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Positive ground states for nonlinear Schrödinger–Kirchhoff equations with periodic potential or potential well in $\mathbf{R}^3$

Wei Chen[1], Zunwei Fu[1] and Yue Wu[1*]

*Correspondence:
wuyue@lyu.edu.cn
[1]School of Mathematics and Statistics, Linyi University, 276000 Linyi, China

Abstract

This work is devoted to the nonlinear Schrödinger–Kirchhoff-type equation

$$-\left(a+b\int_{\mathbb{R}^3}|\nabla u|^2\,\mathrm{d}x\right)\Delta u+V(x)u=f(x,u),\quad \text{in } \mathbb{R}^3,$$

where $a>0$, $b\geq 0$, the nonlinearity $f(x,\cdot)$ is 3-superlinear and the potential V is either periodic or exhibits a finite potential well. By the mountain pass theorem, Lions' concentration-compactness principle, and the energy comparison argument, we obtain the existence of positive ground state for this problem without proving the Palais–Smale compactness condition.

Keywords: Schrödinger–Kirchhoff equations; Ground states; Potential well; Variational methods

1 Introduction and main results

In this work, we study the following Schrödinger–Kirchhoff-type equation:

$$-\left(a+b\int_{\mathbb{R}^3}|\nabla u|^2\,\mathrm{d}x\right)\Delta u+V(x)u=f(x,u),\quad \text{in } \mathbb{R}^3, \tag{1.1}$$

where $a>0$, $b\geq 0$ are constants. We assume that the potential $V(x)$ satisfies

(V_0) $V\in C(\mathbb{R}^3)$ and $\alpha:=\inf_{x\in\mathbb{R}^3}V(x)>0$,

and one of the following conditions:

(V_1) $V(x)$ and $f(x,t)$ are 1-periodic in x_1,x_2,x_3;

(V_2) $V(x)<V_\infty:=\lim_{|x|\to\infty}V(x)<\infty$ for all $x\in\mathbb{R}^3$.

For the nonlinearity $f(x,t)$, we make the following assumptions:

(f_1) $f\in C(\mathbb{R}^3\times\mathbb{R}^+)$ and satisfies

$$\lim_{t\to 0^+}\frac{f(x,t)}{t}=\lim_{t\to+\infty}\frac{f(x,t)}{t^5}=0\quad \text{uniformly in } x\in\mathbb{R}^3;$$

(f_2) $F(x,t)/t^4\to+\infty$ as $t\to+\infty$ uniformly in $x\in\mathbb{R}^3$, where $F(x,t):=\int_0^t f(x,s)\,\mathrm{d}s$;

(f_3) $f(x,t)/t^3$ is nondecreasing in $t \in \mathbb{R}^+$.

Since we intend to look for positive solutions of (1.1), we may assume without restriction that $f(x,t) = 0$ for all $(x,t) \in \mathbb{R}^3 \times (-\infty, 0]$ throughout this paper. We shall work on $E \equiv H^1(\mathbb{R}^3)$ with the norm

$$\|u\| = \left(\int_{\mathbb{R}^3} \left(a|\nabla u|^2 + V(x)u^2\right) \mathrm{d}x \right)^{1/2}, \tag{1.2}$$

which, by virtue of the assumptions on V, is equivalent to the standard $H^1(\mathbb{R}^3)$ norm. The energy functional associated with (1.1) is defined by $\Phi : E \to \mathbb{R}$:

$$\Phi(u) = \frac{a}{2} \int_{\mathbb{R}^3} |\nabla u|^2 \,\mathrm{d}x + \frac{1}{2} \int_{\mathbb{R}^3} V(x)u^2 \,\mathrm{d}x + \frac{b}{4} \left(\int_{\mathbb{R}^3} |\nabla u|^2 \,\mathrm{d}x \right)^2 - \int_{\mathbb{R}^3} F(x,u) \,\mathrm{d}x. \tag{1.3}$$

By (f_1), Φ is of class C^1 on E with the derivative given by

$$\langle \Phi'(u), v \rangle = \left(a + b \int_{\mathbb{R}^3} |\nabla u|^2 \,\mathrm{d}x \right) \int_{\mathbb{R}^3} \nabla u \nabla v \,\mathrm{d}x + \int_{\mathbb{R}^3} V(x)uv \,\mathrm{d}x - \int_{\mathbb{R}^3} f(x,u)v \,\mathrm{d}x \tag{1.4}$$

for all $u, v \in E$. It is well known that the critical points of Φ are weak solutions of problem (1.1). Furthermore, a *ground state solution* is the solution corresponding to the least critical value of Φ, that is, a nontrivial solution u satisfying $\Phi(u) = \inf_{\mathcal{K}} \Phi$, where

$$\mathcal{K} := \left\{ v \in E \backslash \{\mathbf{0}\} : \Phi'(v) = 0 \right\}. \tag{1.5}$$

In (1.1), if $V(x) \equiv 0$ and $\mathbb{R}^3$ is replaced by a bounded domain $\Omega \subset \mathbb{R}^N$, it reduces to the following Dirichlet problem of Kirchhoff type:

$$\begin{cases} -(a + b \int_{\Omega} |\nabla u|^2 \,\mathrm{d}x) \Delta u = f(x,u) & \text{in}\Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.6}$$

Problem (1.6) has a great importance in the study of stationary solutions for the equation

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 \mathrm{d}x \right) \frac{\partial^2 u}{\partial x^2} = 0$$

proposed by Kirchhoff [1] as an extension of the classical D'Alembert's wave equation for free vibrations of elastic strings, where ρ, h, P_0, L are positive constants. For more details on the physical and mathematical background of this problem, we refer to [1–3] for references. Such problems are often called nonlocal since the equation is no longer a pointwise identity due to the presence of integral over Ω. This phenomenon gives rise to some mathematical difficulties and makes (1.6) different from the classical elliptic problems.

Problems on a bounded domain like (1.6) have been studied extensively; see, e.g., [4–6] and the references therein. In recent years, growing attention has been paid to equations of type (1.1) set on the entire space. Some interesting studies by variational methods can be found in, for example, [3, 6–20] and the references therein. The current work is concerned with (1.1) of the subcritical and superlinear case. In the following, we shall focus more on some related results. On account of the 4-order term $(\int_{\mathbb{R}^3} |\nabla u|^2\,dx)^2$, to ensure that the functional Φ is superlinear, one usually assumes that $f(x,t)$ is 4-superlinear at infinity, namely,

$$\lim_{|t|\to\infty} \frac{F(x,t)}{t^4} = +\infty \quad \text{uniformly in } x \in \mathbb{R}^3,$$

or further, satisfies the widely used Ambrosetti–Rabinowitz-type condition in the form of

$$\exists \mu > 4 \quad \text{such that } 0 < \mu F(x,t) \le tf(x,t) \text{ for all } t \ne 0. \tag{AR}$$

It is well known that a main difficulty in studying (1.1) in $\mathbb{R}^3$ is the lack of compactness. In fact, this difficulty can be avoided when problems are considered restricting to the subspace of $H^1(\mathbb{R}^3)$ consisting of radially symmetric functions, usually denoted by $H^1_r(\mathbb{R}^3)$, since in this case the embedding $H^1_r(\mathbb{R}^3) \hookrightarrow L^s(\mathbb{R}^3)$ $(2 \le s < 6)$ is compact. We refer to [11, 14, 21] in this direction, where the potential V is radial or a positive constant.

Besides, one can also add some conditions on V to restore the lacked compact imbedding. In [22], Wu obtained nontrivial solutions of (1.1) by assuming that the potential V satisfies (V_0) and

$$\operatorname{meas}\bigl\{x \in \mathbb{R}^3 : V(x) \le M\bigr\} < \infty, \quad \forall M > 0, \tag{V_3}$$

where meas $(\cdot)$ denotes the Lebesgue measure in $\mathbb{R}^3$; the nonlinearity $f(x,t)$ is subcritical, satisfies (f_2),

$$4F(x,t) \le tf(x,t) \quad \text{for all } t \in \mathbb{R},$$

and other conditions. Note that condition (V_3) implies that the embedding X into $L^s(\mathbb{R}^3)$ is compact (see [23]), where

$$X := \left\{ u \in H^1(\mathbb{R}^3) : \int_{\mathbb{R}^3} V(x)u^2\,dx < \infty \right\}$$

is a linear subspace of $H^1(\mathbb{R}^3)$, equipped with the norm

$$\|u\|_X := \left(\int_{\mathbb{R}^3} \bigl(|\nabla u|^2 + V(x)u^2\bigr)\,dx \right)^{1/2}.$$

Therefore the Palais–Smale condition can be proved. Then Liu and He [24] proved that (1.1) has infinitely many solutions by using a variant version of the fountain theorem under an oddness assumption on f and conditions (V_0), (V_3), (AR), etc. We also note that if $V(x) \to +\infty$ as $|x| \to \infty$, then (V_3) is satisfied.

In [3], for $f(x,t) = |u|^{p-1}u$ $(2 < p < 5)$ and V satisfying (V_0), (V_2), and

(V_4) V is weakly differentiable, $(\nabla V(\cdot), \cdot) \in L^\infty(\mathbb{R}^3) \cap L^{3/2}(\mathbb{R}^3)$ and

$$V(x) - \bigl(\nabla V(x), x\bigr) \ge 0 \quad \text{a.e. } x \in \mathbb{R}^3.$$

Li and Ye proved that (1.1) has a positive ground state solution by using the Pohozaev identity, a monotonicity trick, and a new version of the global compactness lemma. And recently, Liu and Guo [17] extended this result to the general nonlinearity $f(x,t)$ with assuming (V_0), (V_2), (V_4) and some conditions on $f(x,t)$.

In recent years, there have been intensive studies on semiclassical states of Kirchhoff-type problems of the form

$$-\left(\varepsilon^2 a + \varepsilon b \int_{\mathbb{R}^3} |\nabla u|^2 \,\mathrm{d}x\right)\Delta u + V(x)u = f(x,u), \quad x \in \mathbb{R}^3,$$

where $\varepsilon > 0$ is a small parameter. We would like to refer to [8], in which He and Zou proved the existence of a positive ground state solution by using the Nehari manifold and assuming (V_0), (V_2), (AR), f is independent of x, and $f(t)/t^3$ is strictly increasing.

To overcome the difficulties of the lack of compactness, in addition to the methods mentioned above, another effective way is to make use of the period-translation invariance of the energy functional Φ. More specifically, we could assume V and f satisfy (V_1) and (f_1), respectively, and then we apply the concentration-compactness principle discovered by Lions [25] to obtain nontrivial critical points up to certain suitable $\mathbb{Z}^3$ translations. To the best of our knowledge, there exist few results of this case. Motivated by the above facts, and [26, 27] besides, we consider two cases of the potentials, that is, the periodic cases and the bounded potential well case. Our main results read as follows.

Theorem 1.1 *Under assumptions* (V_0), (V_1), *and* (f_1)–(f_3), *problem* (1.1) *has a positive ground state solution*.

Before stating the theorem of potential well case, we need to assume that the nonlinearity $f(x,t) = f(t)$ does not depend on x. That is, the problem is of the form

$$-\left(a + b \int_{\mathbb{R}^3} |\nabla u|^2 \,\mathrm{d}x\right)\Delta u + V(x)u = f(u) \quad \text{in } \mathbb{R}^3. \tag{1.7}$$

In Sect. 4, we will prove the following result.

Theorem 1.2 *Under assumptions* (V_0), (V_2), *and* (f_1)–(f_3) *with f independent of x, problem* (1.7) *has a positive ground state solution*.

Remark 1.3 It is worth to point out that we cannot easily see that Φ' is weakly sequentially continuous in E by direct calculations due to the nonlocal term $\int_{\mathbb{R}^3} |\nabla u|^2 \,\mathrm{d}x$. In fact, in general, we do not know $\int_{\mathbb{R}^3} |\nabla u_n|^2 \,\mathrm{d}x \to \int_{\mathbb{R}^3} |\nabla u|^2 \,\mathrm{d}x$ from $u_n \rightharpoonup u$ in E. Thanks to condition (f_3), we can prove that the weak limit of a bounded (PS) sequence is a weak solution of this problem (see Lemma 3.1). Hence we do not have to verify the (PS) compactness condition. This dramatically simplifies our proof, especially when the embedding $E \hookrightarrow L^2(\mathbb{R}^3)$ is not compact.

Remark 1.4 As have been mentioned previously, [3, 17] are concerned with (1.1) of the potential well case, as well. However, Theorem 1.2 is not strictly comparable to them. Since (V_4) is not assumed, our methods become substantially different and more direct.

Remark 1.5 Theorem 1.2 is different from Theorem 1.1 in the aforementioned [8], which involved a small parameter since it is concerned with semiclassical states. Whereas our results do not require the smallness of such a parameter. Moreover, our assumptions on f are much more general than the ones in [8].

The paper is organized as follows. In Sect. 2, we show that the functional Φ associated with (1.1) has a mountain pass geometry. Moreover, we also prove that the corresponding Palais–Smale sequence $\{u_n\}$ of Φ at the mountain pass level c is bounded. In Sect. 3, we complete the proof of Theorem 1.1 by using the mountain pass lemma and Lions' concentration-compactness principle. Section 4 is devoted to the proof of Theorem 1.2 by applying a comparison argument.

2 A mountain pass geometry

To begin with, we recall that a sequence $\{u_n\} \in H^1(\mathbb{R}^3)$ is called a Palais–Smale sequence of Φ at the level c, a $(\mathrm{PS})_c$ sequence for short, if

$$\Phi(u_n) \to c \quad \text{and} \quad \Phi'(u_n) \to 0.$$

Throughout this paper, we denote by $|\cdot|_s$ the $L^s(\mathbb{R}^3)$ norm for $s \in [1,\infty]$ and by c_i a certain positive constant.

Lemma 2.1 *Suppose that* (V_0), (V_1) (*or* (V_2)), (f_1), *and* (f_2) *are satisfied, then there exist* $r > 0$ *and* $e \in E$ *with* $\|e\| > r$ *such that*

$$b := \inf_{\|u\|=r} \Phi(u) > \Phi(\mathbf{0}) = 0 \geq \Phi(e).$$

Proof From (f_1), given $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$\bigl|F(x,t)\bigr| \leq \varepsilon t^2 + C_\varepsilon t^6, \quad \forall t \geq 0.$$

By the Sobolev inequality, we have that

$$\begin{aligned}\Phi(u) &\geq \frac{1}{2}\|u\|^2 + \frac{b}{4}|\nabla u|_2^4 - \int_{\mathbb{R}^3}\bigl|F(x,u)\bigr|\,\mathrm{d}x \\ &\geq \frac{1}{2}\|u\|^2 - \varepsilon|u|_2^2 - C_\varepsilon|u|_6^6 \geq \left(\frac{1}{2} - \varepsilon c_1\right)\|u\|^2 - c_2 C_\varepsilon\|u\|^6\end{aligned}$$

for some constants $c_1, c_2 > 0$. Then we can choose ε and r small enough such that

$$b = \inf_{\|u\|=r} \Phi(u) > \Phi(\mathbf{0}) = 0.$$

For fixed $v \in E\backslash\{\mathbf{0}\}$, it follows from (f_2) that

$$\begin{aligned}\Phi(tv) &= \frac{t^2}{2}\|v\|^2 + \frac{bt^4}{4}|\nabla v|_2^4 - \int_{\mathbb{R}^3} F(x,tv)\,\mathrm{d}x \\ &\leq t^4\left(\frac{\|v\|}{2t^2} + \frac{b|\nabla v|_2^4}{4} - \int_{v\neq 0}\frac{F(x,tv)}{t^4v^4}v^4\,\mathrm{d}x\right) \to -\infty\end{aligned}$$

as $t \to +\infty$. Then there exists $e := tv$ with t sufficiently large such that $\|e\| > r$ and $\Phi(e) < 0$. $\square$

By Lemma 2.1 we see that Φ has a mountain pass geometry. Namely, setting

$$\Gamma = \{\gamma \in C([0,1],E) : \gamma(0) = \mathbf{0} \text{ and } \Phi(\gamma(1)) < 0\}, \tag{2.1}$$

we have $\Gamma \neq \emptyset$. By a version of the mountain pass lemma [28], for the mountain pass level

$$c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} \Phi(\gamma(t)), \tag{2.2}$$

there exists a $(\mathrm{PS})_c$ sequence $\{u_n\}$ for Φ. Moreover, we see that $c > 0$.

Lemma 2.2 *Suppose that* (V_0) *and* (f_1)–(f_3) *are satisfied, then any* $(\mathrm{PS})_c$ *sequence of* Φ *is bounded in* E.

Proof Let $\{u_n\} \in E$ be a $(\mathrm{PS})_c$ sequence. Given any $t > 0$, by (f_3) we obtain that $f(x,s)/s^3 \leq f(x,t)/t^3$ for $0 < s < t$. Hence

$$F(x,t) = \int_0^t f(x,s)\,\mathrm{d}s \leq \int_0^t \frac{f(x,t)}{t^3} s^3\,\mathrm{d}x = \frac{1}{4} tf(x,t),$$

that is,

$$\frac{1}{4} tf(x,t) - F(x,t) \geq 0, \quad \forall (x,t) \in (\mathbb{R}^3 \times \mathbb{R}^+). \tag{2.3}$$

Then by $\Phi'(u_n) \to 0$ we have

$$\begin{aligned} c + 1 + \|u_n\| &\geq \Phi(u_n) - \frac{1}{4}\langle \Phi'(u_n), u_n\rangle \\ &= \frac{1}{4}\|u_n\|^2 + \int_{\mathbb{R}^3} \left(\frac{1}{4} u_n f(x,u_n) - F(x,u_n)\right) \mathrm{d}x \\ &\geq \frac{1}{4}\|u_n\|^2 \end{aligned}$$

for n large enough. The above inequality implies that $\{u_n\}$ is bounded. $\square$

3 The periodic case

We have proved that the $(\mathrm{PS})_c$ sequence $\{u_n\}$ is bounded. We may assume that $u_n \rightharpoonup u$ in E, up to a subsequence if necessary. We shall show that the weak limit u is a nonzero critical point of Φ. Now the main difficulty we face is that Φ' is not weakly continuous in E. We give the following lemma.

Lemma 3.1 *Suppose that* (f_1)–(f_3) *are satisfied and* $m \leq c$ *is a constant, where* c *is given by* (2.2). *If* $\{u_n\}$ *is a* $(\mathrm{PS})_m$ *sequence of* Φ *and* $u_n \rightharpoonup u$ *in* E, *then* $\Phi'(u) = \mathbf{0}$. *Moreover, if* $u \neq \mathbf{0}$, *then*

$$\int_{\mathbb{R}^3} |\nabla u_n|^2 dx \to \int_{\mathbb{R}^3} |\nabla u|^2\, dx \quad \textit{as } n \to +\infty.$$

Proof Passing to a subsequence if necessary, we assume that

$$\int_{\mathbb{R}^3} |\nabla u_n|^2 \, \mathrm{d}x \to A^2$$

for some $A \in \mathbb{R}^+$. If $u = \mathbf{0}$, the desired conclusion holds obviously. If $u \neq \mathbf{0}$, we see that

$$\int_{\mathbb{R}^3} |\nabla u|^2 \, \mathrm{d}x \leq \varliminf_{n\to\infty} \int_{\mathbb{R}^3} |\nabla u_n|^2 \, \mathrm{d}x = A^2.$$

Suppose by contradiction that

$$\int_{\mathbb{R}^3} |\nabla u|^2 \, \mathrm{d}x < A^2. \tag{3.1}$$

For any $\phi \in C_0^\infty(\mathbb{R}^3)$, by $\Phi'(u_n) \to 0$ we have

$$\begin{aligned} 0 &= \lim_{n\to\infty} \langle \Phi'(u_n), \phi \rangle \\ &= (a + bA^2) \int_{\mathbb{R}^3} \nabla u \nabla \phi \, \mathrm{d}x + \int_{\mathbb{R}^3} V(x) u \phi \, \mathrm{d}x - \int_{\mathbb{R}^3} f(x,u)\phi \, \mathrm{d}x. \end{aligned} \tag{3.2}$$

Then from (3.1) and (3.2) we have

$$\langle \Phi'(u), u \rangle < 0. \tag{3.3}$$

On the other hand, condition (f_1) implies that

$$\begin{aligned} \langle \Phi'(tu), tu \rangle &= t \int_{\mathbb{R}^3} (a|\nabla u|^2 + V(x)u^2) \, \mathrm{d}x + bt^3 \left(\int_{\mathbb{R}^3} |\nabla u|^2 \, \mathrm{d}x \right)^2 - \int_{\mathbb{R}^3} f(x,tu) u \, \mathrm{d}x \\ &\geq t \int_{\mathbb{R}^3} (a|\nabla u|^2 + V(x)u^2) \, \mathrm{d}x - \varepsilon t \int_{\mathbb{R}^3} u^2 \, \mathrm{d}x - c_\varepsilon t^5 \int_{\mathbb{R}^3} u^6 \, \mathrm{d}x. \end{aligned}$$

Choosing ε small enough, there exists $t_1 \in (0,1)$ such that

$$\langle \Phi'(t_1 u), u \rangle > 0. \tag{3.4}$$

Hence there exists $t_0 \in (0,1)$ such that

$$\langle \Phi'(t_0 u), u \rangle = 0. \tag{3.5}$$

For any $u \in E\backslash\{\mathbf{0}\}$, $t > 0$, we consider

$$\begin{aligned} \varphi(t) &:= \Phi(tu) = \frac{t^2}{2} \int_{\mathbb{R}^3} (a|\nabla u|^2 + V(x)u^2) \, \mathrm{d}x + \frac{bt^4}{4} \left(\int_{\mathbb{R}^3} |\nabla u|^2 \, \mathrm{d}x \right)^2 - \int_{\mathbb{R}^3} F(x,tu) \, \mathrm{d}x, \\ \varphi'(t) &= t \int_{\mathbb{R}^3} (a|\nabla u|^2 + V(x)u^2) \, \mathrm{d}x + bt^3 \left(\int_{\mathbb{R}^3} |\nabla u|^2 \, \mathrm{d}x \right)^2 - \int_{\mathbb{R}^3} f(x,tu) u \, \mathrm{d}x. \end{aligned}$$

Let $\varphi'(t) = 0$, which is equivalent to

$$\frac{\int_{\mathbb{R}^3}(a|\nabla u|^2 + V(x)u^2)\,\mathrm{d}x}{t^2} + b\left(\int_{\mathbb{R}^3}|\nabla u|^2\,\mathrm{d}x\right)^2 = \frac{\int_{\mathbb{R}^3} f(x,tu)u\,\mathrm{d}x}{t^3}.$$

It follows from (f_3) that the right-hand side of this equation is nondecreasing in $t \in (0,+\infty)$, while the left-hand side is strictly decreasing in $t \in (0,+\infty)$. Note also that, by (3.5),

$$\varphi'(t_0) = \langle \Phi'(t_0u), u\rangle = 0.$$

Hence $t = t_0$ is the unique solution of $\varphi'(t) = 0$ for $t \in (0,+\infty)$. Therefore, this together with (3.3) and (3.4) yields that

$$\Phi(t_0u) = \max_{t\in[0,1]} \Phi(tu).$$

It can be easily concluded from (f_2) and (f_3) that

$$\mathcal{F}(x,t) := \frac{1}{4}f(x,t)t - F(x,t) > 0$$

and $\mathcal{F}(x,t)$ is nondecreasing in $t > 0$. Noting that $0 < t_0 < 1$ and $\|u\| \neq 0$, we obtain

$$\begin{aligned}
m \le c &\le \Phi(t_0u) - \frac{1}{4}\langle \Phi'(t_0u), t_0u\rangle \\
&= \frac{t_0^2}{4}\|u\|^2 + \int_{\mathbb{R}^3}\left(\frac{1}{4}f(x,t_0u)t_0u - F(x,t_0u)\right)\mathrm{d}x \\
&< \frac{1}{4}\|u\|^2 + \int_{\mathbb{R}^3}\left(\frac{1}{4}f(x,u)u - F(x,u)\right)\mathrm{d}x \\
&\le \varliminf_{n\to\infty}\left[\frac{1}{4}\|u_n\|^2 + \int_{\mathbb{R}^3}\left(\frac{1}{4}f(x,u_n)u_n - F(x,u_n)\right)\mathrm{d}x\right] \\
&= \varliminf_{n\to\infty}\left(\Phi(u_n) - \frac{1}{4}\langle \Phi'(u_n), u_n\rangle\right) = m.
\end{aligned}$$

This is impossible. Hence $\int_{\mathbb{R}^3}|\nabla u|^2\,\mathrm{d}x = A^2$ and then $\Phi'(u) = \mathbf{0}$. □

Proof of Theorem 1.1 *Step 1.* In view of Lemmas 2.1–2.2, Φ has a bounded $(\mathrm{PS})_c$ sequence $\{u_n\}$ in E. Let

$$\delta = \lim_{n\to\infty}\sup_{y\in\mathbb{R}^3}\int_{B_2(y)}|u_n|^2\,\mathrm{d}x. \tag{3.6}$$

If $\delta = 0$, by Lions' lemma (see [25, Lemma I.1]), we obtain that $u_n \to \mathbf{0}$ in $L^s(\mathbb{R}^3)$ for any $s \in (2,6)$. We claim that

$$\lim_{n\to\infty}\int_{\mathbb{R}^3} u_n f(x,u_n)\,\mathrm{d}x = 0. \tag{3.7}$$

Indeed, by (f_1), for any $\varepsilon > 0$, there exist $c_\varepsilon > 0$ and $q \in (2,6)$ such that

$$|f(x,t)| \le \varepsilon\left(t + t^5\right) + c_\varepsilon t^{q-1}, \quad \forall t \ge 0. \tag{3.8}$$

Since $\{u_n\}$ is bounded in E, there exists a constant $M > 0$ such that

$$|u_n|_2^2 + |u_n|_6^6 \le M,$$

and consequently,

$$\overline{\lim_{n\to\infty}}\left|\int_{\mathbb{R}^3} u_n f(x,u_n)\,\mathrm{d}x\right| \le \overline{\lim_{n\to\infty}}\left(\varepsilon\left(|u_n|_2^2 + |u_n|_6^6\right) + c_\varepsilon |u_n|_q^q\right) \le \varepsilon M.$$

By the arbitrariness of ε, we deduce (3.7).

Noting that $\langle \Phi'(u_n), u_n\rangle = o(1)$, by (1.4) and (3.7) we have

$$\|u_n\|^2 \le \|u_n\|^2 + b\left(\int_{\mathbb{R}^3} |\nabla u_n|^2\,\mathrm{d}x\right)^2 = \left\langle \Phi'(u_n), u_n\right\rangle + \int_{\mathbb{R}^3} u_n f(x,u_n)\,\mathrm{d}x \to 0, \tag{3.9}$$

that is, $u_n \to \mathbf{0}$ in E. Then $\Phi(u_n) \to 0$, contrary to $c > 0$. Therefore $\delta > 0$.

Going if necessary to a subsequence, we may assume that there exists a sequence $\{z_n\} \subset \mathbb{R}^3$ such that

$$\int_{B_2(z_n)} |u_n|^2\,\mathrm{d}x \ge \frac{\delta}{2}.$$

Since the number of points in $\mathbb{Z}^3 \cap B_2(z_n)$ is less than 4^3, there exists $y_n \in \mathbb{Z}^3 \cap B_2(z_n)$ such that

$$\int_{B_2(0)} |\tilde{u}_n|^2\,\mathrm{d}x = \int_{B_2(y_n)} |u_n|^2\,\mathrm{d}x \ge \frac{\delta}{2\times 4^3} > 0, \tag{3.10}$$

where $\tilde{u}_n$ is defined as $\tilde{u}_n(x) = u_n(x+y_n)$. By (V_1) and (1.2), $\|\cdot\|$ is invariant under $\mathbb{Z}^3$ translation, that is, $\|\tilde{u}_n\| = \|u_n\|$. Therefore $\{\tilde{u}_n\}$ is also bounded in E. Up to a subsequence if necessary, we may assume that

$$\tilde{u}_n \rightharpoonup \tilde{u} \quad \text{in } E, \qquad \tilde{u}_n \to \tilde{u} \quad \text{in } L^2_{\mathrm{loc}}\left(\mathbb{R}^3\right).$$

It follows from (3.10) that $\tilde{u} \ne \mathbf{0}$. Moreover, by the $\mathbb{Z}^3$ invariance of the problem, $\{\tilde{u}_n\}$ is also a $(\mathrm{PS})_c$ sequence of Φ. Thus, for any $\phi \in C_0^\infty(\mathbb{R}^3)$, by Lemma 2.2 and 3.1 we have

$$\left\langle \Phi'(\tilde{u}), \phi\right\rangle = \lim_{n\to\infty}\left\langle \Phi'(\tilde{u}_n), \phi\right\rangle = 0.$$

Then $\Phi'(\tilde{u}) = 0$ and $\tilde{u}$ is a nontrivial solution of (1.1).

Step 2. Denote by $\mathcal{K}$ as in (1.5) the set of nontrivial critical points of Φ and let

$$m = \inf_{\mathcal{K}} \Phi. \tag{3.11}$$

For any $u \in \mathcal{K}$, it follows from (3.8) and Sobolev's inequality that

$$\|u\|^2 = \left\langle \Phi'(u), u\right\rangle - b\left(\int_{\mathbb{R}^3} |\nabla u|^2\,\mathrm{d}x\right)^2 + \int_{\mathbb{R}^3} u f(x,u)\,\mathrm{d}x$$

$$\leq \varepsilon\left(c_3\|u\|^2 + c_4\|u\|^6\right) + c_\varepsilon c_5\|u\|^q,$$

where c_3, c_4, c_5 are related to the Sobolev constants. Choosing ε small enough and multiplying both sides of the above inequality by $\|u\|^{-2}$, we deduce that there exists $\rho > 0$ such that

$$\|u\| \geq \rho > 0, \quad \forall u \in \mathcal{K}. \tag{3.12}$$

Therefore, for any nontrivial critical point u of Φ, by (2.3) and (3.12) we have

$$\begin{aligned}\Phi(u) &= \Phi(u) - \frac{1}{4}\langle\Phi'(u), u\rangle \\ &= \frac{1}{4}\|u\|^2 + \int_{\mathbb{R}^3}\left(\frac{1}{4}f(x,u)u - F(x,u)\right)\mathrm{d}x \geq \rho > 0.\end{aligned} \tag{3.13}$$

Hence $0 < m \leq \Phi(\tilde{u}) < +\infty$. Let $\{u_n\} \in \mathcal{K}$ such that $\Phi(u_n) \to m$. Noting that $\langle\Phi'(u_n), u_n\rangle = 0$, then $\{u_n\}$ is a $(\mathrm{PS})_m$ sequence. By Lemma 2.2, $\{u_n\}$ is bounded in E. For this sequence $\{u_n\}$, denote δ as in (3.6). If $\delta = 0$, similar to (3.9) we obtain that $\|u_n\| \to 0$, which contradicts (3.12). Therefore $\delta > 0$. Now, by the $\mathbb{Z}^3$ translation invariance of the problem, using the same argument in step 1, we can obtain a suitable $\mathbb{Z}^3$ translation of $\{u_n\}$, denoted by $\{v_n\}$, such that

$$\Phi'(v_n) = 0, \qquad \Phi(v_n) = \Phi(u_n) \to m,$$

and $\{v_n\}$ converges weakly to some $v \neq \mathbf{0}$. By Lemma 3.1, v is a nonzero critical point of Φ. Moreover, it follows from (2.3) and Fatou's lemma that

$$\begin{aligned}m \leq \Phi(v) &= \Phi(v) - \frac{1}{4}\langle\Phi'(v), v\rangle \\ &= \frac{1}{4}\|v\|^2 + \int_{\mathbb{R}^3}\left(\frac{1}{4}f(x,v)v - F(x,v)\right)\mathrm{d}x \\ &\leq \varliminf_{n\to\infty}\left[\frac{1}{4}\|v_n\|^2 + \int_{\mathbb{R}^3}\left(\frac{1}{4}f(x,v_n)v_n - F(x,v_n)\right)\mathrm{d}x\right] \\ &= \varliminf_{n\to\infty}\left(\Phi(v_n) - \frac{1}{4}\langle\Phi'(v_n), v_n\rangle\right) = m.\end{aligned} \tag{3.14}$$

Therefore, $\Phi(v) = m$, and so v is a ground state solution of problem (1.1). Finally, using the strong maximum principle, we can conclude that v is positive. Theorem 1.1 is proved. □

4 The potential well case

In this section, we shall prove Theorem 1.2. Since the nonlinearity f does not depend on x, the functional Φ is now written as

$$\Phi(u) = \frac{a}{2}\int_{\mathbb{R}^3}|\nabla u|^2\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^3}V(x)u^2\,\mathrm{d}x + \frac{b}{4}\left(\int_{\mathbb{R}^3}|\nabla u|^2\,\mathrm{d}x\right)^2 - \int_{\mathbb{R}^3}F(u)\,\mathrm{d}x.$$

By Lemmas 2.1 and 2.2, for the mountain pass level c, Φ has a bounded $(\mathrm{PS})_c$ sequence $\{u_n\}$. Up to a subsequence, we get that $u_n \rightharpoonup u$ in E. To show that u is a nonzero critical point of Φ, we need to consider the limiting functional $\Phi_\infty : E \to \mathbb{R}$,

$$\Phi_\infty(u) = \frac{a}{2}\int_{\mathbb{R}^3} |\nabla u|^2\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^3} V_\infty u^2\,\mathrm{d}x + \frac{b}{4}\left(\int_{\mathbb{R}^3} |\nabla u|^2\,\mathrm{d}x\right)^2 - \int_{\mathbb{R}^3} F(u)\,\mathrm{d}x.$$

For any nontrivial critical point v of Φ_∞, it is standard to prove the following Pohozaev identity corresponding to Φ_∞:

$$\frac{a}{6}\int_{\mathbb{R}^3} |\nabla v|^2\,\mathrm{d}x + \frac{b}{6}\left(\int_{\mathbb{R}^3} |\nabla v|^2\,\mathrm{d}x\right)^2 + \frac{1}{2}\int_{\mathbb{R}^3} V_\infty v^2\,\mathrm{d}x - \int_{\mathbb{R}^3} F(v)\,\mathrm{d}x = 0. \tag{4.1}$$

One can refer to [11, 29] for details.

We assume by contradiction that $u = \mathbf{0}$. Then the above sequence $\{u_n\}$, as we shall see, is also a $(\mathrm{PS})_c$ sequence of Φ_∞.

Lemma 4.1 *If $u = \mathbf{0}$, then $\{u_n\}$ is also a bounded* $(\mathrm{PS})_c$ *sequence of Φ_∞.*

Proof Since $u_n \rightharpoonup \mathbf{0}$ in E, we have that $u_n \to 0$ in $L^2_{\mathrm{loc}}(\mathbb{R}^3)$. Because of

$$\lim_{|x|\to\infty} V(x) = V_\infty,$$

for any $\varepsilon > 0$, there exists $R > 0$ such that

$$\bigl|V(x) - V_\infty\bigr| < \varepsilon, \quad \forall |x| > R.$$

Consequently,

$$\begin{aligned}\bigl|\Phi_\infty(u_n) - \Phi(u_n)\bigr| &= \frac{1}{2}\left(\int_{|x|\geq R} + \int_{|x|<R}\right)\bigl(V_\infty - V(x)\bigr)u_n^2\,\mathrm{d}x \\ &\leq \frac{\varepsilon}{2}|u_n|_2^2 + \frac{V_\infty}{2}\int_{|x|<R} u_n^2\,\mathrm{d}x.\end{aligned}$$

Noting that $u_n \to 0$ in $L^2_{\mathrm{loc}}(\mathbb{R}^3)$ and $\sup_n |u_n|_2^2 < \infty$, letting $n \to \infty$, the above inequality yields

$$\overline{\lim_{n\to\infty}}\bigl|\Phi_\infty(u_n) - \Phi(u_n)\bigr| \leq c_6\varepsilon$$

for some constant $c_6 > 0$. By the arbitrariness of ε, we obtain

$$\bigl|\Phi_\infty(u_n) - \Phi(u_n)\bigr| \to 0 \quad \text{as } n \to \infty.$$

Similarly, we get

$$\bigl\|\Phi'_\infty(u_n) - \Phi'(u_n)\bigr\|_{H^{-1}(\mathbb{R}^3)} = \sup_{\phi\in E, \|\phi\|=1}\left|\int_{\mathbb{R}^3}\bigl(V_\infty - V(x)\bigr)u_n\phi\,\mathrm{d}x\right| \to 0,$$

as $n \to \infty$. Therefore, $\Phi_\infty(u_n) \to c$ and $\Phi'_\infty(u_n) \to 0$, that is, $\{u_n\}$ is a $(\mathrm{PS})_c$ sequence of Φ_∞. □

Lemma 4.2 *If $v \in E$ is a nontrivial critical point of Φ_∞, then there exists $\gamma \in C([0,1],E)$ such that $\gamma(0) = \mathbf{0}$, $\Phi_\infty(\gamma(1)) < 0$, $v \in \gamma([0,1])$, and*

$$\max_{t\in[0,1]} \Phi_\infty\big(\gamma(t)\big) = \Phi_\infty(v) > 0.$$

Proof For $\tau > 0$, set $v_\tau(x) := v(\tau^{-1}x)$. A direct computation shows

$$\int_{\mathbb{R}^3} |\nabla v_\tau|^2\,\mathrm{d}x = \tau \int_{\mathbb{R}^3} |\nabla v|^2\,\mathrm{d}x, \qquad \int_{\mathbb{R}^3} v_\tau^2\,\mathrm{d}x = \tau^3 \int_{\mathbb{R}^3} v^2\,\mathrm{d}x$$

and

$$\int_{\mathbb{R}^3} F(v_\tau)\,\mathrm{d}x = \tau^3 \int_{\mathbb{R}^3} F(v)\,\mathrm{d}x.$$

Then by (4.1) we have

$$\begin{aligned}
\Phi_\infty(v_\tau) &= \frac{a}{2}\int_{\mathbb{R}^3} |\nabla v_\tau|^2\,\mathrm{d}x + \frac{b}{4}\left(\int_{\mathbb{R}^3} |\nabla v_\tau|^2\,\mathrm{d}x\right)^2 + \frac{1}{2}\int_{\mathbb{R}^3} V_\infty v_\tau^2\,\mathrm{d}x - \int_{\mathbb{R}^3} F(v_\tau)\,\mathrm{d}x \\
&= \frac{a\tau}{2}\int_{\mathbb{R}^3} |\nabla v|^2\,\mathrm{d}x + \frac{b\tau^2}{4}\left(\int_{\mathbb{R}^3} |\nabla v|^2\,\mathrm{d}x\right)^2 - \tau^3 \int_{\mathbb{R}^3}\left(F(v) - \frac{1}{2}V_\infty v^2\right)\mathrm{d}x \qquad (4.2) \\
&= \frac{a\tau}{2}|\nabla v|_2^2 + \frac{b\tau^2}{4}|\nabla v|_2^4 - \tau^3\left(\frac{a}{6}|\nabla v|_2^2 + \frac{b}{6}|\nabla v|_2^4\right) \\
&\to -\infty \quad \text{as } \tau \to +\infty.
\end{aligned}$$

Thus, there exists $\tau > 1$ such that $\Phi_\infty(v_\tau) < 0$. Define

$$\gamma(t) = \begin{cases} v_{\tau t}, & 0 < t \le 1, \\ \mathbf{0}, & t = 0. \end{cases}$$

Noting that

$$\|v_t\|^2 \le a|\nabla v_t|_2^2 + V_\infty |v_t|_2^2 = at|\nabla v|_2^2 + V_\infty t^2 |v|_2^2 \to 0, \quad \text{as } t \to 0,$$

we see that $\gamma \in C([0,1],E)$. Moreover, it follows from (4.2) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi_\infty\big(\gamma(t)\big) = \frac{a\tau}{2}|\nabla v|_2^2 + \frac{b\tau^2}{2}|\nabla v|_2^4 \cdot t - \left(\frac{a}{2}|\nabla v|_2^2 + \frac{b}{2}|\nabla v|_2^4\right)\tau^3 t^2.$$

Observing the relationship between the roots and coefficients, we conclude that $t = 1/\tau$ is the unique solution of

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi_\infty\big(\gamma(t)\big) = 0 \quad \text{for } t \in (0,\infty).$$

Remind that $\Phi_\infty(\gamma(1/\tau)) = \Phi_\infty(v) > 0$ (by (4.2)), $\Phi_\infty(\gamma(0)) = \Phi_\infty(\mathbf{0}) = 0$, and

$$\lim_{t\to+\infty} \Phi_\infty\big(\gamma(t)\big) = -\infty.$$

Therefore, we obtain that

$$\max_{t\in[0,1]} \Phi_\infty\bigl(\gamma(t)\bigr) = \Phi_\infty\bigl(\gamma(1/\tau)\bigr) = \Phi_\infty(v) > 0.$$

The lemma has been proved. □

Lemma 4.3 *Suppose that* (V_2) *and* (f_1)–(f_3) *are satisfied.* c *is a constant given by* (2.2). *If* $\{u_n\}$ *is a* $(PS)_c$ *sequence of* Φ_∞ *and* $u_n \rightharpoonup u_0$ *in* E, *then* u_0 *is a critical point of* Φ_∞.

Proof If $u_0 = \mathbf{0}$, the desired conclusion holds obviously. Otherwise, define

$$c_\infty := \max_{t>0} \Phi_\infty(tu_0).$$

By (V_2), we have

$$\Phi(u) \le \Phi_\infty(u), \quad \forall u \in E.$$

In view of (f_2), we obtain

$$\begin{aligned}\Phi_\infty(tu_0) &= \frac{t^2}{2}\int_{\mathbb{R}^3}\bigl(a|\nabla u_0|^2 + V_\infty u_0^2\bigr)\,\mathrm{d}x + \frac{bt^4}{4}\left(\int_{\mathbb{R}^3}|\nabla u_0|^2\,\mathrm{d}x\right)^2 - \int_{\mathbb{R}^3}F(tu_0)\,\mathrm{d}x \\ &\le t^4\left[\frac{1}{2t^2}\int_{\mathbb{R}^3}\bigl(a|\nabla u_0|^2 + V_\infty u_0^2\bigr)\,\mathrm{d}x + \frac{b}{4}\left(\int_{\mathbb{R}^3}|\nabla u_0|^2\,\mathrm{d}x\right)^2 - \int_{\mathbb{R}^3}\frac{F(tu_0)}{t^4}\,\mathrm{d}x\right] \\ &\to -\infty,\end{aligned}$$

as $t \to \infty$. Hence

$$c \le \max_{t>0} \Phi(tu_0) \le \max_{t>0} \Phi_\infty(tu_0) = c_\infty < +\infty.$$

Then using the argument similar to that of the proof of Lemma 3.1, the lemma can be proved. □

Now, we are ready to prove Theorem 1.2 by comparing the energy between Φ and Φ_∞.

Proof of Theorem 1.2 *Step 1.* We first show that Φ has a nonzero critical point.

Recall that u is the weak limit obtained from the bounded $(PS)_c$ sequence $\{u_n\}$ of Φ, which is discussed at the beginning of this section. If $u = \mathbf{0}$, by Lemma 4.1 we see that $\{u_n\}$ is also a bounded $(PS)_c$ sequence of Φ_∞. As in Step 1 of the proof of Theorem 1.1, since Φ_∞ is invariant under $\mathbb{Z}^3$ translation, by Lions' lemma we know that there exists $\{y_n\} \in \mathbb{Z}^3$ such that let

$$v_n(x) := u_n(x + y_n),$$

then $\{v_n\}$ is also a bounded $(PS)_c$ sequence of Φ_∞, and $v_n \rightharpoonup v \ne \mathbf{0}$ in E. By Lemma 4.3 we know v is a nonzero critical point of Φ_∞. Moreover, it follows from (2.3) and Fatou's lemma that

$$\begin{aligned}
\Phi_\infty(v) &= \Phi_\infty(v) - \frac{1}{4}\langle \Phi'_\infty(v), v\rangle \\
&= \frac{1}{4}\int_{\mathbb{R}^3} \left(a|\nabla v|^2 + V_\infty v^2\right) \mathrm{d}x + \int_{\mathbb{R}^3} \left(\frac{1}{4} v f(v) - F(v)\right) \mathrm{d}x \\
&\le \varliminf_{n\to\infty} \left[\frac{1}{4}\int_{\mathbb{R}^3} \left(a|\nabla v_n|^2 + V_\infty v_n^2\right) \mathrm{d}x + \int_{\mathbb{R}^3} \left(\frac{1}{4} v_n f(v_n) - F(v_n)\right) \mathrm{d}x\right] \\
&= \varliminf_{n\to\infty} \left(\Phi_\infty(v_n) - \frac{1}{4}\langle \Phi'_\infty(v_n), v_n\rangle\right) = c.
\end{aligned}$$

Now, by Lemma 4.2, there exists $\gamma \in C([0,1],E)$ such that $\gamma(0) = \mathbf{0}$, $\Phi_\infty(\gamma(1)) < 0$, $v \in \gamma([0,1])$, and

$$0 < \max_{t\in[0,1]} \Phi_\infty\big(\gamma(t)\big) = \Phi_\infty(v) \le c. \tag{4.3}$$

Furthermore, by the construction of γ, we also know that $\mathbf{0} \notin \gamma((0,1])$. Therefore, according to (V_2), we obtain

$$\Phi\big(\gamma(t)\big) < \Phi_\infty\big(\gamma(t)\big), \quad \forall t \in (0,1]. \tag{4.4}$$

In particular, $\Phi(\gamma(1)) < \Phi_\infty(\gamma(1)) < 0$, and hence $\gamma \in \Gamma$, where Γ is defined in (2.1). Note that $\Phi(\mathbf{0}) = \Phi_\infty(\mathbf{0}) = 0$ and $c > 0$. Combining (2.2), (4.3), and (4.4), we deduce that

$$c \le \max_{t\in[0,1]} \Phi\big(\gamma(t)\big) < \max_{t\in[0,1]} \Phi_\infty\big(\gamma(t)\big) = \Phi_\infty(v) \le c.$$

This is a contradiction. Therefore, $u \ne \mathbf{0}$. By Lemma 3.1, u is a nonzero critical point of Φ.

Step 2. We now show that (1.7) has a ground state.

Denote $\mathcal{K}$ and m as in (1.5) and (3.11), respectively. Using the argument similar to (3.13) and (3.14), it is easy to see that

$$0 < m \le \Phi(w) \le c, \quad \forall w \in \mathcal{K}. \tag{4.5}$$

Let $\{w_n\} \in \mathcal{K}$ such that $\Phi(w_n) \to m$. Then $\{w_n\}$ is a $(PS)_m$ sequence of Φ, and hence $\{w_n\}$ is bounded by Lemma 2.2. Passing to a subsequence, we may assume $w_n \rightharpoonup w$ in E.

If $w = \mathbf{0}$, repeating the previous argument of Step 1, we can get a nonzero critical point $\tilde{w}$ of Φ_∞ such that $\Phi_\infty(\tilde{w}) \le m$, and we can construct a path $\gamma \in \Gamma$ such that

$$c \le \max_{t\in[0,1]} \Phi\big(\gamma(t)\big) < \max_{t\in[0,1]} \Phi_\infty\big(\gamma(t)\big) = \Phi_\infty(\tilde{w}) \le m,$$

contradicting (4.5). Therefore $w \ne \mathbf{0}$, and then, by Lemma 3.1, w is a nontrivial critical point of Φ. Furthermore, similar to (3.14), we deduce that $\Phi(w) = m$. Hence w is a ground state solution of (1.7). It is then easy to see that w is positive by the strong maximum principle again. The proof of Theorem 1.2 is completed. □

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