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Fitted numerical method for singularly perturbed Burger–Huxley equation

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Abstract

This paper deals with the numerical treatment of a singularly perturbed unsteady Burger–Huxley equation. We linearize the problem using the Newton–Raphson–Kantorovich approximation method. We discretize the resulting linear problem using the implicit Euler method and specially fitted finite difference method for time and space variables, respectively. We provide the stability and convergence analysis of the method, which is first-order parameter uniform convergent. We present several model examples to illustrate the efficiency of the proposed method. The numerical results depict that the present method is more convergent than some methods available in the literature.

MSC: 35B25; 35K67; 65M12

Keywords: Singularly perturbed problems; Burger–Huxley equation; Implicit Euler method; Fitted operator method; Nonlinear problem

1 Introduction

We consider the following unsteady nonlinear singularly perturbed Burger–Huxley equation:

$$\begin{cases} \pounds_\varepsilon y(s,t) = \frac{\partial y}{\partial t} - \varepsilon \frac{\partial^2 y}{\partial s^2} + \alpha y \frac{\partial y}{\partial s} - \lambda(1-y)(y-\theta) = 0, \\ (s,t) \in \Im = \Im_s \times \Im_t = (0,1) \times (0,T], \\ y(s,0) = y_0(s), \quad s \in \overline{\Im}_s, \\ y(0,t) = \wp_0(t), \qquad y(1,t) = \wp_1(t), \quad t \in (0,T], \end{cases} \tag{1}$$

with left boundary $\Im_l = \{(s,t) : s = 0, t \in \Im_t\}$, initial boundary $\Im_i = \{(s,t) : t = 0, s \in \Im_s\}$, and right boundary $\Im_r = \{(s,t) : s = 1, t \in \Im_t\}$, where ε is a small singular perturbation parameter such that $0 < \varepsilon \ll 1$, $\alpha \geq 1$, $\lambda \geq 0$, $\theta \in (0,1)$, and $\partial \Im = \Im_l \cup \Im_i \cup \Im_r$, The functions $\wp_0(t)$, $\wp_1(t)$, and $y_0(s)$ are assumed to be sufficiently smooth, bounded, and independent of ε. Equation (1) shows a prototype model for describing the interaction between nonlinear convection effects, reaction mechanisms, and diffusion transport. This equation has many intriguing phenomena such as bursting oscillation [1], population genetics [2], interspike [3], bifurcation, and chaos [4]. Several membrane models based on the dynamics of potassium and sodium ion fluxes can be found in [5].

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In [6–16] and references therein, the authors constructed various analytical and numerical methods for the Burger equations. The Burger–Huxley equation, in which the highest order derivative term is affected by a small parameter ε ($0 < \varepsilon \ll 1$), is classified as the singularly perturbed Burger–Huxley equation (SPBHE). The presence of ε and the nonlinearity in the problem lead to severe difficulties in approximating the solution of the problem. For instance, due to the presence of ε, the solution reveals boundary/sharp interior layer(s), and it is tough to find a stable numerical approximation. While solving SPBHE, unless specially designed meshes are used, the presented methods in [6–16] and other standard numerical methods fail to give acceptable results. This limitation of the conventional numerical methods has encouraged researchers to develop robust numerical techniques that perform well enough independently of ε. Kaushik and Sharma [17] investigated problem (1) using the finite difference method (FDM) on the piecewise uniform Shishkin mesh. In [18] a monotone hybrid finite difference operator on a piecewise uniform Shishkin mesh is employed to find the approximate solution for problem (1). An upwind FDM on an adaptive nonuniform grid to find an approximate solution for problem (1) is suggested by Liu et al. [19]. In [20–23] the authors proposed a parameter uniform numerical method based on fitted operator techniques for problem (1).

However, the development of the solution methodologies for problem (1) is at an infant stage. This limitation motivated us to construct a parameter uniform numerical scheme for solving problem (1) based on the fitted operator approach. The proposed method is an ε-uniformly convergent numerical algorithm that does not require a priori knowledge of the location and breadth of the boundary layer(s), which in turn increases the difficulty of finding the free oscillation solution of the problem under consideration. Also, the proposed method requires less computational effort to solve the families of the problem under consideration.

2 A priori estimates for the solution of the continuous problem

Lemma 2.1 (maximum principle) *If $y|_{\partial \Im} \geq 0$ and $(\pounds_\varepsilon)y|_\Im \geq 0$, then $y|_{\overline{\Im}} \geq 0$.*

Proof See [18]. □

Lemma 2.2 (stability estimate) *The solution $y(s,t)$ of Eq. (1) is bounded, that is,*

$$\|y\|_{\overline{\Im}} \leq T\|y_0\|_{\Im_i} + \|y\|_{\partial \Im}.$$

Proof See [18]. □

3 Formulation of the numerical scheme

3.1 Quasi-linearization technique

Equation (1) can be rewritten as

$$\begin{cases} \pounds_\varepsilon y(s,t) = \left(\frac{\partial y}{\partial t} - \varepsilon \frac{\partial^2 y}{\partial s^2}\right)(s,t) = g\left(s,t,y(s,t),\frac{\partial y}{\partial s}(s,t)\right), & (s,t) \in \Im, \\ y(s,0) = y_0(s), & s \in \overline{\Im}_s, \\ y(0,t) = \wp_0(t), \qquad y(1,t) = \wp_1(t), & t \in (0,T], \end{cases} \tag{2}$$

where $g(s,t,y(s,t),\frac{\partial y}{\partial s}(s,t)) = -\alpha y \frac{\partial y}{\partial s} + \lambda(1-y)(y-\theta)$ is a nonlinear function of s, t, $y(s,t)$, $\frac{\partial y}{\partial s}(s,t)$.

To linearize the semilinear term of Eq. (1), we choose a reasonable initial approximation $y^0(s,t)$ for the function $y(s,t)$ in the term $g(s,t,y(s,t),\frac{\partial y}{\partial s}(s,t))$ that satisfies both initial and boundary conditions and is obtained by the separation-of-variables method of the homogeneous part of the problem under consideration; it is given by

$$y^0(s,t) = y_0(s)\exp(-\pi^2 t).$$

Now we apply the Newton–Raphson–Kantorovich approximation technique to the nonlinear term $g(s,t,y(s,t),\frac{\partial y}{\partial s}(s,t))$ of Eq. (2), which can be linearized as

$$
\begin{aligned}
g&\left(s,t,y^{(m+1)}(s,t),\frac{\partial y^{(m+1)}}{\partial s}(s,t)\right) \\
&\cong g\left(s,t,y^{(m)}(s,t),\frac{\partial y^{(m)}}{\partial s}(s,t)\right) + \left(y^{(m+1)}(y,t) - y^{(m)}(s,t)\right)\frac{\partial g}{\partial y^{(m)}}\bigg|_{(s,t,y^{(m)}(s,t),\frac{\partial y^{(m)}}{\partial s}(s,t))} \\
&\quad + \left(\frac{\partial y^{(m+1)}}{\partial s}(s,t) - \frac{\partial y^{(m)}}{\partial s}(s,t)\right)\frac{\partial g}{\partial\left(\frac{\partial y^{(m)}}{\partial s}\right)}\bigg|_{(s,t,y^{(m)}(s,t),\frac{\partial y^{(m)}}{\partial s}(s,t))} + \cdots,
\end{aligned}
\tag{3}
$$

where $\{y^{(m)}\}_{m=0}^{\infty}$ is a sequence of approximate solutions of $g(s,t,y^{(m)}(s,t),\frac{\partial y^{(m)}}{\partial s}(s,t))$.

For simplicity, we denote $y^{(m+1)} = \hat{y}$ and substitute Eq. (3) into Eq. (2), which yields

$$
\begin{cases}
\pounds_\varepsilon \hat{y}(s,t) = \left(\frac{\partial \hat{y}}{\partial t} - \varepsilon\frac{\partial^2 \hat{y}}{\partial s^2} + \gamma\frac{\partial \hat{y}}{\partial s} + \delta\hat{y}\right)(s,t) = \nu(s,t), \\
\hat{y}(s,0) = y_0(s), \quad s \in \overline{\Im}_s, \\
\hat{y}(0,t) = \phi_0(t), \quad t \in \overline{\Im}_t, \\
\hat{y}(1,t) = \phi_1(t), \quad t \in \overline{\Im}_t,
\end{cases}
\tag{4}
$$

where

$$
\gamma(s,t) = -\frac{\partial g}{\partial\left(\frac{\partial y^{(m)}}{\partial s}\right)}\bigg|_{(s,t,y^{(m)}(s,t),\frac{\partial y^{(m)}}{\partial s}(s,t))},
$$

$$
\delta(s,t) = -\frac{\partial m}{\partial s^{(m)}}\bigg|_{(s,t,y^{(m)}(s,t),\frac{\partial y^{(m)}}{\partial s}(s,t))},
$$

$$
\begin{aligned}
\nu(s,t) &= g\left(s,t,y^{(m)}(s,t),\frac{\partial y^{(m)}}{\partial s}(s,t)\right) - y^{(m)}\frac{\partial g}{\partial y^{(m)}}\bigg|_{(s,t,y^{(m)}(s,t),\frac{\partial y^{(m)}}{\partial s}(s,t))} \\
&\quad - \frac{\partial y^{(m)}}{\partial s}(s,t)\frac{\partial g}{\partial\left(\frac{\partial y^{(m)}}{\partial s}\right)}\bigg|_{(s,t,y^{(m)}(s,t),\frac{\partial y^{(m)}}{\partial s}(s,t))}.
\end{aligned}
$$

3.2 Temporal semidiscretization

Now we apply the implicit Euler method with uniform mesh $\Im_\tau^M = \{j\tau, 0 < j \le M, \tau = T/M\}$ to Eq. (4) in temporal variable:

$$
\begin{cases}
(1 + \tau\pounds_\varepsilon^M)\hat{Y}(s,t_{j+1}) = \left(-\varepsilon\frac{\partial^2 \hat{Y}}{\partial s^2} + \gamma\frac{\partial \hat{Y}}{\partial s} + \delta\hat{Y}\right)(s,t_{j+1}) - \nu(s,t_{j+1}) = \hat{Y}(s,t_j), \\
\hat{Y}(s,0) = Y_0(s), \quad s \in \overline{\Im}_s, \\
\hat{Y}(0,t_{j+1}) = \wp_0(t_{j+1}), \quad 0 \le j \le M-1, \\
\hat{Y}(1,t_{j+1}) = \wp_1(t_{j+1}), \quad 0 \le j \le M-1.
\end{cases}
\tag{5}
$$

Clearly, the operator $(I + \tau \pounds_\varepsilon^M)$ satisfies the maximum principle, which confirms the stability of the semidiscrete equation (5).

Lemma 3.1 (Local Error Estimate) *The local truncation error estimate $e_{j+1} = \hat{y}(s, t_{j+1}) - \hat{Y}(s, t_{j+1})$ of the solution of Eq. (5) is bounded by*

$$\|e_{j+1}\|_\infty \leq C\tau^2,$$

and the global error estimate in the temporal direction is given by

$$\|E_j\|_\infty \leq C\tau, \quad j \leq T/\tau,$$

where C is a positive constant independent of ε and τ.

Proof See [23].   □

Lemma 3.2 *The derivative of the solution $Y^{j+1}(s)$ of Eq. (5) is bounded by*

$$\left\| \frac{\partial^i Y^{j+1}(s)}{\partial s^i} \right\|_{\overline{\Im}_s} \leq C\left(1 + \varepsilon^{-i} \exp\left(\frac{-(\gamma^*(1-s))}{\varepsilon}\right)\right), \quad 0 \leq i \leq 4.$$

Proof See [18].   □

Rewrite Eq. (5) as

$$
\begin{cases}
-\varepsilon \frac{d^2 Y^{j+1}(s)}{ds^2} + \gamma(s)\frac{dY^{j+1}(s)}{ds} + Q(s)Y^{j+1}(s) = \vartheta^{j+1}(s), \quad 0 \leq s \leq 1, \\
Y^{j+1}(0) = \wp_0^{j+1}, \qquad Y^{j+1}(1) = \wp_1^{j+1}, \quad 0 < j < M-1,
\end{cases}
\tag{6}
$$

where

$$Y^{j+1}(s) = \hat{Y}^{j+1}(s), \qquad Q(s) = \left(\delta(s) + \frac{1}{\tau}\right), \qquad \vartheta^{j+1}(s) = \left(\nu^{j+1}(s) + \frac{Y^j(s)}{\tau}\right).$$

3.3 Spatial semidiscretization

In this section, we use the finite difference method for the spatial discretization of problem (6) with a uniform step size. For right boundary layer problem, by the theory of singular perturbations [24] the asymptotic solution of the zeroth-order approximation for problem (6) is given as

$$Y^{j+1}(s) \approx Y_0^{j+1}(s) + \frac{\gamma(1)}{\gamma(s)}\left(\wp_1^{j+1} - Y_0^{j+1}(s)\right) \exp\left(-\gamma(s)\frac{1-s}{\varepsilon}\right) + O(\varepsilon), \tag{7}$$

where $Y_0^{j+1}(s)$ is the solution of the reduced problem

$$\gamma(s)\frac{dY_0^{j+1}(s)}{ds} + Q(s)Y_0^{j+1}(s) = \vartheta^{j+1}(s) \quad \text{with } Y_0^{j+1}(1) = \wp_1^{j+1}.$$

Taking the first terms in Taylor's series expansion for $\gamma(s)$ about the point 1, Eq. (7) becomes

$$Y^{j+1}(s) \approx Y_0^{j+1}(s) + \left(\wp_1^{j+1} - Y_0^{j+1}(s)\right) \exp\left(-\gamma(1)\frac{1-s}{\varepsilon}\right) + O(\varepsilon). \tag{8}$$

Now we divide the interval $[0, 1]$ into N equal parts with $\ell = 1/N$ yielding a space mesh $\Im_s^N = \{0 = s_0, s_1, s_2, \ldots, s_N = 1\}$. Then we have $s_i = i\ell$, $i = 0, 1, 2, \ldots, N$. By considering Eq. (8) at $s_i = i\ell$ as $\ell \to 0$ we obtain

$$\lim_{\ell \to 0} Y^{j+1}(i\ell) \approx Y_0^{j+1}(0) + \left(\wp_1^{j+1} - Y_0^{j+1}(1)\right) \exp\left(-\gamma(1)\left(\frac{1}{\varepsilon^2} - i\rho\right)\right) + O(\varepsilon), \tag{9}$$

where $\rho = \frac{\ell}{\varepsilon^2}$.

Let $Y^{j+1}(s)$ be a smooth function in the interval $[0, 1]$. Then by applying Taylor's series we have

$$Y^{j+1}(s_{i+1}) \approx Y_{i+1}^{j+1}$$

$$\approx Y_i^{j+1} + \ell \frac{dY_i^{j+1}}{ds} + \frac{\ell^2}{2!} \frac{d^2 Y_i^{j+1}}{ds^2} + \frac{\ell^3}{3!} \frac{d^3 Y_i^{j+1}}{ds^3} + \frac{\ell^4}{4!} \frac{d^4 Y_i^{j+1}}{ds^4} \tag{10}$$

$$+ \frac{\ell^5}{5!} \frac{d^5 Y_i^{j+1}}{ds^5} + \frac{\ell^6}{2!} \frac{d^6 Y_i^{j+1}}{ds^6} + \frac{\ell^7}{7!} \frac{d^7 Y_i^{j+1}}{ds^7} + \frac{\ell^8}{8!} \frac{d^8 Y_i^{j+1}}{ds^8} + O(\ell^9)$$

and

$$Y^{j+1}(s_{i-1}) \approx Y_{i-1}^{j+1}$$

$$\approx Y_i^{j+1} - \ell \frac{dY_i^{j+1}}{ds} + \frac{\ell^2}{2!} \frac{d^2 Y_i^{j+1}}{ds^2} - \frac{\ell^3}{3!} \frac{d^3 Y_i^{j+1}}{ds^3} + \frac{\ell^4}{4!} \frac{d^4 Y_i^{j+1}}{ds^4} \tag{11}$$

$$- \frac{\ell^5}{5!} \frac{d^5 Y_i^{j+1}}{ds^5} + \frac{\ell^6}{2!} \frac{d^6 Y_i^{j+1}}{ds^6} - \frac{\ell^7}{7!} \frac{d^7 Y_i^{j+1}}{ds^7} + \frac{\ell^8}{8!} \frac{d^8 Y_i^{j+1}}{ds^8} - O(\ell^9).$$

Adding Eq. (10) and Eq. (11), we get

$$Y_{i-1}^{j+1} - 2Y_i^{j+1} + Y_{i+1}^{j+1} = \frac{2\ell^2}{2!} \frac{d^2 Y_i^{j+1}}{ds^2} + \frac{2\ell^4}{4!} \frac{d^4 Y_i^{j+1}}{ds^4} + \frac{2\ell^6}{2!} \frac{d^6 Y_i^{j+1}}{ds^6} + \frac{2\ell^8}{8!} \frac{d^8 Y_i^{j+1}}{ds^8} + O(\ell^{10}) \tag{12}$$

and

$$\frac{d^2 Y_{i-1}^{j+1}}{ds^2} - \frac{d^2 Y_i^{j+1}}{ds^2} + \frac{d^2 Y_{i+1}^{j+1}}{ds^2}$$

$$= \frac{2\ell^2}{2!} \frac{d^4 Y_i^{j+1}}{ds^4} + \frac{2\ell^4}{4!} \frac{d^6 Y_i^{j+1}}{ds^6} + \frac{2\ell^6}{6!} \frac{d^8 Y_i^{j+1}}{ds^8} + \frac{2\ell^8}{8!} \frac{d^{10} Y_i^{j+1}}{ds^{10}} + O(\ell^{12}). \tag{13}$$

Substituting $\frac{\ell^4}{12} \frac{d^6 Y_i^{j+1}}{ds^6}$ from Eq. (13) into Eq. (12), we obtain

$$Y_{i-1}^{j+1} - 2Y_i^{j+1} + Y_{i+1}^{j+1} = \frac{\ell^2}{30} \left(\frac{d^2 Y_{i-1}^{j+1}}{ds^2} + 28\frac{d^2 Y_i^{j+1}}{ds^2} + \frac{d^2 Y_{i+1}^{j+1}}{ds^2}\right) + R, \tag{14}$$

where $R = \frac{\ell^4}{20} \frac{d^4 Y_i^{j+1}}{ds^4} - \frac{13\ell^6}{302400} \frac{d^8 Y_i^{j+1}}{ds^8} + O(\ell^{(10)})$.

Now from Eq. (6) we have

$$
\begin{cases}
\varepsilon \frac{d^2 Y_{i-1}^{j+1}}{ds^2} = \gamma_{i-1} \frac{dY_{i-1}^{j+1}}{ds} + Q_{i-1} Y_{i-1}^{j+1} - \vartheta_{i-1}^{j+1}, \\
\varepsilon \frac{d^2 Y_i^{j+1}}{ds^2} = \gamma_i \frac{dY_i^{j+1}}{ds} + Q_i Y_i^{j+1} - \vartheta_i^{j+1}, \\
\varepsilon \frac{d^2 Y_{i+1}^{j+1}}{ds^2} = \gamma_{i+1} \frac{dY_{i+1}^{j+1}}{ds} + Q_{i+1} Y_{i+1}^{j+1} - \vartheta_{i+1}^{j+1},
\end{cases}
\tag{15}
$$

where we approximate $\frac{dY_{i-1}^{j+1}}{ds}$, $\frac{dY_i^{j+1}}{ds}$, and $\frac{dY_{i+1}^{j+1}}{ds}$ using nonsymmetric finite differences [25]:

$$
\begin{cases}
\frac{dY_{i-1}^{j+1}}{ds} \approx \frac{-Y_{i+1}^{j+1} + 4Y_i^{j+1} - 3Y_{i-1}^{j+1}}{2\ell} + \ell \frac{d^2 Y_i^{j+1}}{ds^2} + O(\ell^2), \\
\frac{dY_i^{j+1}}{ds} \approx \frac{Y_{i+1}^{j+1} - Y_{i-1}^{j+1}}{2\ell} + O(\ell^2), \\
\frac{dY_{i+1}^{j+1}}{ds} \approx \frac{3Y_{i+1}^{j+1} - 4Y_i^{j+1} + Y_{i-1}^{j+1}}{2\ell} - \ell \frac{d^2 Y_i^{j+1}}{ds^2} + O(\ell^2).
\end{cases}
\tag{16}
$$

Substituting Eq. (16) into Eq. (15), we obtain

$$
\begin{cases}
\varepsilon \frac{d^2 Y_{i-1}^{j+1}}{ds^2} = \gamma_{i-1} \left(\frac{-Y_{i+1}^{j+1} + 4Y_i^{j+1} - 3Y_{i-1}^{j+1}}{2\ell} \right) + Q_{i-1} Y_{i-1}^{j+1} - \vartheta_{i-1}^{j+1}, \\
\varepsilon \frac{d^2 Y_i^{j+1}}{ds^2} = \gamma_i \left(\frac{Y_{i+1}^{j+1} - Y_{i-1}^{j+1}}{2\ell} \right) + Q_i Y_i^{j+1} - \vartheta_i^{j+1}, \\
\varepsilon \frac{d^2 Y_{i+1}^{j+1}}{ds^2} = \gamma_{i+1} \left(\frac{3Y_{i+1}^{j+1} - 4Y_i^{j+1} + Y_{i-1}^{j+1}}{2\ell} \right) + Q_{i+1} Y_{i+1}^{j+1} - \vartheta_{i+1}^{j+1}.
\end{cases}
\tag{17}
$$

Inserting Eq. (17) into Eq. (14) and rearranging, we get

$$
\left(\varepsilon + \frac{\gamma_{i+1}\ell}{30} - \frac{\gamma_{i-1}\ell}{30} \right) \left(\frac{Y_{i-1}^{j+1} - 2Y_i^{j+1} + Y_{i+1}^{j+1}}{\ell^2} \right)
$$
$$
= \left(\frac{-\gamma_{i-1}}{20\ell} + \frac{Q_{i-1}}{30} - \frac{7\gamma_i}{15\ell} + \frac{\gamma_{i+1}}{60\ell} \right) Y_{i-1}^{j+1} + \left(\frac{\gamma_{i-1}}{15\ell} + \frac{14Q_i}{15\ell} - \frac{\gamma_{i+1}}{15\ell} \right) Y_i^{j+1}
$$
$$
+ \left(\frac{-\gamma_{i-1}}{60\ell} + \frac{7\gamma_i}{15\ell} + \frac{\gamma_{i+1}}{20\ell} + \frac{Q_{i+1}}{30} \right) Y_{i+1}^{j+1} - \frac{1}{30} \left(\vartheta_{i-1}^{j+1} + 28\vartheta_i^{j+1} + \vartheta_{i+1}^{j+1} \right).
\tag{18}
$$

Introducing a constant fitting factor $\sigma(\rho)$ in Eq. (18), we obtain

$$
\left(\sigma(\rho)\varepsilon + \frac{\gamma_{i+1}\ell}{30} - \frac{\gamma_{i-1}\ell}{30} \right) \left(\frac{Y_{i-1}^{j+1} - 2U_i^{j+1} + Y_{i+1}^{j+1}}{\ell^2} \right)
$$
$$
= \left(\frac{-\gamma_{i-1}}{20\ell} + \frac{Q_{i-1}}{30} - \frac{7\gamma_i}{15h} + \frac{\gamma_{i+1}}{60\ell} \right) Y_{i-1}^{j+1} + \left(\frac{\gamma_{i-1}}{15\ell} + \frac{14Q_i}{15} - \frac{\gamma_{i+1}}{15h} \right) Y_i^{j+1}
$$
$$
+ \left(\frac{-\gamma_{i-1}}{60\ell} + \frac{7\gamma_i}{15\ell} + \frac{\gamma_{i+1}}{20\ell} + \frac{Q_{i+1}}{30} \right) Y_{i+1}^{j+1} - \frac{1}{30} \left(\vartheta_{i-1}^{j+1} + 28\vartheta_i^{j+1} + \vartheta_{i+1}^{j+1} \right).
\tag{19}
$$

Multiplying (19) by ℓ and taking the limit as $\ell \to 0$, we get

$$
\lim_{\ell \to 0} \sigma(\rho) \left(\frac{Y_{i-1}^{j+1} - 2Y_i^{j+1} + Y_{i+1}^{j+1}}{\rho} \right) = \frac{\gamma(0)}{2} \left(Y_{i+1}^{j+1} - Y_{i-1}^{j+1} \right).
\tag{20}
$$

Using Eq. (9), we have

$$
\begin{cases}
\frac{\sigma(\rho)}{\rho} \lim_{\ell \to 0}(Y^{j+1}(i\ell - \ell) - 2Y^{j+1}(i\ell) + Y^{j+1}(i\ell + \ell)) \\
\quad \approx (\wp_1^{j+1} - Y_0^{j+1}(1)) \exp(-\gamma(1)(\frac{1}{\varepsilon} - i\rho))(\exp(\gamma(1)\rho) - 2 + \exp(-\gamma(1)\rho)), \\
\frac{\sigma(\rho)}{\rho} \lim_{\ell \to 0}(Y^{j+1}(i\ell + \ell) - Y^{j+1}(i\ell - \ell)) \\
\quad \approx (\wp_1^{j+1} - Y_0^{j+1}(1)) \exp(-\gamma(1)(\frac{1}{\varepsilon} - i\rho))(\exp(-\gamma(1)\rho) - \exp(\gamma(1)\rho)).
\end{cases}
$$

Using the above expressions in Eq. (20), we get

$$
\frac{\sigma(\rho)}{\rho}\left(e^{\gamma(1)\rho} - 2 + e^{-\gamma(1)\rho}\right) = \frac{\gamma(0)}{2}\left(e^{\gamma(1)\rho} - e^{-\gamma(1)\rho}\right).
$$

On simplifying, we get

$$
\sigma(\rho) = \frac{\gamma(0)\rho}{2} \coth\left(\frac{\gamma(1)\rho}{2}\right), \tag{21}
$$

which is the required value of the constant fitting factor $\sigma(\rho)$. Finally, using Eq. (19) and the value of $\sigma(\rho)$ given by Eq. (21), we get

$$
\begin{aligned}
&\pounds^{N,M} Y_i^{j+1} = \chi_i^- Y_{i-1}^{j+1} + \chi_i^c Y_i^{j+1} + \chi_i^+ Y_{i+1}^{j+1} = \mu_i^{j+1}, \\
&\quad i = 1, 2, \dots, N-1, j = 0, 1, \dots, M-1,
\end{aligned} \tag{22}
$$

where

$$
\begin{cases}
\chi_i^- = -\frac{\sigma(\rho)\varepsilon}{\ell^2} - \frac{\gamma_{i-1}}{60\ell} - \frac{28\gamma_i}{60\ell} - \frac{\gamma_{i+1}}{60\ell} + \frac{Q_{i-1}}{30}, \\
\chi_i^c = \frac{2\sigma(\rho)\varepsilon}{\ell^2} + \frac{28Q_i}{30}, \\
\chi_i^+ = -\frac{\sigma(\rho)\varepsilon}{\ell^2} + \frac{\gamma_{i-1}}{60h} + \frac{\gamma_{i+1}}{60h} + \frac{28\gamma_i}{60h} + \frac{Q_{i+1}}{30}, \\
\mu_i^{j+1} = \frac{1}{30}(\vartheta_{i-1}^{j+1} + 28\vartheta_i^{j+1} + \vartheta_{i+1}^{j+1}).
\end{cases}
$$

For sufficiently small mesh sizes, the above matrix is nonsingular, and $|\chi_i^c| \geq |\chi_i^c| + |\chi_i^+|$. Hence by [26] the matrix χ is an M-matrix and has an inverse. Therefore Eq. (22) can be solved by the matrix inverse with given boundary conditions.

4 Convergence analysis

Lemma 4.1 *If $Y_i^{j+1} \geq 0$ on $i = 0, N$ and $\pounds^{N,M} Y_i^{j+1} \geq 0$ on $\overline{\Im}^{N,M}$, then $Y_i^{j+1} \geq 0$ at each point of $\overline{\Im}^{N,M}$.*

Lemma 4.2 *The solution Y_i^{j+1} of the discrete scheme in (22) on $\overline{\Im}^{N,M}$ satisfies the following bound*:

$$
\left\| Y_i^{j+1} \right\| \leq \max\left\{\left|Y_0^{j+1}\right|, \left|Y_N^{j+1}\right|\right\} + \frac{\|\pounds^{N,M}\|}{Q^*},
$$

where $Q(s_i) \geq Q^* > 0$.

Hence Lemma 4.2 confirms that the discrete scheme (22) is uniformly stable in supremum norm.

Lemma 4.3 *If $Y \in C^3(I)$, then the local truncation error in space discretization is given as*

$$|\mathsf{T}_i| \leq \max_{s_{i-1} \leq s \leq s_{i+1}} \left\{ \frac{28\gamma \ell^2}{180} \left| \frac{d^3 Y^{j+1}(s)}{ds^3} \right| \right\} + O(\ell^3), \quad i = 1, 2, \ldots, N-1.$$

Proof By definition

$$
\begin{aligned}
\mathsf{T}_i = {} & -\sigma \varepsilon \left\{ \frac{Y_{i-1}^{j+1} - 2Y_i^{j+1} + Y_{i+1}^{j+1}}{\ell^2} - \frac{d^2 Y_i^{j+1}}{ds^2} \right\} \\
& + \frac{\gamma_{i-1}}{30} \left\{ \left(\frac{-3Y_{i-1}^{j+1} + 4Y_i^{j+1} - Y_{i-1}^{j+1}}{2\ell} + \ell \frac{d^2 Y_i^{j+1}}{ds^2} \right) - \frac{dY_{i-1}^{j+1}}{ds} \right\} \\
& + \frac{28\gamma_i}{30} \left\{ \frac{Y_{i+1}^{j+1} - Y_{i-1}^{j+1}}{2\ell} - \frac{dY_i^{j+1}}{ds} \right\} \\
& + \frac{\gamma_{i+1}}{30} \left\{ \left(\frac{Y_{i+1}^{j+1} - 4Y_i^{j+1} + 3Y_{i-1}^{j+1}}{2\ell} - \ell \frac{d^2 Y_i^{j+1}}{ds^2} \right) - \frac{dY_{i+1}^{j+1}}{ds} \right\},
\end{aligned}
$$

$$i = 1(1)N - 1.$$

$$
\begin{aligned}
\Rightarrow \quad \mathsf{T}_i = {} & -\sigma \varepsilon \left\{ \frac{\ell^2}{12} \frac{d^4 Y_i^{j+1}}{ds^4} + \frac{\ell^4}{360} \frac{d^6 Y_i^{j+1}}{ds^6} + \cdots \right\} + \frac{\gamma_{i-1}}{30} \left\{ \ell \frac{d^2 Y_i^{j+1}}{ds^2} - \frac{2\ell^2}{3} \frac{d^3 Y_i^{j+1}}{ds^3} + \right\} \\
& + \frac{28\gamma_i}{30} \left\{ \frac{\ell^2}{6} \frac{d^3 Y_i^{j+1}}{ds^3} + \frac{\ell^4}{120} \frac{d^5 Y_i^{j+1}}{ds^5} + \cdots \right\} \\
& + \frac{\gamma_{i+1}}{30} \left\{ -\ell \frac{d^2 Y_i^{j+1}}{ds^2} - \frac{2\ell^2}{3} \frac{d^3 Y_i^{j+1}}{ds^3} + \cdots \right\}
\end{aligned}
$$

$$\Rightarrow \quad |\mathsf{T}_i| \leq \max_{s_{i-1} \leq s \leq s_{i+1}} \left\{ \frac{\sigma \ell^2 \varepsilon}{12} \left| \frac{d^4 Y^{j+1}(s)}{ds^4} \right| \right\} + \max_{s_{i-1} \leq s \leq s_{i+1}} \left\{ \frac{28}{180} \gamma \ell^2 \left| \frac{d^3 Y^{j+1}(s)}{ds^3} \right| \right\}.$$

Using relation (22) with $W = \frac{\gamma(0)}{2} \coth(\frac{\gamma(1)\rho}{2})$, we get

$$\Rightarrow \quad |\mathsf{T}_i| \leq \max_{s_{i-1} \leq s \leq s_{i+1}} \left\{ \frac{W\ell^3}{12} \left| \frac{d^4 Y^{j+1}(s)}{ds^4} \right| \right\} + \max_{s_{i-1} \leq s \leq s_{i+1}} \left\{ \frac{28}{180} \gamma \ell^2 \left| \frac{d^3 Y^{j+1}(s)}{ds^3} \right| \right\}$$

$$\Rightarrow \quad |\mathsf{T}_i| \leq \max_{s_{i-1} \leq s \leq s_{i+1}} \left\{ \frac{28}{180} \gamma \ell^2 \left| \frac{d^3 Y^{j+1}(s)}{ds^3} \right| \right\} + O(\ell^3)$$

$$\Rightarrow \quad |\mathsf{T}_i| \leq O(\ell^2), \quad i = 1, 2, \ldots, N-1.$$

Thus we obtain the desired result. □

Lemma 4.4 *Let $Y(s_i, t_{j+1})$ be the solution of problem (6), and let Y_i^{j+1} be the solution of the discrete problem (22). Then we have the following estimate:*

$$\left| Y(s_i, t_{j+1}) - Y_i^{j+1} \right| \leq O(\ell^2).$$

Proof Rewrite Eq. (22) in matrix vector form as

$$ZY = H, \tag{23}$$

where $Z = (\chi_{i,j})$, $0 \leq j \leq M-1$, $1 \leq i \leq N-1$, is the tridiagonal matrix with

$$\chi_{i-1,j+1} = -\frac{\sigma(\rho)\varepsilon}{\ell^2} - \frac{\gamma_{i-1}}{60\ell} - \frac{28\gamma_i}{60\ell} - \frac{\gamma_{i+1}}{60\ell} + \frac{Q_{i-1}}{30},$$

$$\chi_{i,j+1} = \frac{2\sigma(\rho)\varepsilon}{\ell^2} + \frac{28Q_i}{30},$$

$$\chi_{i+1,j+1} = -\frac{\sigma(\rho)\varepsilon}{\ell^2} + \frac{\gamma_{i-1}}{60\ell} - \frac{\gamma_{i+1}}{60\ell} + \frac{28\gamma_i}{60\ell} + \frac{Q_{i+1}}{30},$$

and $H = (\mu_i^{j+1})$ is the column vector with $(\mu_i^{j+1}) = \frac{1}{30}(\vartheta_{i-1}^{j+1} + 28\vartheta_i^{j+1} + \vartheta_{i+1}^{j+1})$, $i = 1, 2, \ldots, N-1$, with local truncation error

$$|\top_i| \leq C(\ell^2).$$

We also have

$$Z\overline{Y} - \top(\ell) = H, \tag{24}$$

where $\overline{Y} = (\overline{Y}_0, \overline{Y}_1, \ldots, \overline{Y}_N)^t$ and $\top(\ell) = (\top_1(\ell), \top_2(\ell), \ldots, \top_N(\ell))^t$ are the actual solution and the local truncation error, respectively.

From Eqs. (23) and (24) we get

$$Z(\overline{Y} - Y) = \top(\ell). \tag{25}$$

Then Eq. (25) can be written as

$$ZE = \top(\ell), \tag{26}$$

where $E = \overline{Y} - Y = (\top_0, \top_1, \top_2, \ldots, \top_N)^t$. Let S be the sum of elements of the ith row of Z. Then we have

$$\Gamma_1 = \sum_{j=1}^{N-1} \chi_{1,j} = \frac{\sigma\varepsilon}{\ell^2} + \frac{\gamma_{i+1}}{60\ell} + \frac{\gamma_{i-1}}{60\ell} + \frac{28Q_i}{30} + \frac{Q_{i+1}}{30} + \frac{28\gamma_i}{60\ell},$$

$$\Gamma_{N-1} = \sum_{j=1}^{N-1} \chi_{N-1,j} = \frac{\sigma\varepsilon}{\ell^2} - \frac{\gamma_{i+1}}{60\ell} - \frac{\gamma_{i-1}}{60\ell} + \frac{28Q_i}{30} + \frac{Q_{i-1}}{30} - \frac{28\gamma_i}{60\ell},$$

$$\Gamma_i = \sum_{j=1}^{N-1} \chi_{i,j} = \frac{1}{30}(\vartheta_{i-1}^{j+1} + 28\vartheta_i^{j+1} + \vartheta_{i+1}^{j+1})$$

$$= \Gamma_i + O(\ell^2) = B_{i0}; \quad i = 2(1)N-2,$$

where $B_{i0} = \Gamma_i = \frac{1}{30}(\vartheta_{i-1}^{j+1} + 28\vartheta_i^{j+1} + \vartheta_{i+1}^{j+1})$.

Since $0 < \varepsilon \ll 1$, for sufficiently small ℓ, the matrix Y is irreducible and monotone. Then it follows that Z^{-1} exists and its elements are nonnegative [27]. Hence from Eq. (26) we obtain

$$E = Z^{-1}\top(\ell)$$

and

$$\|E\| \le \|Z^{-1}\| \|\top(\ell)\|. \tag{27}$$

Let $\overline{\chi}_{ki}$ be the (ki)th element of Z^{-1}. Since $\overline{\chi}_{ki} \ge 0$, by the definition of multiplication of matrices with its inverses we have

$$\sum_{i=1}^{N-1} \overline{\chi}_{ki} \Gamma_i = 1, \quad k = 1, 2, \dots, N-1.$$

Therefore it follows that

$$\sum_{i=1}^{N-1} \overline{\chi}_{ki} \le \frac{1}{\min_{0 \le i \le N-1} \Gamma_i} = \frac{1}{B_{i,0}} \le \frac{1}{|B_{i0}|} \tag{28}$$

for some $i0$ between 1 and $N-1$, and $B_{i0} = \Gamma_i$. From equations (23), (27), and (28) we obtain

$$E_i = \sum_{i=1}^{N-1} \overline{\chi}_{ki} \top(\ell), \quad i = 1(1)N-1,$$

which implies

$$E_i \le \frac{C(\ell^2)}{|\Gamma_i|}, \quad i = 1(1)N-1.$$

Therefore

$$\|E\| \le C(\ell^2).$$

This implies that the spatial semidiscretization process is convergent of second order. $\square$

Theorem 4.5 *Let $y(s, t)$ be the solution of problem* (1), *and let Y_i^j be the numerical solution obtained by the proposed scheme* (22). *Then we have the following error estimate for the totally discrete scheme*:

$$\sup_{0 < \varepsilon \ll 1} \max_{s_i, t_j} |y(s_i, t_j) - Y_i^j| \le C(\tau + (\ell)^2).$$

Proof By combining the result of Lemmas 3.1 and 4.4 we obtain the required bound. $\square$

5 Numerical examples, results, and discussion

In this section, we consider three model problems to verify the theoretical findings of the proposed method. As the exact solutions of the considered examples are not known, we calculate the maximum absolute error for each ε given in [28] by

$$E_\varepsilon^{N,\tau} = \max_{(s_i, t_{j+1}) \in \Im^{N,M}} |Y^{N,\tau}(s_i, t_{j+1}) - Y^{2N,\tau/2}(s_i, t_{j+1})|$$

and the corresponding order of convergence for each ε by

$$r_\varepsilon^{N,\tau} = \log_2\left(E_\varepsilon^{N,\tau}/E_\varepsilon^{2N,\tau/2}\right).$$

For all N and τ, the ε-uniform maximum error and the corresponding ε-uniform order of convergence are calculated using

$$E^{N,\tau} = \max_\varepsilon E_\varepsilon^{N,\tau} \quad \text{and} \quad r^{N,\tau} = \log_2\left(E^{N,\tau}/E^{2N,\tau/2}\right), \quad \text{respectively.}$$

Example 5.1 Consider the following SPBHE:

$$\begin{cases} \frac{\partial y}{\partial t} - \varepsilon \frac{\partial^2 y}{\partial s^2} + y \frac{\partial y}{\partial s} - (1-y)(y-0.5) = 0, & (s,t) \in \Im, \\ y(s,0) = s(1-s^2), & 0 \le s \le 1, \\ y(0,t) = 0 = y(1,t) = 0, & t \in (0,T]. \end{cases}$$

Example 5.2 Consider the following SPBHE:

$$\begin{cases} \frac{\partial y}{\partial t} - \varepsilon \frac{\partial^2 y}{\partial s^2} + y \frac{\partial y}{\partial s} = 0, (s,t) \in \Im, \\ y(s,0) = s(1-s^2), & 0 \le s \le 1, \\ y(0,t) = 0 = y(1,t) = 0, & t \in (0,T]. \end{cases}$$

Example 5.3 Consider the following SPBHE:

$$\begin{cases} \frac{\partial y}{\partial t} - \varepsilon \frac{\partial^2 y}{\partial s^2} + y \frac{\partial y}{\partial s} = (1-y)(y-0.5), & (s,t) \in \Im, \\ y(s,0) = \sin(\pi s), & 0 \le s \le 1, \\ y(0,t) = 0 = y(1,t) = 0, & t \in (0,T]. \end{cases}$$

The $E_\varepsilon^{N,\tau}$ $E^{N,\tau}$, and $r^{N,\tau}$ for Examples 5.1, 5.2, and 5.3 are tabulated for various values of ε, M, and N in Tables 1–4. These results show that the proposed scheme reveals the parameter uniform convergence of first order. Besides, the numerical results depict that the proposed method gives better results than those in [17–19, 23]. The 3D view of the numerical solution of Examples 5.1 and 5.2 at $N = 64$, $M = 40$, and $\varepsilon = 2^{-18}$ are plotted in Fig. 1. The effects of ε and the time step on the solution profile for the considered problems

Table 1 $E_\varepsilon^{N,\tau}$, $E^{N,\tau}$, and $r^{N,\tau}$ for Example 5.1 with $M = N$

$\varepsilon \downarrow N \rightarrow$	16	32	64	128	256	512
2^{-6}	4.1511e–03	2.1635e–03	1.1017e–03	5.5395e–04	2.7766e–04	1.3899e–04
2^{-8}	5.1926e–03	2.6194e–03	1.2873e–03	6.3309e–04	3.1331e–04	1.5576e–04
2^{-10}	5.9142e–03	3.0773e–03	1.4933e–03	7.0466e–04	3.3633e–04	1.7942e–04
2^{-12}	6.1411e–03	3.3268e–03	1.6982e–03	7.0466e–04	3.8724e–04	1.7942e–04
2^{-14}	6.1690e–03	3.3829e–03	1.7627e–03	8.9236e–04	4.4031e–04	2.1086e–04
2^{-16}	6.1691e–03	3.3865e–03	1.7723e–03	9.0674e–04	4.5674e–04	2.2707e–04
2^{-18}	6.1691e–03	3.3865e–03	1.7724e–03	9.0765e–04	4.5916e–04	2.3072e–04
2^{-20}	6.1691e–03	3.3865e–03	1.7724e–03	9.0765e–04	4.5918e–04	2.3095e–04
2^{-22}	6.1691e–03	3.3865e–03	1.7724e–03	9.0765e–04	4.5918e–04	2.3095e–04
$E^{N,\tau}$	6.1691e–03	3.3865e–03	1.7724e–03	9.0765e–04	4.5918e–04	2.3095e–04
$r^{N,\tau}$	0.86527	0.93409	0.96550	0.98308	0.99148	

Table 2 $E_\varepsilon^{N,\tau}$, $E^{N,\tau}$, and $r^{N,\tau}$ for Example 5.1

ε ↓	$N = 32$ $M = 20$	64 40	128 80	256 160	512 320	1024 640
Present Method						
2^{-6}	3.3862e−03	1.7451e−03	8.8199e−04	4.9830e−04	2.2211e−04	1.1118e−04
	0.95636	0.98448	0.82375	1.1657	0.99838	
2^{-8}	4.0057e−03	2.0129e−03	1.0010e−03	4.4317e−04	2.4846e−04	1.2404e−04
	0.99278	1.0078	1.1755	0.834851	1.0022	
2^{-10}	4.4504e−03	2.2209e−03	1.0791e−03	5.2577e−04	2.5865e−04	1.2819e−04
	1.0028	1.0413	1.0373	1.0234	1.0127	
2^{-12}	4.6896e−03	2.4227e−03	1.2006e−03	5.7639e−04	2.7481e−04	1.3289e−04
	0.95285	1.0129	1.0586	1.0686	1.0482	
2^{-14}	4.7443e−03	2.4871e−03	1.2644e−03	6.2915e−04	3.0594e−04	1.4544e−04
	0.93173	0.97601	1.0070	1.0402	1.0728	
2^{-16}	4.7485e−03	2.4971e−03	1.2792e−03	6.4578e−04	3.2224e−04	1.5880e−04
	0.92722	0.96501	0.98613	1.0029	1.0209	
2^{-18}	4.7485e−03	2.4972e−03	1.2802e−03	6.4830e−04	3.2598e−04	1.6299e−04
$E^{N,\tau}$	4.7485e−03	2.4972e−03	1.2802e−03	6.4830e−04	3.2598e−04	1.6299e−04
$r^{N,\tau}$	0.92716	0.96394	0.98164	0.99188	1.0000	
Results in [19]						
$E^{N,\tau}$	2.8299e−01	1.7036e−01	1.1805e−01	7.4251e−02	4.1879e−02	2.2413e−02
$r^{N,\tau}$	0.2844	0.6496	0.6469	0.8100	0.8973	
Results in [23]						
$E^{N,\tau}$	5.4827e−03	2.9644e−03	1.5416e−03	7.8591e−04	3.9638e−04	1.9820e−04
$r^{N,\tau}$	0.8874	0.9447	0.9756	0.9931	1.0069	

Table 3 $E_\varepsilon^{N,\tau}$, $E^{N,\tau}$, and $r^{N,\tau}$ for Example 5.2

ε ↓	$N = 32$ $M = 20$	64 40	128 80	256 160	512 320	1024 640
Present Method						
2^{-6}	2.7711e−03	2.2111e−03	7.2450e−04	3.6445e−04	1.8272e−04	9.1482e−05
	0.32569	1.6097	0.99126	0.99609	0.99807	
2^{-8}	3.4720e−03	1.7347e−03	8.6045e−04	4.2759e−04	2.1302e−04	1.0630e−04
	1.0011	1.0115	1.0089	1.0052	1.0028	
2^{-10}	3.9921e−03	1.9665e−03	9.4700e−04	4.5914e−04	2.2533e−04	1.1154e−04
	1.0215	1.0542	1.0444	1.0269	1.0145	
2^{-12}	4.3177e−03	2.2111e−03	1.0828e−03	5.1318e−04	2.4267e−04	1.1690e−04
	0.95285	1.0129	1.0586	1.0686	1.0482	
2^{-14}	4.4129e−03	2.3029e−03	1.1644e−03	5.7521e−04	2.9685e−04	1.2987e−04
	0.93828	0.98387	1.0174	0.95435	1.1927	
2^{-16}	4.4375e−03	2.3029e−03	1.1882e−03	5.9758e−04	3.0263e−04	1.4530e−04
	0.94630	0.95467	0.99157	0.98158	1.0585	
2^{-18}	4.4436e−03	2.3335e−03	1.1944e−03	6.0352e−04	3.0263e−04	1.5081e−04
$E^{N,\tau}$	4.4436e−03	2.3335e−03	1.1944e−03	6.0352e−04	3.0263e−04	1.5081e−04
$r^{N,\tau}$	0.92923	0.96621	0.98481	0.99585	1.0048	
Results in [19]						
$E^{N,\tau}$	2.5614e−01	2.1031e−01	1.3406e−01	8.5618e−02	4.8834e−02	2.6219e−02
$r^{N,\tau}$	0.7322	0.5292	0.6689	0.8262	0.9019	
Results in [23]						
$E^{N,\tau}$	5.3005e−03	2.8654e−03	1.4887e−03	7.5704e−04	3.8034e−04	1.8926e−04
$r^{N,\tau}$	0.7322	0.5292	0.6689	0.8262	0.9019	

are displayed in Figs. 2 and 3, respectively. The log-log plots of maximum absolute errors for Examples 5.1–5.3 are plotted in Fig. 4. This figure shows that the obtained theoretical rate of convergence of the proposed method agrees with numerical experiments.
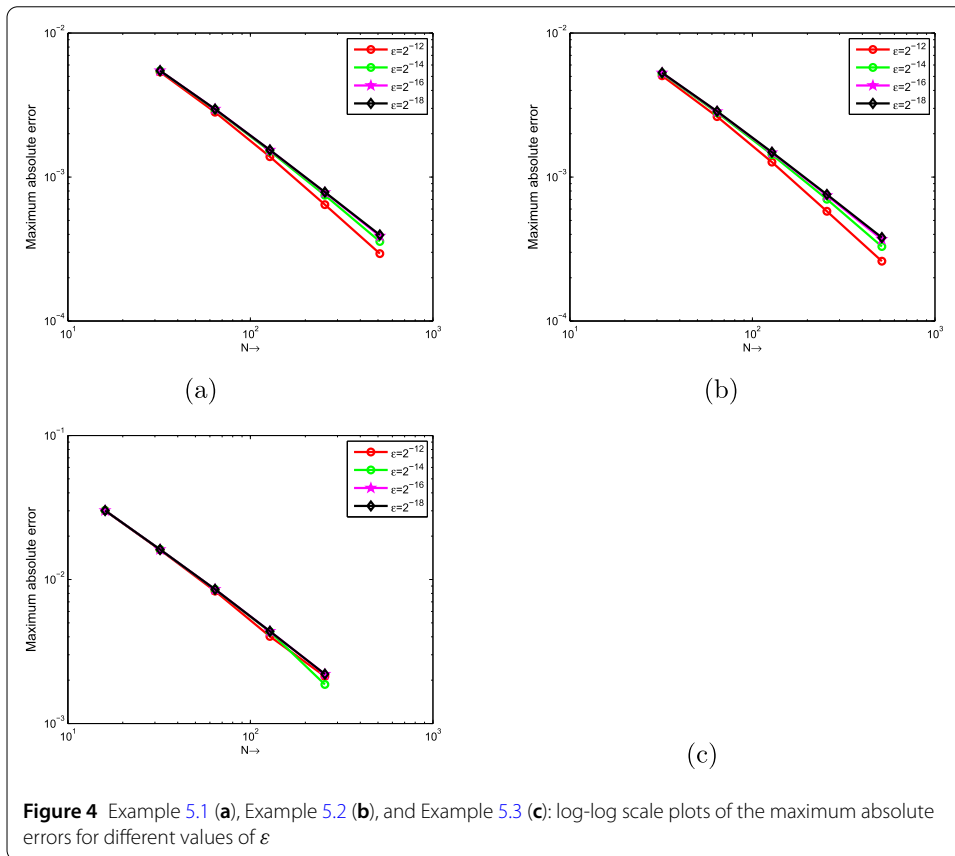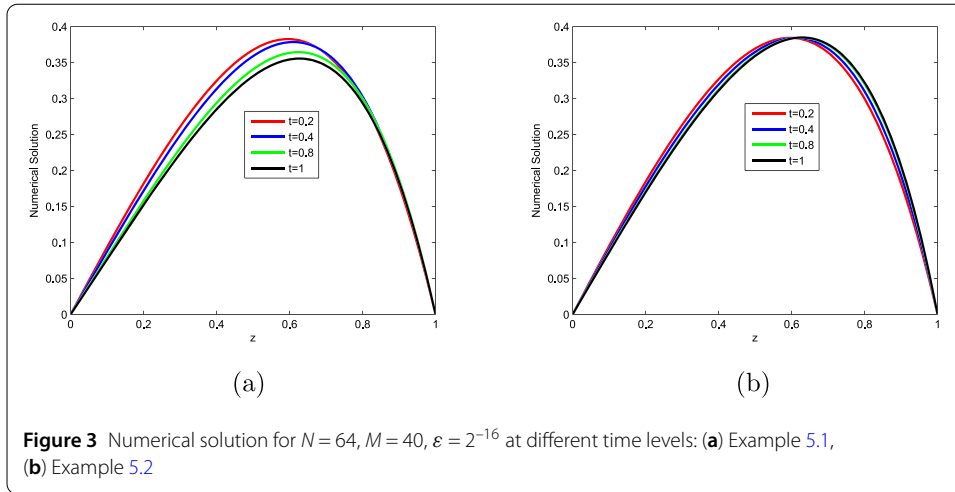
Table 4 $E_\varepsilon^{N,\tau}$, $E^{N,\tau}$, and $r^{N,\tau}$ for Example 5.3 with $\tau = 0.0001$

$\varepsilon \downarrow N \rightarrow$	16	32	64	128	256
2^{-6}	5.8282e–03	1.6509e–03	4.4458e–04	1.3541e–04	6.4898e–05
2^{-8}	1.2323e–02	5.1156e–03	1.7053e–03	4.9535e–04	1.5495e–04
2^{-10}	1.4869e–02	7.3963e–03	3.4059e–03	1.3771e–03	4.7100e–04
2^{-12}	1.5284e–02	7.9861e–03	4.0270e–03	1.9576e–03	9.0048e–04
2^{-14}	1.5294e–02	8.0395e–03	4.1387e–03	2.1050e–03	1.0594e–03
2^{-16}	1.5294e–02	8.0396e–03	4.1419e–03	2.1194e–03	1.0682e–03
2^{-18}	1.5294e–02	8.0396e–03	4.1419e–03	2.1195e–03	1.0791e–03
2^{-20}	1.5294e–02	8.0396e–03	4.1419e–03	2.1195e–03	1.0791e–03
2^{-22}	1.5294e–02	8.0396e–03	4.1419e–03	2.1195e–03	1.0791e–03
$E^{N,\tau}$	1.5294e–02	8.0396e–03	4.1419e–03	2.1195e–03	1.0791e–03
$r^{N,\tau}$	0.92777	0.95683	0.96657	0.97390	
$E^{N,\tau}$ [18]	4.0948e–2	1.8548e–2	7.5517e–3	3.3634e–3	1.6113e–3
$E^{N,\tau}$ [17]	4.0835e–2	2.2530e–2	1.1907e–2	6.128e–3	3.110e–3

Figure 1 3D view of numerical solution for $N = 64$, $M = 40$, $\varepsilon = 2^{-18}$: (**a**) Example 5.1, (**b**) Example 5.2

Figure 2 Effect of ε on the behavior of the solution with layer formation: (**a**) Example 5.1, (**b**) Example 5.2
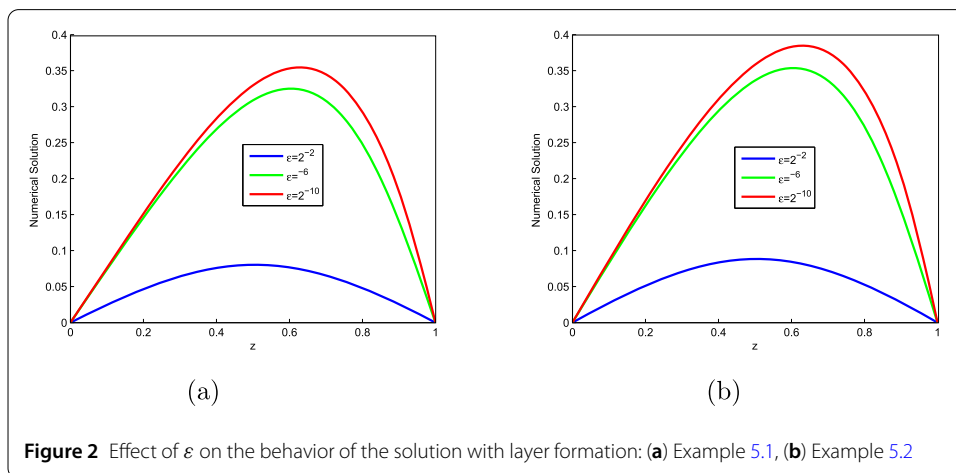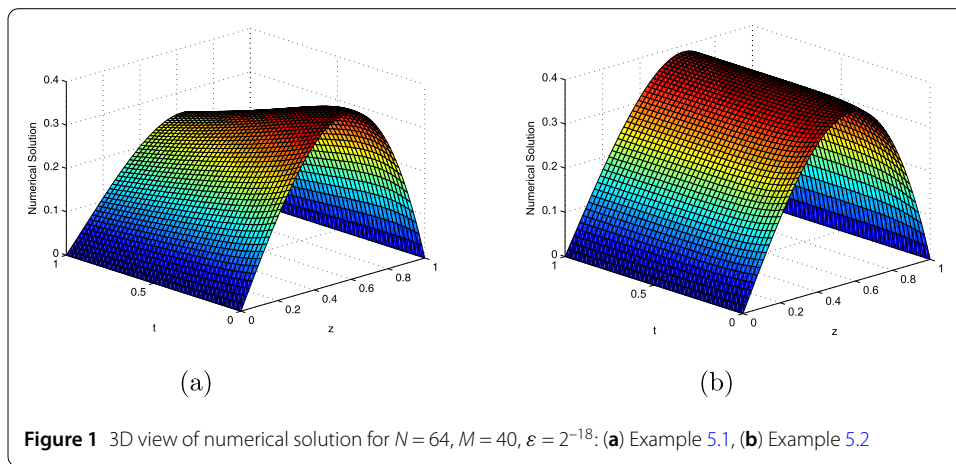
6 Conclusion

We have presented a parameter uniform numerical scheme for the singularly perturbed unsteady Burger–Huxley equation. The developed scheme constitutes the implicit Euler in the time direction and specially fitted finite difference method in the space direction. Theoretical and numerical stability and parameter uniform convergence analysis of the

Figure 3 Numerical solution for $N = 64$, $M = 40$, $\varepsilon = 2^{-16}$ at different time levels: (**a**) Example 5.1, (**b**) Example 5.2

Figure 4 Example 5.1 (**a**), Example 5.2 (**b**), and Example 5.3 (**c**): log-log scale plots of the maximum absolute errors for different values of ε

developed scheme is presented. The presented method is shown to be ε-uniformly convergent with convergence order $O(\tau + \ell^2)$. Several model examples are presented to illustrate the efficiency of the proposed method. The proposed scheme gives more accurate numerical results than those in [17–19, 23].

Declarations

Ethics approval and consent to participate
Not applicable.

Consent for publication
Not applicable.

Competing interests
The authors declare no competing interests.

Author contributions
Conceptualization: I.T.Daba & G.F. Duressa; Investigation and formal analysis: I.T.Daba & G.F. Duressa; Software programming: I.T.Daba; Visualization: I.T.Daba & G.F. Duressa; Writing- original draft: I.T.Daba; Writing- review & editing: G.F. Duressa.

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