_\beta(t,\cdot) - u(t,\cdot)\|_{\mathbb{H}^r(\Omega)} \\
&\quad \leq (R_\beta)^r\|Z_\beta(t,\cdot) - \mathbb{S}_{R_\beta}u(t,\cdot)\|_{L^2(\Omega)} + (R_\beta)^{-\delta}\|u\|_{L^\infty(0,T;\mathbb{H}^{r+\delta}(\Omega))} \\
&\quad \leq \exp\bigl(R_\beta^{\frac{1}{\alpha}}(t-T)\bigr)\left[\frac{\overline{M}(\lambda_1^{-1/\alpha}+1)}{\alpha}(R_\beta)^{r-\frac{dp-2d}{4p}}\exp\bigl(R_\beta^{\frac{1}{\alpha}}T\bigr)\beta + (R_\beta)^{r-s}\overline{B}\right] \\
&\qquad + (R_\beta)^{-\delta}\|u\|_{L^\infty(0,T;\mathbb{H}^{r+\delta}(\Omega))}.
\end{aligned} \tag{5.39}$$

Using the embedding $\mathbb{H}^r(\Omega) \hookrightarrow L^{\frac{2d}{d-4r}}(\Omega)$, we deduce that

$$\begin{aligned}
\|Z_\beta(t,\cdot) - u(t,\cdot)\|_{L^{\frac{2d}{d-4r}}(\Omega)} &\lesssim \|Z_\beta(t,\cdot) - u(t,\cdot)\|_{\mathbb{H}^r(\Omega)} \\
&\lesssim \max\bigl((R_\beta)^{r-\frac{dp-2d}{4p}}\exp\bigl(R_\beta^{\frac{1}{\alpha}}T\bigr)\beta, (R_\beta)^{r-s}, (R_\beta)^{-\delta}\bigr).
\end{aligned} \tag{5.40}$$

□

6 Conclusion

We consider the Cauchy problem for an evolution equation with the Caputo fractional derivative. The most important feature of the paper is that the problem is not well posed in the sense of Hadamard. Throughout this work, the homogeneous case, the inhomogeneous case, and the nonlinear case are investigated in turn. For each case, we construct a regularized solution by using the Fourier truncation method. In addition, using Cauchy data in L^p spaces, $p \neq 2$, by some embeddings linking the Hilbert scale-spaces and the Lebesgue spaces, we derive error estimates between regularized solutions and exact solutions.

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Author details
[1]Industrial University of Ho Chi Minh City, Ho Chi Minh City, Vietnam. [2]Department of Mathematics, Cankaya University, 06530 Balgat, Ankara, Turkey. [3]Institute of Space Sciences, R76900 Magurele-Bucharest, Romania. [4]Department of Mathematics, Texas AM University-Kingsville, 700 University Blvd., MSC 172, Kingsville, TX, USA. [5]Division of Applied Mathematics, Science and Technology Advanced Institute, Van Lang University, Ho Chi Minh City, Vietnam. [6]Faculty of Applied Technology, School of Engineering and Technology, Van Lang University, Ho Chi Minh City, Vietnam.

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