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Existence and multiplicity of radially symmetric k-admissible solutions for a k-Hessian equation

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Abstract
In this paper, we show that the radially symmetric k-admissible solutions set of a k-Hessian equation Dirichlet problem with homogeneous boundary condition contains a reversed S-shaped connected component. By determining the shape of unbounded continua of the solutions, we obtain the existence and multiplicity of radially symmetric k-admissible solutions with respect to the bifurcation parameter λ. The proof is based on the bifurcation technique.

MSC: 34C23; 35J60

Keywords: Hessian equation; Admissible solutions; Positive solutions; Bifurcation

1 Introduction

Consider the following k-Hessian equation Dirichlet problem with homogeneous boundary condition

$$\begin{cases} S_k(D^2u) = \lambda^k f(-u) & \text{in } B_1, \\ u = 0 & \text{on } \partial B_1, \end{cases} \tag{1.1}$$

where $B_1 = \{x \in \mathbb{R}^n : |x| < 1\}$, λ is a positive parameter, $S_k(D^2u)$ is the k-Hessian operator of $u, k \in \{1, \dots, n\}$, $f : [0,\infty) \to [0,\infty)$ is a continuous function with $f(s) > 0$ for all $s > 0$. Let $\lambda(D^2u) := (\lambda_1, \lambda_2, \dots, \lambda_n)$ be the eigenvalues of the Hessian matrix D^2u and $\sigma_k(\lambda)$ be the kth elementary symmetric function of λ defined as follows:

$$\sigma_k\big(\lambda(D^2u)\big) = \sum_{1\le i_1<i_2<\cdots<i_k\le n} \lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_k}.$$

Then, $S_k(D^2u) := \sigma_k(\lambda(D^2u))$.

To work with elliptic operators, we always need u to be a k-admissible function. A function $u \in C^2(B_1) \cap C^0(\overline{B_1})$ is called k-admissible if $\lambda(D^2u) \in \Gamma_k$, where

$$\Gamma_k = \big\{\lambda \in \mathbb{R}^n : \sigma_j(\lambda) > 0, j = 1, 2, \dots k\big\}.$$

It is worth noting that the k-Hessian operator is elliptic for any k-admissible function u. Moreover, a k-admissible solution is subharmonic and, by the maximum principle, is negative in B_1, see [1, p. 30]. The k-Hessian operator includes the following special examples:

- when $k = 1$, $S_1(D^2u) = \Delta u = \sum \partial^2 u/\partial x_i^2$ and
- when $k = n$, $S_n(D^2u) = \det D^2u$.

Hence, the k-Hessian equation can be regarded as an extension of the semilinear elliptic equation and the Monge–Ampère equation. The study of the Monge–Ampère equation began with Monge [2] in 1784 and was continued by Ampère [3] in 1820. Since then, many scholars have studied this equation. For a full discussion, we recommend [4] and references therein.

The study of k-admissible solutions for a k-Hessian equation by the bifurcation technique can be traced back to Jacobsen [5]. In 1999, Jacobsen established the global bifurcation result for problem (1.1). Let μ_1 be the first eigenvalue of boundary value problem

$$\begin{cases} S_k(D^2u) = \lambda^k(-u)^k & \text{in } B_1, \\ u = 0 & \text{on } \partial B_1, \end{cases} \tag{1.2}$$

which was first obtained in [1]. Jacobsen [5] proved that problem (1.1) with $f(s) = s^k + g(s)$ and $\lim_{s\to 0^+} g(s)/s^k = 0$ possesses an unbounded continuum $\mathcal{C}$ of nontrivial admissible solutions, which bifurcates from $(\mu_1, 0)$ and lies in the strip $\{(\lambda, u) : 0 \le \lambda \le \mu_1\}$. Dai and Luo [6] in 2018 pointed out that the conclusion of "$\mathcal{C}$ lies in the strip $\{(\lambda, u) : 0 \le \lambda \le \mu_1\}$" is not true. By using the bifurcation technique, they corrected this result and studied the global behavior of admissible solutions for problem (1.1). In another paper, Dai [7] established the existence, nonexistence, uniqueness, and multiplicity of radial symmetric k-admissible solutions of problem (1.1) by using the bifurcation technique according to the asymptotic behavior of f at 0 and ∞. However, the sublinear and superlinear conditions imposed on the nonlinearities only deduce a relatively simple *shape of the component*, and they provided no information on at least two direction turns of the connected component. In 2019, by the bifurcation technique, Ma, He and Yan [8] improved the result of [7], showing in their Theorem 1.1 that problem (1.1) has at least three radially symmetric k-admissible solutions under suitable conditions on the nonlinearity.

Very recently, He and Miao [9] showed that the k-admissible solutions of (1.1) are in general not convex, and they constructed a new cone and obtained the existence of three radially symmetric k-admissible solutions via the Leggett–Williams' fixed-point theorem. Zhang, Xu and Wu [10] studied the k-admissible solutions for the eigenvalue problem of a singular k-Hessian equation. By constructing the upper and lower solutions of the k-Hessian equation, the existence of a radially symmetric solution for the eigenvalue problem is established via Schauder's fixed-point theorem.

For other results concerning the existence, nonexistence, and multiplicity of k-admissible solutions for a k-Hessian equation, we refer the reader to [11–21] and the references therein.

Inspired by [5–10] and by a modified version of the global bifurcation result of problem (1.1) in [6], we show in this paper the existence of a *reversed S-shaped connected component* of radially symmetric k-admissible solutions of (1.1). As a byproduct, we assert further that (1.1) has one, two or three radially symmetric k-admissible solutions under the suitable conditions on the nonlinearity.

Throughout the paper, we make the following hypotheses on the nonlinearity f:

(F1) there exist $\alpha > 0$, $f_0 > 0$ and $f_1 > 0$ such that

$$\lim_{s\to 0^+}\frac{f(s)-f_0^k s^k}{s^{k+\alpha}} = f_1^k; \tag{1.3}$$

(F2)

$$f_\infty := \lim_{s\to\infty}\frac{f(s)}{s^k} = \infty; \tag{1.4}$$

(F3) there exists $s_0 > 0$ such that if $0 \le s \le s_0$ we have

$$0 \le f(s) \le \frac{C_n^k f_0^k}{\mu_1^k n} s_0^k,$$

where $\mu_1 > 0$ is the first eigenvalue of (1.2). Moreover, μ_1 is simple (see [1]).

Let $X = C[0,1]$ with the normal $\|u\| = \max_{r\in[0,1]}|u(r)|$. Let $E = \{u \in C^1[0,1] : u'(0) = u(1) = 0\}$ with the norm $\|u\|_1 = \max_{r\in[0,1]}|u(r)| + \max_{r\in[0,1]}|u'(r)|$. Let $X^+ = \{u \in X : u \ge 0\}$ and P^+ be the set of functions in X^+ that are positive in $[0,1)$. Also, set $K^+ = \mathbb{R}\times P^+$ under the product topology.

Our main result is the following theorem.

Theorem 1.1 (see Fig. 1) *Assume that* (*F*1)–(*F*3) *hold. Then, there exist* $\lambda_* \in (0, \frac{\mu_1}{f_0})$ *and* $\lambda^* > \frac{\mu_1}{f_0}$ *such that*:

(i) (1.1) *has at least one radially symmetric k-admissible solution if* $0 < \lambda < \lambda_*$;

(ii) (1.1) *has at least two radially symmetric k-admissible solutions if* $\lambda = \lambda_*$;

(iii) (1.1) *has at least three radially symmetric k-admissible solutions if* $\lambda_* < \lambda < \mu_1/f_0$;

(iv) (1.1) *has at least two radially symmetric k-admissible solutions if* $\mu_1/f_0 < \lambda \le \lambda^*$;

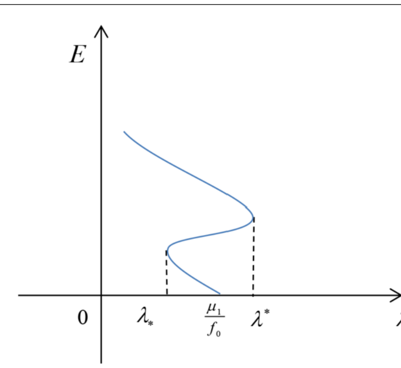
(v) (1.1) *has at least one radially symmetric k-admissible solution if* $\lambda = \lambda^*$;

(vi) (1.1) *has no radially symmetric k-admissible solution if* $\lambda > \lambda^*$.

Remark 1.1 Condition (F1) implies

$$\lim_{s\to 0^+}\frac{f(s)}{s^k} = f_0^k.$$

Figure 1 Reversed *S*-shaped connected component



Remark 1.2 Let us consider the function

$$f(s) = ms^k + s^k \ln(1+s), \quad m > 0, s \in [0,\infty).$$

Obviously, f satisfies (F1) and (F2) with

$$\alpha = 1, \qquad f_0 = m^{1/k}, \qquad f_1 = 1.$$

It is easy to see that if $m > \frac{mC_n^k}{\mu_1^k n}$ is sufficiently large, then the function f also satisfies (F3).

Remark 1.3 Note that Condition (F1) has never been used before, as far as the authors know. Indeed, under (F1), we have an unbounded subcontinuum that is bifurcating from $(\mu_1/f_0, 0)$ and goes leftward. Conditions (F2) and (F3) lead the unbounded subcontinuum to the right at some point, and finally to the left near $\lambda = 0$.

The paper is organized as follows. In Sect. 2, we show global bifurcation phenomena from the trivial branch with the leftward direction near the initial point. Section 3 is devoted to showing that there are at least two direction turns of the component and completing the proof of Theorem 1.1.

2 Preliminaries

In this section, we give some lemmas and show a global bifurcation phenomenon from the trivial branch.

Lemma 2.1 ([11]) *Assume $z(r) \in C^2[0,R]$ is radially symmetric and $z'(0) = 0$. Then, the function $u(|x|) = z(r)$ with $r = |x| < R$ belongs to $C^2(B_R)$, and*

$$\lambda(D^2u) = \begin{cases} (z''(r), \frac{z'(r)}{r}, \dots, \frac{z'(r)}{r}), & r \in (0,R), \\ (z''(0), z''(0), \dots, z''(0)), & r = 0, \end{cases}$$

$$S_k(\lambda(D^2u)) = \begin{cases} C_{n-1}^{k-1} z''(r)(\frac{z'(r)}{r})^{k-1} + C_{n-1}^k(\frac{z'(r)}{r})^k, & r \in (0,R), \\ C_n^k(z''(0))^k, & r = 0, \end{cases}$$

where $B_R = \{x \in \mathbb{R}^n : |x| < R\}$ and $C_n^m = \frac{n!}{(n-m)!m!}$.

Lemma 2.2 *The function $u \in C^2(B_1)$ is a radially symmetric k-admissible solution of the problem (1.1) if and only if $z(r)$ is a solution of the boundary value problem*

$$\begin{cases} (r^{n-k}(z')^k)' = \lambda^k \frac{n}{C_n^k} r^{n-1} f(-z), & r \in (0,1), \\ z'(0) = z(1) = 0. \end{cases} \tag{2.1}$$

Let $v = -z$. The problem (2.1) can be written as

$$\begin{cases} (r^{n-k}(-v')^k)' = \lambda^k \frac{n}{C_n^k} r^{n-1} f(v), & r \in (0,1), \\ v'(0) = v(1) = 0. \end{cases} \tag{2.2}$$

Next, we will establish the global bifurcation result for the problem (2.2) with $f(s) = s^k + g(s)$, i.e.,

$$\begin{cases} (r^{n-k}(-v')^k)' = \lambda^k \frac{n}{C_n^k} r^{n-1}(v^k + g(v)), & r \in (0,1), \\ v'(0) = v(1) = 0. \end{cases} \tag{2.3}$$

Here, $g : \mathbb{R}^+ \to \mathbb{R}$ satisfies $\lim_{s\to 0^+} g(s)/s^k = 0$ and the following subcritical growth restriction

$$|g(s)| \le C\big(1 + |s|^{q-1}\big) \tag{2.4}$$

for some $q \in (0, k^*)$, where

$$k^* = \begin{cases} \frac{(n+2)k}{n-2k} & \text{if } 1 \le k < \frac{n}{2}, \\ \infty & \text{if } \frac{n}{2} \le k \le n \end{cases}$$

is the critical exponent for the k-Hessian operator [12]. In particular, in [12], the author proved that the boundary value problem

$$\begin{cases} S_k(D^2 u) = (-u)^p & \text{in } B_R, \\ u = 0 & \text{on } \partial B_R \end{cases}$$

has no solution in $C^1(\bar{B}_R) \cap C^4(B_R)$ for the supercritical (or critical) case $p \ge k^*$ and it admits a solution that is radially symmetric and is in $C^2(\bar{B}_R)$ in the subcritical case $0 < p < k^*$, with $p \ne k$. Moreover, if $k = 1$, the 1-Hessian operator is the Laplacian and $k^* = (n+2)/(n-2)$ is the critical Sobolev exponent.

Lemma 2.3 ([7, Theorem 1.1]) *The pair* $(\mu_1, 0)$ *is a bifurcation point of problem* (2.3) *and the associated bifurcation branch* $\mathcal{C} \subseteq (K^+ \cup \{(\mu_1, 0)\})$ *is unbounded in* $[0,\infty) \times X$.

Lemma 2.4 ([7, Theorem 6.5]) *If* $f_0 \in (0,\infty)$, $f_\infty = \infty$ *and* (2.4) *holds, there is an unbounded component* $\mathcal{C}$ *of the set of positive solutions of problem* (2.2) *bifurcating from* $(\mu_1/f_0, 0)$ *such that* $\mathcal{C} \subseteq (K^+ \cup \{(\mu_1/f_0, 0)\})$. *Moreover,* $\mathcal{C}$ *joins* $(\mu_1/f_0, 0)$ *to* $(0,\infty)$.

Lemma 2.5 *Let the hypotheses of Lemma* 2.4 *hold. Suppose* $(\lambda_j, u_j) \subset \mathcal{C}$ *is a sequence of positive solutions of* (2.2) *that satisfies*

$$\|u_j\| \to 0 \quad \textit{and} \quad \lambda_j \to \mu_1/f_0 \quad \textit{as } n \to \infty.$$

Then, there exists a subsequence of $\{u_j\}$, *again denoted by* $\{u_j\}$, *such that* $u_j/\|u_j\|$ *converges uniformly to* ϕ_1 *on* $[0,1]$. *Moreover,* $u_j/\|u_j\|_1$ *converges to* ϕ_1 *in* $C^1[0,1]$. *Here,* $\phi_1(r)$ *is the eigenfunction of* (1.2) *corresponding to* μ_1.

Proof As the proof is very similar to that in [8, Lemma 2.3] we omit it here. □

3 Direction turn of bifurcation

Lemma 3.1 *Let the hypotheses of Lemma 2.4 hold. Then, there exists $\delta > 0$ such that $(\lambda, u) \in \mathcal{C}$ and $|\lambda - \mu_1/f_0| + \|u\| \leq \delta$ implies $\lambda < \mu_1/f_0$.*

Proof For contradiction, we assume that there exists a sequence $\{(\lambda_j, u_j)\} \subset \mathcal{C}$ satisfying

$$\lambda_j \to \mu_1/f_0, \qquad \|u_j\| \to 0 \quad \text{and} \quad \lambda_j \geq \mu_1/f_0.$$

By Lemma 2.5, there exists a subsequence of $\{u_j\}$, again denoted by $\{u_j\}$, such that $u_j/\|u_j\|$ converges uniformly to ϕ_1 on $[0,1]$, where $\phi_1(r) > 0$ is the first eigenfunction of (1.2) that satisfies $\|\phi_1\| = 1$. Multiplying Eq. (2.2) with $(\lambda, v) = (\lambda_j, u_j)$ by u_j and integrating it on $[0,1]$, we obtain

$$(\lambda_j)^k \frac{n}{C_n^k} \int_0^1 r^{n-1} f\big(u_j(r)\big) u_j(r)\, dr = \int_0^1 r^{n-k} \big(-u_j'(r)\big)^{k+1}\, dr,$$

that is

$$(\lambda_j)^k \frac{n}{C_n^k} \int_0^1 r^{n-1} \frac{f(u_j(r))}{\|u_j\|^k} \frac{u_j(r)}{\|u_j\|}\, dr = \int_0^1 r^{n-k} \frac{(-u_j'(r))^{k+1}}{\|u_j\|^{k+1}}\, dr.$$

From Lemma 2.5, after taking a subsequence and relabeling if necessary, $u_j/\|u_j\|$ converges to ϕ_1 in $C^1[0,1]$. Since

$$\int_0^1 r^{n-k} \big(-\phi_1'(r)\big)^{k+1}\, dr = \mu_1^k \frac{n}{C_n^k} \int_0^1 r^{n-1} \phi_1^{k+1}\, dr,$$

it follows that

$$(\lambda_j)^k \frac{n}{C_n^k} \int_0^1 r^{n-1} \frac{f(u_j(r))}{\|u_j\|^k} \frac{u_j(r)}{\|u_j\|}\, dr = \mu_1^k \frac{n}{C_n^k} \int_0^1 r^{n-1} \frac{(u_j(r))^{k+1}}{\|u_j\|^{k+1}}\, dr - \hat{\zeta}(j),$$

and accordingly,

$$(\lambda_j)^k \frac{n}{C_n^k} \int_0^1 r^{n-1} f\big(u_j(r)\big) u_j(r)\, dr = \mu_1^k \frac{n}{C_n^k} \int_0^1 r^{n-1} \big(u_j(r)\big)^{k+1} - \hat{\zeta}(j) \|u_j\|^{k+1},$$

with $\hat{\zeta} : \mathbb{N} \to \mathbb{R}$ satisfying $\lim_{j\to\infty} \hat{\zeta}(j) = 0$.

That is,

$$\begin{aligned}
&\frac{n}{C_n^k} \int_0^1 r^{n-1} \frac{f(u_j(r)) - f_0^k |u_j(r)|^k}{|u_j(r)|^{k+\alpha}} \left| \frac{u_j(r)}{\|u_j\|} \right|^{k+1+\alpha} dr \\
&\quad = \frac{\frac{n\mu_1^k}{C_n^k} - \frac{nf_0^k}{C_n^k}\lambda_j^k}{\lambda_j^k \|u_j\|^\alpha} \int_0^1 r^{n-1} \left| \frac{u_j(r)}{\|u_j\|} \right|^{k+1} dr - \hat{\zeta}(j).
\end{aligned}$$

Lebesgue's dominated convergence theorem, condition (F1) implies that

$$\int_0^1 r^{n-1} \frac{f(u_j(r)) - f_0^k |u_j(r)|^k}{|u_j(r)|^{k+\alpha}} \left| \frac{u_j(r)}{\|u_j\|} \right|^{k+1+\alpha} dr \to f_1^k \int_0^1 r^{n-1} \big|\phi_1(r)\big|^{k+1+\alpha}\, dr > 0$$

and

$$\int_0^1 r^{n-1}\left|\frac{u_j(r)}{\|u_j\|}\right|^{k+1} dr \to \int_0^1 r^{n-1}\left|\phi_1(r)\right|^{k+1} dr > 0.$$

This contradicts $\lambda_j \geq \mu_1/f_0$. □

Remark 3.1 Lemma 3.1 implies that the bifurcation branch $\mathcal{C}$ has the leftward direction from the bifurcation point $(\mu_1/f_0, 0)$.

Lemma 3.2 *Assume that* $(F1)$–$(F3)$ *hold. Let u be a positive solution of* (2.2) *with* $0 < f(s) \leq f_* s^k$ *for some* $f_* > 0$. *Then, there exists a constant* $C > 0$ *independently of u such that*

$$\left|u'(r)\right| \leq \lambda C\|u\|, \quad r \in [0,1].$$

Proof Integrating Eq. (2.2) on $[0,r]$ and recalling that $f(s) \leq f_* s^k$, we have

$$\left(-u'(r)\right)^k = r^{k-n}\int_0^r \left(\lambda^k \frac{n}{C_n^k} t^{n-1} f\left(u(t)\right)\right) dt \leq \int_0^1 \left(\lambda^k \frac{n}{C_n^k} f_* \|u\|^k\right) dt,$$

that is,

$$\left|u'\right| \leq \lambda\left(\int_0^1 \frac{n}{C_n^k} f_* \, dt\right)^{1/k} \|u\|.$$ □

Lemma 3.3 *Assume that* $(F3)$ *holds. Suppose u is a positive solution of* (2.2) *with* $\|u\| = s_0$. *Then,*

$$\lambda > \mu_1/f_0.$$

Proof Let (λ, u) be a positive solution of (2.2). By Lemma 3.2 and condition (F3), it follows that

$$s_0 = \|u\| = -\int_0^1 u'(x)\,dx \leq \int_0^1 \left|u'(x)\right| dx < \lambda\left(\int_0^1 \frac{n}{C_n^k}\frac{C_n^k f_0^k}{\mu_1^k n}\, dt\right)^{1/k} \|u\|.$$

That is, $\lambda > \frac{\mu_1}{f_0}$. □

Remark 3.2 By Lemma 3.3, we know that there exists a direction turn of the bifurcation branch $\mathcal{C}$ that grows to the right at some point $(\lambda^*, u_{\lambda^*}) \in \mathcal{C}$, where $\|u_{\lambda^*}\| = s_0$.

Proof of Theorem 1.1 Let $\mathcal{C}$ be as in Lemma 2.4. By Lemma 2.4, $\mathcal{C}$ is bifurcating from $(\frac{\mu_1}{f_0}, 0)$ and joins $(\mu_1/f_0, 0)$ to $(0, \infty)$.

Since $\mathcal{C}$ is unbounded, there exists $\{(\lambda_n, u_n)\}$ such that $(\lambda_n, u_n) \in \mathcal{C}$ and $\lambda_n + \|u_n\| \to \infty$. By Lemma 2.4, we have that $\|u_n\| \to \infty$ and $\lambda_n \to 0$, then there exists $(\lambda_0, u_0) \in \mathcal{C}$ such that $\|u_0\| = s_0$ and Lemma 3.3 shows that $\lambda_0 > \frac{\mu_1}{f_0}$.

By Lemmas 3.1 and 3.3, $\mathcal{C}$ passes through some points $(\frac{\mu_1}{f_0}, v_1)$ and $(\frac{\mu_1}{f_0}, v_2)$ with $\|v_1\| < s_0 < \|v_2\|$, and there exist $\underline{\lambda}$ and $\overline{\lambda}$ that satisfy $0 < \underline{\lambda} < \frac{\mu_1}{f_0} < \overline{\lambda}$ and both (i) and (ii):

(i) if $\lambda \in (\frac{\mu_1}{f_0}, \overline{\lambda}]$, then there exist u and v such that $(\lambda, u), (\lambda, v) \in \mathcal{C}$ and $\|u\| < s_0 < \|v\|$;
(ii) if $\lambda \in (\underline{\lambda}, \frac{\mu_1}{f_0}]$, then there exist u and v such that $(\lambda, u), (\lambda, v) \in \mathcal{C}$ and $\|u\| < \|v\| < s_0$.

Define $\lambda^* = \sup\{\overline{\lambda} : \overline{\lambda}$ satisfies (i)$\}$ and $\lambda_* = \inf\{\underline{\lambda} : \underline{\lambda}$ satisfies (ii)$\}$. Then, by the standard arguments, (2.2) has a positive solution at $\lambda = \lambda_*$ and $\lambda = \lambda^*$, respectively.

Clearly, $\mathcal{C}$ turns to the right at $(\lambda_*, \|u_{\lambda_*}\|)$ and to the left at $(\lambda^*, \|u_{\lambda^*}\|)$, and finally to the left near $\lambda = 0$. This complete the proof of Theorem 1.1. □

Now, we strengthen the assumptions on f. Assume, in addition to (F1)–(F3), that f satisfies:

(F4) there exists $s_1 > 2s_0 > 0$ such that $0 \le s \le s_1$ implies that

$$\min_{s_1 \le s \le 2s_1} \frac{f(s)}{s^k} > \frac{f_0^k}{\mu_1^k}\eta_1^k,$$

where η_1 is the first positive eigenvalue of the following problem

$$\begin{cases} (r^{n-k}(-v')^k)' = \eta^k \frac{n}{C_n^k} r^{n-1} v^k, & r \in (0,1), \\ v'(0) = v(\frac{1}{2}) = 0. \end{cases}$$

By an argument similar to proving [8, Lemma 3.5] with small modification, we may obtain the following result.

Lemma 3.4 *Assume* (F4) *holds. Let* (λ, u) *be a positive solution of* (2.2) *with* $\|u\| = s_1$. *Then,* $\lambda < \frac{\mu_1}{f_0}$.

Remark 3.3 Lemma 3.4 means that there exists a direction turn of the bifurcation continuum $\mathcal{C}$ that grows to the left at some point $(\lambda^{**}, u_{\lambda^{**}}) \in \mathcal{C}$, where $\|u_{\lambda^{**}}\| = s_1$.

Using the method similar to that used in proving Lemmas 3.3 and 3.4 infinite time, we have

Theorem 3.1 *Assume that* (F1)–(F4) *hold. Then, the continuum* $\mathcal{C}$ *is unbounded, joins* $(\frac{\mu_1}{f_0}, 0)$ *to* $(0, \infty)$, *and oscillates around the axis* $\{\lambda = \frac{\mu_1}{f_0}\}$ *an infinite number of times.*

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Author contributions
LYM completed the main study and wrote the manuscript, ZQH checked the proofs process and verified the calculation. Moreover, all the authors read and approved the last version of the manuscript.

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