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Infinitely many solutions for a class of fractional Schrödinger equations with sign-changing weight functions

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Abstract

In this paper, we study the fractional Schrödinger equation

$$\begin{cases} (-\Delta)^s u + u = a(x)|u|^{p-2}u + b(x)|u|^{q-2}u, \\ u \in H^s(\mathbb{R}^N), \end{cases}$$

where $(-\Delta)^s$ denotes the fractional Laplacian of order $s \in (0,1)$, $N > 2s$, $2 < p < q < 2_s^*$, and 2_s^* is the fractional critical Sobolev exponent. The weight potentials a or b is a sign-changing function and satisfies some valid assumptions. We obtain the existence of infinitely many solutions to the problem by the Nehari manifold.

MSC: 35J20; 35J70; 58E05

Keywords: Fractional Schrödinger equation; Sign-changing weight functions; Nehari manifold

1 Introduction and the main results

In this paper, we study the fractional Schrödinger equation

$$\begin{cases} (-\Delta)^s u + u = a(x)|u|^{p-2}u + b(x)|u|^{q-2}u, \\ u \in H^s(\mathbb{R}^N), \end{cases} \tag{1.1}$$

where the fractional Laplacian $(-\Delta)^s$ is defined by

$$(-\Delta)^s \Psi(x) = C_{N,s} P.V. \int_{\mathbb{R}^N} \frac{\Psi(x) - \Psi(y)}{|x-y|^{N+2s}}\, dy, \quad \Psi \in \mathcal{S}(\mathbb{R}^N),$$

$P.V.$ stands for the Cauchy principal value, $C_{N,s}$ is a normalizing constant, $\mathcal{S}(\mathbb{R}^N)$ is the Schwartz space of rapidly decaying functions, $s \in (0,1)$, $N > 2s$, $2 < p < q < 2_s^*$, and 2_s^* is the fractional critical Sobolev exponent.

In the last few years, the time-dependent fractional Schrödinger equation has been studied extensively in the literature. It appears widely in optimization, finance, phase transi-

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tions, stratified materials, crystal dislocation, flame propagation, conservation laws, materials science, and water waves (see [5]). The standing waves have a wide range of applications in the real world. Musical instruments generally emit sound due to the standing wave generated by the vibration of a string. In the nonlinear fractional Schrödinger equation, the question of the existence and stability of the standing wave solution is an important research topic. It has been applied in many areas of physics, such as constructive quantum field theory, plasma physics, nonlinear optics, and so on. A basic motivation for the study of Eq. (1.1) arises in looking for the standing wave solutions of the type

$$\Psi(x,t) = e^{-iEt/\varepsilon}u(x)$$

for the following time-dependent fractional Schrödinger equation:

$$i\varepsilon\frac{\partial\Psi}{\partial t} = \varepsilon^{2s}(-\Delta)^s\Psi + \bigl(V(x)+E\bigr)\Psi - f(x,\Psi), \quad (x,t)\in\mathbb{R}^N\times\mathbb{R}. \tag{1.2}$$

Equation (1.2), introduced by Laskin [16, 17], describes how the wave function of a physical system evolves over time. Unlike the classical Laplacian operator, the usual analysis tools for elliptic PDEs cannot be directly applied to (1.2) since $(-\Delta)^s$ is a nonlocal operator. Cafferelli and Silvestre [6] developed a powerful extension method, which transfers the nonlocal equation (1.2) into a local one on a half-space. Recently, Di Nezza, Palatucci, and Valdinoci [9] gave a survey on the fractional Sobolev spaces, which are more convenient for fractional Laplacian equations. Lee, Kim, Kim, and Scapellato [18] examined the existence of at least two distinct nontrivial solutions to a Schrödinger-type problem involving the nonlocal fractional p-Laplacian with concave–convex nonlinearities. Since then, there have been some works on the existence, multiplicity, and concentration phenomenon of solutions to the nonlinear fractional Schrödinger equation (1.2) and other differential problems driven by Laplace-type operators; see [1, 2, 7, 10–13, 15, 19–21, 25, 28–32].

When $s = 1$, (1.1) is the classical semilinear elliptic equation in $\mathbb{R}^N$ with sign-changing weight functions. Equations of this type have been studied extensively in recent years, mainly on bounded domains. Below we briefly describe some of this work. Berestycki et al. [3] studied the existence and nonexistence of positive solutions to the problem

$$\begin{cases} -\Delta u + m(x)u = a(x)u^p, & x\in\Omega, \\ Bu(x) = 0, & x\in\partial\Omega. \end{cases} \tag{1.3}$$

Here m and a may be sign-changing functions, $1 < p < 2^*$, $Bu = u$, and $Bu = \partial_\nu u$ (respectively, the Dirichlet and Neumann boundary conditions). Brown and Zhang [4] considered a problem similar to (1.3) by splitting the Nehari manifold into three parts corresponding to local minima, local maxima, and points of inflection of the fibering map and then looked for minimizers of the energy functional on the first two parts. De Paiva [8] studied the problem

$$\begin{cases} -\Delta u = a(x)u^p + \lambda b(x)u^q, & x\in\Omega, \\ u(x) = 0, & x\in\partial\Omega, \end{cases} \tag{1.4}$$

where $0 < p < 1 < q \leq 2^* - 1$, a changes sign, and $b \geq 0$. He proved that there exists $\lambda^* \in (0, +\infty)$ such that (1.4) has at least one nonnegative solution for $0 < \lambda < \lambda^*$ and there is no such solution for $\lambda > \lambda^*$. For an unbounded domain, we can mention Wu [27], who studied the multiplicity of positive solutions for the concave–convex elliptic equation

$$\begin{cases} -\Delta u + u = f_\lambda(x)u^{p-1} + g_\mu(x)u^{q-1}, & x \in \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{cases} \tag{1.5}$$

where $1 < p < 2 < q < 2^*$, the parameters $\lambda, \mu \geq 0$, $f_\lambda = \lambda f_+ + f_-$, $(f_\pm := \max\{0, \pm f\})$ is sign-changing, and $g_\mu(x) = a(x) + \mu b(x)$. Here the author used the idea of Brown and Zhang [4] mentioned above and has split the Nehari manifold into three parts considered separately. See also [14, 24], where related (singular) problems with sign-changing weights in $\mathbb{R}^N$ are studied by similar techniques. To our best knowledge, there is no similar result for nonlocal problem (1.2) with sign-changing weight functions, so in this paper, we will fill this gap.

Our purpose here is to study the fractional Schrödinger equation (1.1). We are interested in the situation where one of the weight functions a and b is sign-changing, and unlike the papers mentioned above, we are mainly concerned with the existence of infinitely many solutions. In what follows, we assume that a and b satisfy some of the following hypotheses:

(H_1) $a \in L^r(\mathbb{R}^N)$ and $b \in L^t(\mathbb{R}^N)$, where $1 < \frac{r}{r-1} < \frac{2_s^*}{p}$, $1 < \frac{t}{t-1} < \frac{2_s^*}{q}$;
(H_2) $a, b \in L^\infty(\mathbb{R}^N)$, $\limsup_{|x|\to\infty} a(x) \leq 0$, and $\limsup_{|x|\to\infty} b(x) \leq 0$.
(H_3) $b \geq 0$ in $\mathbb{R}^N$, and the set $\{x \in \mathbb{R}^N : b(x) > 0\}$ has a nonempty interior.
(H_4) $a \leq 0$ in $\mathbb{R}^N$, and the set $\{x \in \mathbb{R}^N : b(x) > 0\}$ has a nonempty interior.

For example, $a(x) = e^{-|x|} \sin|x|$ and $b(x) = e^{-|x|}$ satisfy assumptions (H_1) or (H_2) and (H_3). $a(x) = -e^{-|x|}$ and $b(x) = e^{-|x|}\sin|x|$ satisfy assumptions (H_1) or (H_2) and (H_4).

Our main results of this paper is the following:

Theorem 1.1 *Assume that* (H_1) *or* (H_2) *and* (H_3) *or* (H_4) *hold. Then problem* (1.1) *has infinitely many solutions.*

Under the assumptions above, we prove that the Nehari manifold is closed and of class C^2 and that the energy functional corresponding to problem (1.1) is bounded below. When (H_1) or (H_2) is satisfied, we show that the energy functional satisfies the Palais–Smale condition on the Nehari manifold, and then using some arguments based on the Krasnoselskii genus, we establish the existence of infinitely many solutions for problem (1.1).

This paper is organized as follows. In Sect. 2, we describe the functional setting to study problem (1.1) and prove some preliminary lemmas. In Sect. 3, we complete the proof of Theorem 1.1.

2 Variational settings and preliminary results

We denote by $|\cdot|_p$ the usual norm of the space $L^p(\mathbb{R}^3)$, $1 \leq p < \infty$, by $B_r(x)$ the open ball with center at x and radius r, and by C or C_i $(i = 1, 2, \dots)$ positive constants that may change from line to line. By $a_n \rightharpoonup a$ and $a_n \to a$ we mean the weak and strong convergence, respectively, as $n \to \infty$.

2.1 The functional space setting

Firstly, fractional Sobolev spaces are convenient for our problem, so we will give some sketches on them; a complete introduction can be found in [9]. We recall that for $s \in (0,1)$, the fractional Sobolev space $H^s(\mathbb{R}^3) = W^{s,2}(\mathbb{R}^3)$ is defined as follows:

$$H^s(\mathbb{R}^3) = \left\{ u \in L^2(\mathbb{R}^3) : \int_{\mathbb{R}^3} \left(|\xi|^{2s} \left|\mathcal{F}(u)\right|^2 + \left|\mathcal{F}(u)\right|^2 \right) d\xi < \infty \right\}$$

with the norm

$$\|u\|^2_{H^s(\mathbb{R}^3)} = \int_{\mathbb{R}^3} \left(|\xi|^{2s} \left|\mathcal{F}(u)\right|^2 + \left|\mathcal{F}(u)\right|^2 \right) d\xi,$$

where $\mathcal{F}$ denotes the Fourier transform. We also define the homogeneous fractional Sobolev space $\mathcal{D}^{s,2}(\mathbb{R}^3)$ as the completion of $C_0^\infty(\mathbb{R}^3)$ with respect to the norm

$$\|u\|_{\mathcal{D}^{s,2}(\mathbb{R}^3)} := \left(\iint_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{|u(x) - u(y)|^2}{|x-y|^{3+2s}} \, dx \, dy \right)^{\frac{1}{2}} = [u]_{H^s(\mathbb{R}^3)}.$$

The fractional Laplacian $(-\Delta)^s u$ of a smooth function $u : \mathbb{R}^3 \to \mathbb{R}$ is defined by

$$\mathcal{F}\left((-\Delta)^s u\right)(\xi) = |\xi|^{2s} \mathcal{F}(u)(\xi), \quad \xi \in \mathbb{R}^3,$$

that is,

$$\mathcal{F}(\phi)(\xi) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{\mathbb{R}^3} e^{-i\xi \cdot x} \phi(x) \, dx$$

for functions ϕ in the Schwartz class. Also, $(-\Delta)^s u$ can be equivalently represented as (see [9])

$$(-\Delta)^s u(x) = -\frac{1}{2} C(s) \int_{\mathbb{R}^3} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{3+2s}} \, dy, \quad x \in \mathbb{R}^3,$$

where

$$C(s) = \left(\int_{\mathbb{R}^3} \frac{(1 - \cos \xi_1)}{|\xi|^{3+2s}} \, d\xi \right)^{-1}, \quad \xi = (\xi_1, \xi_2, \xi_3).$$

Also, by the Plancherel formula in Fourier analysis we have

$$[u]^2_{H^s(\mathbb{R}^3)} = \frac{2}{C(s)} \left\| (-\Delta)^{\frac{s}{2}} u \right\|_2^2.$$

As a consequence, the norms on $H^s(\mathbb{R}^3)$

$$u \longmapsto \left(\int_{\mathbb{R}^3} |u|^2 \, dx + \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{|u(x) - u(y)|^2}{|x-y|^{3+2s}} \, dx \, dy \right)^{\frac{1}{2}},$$

$$u \longmapsto \left(\int_{\mathbb{R}^3} \left(|\xi|^{2s} \left|\mathcal{F}(u)\right|^2 + \left|\mathcal{F}(u)\right|^2 \right) d\xi \right)^{\frac{1}{2}},$$

$$u \longmapsto \left(\int_{\mathbb{R}^3} |u|^2 \, dx + \left\| (-\Delta)^{\frac{s}{2}} u \right\|_2^2 \right)^{\frac{1}{2}}$$

are equivalent.

For the reader's convenience, we consider the space $X := H^s(\mathbb{R}^N)$ with the norm

$$\|u\| = \left(\int_{\mathbb{R}^3} |u|^2 \, dx + \left\| (-\Delta)^{\frac{s}{2}} u \right\|_2^2 \right)^{\frac{1}{2}},$$

and we review the main embedding result for this class of fractional Sobolev spaces.

Lemma 2.1 ([9]) *Let* $0 < s < 1$. *Then there exists a constant* $C = C(s) > 0$ *such that*

$$\|u\|^2_{L^{2^*_s}(\mathbb{R}^3)} \leq C[u]^2_{H^s(\mathbb{R}^3)}$$

for every $u \in H^s(\mathbb{R}^3)$, *where* $2^*_s = \frac{6}{3-2s}$ *is the fractional critical exponent. Moreover, the embedding* $X \hookrightarrow L^r(\mathbb{R}^3)$ *is continuous for any* $r \in [2, 2^*_s]$ *and is locally compact whenever* $r \in [2, 2^*_s)$.

Lemma 2.2 ([22]) *If* $\{u_n\}$ *is bounded in* $H^s(\mathbb{R}^3)$ *and for some* $R > 0$,

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^3} \int_{B_R(y)} |u_n|^2 \, dx = 0,$$

then $u_n \to 0$ *in* $L^r(\mathbb{R}^3)$ *for all* $2 < r < 2^*_s$.

2.2 Properties of the Nehari manifold

Set $X := H^s(\mathbb{R}^N)$, and let $A, B : X \to \mathbb{R}$ be defined by

$$A(u) := \int_{\mathbb{R}^N} a(x)|u|^p \, dx, \tag{2.1}$$

$$B(u) := \int_{\mathbb{R}^N} b(x)|u|^q \, dx. \tag{2.2}$$

Remark 2.1 Since $2 < p < q < 2^*_s$, it is easy to see that if (H_1) or (H_2) is satisfied, then $A, B \in C^2(X, \mathbb{R})$.

It is clear that problem (1.1) is the Euler–Lagrange equation for the functional $J : X \to \mathbb{R}$ defined by

$$J(u) = \frac{1}{2}\|u\|^2 - \frac{1}{p}\int_{\mathbb{R}^N} a(x)|u|^p \, dx - \frac{1}{q}\int_{\mathbb{R}^N} b(x)|u|^q \, dx. \tag{2.3}$$

By this remark the action functional $J \in C^2(X, \mathbb{R})$, and its critical points are weak solutions of problem (1.1). Moreover, for all $u, v \in X$, we have

$$\langle J'(u), v \rangle = \int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} v \, dx - \int_{\mathbb{R}^N} a(x)|u|^{p-2} uv dx - \int_{\mathbb{R}^3} b(x)|u|^{q-2} uv dx.$$

Hence in the following, we consider critical points of I using the variational method.

We first introduce the Nehari manifold associated with the functional J:

$$\mathcal{N} := \{u \in X\backslash\{0\} : \langle J'(u), u\rangle = 0\} = \{u \in X\backslash\{0\} : \|u\|^2 = A(u) + B(u)\}.$$

Now we define the fibering map corresponding to $u \in X\backslash\{0\}$ by setting $\alpha_u(t) = J(tu)$, $t > 0$. Then

$$\alpha_u(t) = \frac{t^2}{2}\|u\|^2 - \frac{t^p}{p}A(u) - \frac{t^q}{q}B(u),$$

and

$$\alpha_u'(t) = t\|u\|^2 - t^{p-1}A(u) - t^{q-1}B(u).$$

Moreover, $tu \in \mathcal{N}$ if and only if $\alpha_u'(t) = 0$.

Lemma 2.3 *Suppose that either* (H_1) *or* (H_2) *is satisfied.*

(*i*) *If* (H_3) *holds, then the equation* $\alpha_u'(t) = 0$ *has only one solution* $t_u > 0$, *provided that* $A(u) > 0$ *or* $B(u) > 0$. *Moreover,* $J(u) > 0$ *for all* $u \in \mathcal{N}$.

(*ii*) *If* (H_4) *holds, then the equation* $\alpha_u'(t) = 0$ *has only one solution* $t_u > 0$, *provided that* $B(u) > 0$. *Moreover,* $J(u) > 0$ *for all* $u \in \mathcal{N}$.

Proof Suppose (H_3) holds. If $A(u) \le 0$ and $B(u) = 0$, then $\alpha_u'(t) > 0$ for all $t > 0$. If $B(u) > 0$, or $B(u) = 0$ and $A(u) > 0$, then α_u has a positive maximum.

Suppose (H_4) holds. Since now $A(u) \le 0$, $\alpha_u'(t) > 0$ for all $t > 0$ if $B(u) \le 0$, and α_u has a positive maximum if $B(u) > 0$.

It remains to show that the equation $\alpha_u'(t) = 0$ has at most one solution. Since $\alpha_u(t_u) > 0$, it will then follow that $J(u) > 0$ for all $u \in \mathcal{N}$. Let $\alpha_u'(t_1) = \alpha_u'(t_2) = 0$ for $t_1, t_2 > 0$. We have

$$\|u\|^2 = t_1^{p-2}A(u) + t_2^{q-2}B(u)$$

and

$$\|u\|^2 = t_2^{p-2}A(u) + t_2^{q-2}B(u).$$

Therefore

$$\left(t_1^{p-2} - t_2^{p-2}\right)\|u\|^2 = (t_1t_2)^{p-2}\left(t_1^{q-p} - t_2^{q-p}\right)B(u). \tag{2.4}$$

From (2.4) we see that either $t_1 = t_2$ or $B(u) < 0$. However, in the second case, α_t' is never 0 under our hypotheses. □

Lemma 2.4 *Suppose that* (H_1) *and either* (H_3) *or* (H_4) *is satisfied. Then the Nehari manifold* $\mathcal{N}$ *is a closed* C^2*-manifold. Moreover,* $\mathcal{N}$ *is bounded away from zero.*

Proof Let $u \in \mathcal{N}$. Then from the Sobolev and Hölder inequalities and (H_1) we have

$$\|u\|^2 = A(u) + B(u) \le |a|_r|u|_{pr'}^p + |b|_s|u|_{qs'}^q \le C_1|a|_r\|u\|^p + C_2|b|_s\|u\|^q, \tag{2.5}$$

where $r' = \frac{r}{r-1}$, $s' = \frac{s}{s-1}$, and C_1, C_2 are positive constants. Since $2 < p < q < 2_s^*$, inequality (2.5) implies that the Nehari manifold is bounded away from 0.

Now we show that it is a closed C^2-manifold. Define $\psi : X \to \mathbb{R}$ as

$$\psi(u) := \langle J'(u), u \rangle = \|u\|^2 - A(u) - B(u).$$

By Remark (2.1), $\psi \in C^2(X, \mathbb{R})M$ and by the definition of ψ, $\mathcal{N} = \psi^{-1}(0) \backslash \{0\}$. Since $\mathcal{N}$ is bounded away from 0, $\mathcal{N}$ is closed. If we show that every point of $\mathcal{N}$ is regular for ψ, THEN the proof will be complete. Let $u \in \mathcal{N} = \psi^{-1}(0) \backslash \{0\}$. Then

$$\|u\|^2 = A(u) + B(u) \tag{2.6}$$

and

$$\langle \psi'(u), u \rangle = 2\|u\|^2 - pA(u) - qB(u). \tag{2.7}$$

It follows from (2.6) and (2.7) that

$$\langle \psi'(u), u \rangle = (2 - p)\|u\|^2 + (p - q)B(u). \tag{2.8}$$

Since, according to Lemma 2.3, $B(u) \geq 0$ if $u \in \mathcal{N}$, the right-hand side of (2.8) is negative. Hence every point of $\mathcal{N}$ is regular for ψ. □

Lemma 2.5 *Suppose that* (H_2) *and either* (H_3) *or* (H_4) *are satisfied. Then the Nehari manifold* $\mathcal{N}$ *is a closed* C^2*-manifold. Moreover,* $\mathcal{N}$ *is bounded away from zero.*

Proof Consider $u \in \mathcal{N}$ and assume that $a, b \in L^\infty(\mathbb{R}^N)$. Then by the Sobolev inequality we have

$$\|u\|^2 = A(u) + B(u) \leq |a|_\infty |u|_p^p + |b|_\infty |u|_q^q \leq C_1 |a|_\infty \|u\|^p + C_2 |b|_\infty \|u\|^q, \tag{2.9}$$

where C_1, C_2 are positive constants. Using (2.9), we deduce that $\mathcal{N}$ is bounded away from 0. The rest of the proof is the same as that of Lemma 2.4. □

A functional $I \in C^1(X, \mathbb{R})$ is said to satisfy the Palais–Smale condition at the level $c \in \mathbb{R}$ (the $(PS)_c$-condition for short) if every sequence $\{u_n\} \subset X$ such that

$$I(u_n) \to c \quad \text{and} \quad I'(u_n) \to 0 \tag{2.10}$$

admits a convergent subsequence. A sequence satisfying (2.10) is called a $(PS)_c$-sequence.

Lemma 2.6 *Suppose that the assumptions of Lemma 2.4 or 2.5 are satisfied. Then* $u \neq 0$ *is a critical point of* J *if and only if it is a critical point of* $J|_{\mathcal{N}}$*, and* $\{u_n\} \subset \mathcal{N}$ *is a* $(PS)_c$*-sequence for* J *if and only if it is a* $(PS)_c$*-sequence for* $J|_{\mathcal{N}}$*.*

Proof It is clear that if $u \neq 0$ is a critical point of J, then $u \in \mathcal{N}$. Let $u \in \mathcal{N}$. By (2.8) we know that $\langle \psi'(u), u \rangle < 0$, and therefore $X = T_u\mathcal{N} \oplus \mathbb{R}u$. Since $J'(u)|_{\mathbb{R}u} \equiv 0$ by the definition of $\mathcal{N}$, the conclusion follows. □

3 Proof of Theorem 1.1

To prove Theorem 1.1, we need to recall the definition of the Krasnoselskii genus and an abstract multiplicity result in [23]. A set $F \subset X$ is said to be symmetric if $F = -F$. Let

$$\Sigma := \{F \subset X : F \text{ is closed and symmetric}\}.$$

For $F \neq \emptyset$ and $F \in \Sigma$, the Krasnoselskii genus of F is the least integer n such that there exists an odd function $f \in C(F, \mathbb{R}^N \setminus \{0\})$. The genus of F is denoted by $\gamma(F)$. Set $\gamma(\emptyset) := 0$ and $\gamma(F) := \infty$ if for all n, there exists no f with the above property.

Theorem 3.1 ([23]) *Suppose $J \in C^1(M, \mathbb{R})$ is an even functional on a complete symmetric $C^{1,1}$-manifold $M \subset V \setminus \{0\}$ in some Banach space V. Suppose also that J satisfies the $(PS)_c$-condition for all $c \in \mathbb{R}$ and is bounded from below on M. Let*

$$\hat{\gamma}(M) := \sup\{\gamma(F) : F \subset M \text{ is compact and symmetric}\}.$$

Then the functional J possesses at least $\hat{\gamma}(M) \leq \infty$ pairs of critical points.

We first need some auxiliary results.

Lemma 3.1 *Suppose that assumption (H_1) holds. Then the functionals A and B defined by (2.1) and (2.2) are weakly continuous:*

$$A(u_n) \to A(u) \quad \text{and} \quad B(u_n) \to B(u) \quad \text{as } u_n \rightharpoonup u.$$

Moreover, $A', B' : X \to X^*$ are completely continuous:

$$A'(u_n) \to A'(u) \quad \text{and} \quad B'(u_n) \to B'(u) \quad \text{as } u_n \rightharpoonup u.$$

Proof We only prove the lemma for A and A'; for B, B', the proof is similar. Let $u_n \in X$ and $u_n \rightharpoonup u$. Using the Rellich–Kondrachov theorem, up to a subsequence, we have

$$\begin{aligned} &u_n \rightharpoonup u \quad \text{in } X, \\ &u_n \to u \quad \text{in } L^l_{\text{loc}}(\mathbb{R}^N), 2 \leq l < 2_s^*, \\ &u_n(x) \to u(x) \quad \text{a.e. in } \mathbb{R}^N. \end{aligned} \tag{3.1}$$

Let $w_n := |u_n|^{p-2}u_n - |u|^{p-2}u$. By (3.1) we get

$$w_n(x) \to 0 \quad \text{a.e. in } \mathbb{R}^N. \tag{3.2}$$

Assumption (H_1) and the Sobolev inequality imply

$$|w_n|^{\frac{p}{p-1}} \leq C_1\big(|u_n|^p + |u|^p\big) \in L^{\frac{r}{r-1}}\big(\mathbb{R}^N\big). \tag{3.3}$$

The boundedness of $\{u_n\}$ in X, (3.2), and (3.3) imply that $\{|w_n|^{\frac{p}{p-1}}\}$ is bounded in $L^{\frac{r}{r-1}}(\mathbb{R}^N)$ and up to a subsequence,

$$|w_n|^{\frac{p}{p-1}} \rightharpoonup 0 \quad \text{in } L^{\frac{r}{r-1}}\big(\mathbb{R}^N\big). \tag{3.4}$$

Now let $v \in X$ with $\|v\| \le 1$. Using the Hölder and the Sobolev inequalities, we deduce that

$$\begin{aligned}
\left|\langle A'(u_n) - A'(u), v\rangle\right| &= \left|\int_{\mathbb{R}^N} a(x) w_n \, dx\right| \\
&\le \left(\int_{\mathbb{R}^N} |a(x)||v|^p \, dx\right)^{\frac{1}{p}} \left(\int_{\mathbb{R}^N} |a(x)||w_n|^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}} \\
&\le |a|_r^{\frac{1}{p}} |v|_{\frac{pr}{r-1}} \left(\int_{\mathbb{R}^N} |a(x)||w_n|^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}} \\
&\le C|a|_r^{\frac{1}{p}} \|v\| \left(\int_{\mathbb{R}^N} |a(x)||w_n|^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}} \\
&\le C|a|_r^{\frac{1}{p}} \left(\int_{\mathbb{R}^N} |a(x)||w_n|^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}},
\end{aligned}$$

where $C > 0$ is a constant. Since $|w_n|^{\frac{p}{p-1}} \rightharpoonup 0$ in $L^{\frac{r}{r-1}}(\mathbb{R}^N)$ and a is in the dual space of $L^{\frac{r}{r-1}}(\mathbb{R}^N)$, the right-hand side above goes to 0 uniformly with respect to $\|v\| \le 1$, which this implies that A' is completely continuous. By the definition of A,

$$A(u) = \frac{1}{p}\langle A'(u), u\rangle.$$

Thus

$$A(u) - A(u_n) = \frac{1}{p}\langle A'(u), u\rangle - \frac{1}{p}\langle A'(u_n), u_n\rangle \to 0$$

by the complete continuity of A'. This proves the weak continuity of A. □

Let

$$a^+(x) := \max\{0, a(x)\}, \qquad a^-(x) := \max\{0, -a(x)\}, \tag{3.5}$$

and define $b^{\pm}(x)$ similarly. Also, put

$$A_{\pm}(u) := \int_{\mathbb{R}^N} a^{\pm}(x)|u|^p \, dx, \qquad B_{\pm}(u) := \int_{\mathbb{R}^N} b^{\pm}(x)|u|^q \, dx.$$

Lemma 3.2 *Suppose that assumption (H_2) holds. Then A_+, B_+ are weakly continuous, and $A'_+, B'_+ : X \to X^*$ are completely continuous.*

Proof We only prove the lemma for A_+, A'_+; for B_+, B'_+, the proof is similar. Let $u_n \in X$ and $u_n \rightharpoonup u$. Using the Rellich–Kondrachov theorem, up to a subsequence, we have

$$\begin{aligned}
&u_n \rightharpoonup u \quad \text{in } X, \\
&u_n \to u \quad \text{in } L^l_{\mathrm{loc}}(\mathbb{R}^N), 2 \le l < 2^*_s, \\
&u_n(x) \to u(x) \quad \text{a.e. in } \mathbb{R}^N.
\end{aligned} \tag{3.6}$$

As in the preceding proof, let $w_n := |u_n|^{p-2}u_n - |u|^{p-2}u$. We see from the Krasnoselskii theorem (see [26]) and (3.6) that

$$w_n \to 0 \quad \text{in } L^{\frac{p}{p-1}}_{\text{loc}}(\mathbb{R}^N). \tag{3.7}$$

It follows from (H_2) that for any $\varepsilon > 0$, there exists $R > 0$ such that

$$a^+(x) < \varepsilon \quad \text{whenever } |x| \geq R. \tag{3.8}$$

Using the Hölder and the Sobolev inequalities and (3.7), we obtain

$$\begin{aligned} \sup_{\|v\|\leq 1} \left| \int_{|x|\leq R} a^+(x) w_n v \, dx \right| &\leq |a^+|_\infty \left(\int_{|x|\leq R} |w_n|^{\frac{p}{p-1}} dx \right)^{\frac{p-1}{p}} \left(\int_{|x|\leq R} |v|^p \, dx \right)^{\frac{1}{p}} \\ &\leq C_1 \left(\int_{|x|\leq R} |w_n|^{\frac{p}{p-1}} dx \right)^{\frac{p-1}{p}} \\ &\to 0 \end{aligned} \tag{3.9}$$

as $n \to \infty$. By inequality (3.3), $\{w_n\}$ is bounded in $L^{\frac{p}{p-1}}(\mathbb{R}^N)$, hence inequality (3.8), the Hölder and the Sobolev inequalities imply that there exists a constant $C_2 > 0$, independent of $\varepsilon > 0$, such that

$$\sup_{\|v\|\leq 1} \left| \int_{|x|>R} a^+(x) w_n v \, dx \right| \leq C_2 \varepsilon. \tag{3.10}$$

Using (3.9) and (3.10), we deduce that

$$\sup_{\|v\|\leq 1} \left| \langle A'_+(u_n) - A'_+(u), v \rangle \right| = \sup_{\|v\|\leq 1} \left| \int_{\mathbb{R}^N} a^+(x) w_n v \, dx \right| \to 0.$$

Since

$$A_+(u) = \frac{1}{p} \langle A'_+(u), u \rangle,$$

we see as in the proof of Lemma 3.2 that A_+ is weakly continuous. □

Lemma 3.3 *Suppose that* (H_1) *or* (H_2) *and* (H_3) *or* (H_4) *are satisfied. Then the functional* J *satisfies the* $(PS)_c$*-condition on* $\mathcal{N}$ *for all* $c \in \mathbb{R}$.

Proof Let $c \in \mathbb{R}$, and let $\{w_n\} \subset \mathcal{N}$ be a $(PS)_c$-sequence. Then

$$\|u_n\|^2 = A(u_n) + B(u_n) \tag{3.11}$$

and

$$J'(u_n) \to 0, \qquad J(u_n) \to c.$$

If (H_3) holds, then $B(u) \geq 0$, and we have, using (3.11) and the boundedness of $J(u_n)$,

$$\begin{aligned} c+1 &\geq J(u) \\ &= \frac{1}{2}\|u\|^2 - \frac{1}{p}A(u) - \frac{1}{q}B(u) \\ &= \left(\frac{1}{2}-\frac{1}{p}\right)\|u\|^2 + \left(\frac{1}{p}-\frac{1}{q}\right)B(u) \\ &\geq \left(\frac{1}{2}-\frac{1}{p}\right)\|u\|^2 \end{aligned} \tag{3.12}$$

for all n large enough.

If (H_4) holds, then $A(u_n) \leq 0$, and using (3.11) again, we have

$$\begin{aligned} c+1 &\geq J(u) \\ &= \frac{1}{2}\|u\|^2 - \frac{1}{p}A(u) - \frac{1}{q}B(u) \\ &= \left(\frac{1}{2}-\frac{1}{q}\right)\|u\|^2 - \left(\frac{1}{p}-\frac{1}{q}\right)A(u) \\ &\geq \left(\frac{1}{2}-\frac{1}{q}\right)\|u\|^2 \end{aligned} \tag{3.13}$$

for all n large enough. Hence in both cases, $\{u_n\}$ is a bounded sequence. So there exists $u \in X$ such that passing to a subsequence, $u_n \rightharpoonup u$. Since $J(u_n) \to 0$, it is easy to see that $J(u) = 0$, and it follows that

$$\langle J'(u_n) - J'(u), u_n - u\rangle \to 0$$

or, equivalently,

$$\|u_n - u\|^2 - \langle A'(u_n) - A'(u), u_n - u\rangle - \langle B'(u_n) - B'(u), u_n - u\rangle \to 0. \tag{3.14}$$

If (H_1) holds, then by Lemma 3.1, $A'(u_n) \to A'(u)$ and $B'(u_n) \to B'(u)$. Thus from (3.14) we get $u_n \to u$ in X.

If (H_2) holds, then since the function $u \mapsto |u|^t$ is convex for $t \geq 2$ (in particular, for $t = p$ and q), we have $(|v|^{t-2}v - |u|^{t-2}u)(v-u) \geq 0$. Therefore, using (3.14), we have

$$\begin{aligned} &\|u_n - u\|^2 - \langle A'_+(u_n) - A'_+(u), u_n - u\rangle - \langle B'_+(u_n) - B'_+(u), u_n - u\rangle \\ &\quad \leq \|u_n - u\|^2 - \langle A'(u_n) - A'(u), u_n - u\rangle - \langle B'(u_n) - B'(u), u_n - u\rangle \\ &\quad \to 0. \end{aligned}$$

Since $A'_+(u_n) \to A'_+(u)$ and $B'_+(u_n) \to B'_+(u)$, we see that $u_n \to u$ in X again. □

Proof of Theorem 1.1 By Lemmas 2.3–2.5 and 3.3, $\mathcal{N}$ is a closed symmetric C^2-manifold, $J(u) > 0$ for all $u \in \mathcal{N}$, and J satisfies the $(PS)_c$-condition on $\mathcal{N}$ for all $c \in \mathbb{R}$. If we show that for any $j \geq 1$, there exists a symmetric compact set $F_j \subset \mathcal{N}$ such that $\gamma(F_j) \geq j$, then the conclusion will follow from Lemma 2.6 and Theorem 3.1.

Let $j \geq 1$, let X_j be a subspace spanned by j linearly independent functions $v_k \in C_0^\infty(\mathbb{R}^N)$ with $\operatorname{supp} v_k \subset \{x \in \mathbb{R}^N : b(x) > 0\}$, and let

$$S^{j-1} := X_j \cap \{x \in X : \|u\| = 1\}.$$

Then we have $B(u) > 0$ for all $u \in S^{j-1}$, and it follows from Lemma 2.3 that the equation $\alpha_u'(t) = 0$ has exactly one solution $t_u \in (0, \infty)$. Hence the mapping $\varphi : S^{j-1} \to \mathcal{N}$ given by $\varphi(u) := t_u u$ is well defined, and it is obviously odd. If it is continuous, then $F_j := \varphi(S^{j-1})$ is homeomorphic to S^{j-1}, and it follows from the properties of genus that $\gamma(F_j) = \gamma(S^{j-1}) = j$.

We need to show that $u \mapsto t_u$ is continuous. An easy computation shows that if the (necessary and sufficient) conditions for the existence of t_u given in Lemma 2.3 are satisfied, then $\alpha_u''(t) < 0$ for $t = t_u$. Hence the continuity of t_u follows from the implicit function theorem. □

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