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A new version of (p,q)-Hermite–Hadamard's midpoint and trapezoidal inequalities via special operators in (p,q)-calculus

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Abstract

In this paper, we conduct a research on a new version of the (p,q)-Hermite–Hadamard inequality for convex functions in the framework of postquantum calculus. Moreover, we derive several estimates for (p,q)-midpoint and (p,q)-trapezoidal inequalities for special (p,q)-differentiable functions by using the notions of left and right (p,q)-derivatives. Our newly obtained inequalities are extensions of some existing inequalities in other studies. Lastly, we consider some mathematical examples for some (p,q)-functions to confirm the correctness of newly established results.

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1 Introduction

The brilliant results of Charles Hermite and Jacques Hadamard's studies, which ended in Hermite–Hadamard inequality, commonly known as Hadamard's inequality, indicate the fact that if $\hbar:[\nu,\omega]\to\mathbb{R}$ is convex, we have the following double inequality:

$$\hbar\left(\frac{\nu+\omega}{2}\right)\leq\frac{1}{\omega-\nu}\int_{\nu}^{\omega}\hbar(x)\,dx\leq\frac{\hbar(\nu)+\hbar(\omega)}{2}. \tag{1}$$

When $\hbar$ is a concave mapping, the inequality holds in the opposite direction. There has been much research done in the Hermite–Hadamard direction for different kinds of convexities. For example, in [1, 2] the authors established some inequalities linked with midpoint and trapezoidal formulas of numerical integration for convex functions. For more results related to the above inequality and convex functions, the reader can consult [3–7]. There are many generalizations of convex functions, like h-convex functions, preinvex functions, m-convex functions, harmonically convex functions, (α,m)-convex functions, convexity with respect to a pair of functions, etc. These kinds of convexities have a very large role in functional analysis, optimization theory, approximation theory, and fractional mathematical modeling [8–24].

Due to its immediate applications in numerical integration, probability theory, information theory, and integral operator theory, the Hermite–Hadamard inequality is of utmost significance. The field of inequalities has seen a tremendous flow of this remarkable inequality and related outstanding (Hadamard-type) inequalities over the past millennium. These inequalities are partially inspired by the results mentioned above but perhaps even more, so by the difficulty of conducting research in a variety of mathematical subdisciplines like mathematical programming, control theory, variational methods, operation research, probability, and statistics.

Besides this, quantum and postquantum calculus are very important branches of calculus having a vast range of applications in the fields of mathematics and physics. Because of numerous applications of quantum calculus (shortly, q-calculus) and postquantum calculus (shortly, (p,q)-calculus) without limit calculus, many researchers began working on them and applying their concepts in differential equations, integral equalities, mathematical modeling, and integral inequalities [25–32].

Alp et al. [33] and Bermudo et al. [34] used q-integrals to prove two different versions of q-Hermite–Hadamard inequalities along with some relevant estimates. The q-Hermite–Hadamard inequalities are described as follows.

Theorem 1.1 ([33, 34]) *For a convex map* $\hbar : [\nu,\omega] \to \mathbb{R}$, *we have the inequalities*

$$\hbar\left(\frac{q\nu+\omega}{[2]_q}\right) \le \frac{1}{\omega-\nu}\int_\nu^\omega \hbar(x)\,{}_\nu d_q x \le \frac{q\hbar(\nu)+\hbar(\omega)}{[2]_q}, \tag{2}$$

$$\hbar\left(\frac{\nu+q\omega}{[2]_q}\right) \le \frac{1}{\omega-\nu}\int_\nu^\omega \hbar(x)\,{}^\omega d_q x \le \frac{\hbar(\nu)+q\hbar(\omega)}{[2]_q}. \tag{3}$$

Remark 1.2 It is very easy to observe that by adding (3) and (4) we derive the q-Hermite–Hadamard inequality (see [34])

$$\hbar\left(\frac{\nu+\omega}{2}\right) \le \frac{1}{2(\omega-\nu)}\left[\int_\nu^\omega \hbar(x)\,{}_\nu d_q x + \int_\nu^\omega \hbar(x)\,{}^\omega d_q x\right] \le \frac{\hbar(\nu)+\hbar(\omega)}{2}. \tag{4}$$

Recently, Ali et al. [35] and Sitthiwirattham et al. [36] used new techniques to prove the following two different and new versions of Hermite–Hadamard-type inequalities in the context of q-operators.

Theorem 1.3 ([35, 36]) *For a convex map* $\hbar : [\nu,\omega] \to \mathbb{R}$, *we have the inequalities*

$$\hbar\left(\frac{\nu+\omega}{2}\right) \le \frac{1}{\omega-\nu}\left[\int_\nu^{\frac{\nu+\omega}{2}} \hbar(x)\,{}^{\frac{\nu+\omega}{2}} d_q x + \int_{\frac{\nu+\omega}{2}}^\omega \hbar(x)\,{}_{\frac{\nu+\omega}{2}} d_q x\right] \le \frac{\hbar(\nu)+\hbar(\omega)}{2}, \tag{5}$$

$$\hbar\left(\frac{\nu+\omega}{2}\right) \le \frac{1}{\omega-\nu}\left[\int_\nu^{\frac{\nu+\omega}{2}} \hbar(x)\,{}_\nu d_q x + \int_{\frac{\nu+\omega}{2}}^\omega \hbar(x)\,{}^\omega d_q x\right] \le \frac{\hbar(\nu)+\hbar(\omega)}{2}. \tag{6}$$

Remark 1.4 When $q \to 1^-$ in (3)–(7), we recapture the traditional Hermite–Hadamard inequality (1).

Kunt et al. [37] and Vivas-Cortez et al. [38] extended the previous studies and derived several Hermite–Hadamard-type inequalities with new structures for convex functions using the (p,q)-integrals.

Theorem 1.5 ([37, 38]) *For a convex mapping* $\hbar : [\nu, \omega] \to \mathbb{R}$, *we have the inequalities*

$$\hbar\left(\frac{q\nu + p\omega}{[2]_{p,q}}\right) \leq \frac{1}{p(\omega - \nu)} \int_{\nu}^{\omega} \hbar(x)\, {}_{\nu}d_{p,q}x \leq \frac{q\hbar(\nu) + p\hbar(\omega)}{[2]_{p,q}} \tag{7}$$

and

$$\hbar\left(\frac{p\nu + q\omega}{[2]_{p,q}}\right) \leq \frac{1}{p(\omega - \nu)} \int_{\nu}^{\omega} \hbar(x)\, {}^{\omega}d_{p,q}x \leq \frac{p\hbar(\nu) + q\hbar(\omega)}{[2]_{p,q}}. \tag{8}$$

Remark 1.6 It is also very easy to observe that by adding (7) and (8) we obtain the (p,q)-Hermite–Hadamard inequality (see [38])

$$\hbar\left(\frac{\nu + \omega}{2}\right) \leq \frac{1}{2p(\omega - \nu)} \left[\int_{\nu}^{\omega} \hbar(x)\, {}_{\nu}d_{p,q}x + \int_{\nu}^{\omega} \hbar(x)\, {}^{\omega}d_{p,q}x\right] \leq \frac{\hbar(\nu) + \hbar(\omega)}{2}. \tag{9}$$

Remark 1.7 It is worth mentioning that inequalities (7)–(9) are generalizations of inequalities (2)–(4), respectively, and for $p = 1$, we can obtain the q-Hermite–Hadamard inequalities.

There has been much research done in the direction of q and (p,q)-integral inequalities for different kinds of convexity. For instance, in [37–40], some new midpoint and trapezoidal inequalities via q and (p,q)-integrals were established. The authors of [41–48] used q and (p,q)-integrals and established Simpson-type inequalities for functions with different forms of convexity. For more recent inequalities in q-calculus, see [49–54].

By considering such advanced level studies we consider the convexity of functions and derive a new variant of Hermite–Hadamard inequality in the setting of (p,q)-calculus. Furthermore, we derive some new midpoint and trapezoidal type inequalities for the special class of functions called (p,q)-differentiable convex functions in the framework of (p,q)-calculus. We also show that our newly established results are an extension of [36], which states the novelty of our research. The results presented here can be helpful in finding the error bounds of numerical integration formulas and variety of mathematical subdisciplines like mathematical programming, control theory, variational methods, operation research, probability, and statistics.

The structure of the paper is as follows. In Sect. 2, we recall some basics of q- and (p,q)-calculus. In Sect. 3, we establish a new variant of q-Hermite–Hadamard-type inequality for some special convex (p,q)-functions. In Sects. 4 and 5, we derive some new midpoint and trapezoidal inequalities for q-differentiable convexity, respectively. Section 6 briefly concludes our work.

2 q- and (p,q)-calculus

We recall some basics of quantum calculus in this section. For $0 < q < p \leq 1$, we denote [38, 55]

$$[n]_q = \frac{1 - q^n}{1 - q}$$

and

$$[n]_{p,q} = \frac{p^n - q^n}{p - q}.$$

Definition 2.1 ([56]) The left or q_ν-derivative of $\hbar : [\nu,\omega] \to \mathbb{R}$ at $x \in [\nu,\omega]$ is defined as

$$ {}_\nu\mathfrak{D}_q \hbar(x) = \frac{\hbar(x) - \hbar(qx + (1-q)\nu)}{(1-q)(x-\nu)}, \quad x \neq \nu. \tag{10} $$

Definition 2.2 ([34]) The right or q^ω-derivative of $\hbar : [\nu,\omega] \to \mathbb{R}$ at $x \in [\nu,\omega]$ is defined as

$$ {}^\omega\mathfrak{D}_q \hbar(x) = \frac{\hbar(qx + (1-q)\omega) - \hbar(x)}{(1-q)(\omega - x)}, \quad x \neq \omega. $$

Definition 2.3 ([57]) The left or $(p,q)_\nu$-derivative of $\hbar : [\nu,\omega] \to \mathbb{R}$ at $x \in [\nu,\omega]$ is defined as

$$ {}_\nu\mathfrak{D}_{p,q} \hbar(x) = \frac{\hbar(px + (1-p)\nu) - \hbar(qx + (1-q)\nu)}{(p-q)(x-\nu)}, \quad x \neq \nu. $$

Definition 2.4 ([38]) The right or $(p,q)^\omega$-derivative of $\hbar : [\nu,\omega] \to \mathbb{R}$ at $x \in [\nu,\omega]$ is defined as

$$ {}^\omega\mathfrak{D}_{p,q} \hbar(x) = \frac{\hbar(qx + (1-q)\omega) - \hbar(px + (1-p)\omega)}{(p-q)(\omega - x)}, \quad x \neq \omega. $$

Definition 2.5 ([56]) The left or q_ν-integral of $\hbar : [\nu,\omega] \to \mathbb{R}$ at $x \in [\nu,\omega]$ is defined as

$$ \int_\nu^x \hbar(t)\, {}_\nu d_q t = (1-q)(x-\nu) \sum_{n=0}^{\infty} q^n \hbar\bigl(q^n x + (1-q^n)\nu\bigr). $$

Definition 2.6 ([34]) The right or q^ω-integral of $\hbar : [\nu,\omega] \to \mathbb{R}$ at $x \in [\nu,\omega]$ is defined as

$$ \int_x^\omega \hbar(t)\, {}^\omega d_q t = (1-q)(\omega - x) \sum_{n=0}^{\infty} q^n \hbar\bigl(q^n x + (1-q^n)\omega\bigr). $$

Definition 2.7 ([57]) The left or $(p,q)_\nu$-integral of $\hbar : [\nu,\omega] \to \mathbb{R}$ is defined as

$$ \int_\nu^x \hbar(t)\, {}_\nu d_{p,q} t = (p-q)(x-\nu) \sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} \hbar\left(\frac{q^n}{p^{n+1}} x + \left(1 - \frac{q^n}{q^{n+1}}\right)\nu \right) $$

for $x \in [\nu, p\omega + (1-p)\nu]$

Definition 2.8 ([38]) The right or q^ω-integral of $\hbar : [\nu,\omega] \to \mathbb{R}$ is defined as

$$ \int_x^\omega \hbar(t)\, {}^\omega d_{p,q} t = (p-q)(\omega - x) \sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} \hbar\left(\frac{q^n}{p^{n+1}} x + \left(1 - \frac{q^n}{p^{n+1}}\right)\omega \right) $$

for $x \in [p\nu + (1-p)\omega, \omega]$

For more properties and details about q- and (p,q)-calculus, the reader can consult [34, 38, 56–58].

3 (p,q)-Hermite–Hadamard inequality

In this section, we establish a new version of Hermite–Hadamard inequality for convex functions and the special (p,q)-operators defined in (p,q)-calculus.

Theorem 3.1 *Let $\hbar:[\nu,\omega]\subset\mathbb{R}\to\mathbb{R}$ be convex. Then*

$$\begin{aligned}\hbar\left(\frac{\nu+\omega}{2}\right) &\le \frac{1}{p(\omega-\nu)}\left[\int_{p\nu+(1-p)\frac{\nu+\omega}{2}}^{\frac{\nu+\omega}{2}} \hbar(x)\ {}^{\frac{\nu+\omega}{2}}d_{p,q}x + \int_{\frac{\nu+\omega}{2}}^{p\omega+(1-p)\frac{\nu+\omega}{2}} \hbar(x)\ {}_{\frac{\nu+\omega}{2}}d_{p,q}x\right] \\ &\le \frac{\hbar(\nu)+\hbar(\omega)}{2}. \end{aligned} \tag{11}$$

Proof The convexity of $\hbar$ implies that

$$\hbar\left(\frac{x+y}{2}\right) \le \frac{1}{2}\left[\hbar(x)+\hbar(y)\right].$$

Setting $x=\frac{1-t}{2}\nu+\frac{1+t}{2}\omega$ and $y=\frac{1+t}{2}\nu+\frac{1-t}{2}\omega$, we get

$$\hbar\left(\frac{\nu+\omega}{2}\right) \le \frac{1}{2}\left[\hbar\left(\frac{1-t}{2}\nu+\frac{1+t}{2}\omega\right)+\hbar\left(\frac{1+t}{2}\nu+\frac{1-t}{2}\omega\right)\right]. \tag{12}$$

By (p,q)-integrating (12) with respect to t on $[0,p]$ we get

$$p\hbar\left(\frac{\nu+\omega}{2}\right) \le \frac{1}{2}\left[\int_0^p \hbar\left(\frac{1-t}{2}\nu+\frac{1+t}{2}\omega\right)d_{p,q}t+\int_0^p \hbar\left(\frac{1+t}{2}\nu+\frac{1-t}{2}\omega\right)d_{p,q}t\right].$$

From Definitions 2.7 and 2.8 we have

$$\hbar\left(\frac{\nu+\omega}{2}\right) \le \frac{1}{p(\omega-\nu)}\left[\int_{p\nu+(1-p)\frac{\nu+\omega}{2}}^{\frac{\nu+\omega}{2}} \hbar(x)\ {}^{\frac{\nu+\omega}{2}}d_{p,q}x + \int_{\frac{\nu+\omega}{2}}^{p\omega+(1-p)\frac{\nu+\omega}{2}} \hbar(x)\ {}_{\frac{\nu+\omega}{2}}d_{p,q}x\right].$$

Thus the first inequality in (11) is proved. We again use the convexity to prove the second inequality in (11):

$$\hbar\left(\frac{1-t}{2}\nu+\frac{1+t}{2}\omega\right)+\hbar\left(\frac{1+t}{2}\nu+\frac{1-t}{2}\omega\right) \le \frac{\hbar(\nu)+\hbar(\omega)}{2}. \tag{13}$$

By (p,q)-integrating (13) with respect to t on $[0,p]$ we get

$$\int_0^p \hbar\left(\frac{1-t}{2}\nu+\frac{1+t}{2}\omega\right)d_{p,q}t+\int_0^p \hbar\left(\frac{1+t}{2}\nu+\frac{1-t}{2}\omega\right)d_{p,q}t \le p\frac{\hbar(\nu)+\hbar(\omega)}{2}.$$

By applying Definitions 2.7 and 2.8 we obtain the required inequality. □

Remark 3.2 By assuming $p=1$ in Theorem 3.1, we regain inequality (5).

Remark 3.3 By setting $p=1$ and taking the limit $q\to 1^-$ we regain the traditional Hermite–Hadamard inequality (1) for the classical convex functions.

Example 3.4 Consider the convex function $\hbar : [0,1] \to \mathbb{R}$ defined as $\hbar(x) = x^2$ with $q = \frac{1}{3}$ and $p = \frac{2}{3}$. Then

$$\hbar\left(\frac{\nu+\omega}{2}\right) = \frac{1}{4}$$

and

$$\frac{\hbar(\nu)+\hbar(\omega)}{2} = \frac{1}{2}.$$

On the other hand, by Definitions 2.7 and 2.8 we have

$$\begin{aligned}
\int_{p\nu+(1-p)\frac{\nu+\omega}{2}}^{\frac{\nu+\omega}{2}} \hbar(x)\ {}^{\frac{\nu+\omega}{2}}d_{p,q}x &= \int_{\frac{1}{6}}^{\frac{1}{2}} x^2\ {}^{\frac{1}{2}}d_{\frac{2}{3},\frac{1}{3}}x \\
&= \frac{1}{3}\cdot\frac{1}{3}\sum_{n=0}^{\infty}\frac{3}{2}\left(\frac{1}{2}\right)^n\left(\frac{3}{2}\left(\frac{1}{2}\right)^n\frac{1}{6}+\left(1-\frac{3}{2}\left(\frac{1}{2}\right)^n\right)\frac{1}{2}\right)^2 \\
&= \frac{1}{24}\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n\left(1-\left(\frac{1}{2}\right)^n\right)^2 \\
&= \frac{1}{24}\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n\left(1-2\left(\frac{1}{2}\right)^n+\left(\frac{1}{2}\right)^{2n}\right) \\
&= \frac{1}{24}\left[2-\frac{8}{3}+\frac{8}{7}\right] \\
&= \frac{5}{252}
\end{aligned}$$

and

$$\begin{aligned}
\int_{\frac{\nu+\omega}{2}}^{p\omega+(1-p)\frac{\nu+\omega}{2}} \hbar(x)\ {}_{\frac{\nu+\omega}{2}}d_{p,q}x &= \int_{\frac{1}{2}}^{\frac{5}{6}} x^2\ {}_{\frac{1}{2}}d_{\frac{2}{3},\frac{1}{3}}x \\
&= \frac{1}{3}\cdot\frac{1}{3}\sum_{n=0}^{\infty}\frac{3}{2}\left(\frac{1}{2}\right)^n\left(\frac{3}{2}\left(\frac{1}{2}\right)^n\frac{5}{6}+\left(1-\frac{3}{2}\left(\frac{1}{2}\right)^n\right)\frac{1}{2}\right)^2 \\
&= \frac{1}{6}\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n\left(\frac{1}{2}+\frac{1}{2}\left(\frac{1}{2}\right)^n\right)^2 \\
&= \frac{1}{24}\left[2+\frac{8}{3}+\frac{8}{7}\right] \\
&= \frac{61}{252}.
\end{aligned}$$

Thus

$$\begin{aligned}
&\frac{1}{p(\omega-\nu)}\left[\int_{p\nu+(1-p)\frac{\nu+\omega}{2}}^{\frac{\nu+\omega}{2}} \hbar(x)\ {}^{\frac{\nu+\omega}{2}}d_{p,q}x + \int_{\frac{\nu+\omega}{2}}^{p\omega+(1-p)\frac{\nu+\omega}{2}} \hbar(x)\ {}_{\frac{\nu+\omega}{2}}d_{p,q}x\right] \\
&\quad = \frac{3}{2}\left[\frac{5}{252}+\frac{61}{252}\right]
\end{aligned}$$

$$= \frac{99}{252}.$$

It is clear that

$$\frac{1}{4} < \frac{99}{252} < \frac{1}{2}.$$

4 (p,q)-Midpoint inequalities

In this section, we establish some new inequalities of midpoint type for (p,q)-differentiable functions in the setting of (p,q)-calculus. We begin with a lemma, which has a great role in establishing the inequalities of this section.

Lemma 4.1 *For $\hbar : [\nu,\omega] \subset \mathbb{R} \to \mathbb{R}$, if ${}_{\nu}\mathfrak{D}_{p,q}\hbar$ and ${}^{\omega}\mathfrak{D}_{p,q}\hbar$ are continuous and integrable mappings over $[\nu,\omega]$, then we have the following identity*:

$$\begin{aligned} &\frac{1}{p(\omega-\nu)}\left[\int_{p\nu+(1-p)\frac{\nu+\omega}{2}}^{\frac{\nu+\omega}{2}} \hbar(x)\,{}^{\frac{\nu+\omega}{2}}d_{p,q}x + \int_{\frac{\nu+\omega}{2}}^{p\omega+(1-p)\frac{\nu+\omega}{2}} \hbar(x)\,{}_{\frac{\nu+\omega}{2}}d_{p,q}x\right] - \hbar\left(\frac{\nu+\omega}{2}\right) \qquad (14) \\ &= \frac{\omega-\nu}{4p^2}\left[\int_0^p (1-qt)\left({}_{\nu}\mathfrak{D}_{p,q}\hbar\left(\frac{p-t}{2p}\nu + \frac{p+t}{2p}\omega\right)\right.\right. \\ &\quad \left.\left. - {}^{\omega}\mathfrak{D}_{p,q}\hbar\left(\frac{p+t}{2p}\nu + \frac{p-t}{2p}\omega\right)\right) d_{p,q}t\right]. \end{aligned}$$

Proof Definitions 2.3 and 2.4 give

$${}^{\omega}\mathfrak{D}_{p,q}\hbar\left(\frac{p+t}{2p}\nu + \frac{p-t}{2p}\omega\right) = 2p\left[\frac{\hbar(\frac{q}{p}t\nu + (1-\frac{q}{p}t)\frac{\nu+\omega}{2}) - \hbar(t\nu + (1-t)\frac{\nu+\omega}{2})}{(p-q)(\omega-\nu)t}\right] \qquad (15)$$

and

$${}_{\nu}\mathfrak{D}_{p,q}\hbar\left(\frac{p-t}{2p}\nu + \frac{p+t}{2p}\omega\right) = 2p\left[\frac{\hbar(t\omega + (1-t)\frac{\nu+\omega}{2}) - \hbar(\frac{q}{p}t\omega + (1-\frac{q}{p}t)\frac{\nu+\omega}{2})}{(p-q)(\omega-\nu)t}\right]. \qquad (16)$$

By Definition 2.8 from (15) we have

$$\begin{aligned} &\int_0^p (1-qt)\,{}^{\omega}\mathfrak{D}_{p,q}\hbar\left(\frac{p+t}{2p}\nu + \frac{p-t}{2p}\omega\right) d_{p,q}t \\ &= \int_0^p (1-qt)p\frac{\hbar(\frac{q}{p}t\nu + (1-\frac{q}{p}t)\frac{\nu+\omega}{2}) - \hbar(t\nu + (1-t)\frac{\nu+\omega}{2})}{(p-q)(\frac{\omega-\nu}{2})t}\,d_{p,q}t \\ &= \frac{2p^2}{\omega-\nu}\left[\sum_{n=0}^{\infty}\hbar\left(\frac{q^{n+1}}{p^{n+1}}\nu + \left(1-\frac{q^{n+1}}{p^{n+1}}\right)\frac{\nu+\omega}{2}\right)\right. \\ &\quad \left. - \sum_{n=0}^{\infty}\hbar\left(\frac{q^n}{p^n}\nu + \left(1-\frac{q^n}{p^n}\right)\frac{\nu+\omega}{2}\right)\right] \\ &\quad - \frac{2p^2q}{\omega-\nu}\left[\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\hbar\left(\frac{q^{n+1}}{p^{n+1}}\nu + \left(1-\frac{q^{n+1}}{p^{n+1}}\right)\frac{\nu+\omega}{2}\right)\right. \\ &\quad \left. - \sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\hbar\left(\frac{q^n}{p^n}\nu + \left(1-\frac{q^n}{p^n}\right)\frac{\nu+\omega}{2}\right)\right] \end{aligned}$$

$$= \frac{2p^2}{\omega-\nu}\left[\hbar\left(\frac{\nu+\omega}{2}\right)-\hbar(\nu)\right]$$
$$-\frac{2p^2q}{\omega-\nu}\left[\frac{1}{q}\sum_{n=0}^{\infty}\frac{q^n}{p^n}\hbar\left(\frac{q^n}{p^n}\nu+\left(1-\frac{q^n}{p^n}\right)\frac{\nu+\omega}{2}\right)-\frac{1}{q}\hbar(\nu)\right.$$
$$\left.-\frac{1}{p}\sum_{n=0}^{\infty}\frac{q^n}{p^n}\hbar\left(\frac{q^n}{p^n}\nu+\left(1-\frac{q^n}{p^n}\right)\frac{\nu+\omega}{2}\right)\right]$$
$$= \frac{2p^2}{\omega-\nu}\hbar\left(\frac{\nu+\omega}{2}\right)-\frac{2p^2q}{\omega-\nu}\left[\frac{p-q}{pq}\sum_{n=0}^{\infty}\frac{q^n}{p^n}\hbar\left(\frac{q^n}{p^n}\nu+\left(1-\frac{q^n}{p^n}\right)\frac{\nu+\omega}{2}\right)\right]$$
$$= \frac{2p^2}{\omega-\nu}\hbar\left(\frac{\nu+\omega}{2}\right)-\frac{4p}{(\omega-\nu)^2}\int_{p\nu+(1-p)\frac{\nu+\omega}{2}}^{\frac{\nu+\omega}{2}}\hbar(x)\,{}^{\frac{\nu+\omega}{2}}d_{p,q}x,$$

which implies that

$$\frac{\omega-\nu}{4p^2}\int_0^p(1-qt)^{\omega}\mathfrak{D}_{p,q}\hbar\left(\frac{p+t}{2p}\nu+\frac{p-t}{2p}\omega\right)d_{p,q}t$$
$$= \frac{1}{2}\hbar\left(\frac{\nu+\omega}{2}\right)-\frac{1}{p(\omega-\nu)}\int_{p\nu+(1-p)\frac{\nu+\omega}{2}}^{\frac{\nu+\omega}{2}}\hbar(x)\,{}^{\frac{\nu+\omega}{2}}d_{p,q}x. \tag{17}$$

Similarly, from Definition 2.7 and relation (16) we have

$$\frac{\omega-\nu}{4p^2}\int_0^p(1-qt)_{\nu}\mathfrak{D}_{p,q}\hbar\left(\frac{p-t}{2p}\nu+\frac{p+t}{2p}\omega\right)d_{p,q}t$$
$$= \frac{1}{p(\omega-\nu)}\int_{\frac{\nu+\omega}{2}}^{p\omega+(1-p)\frac{\nu+\omega}{2}}\hbar(x)\,{}_{\frac{\nu+\omega}{2}}d_{p,q}x-\frac{1}{2}\hbar\left(\frac{\nu+\omega}{2}\right). \tag{18}$$

Thus we derive the required identity (14) by subtracting (17) from (18). □

Remark 4.2 In Lemma 4.1, for $p=1$, we obtain the following identity:

$$\frac{1}{\omega-\nu}\left[\int_{\nu}^{\frac{\nu+\omega}{2}}\hbar(x)\,{}^{\frac{\nu+\omega}{2}}d_qx+\int_{\frac{\nu+\omega}{2}}^{\omega}\hbar(x)\,{}_{\frac{\nu+\omega}{2}}d_qx\right]-\hbar\left(\frac{\nu+\omega}{2}\right)$$
$$= \frac{\omega-\nu}{4}\left[\int_0^1(1-qt)\left({}_{\nu}\mathfrak{D}_q\hbar\left(\frac{1-t}{2}\nu+\frac{1+t}{2}\omega\right)\right.\right.$$
$$\left.\left.-{}^{\omega}\mathfrak{D}_q\hbar\left(\frac{1+t}{2}\nu+\frac{1-t}{2}\omega\right)\right)d_qt\right],$$

which was obtained by Sitthiwirattham et al. [36].

Theorem 4.3 *If Lemma 4.1 holds and* $|{}_{\nu}\mathfrak{D}_{p,q}\hbar|$ *and* $|{}^{\omega}\mathfrak{D}_{p,q}\hbar|$ *are convex, then*

$$\left|\frac{1}{p(\omega-\nu)}\left[\int_{p\nu+(1-p)\frac{\nu+\omega}{2}}^{\frac{\nu+\omega}{2}}\hbar(x)\,{}^{\frac{\nu+\omega}{2}}d_{p,q}x+\int_{\frac{\nu+\omega}{2}}^{p\omega+(1-p)\frac{\nu+\omega}{2}}\hbar(x)\,{}_{\frac{\nu+\omega}{2}}d_{p,q}x\right]-\hbar\left(\frac{\nu+\omega}{2}\right)\right|$$
$$\leq \frac{\omega-\nu}{8p^3}\left[\left(\frac{p^3+p^2q-p^2}{[2]_{p,q}}-p^3q\frac{[3]_{p,q}-[2]_{p,q}}{[2]_{p,q}[3]_{p,q}}\right)\left|{}_{\nu}\mathfrak{D}_{p,q}\hbar(\nu)\right|\right. \tag{19}$$

$$
\begin{aligned}
&+\left(\frac{p^3+p^2q+p^2}{[2]_{p,q}}-p^3q\frac{[3]_{p,q}+[2]_{p,q}}{[2]_{p,q}[3]_{p,q}}\right)\left|{}_{\nu}\mathfrak{D}_{p,q}\hbar(\omega)\right| \\
&+\left(\frac{p^3+p^2q+p^2}{[2]_{p,q}}-p^3q\frac{[3]_{p,q}+[2]_{p,q}}{[2]_{p,q}[3]_{p,q}}\right)\left|{}^{\omega}\mathfrak{D}_{p,q}\hbar(\nu)\right| \\
&+\left(\frac{p^3+p^2q-p^2}{[2]_{p,q}}-p^3q\frac{[3]_{p,q}-[2]_{p,q}}{[2]_{p,q}[3]_{p,q}}\right)\left|{}^{\omega}\mathfrak{D}_{p,q}\hbar(\omega)\right|\Bigg].
\end{aligned}
$$

Proof Taking the modulus in (14), by the convexity properties of $|{}_{\nu}\mathfrak{D}_{p,q}\hbar|$ and $|{}^{\omega}\mathfrak{D}_{p,q}\hbar|$, we estimate

$$
\begin{aligned}
&\left|\frac{1}{p(\omega-\nu)}\left[\int_{p\nu+(1-p)\frac{\nu+\omega}{2}}^{\frac{\nu+\omega}{2}}\hbar(x)\,{}^{\frac{\nu+\omega}{2}}d_{p,q}x+\int_{\frac{\nu+\omega}{2}}^{p\omega+(1-p)\frac{\nu+\omega}{2}}\hbar(x)\,{}_{\frac{\nu+\omega}{2}}d_{p,q}x\right]-\hbar\left(\frac{\nu+\omega}{2}\right)\right| \\
&\leq\frac{\omega-\nu}{4p^2}\left[\int_0^p(1-qt)\left(\left|{}_{\nu}\mathfrak{D}_{p,q}\hbar\left(\frac{p-t}{2p}\nu+\frac{p+t}{2p}\omega\right)\right|\right.\right. \\
&\quad\left.\left.+\left|{}^{\omega}\mathfrak{D}_{p,q}\hbar\left(\frac{p+t}{2p}\nu+\frac{p-t}{2p}\omega\right)\right|\right)d_{p,q}t\right] \\
&\leq\frac{\omega-\nu}{4p^2}\left[\int_0^p(1-qt)\left(\frac{p-t}{2p}\left|{}_{\nu}\mathfrak{D}_{p,q}\hbar(\nu)\right|+\frac{p+t}{2p}\left|{}_{\nu}\mathfrak{D}_{p,q}\hbar(\omega)\right|\right)d_{p,q}t\right. \\
&\quad\left.+\int_0^p(1-qt)\left(\frac{p+t}{2p}\left|{}^{\omega}\mathfrak{D}_{p,q}\hbar(\nu)\right|+\frac{p-t}{2p}\left|{}^{\omega}\mathfrak{D}_{p,q}\hbar(\omega)\right|\right)d_{p,q}t\right] \\
&=\frac{\omega-\nu}{8p^3}\left[\left(\frac{p^3+p^2q-p^2}{[2]_{p,q}}-p^3q\frac{[3]_{p,q}-[2]_{p,q}}{[2]_{p,q}[3]_{p,q}}\right)\left|{}_{\nu}\mathfrak{D}_{p,q}\hbar(\nu)\right|\right. \\
&\quad+\left(\frac{p^3+p^2q+p^2}{[2]_{p,q}}-p^3q\frac{[3]_{p,q}+[2]_{p,q}}{[2]_{p,q}[3]_{p,q}}\right)\left|{}_{\nu}\mathfrak{D}_{p,q}\hbar(\omega)\right| \\
&\quad+\left(\frac{p^3+p^2q+p^2}{[2]_{p,q}}-p^3q\frac{[3]_{p,q}+[2]_{p,q}}{[2]_{p,q}[3]_{p,q}}\right)\left|{}^{\omega}\mathfrak{D}_{p,q}\hbar(\nu)\right| \\
&\quad\left.+\left(\frac{p^3+p^2q-p^2}{[2]_{p,q}}-p^3q\frac{[3]_{p,q}-[2]_{p,q}}{[2]_{p,q}[3]_{p,q}}\right)\left|{}^{\omega}\mathfrak{D}_{p,q}\hbar(\omega)\right|\right],
\end{aligned}
$$

and our proof is completed. □

Remark 4.4 If we set $p=1$ in the previous theorem, then

$$
\begin{aligned}
&\left|\frac{1}{\omega-\nu}\left[\int_{\nu}^{\frac{\nu+\omega}{2}}\hbar(x)\,{}^{\frac{\nu+\omega}{2}}d_qx+\int_{\frac{\nu+\omega}{2}}^{\omega}\hbar(x)\,{}_{\frac{\nu+\omega}{2}}d_qx\right]-\hbar\left(\frac{\nu+\omega}{2}\right)\right| \\
&\leq\frac{\omega-\nu}{8}\left[\left(\frac{q}{[2]_q}-\frac{q^3}{[2]_q[3]_q}\right)\left|{}_{\nu}\mathfrak{D}_q\hbar(\nu)\right|+\left(\frac{2+q}{[2]_q}-q\frac{[3]_q+[2]_q}{[2]_q[3]_q}\right)\left|{}_{\nu}\mathfrak{D}_q\hbar(\omega)\right|\right. \\
&\quad\left.+\left(\frac{2+q}{[2]_q}-q\frac{[3]_q+[2]_q}{[2]_q[3]_q}\right)\left|{}^{\omega}\mathfrak{D}_q\hbar(\nu)\right|+\left(\frac{q}{[2]_q}-\frac{q^3}{[2]_q[3]_q}\right)\left|{}^{\omega}\mathfrak{D}_q\hbar(\omega)\right|\right],
\end{aligned}
$$

which was obtained by Sitthiwirattham et al. [36].

Example 4.5 Consider $\hbar : [0,1] \to \mathbb{R}$ defined by $\hbar(x) = x^3$. Let also $q = \frac{1}{3}$ and $p = \frac{2}{3}$. Then we have the convex functions ${}_{\nu}\mathfrak{D}_{p,q}\hbar(x) = \frac{7x^2}{27}$ and ${}^{\omega}\mathfrak{D}_{p,q}\hbar(x) = \frac{1}{3}(7x^2 + 13x + 7)$, which gives

$$\begin{aligned}
\int_{pa+(1-p)\frac{\nu+\omega}{2}}^{\frac{\nu+\omega}{2}} \hbar(x)\ {}^{\frac{\nu+\omega}{2}}d_{p,q}x &= \int_{\frac{1}{6}}^{\frac{1}{2}} x^3\ {}^{\frac{1}{2}}d_{\frac{2}{3},\frac{1}{3}}x \\
&= \frac{1}{3}\frac{1}{3}\sum_{n=0}^{\infty}\frac{3}{2}\left(\frac{1}{2}\right)^n\left(\frac{3}{2}\left(\frac{1}{2}\right)^n\frac{1}{6} + \left(1 - \frac{3}{2}\left(\frac{1}{2}\right)^n\right)\frac{1}{2}\right)^3 \\
&= \frac{1}{48}\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n\left(1 - \left(\frac{1}{2}\right)^n\right)^3 \\
&= \frac{1}{48}\left[2 - 4 + \frac{24}{7} - \frac{16}{15}\right] \\
&= \frac{19}{2520}
\end{aligned}$$

and

$$\begin{aligned}
\int_{\frac{\nu+\omega}{2}}^{p\omega+(1-p)\frac{\nu+\omega}{2}} \hbar(x)\ {}_{\frac{\nu+\omega}{2}}d_{p,q}x &= \int_{\frac{1}{2}}^{\frac{5}{6}} x^3\ {}_{\frac{1}{2}}d_{\frac{2}{3},\frac{1}{3}}x \\
&= \frac{1}{3}\frac{1}{3}\sum_{n=0}^{\infty}\frac{3}{2}\left(\frac{1}{2}\right)^n\left(\frac{3}{2}\left(\frac{1}{2}\right)^n\frac{5}{6} + \left(1 - \left(\frac{1}{2}\right)^n\right)\frac{1}{2}\right)^3 \\
&= \frac{1}{48}\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n\left(1 + \left(\frac{1}{2}\right)^n\right)^3 \\
&= \frac{1}{48}\left[2 + 4 + \frac{24}{7} + \frac{16}{15}\right] \\
&= \frac{551}{2520}.
\end{aligned}$$

Thus the left-hand side of inequality (19) reduces to

$$\begin{aligned}
&\left|\frac{1}{p(\omega-\nu)}\left[\int_{p\nu+(1-p)\frac{\nu+\omega}{2}}^{\frac{\nu+\omega}{2}} \hbar(x)\ {}^{\frac{\nu+\omega}{2}}d_{p,q}x + \int_{\frac{\nu+\omega}{2}}^{p\omega+(1-p)\frac{\nu+\omega}{2}} \hbar(x)\ {}_{\frac{\nu+\omega}{2}}d_{p,q}x\right] - \hbar\left(\frac{\nu+\omega}{2}\right)\right| \\
&= \left|\frac{3}{2}\left[\frac{18}{2520} + \frac{551}{2520}\right] - \frac{1}{8}\right| \\
&= \frac{9}{42}.
\end{aligned}$$

On the other hand, since ${}_{\nu}\mathfrak{D}_{p,q}\hbar(a) = 0$, $|{}_{\nu}\mathfrak{D}_{p,q}\hbar(\omega)| = \frac{7}{27}$, $|{}^{\omega}\mathfrak{D}_{p,q}\hbar(\nu)| = \frac{7}{3}$, and $|{}^{\omega}\mathfrak{D}_{p,q}\hbar(\omega)| = 9$, The righ-hand side of inequality (19) becomes

$$\begin{aligned}
&\frac{\omega-\nu}{8p^3}\left[\left(\frac{p^3+p^2q-p^2}{[2]_{p,q}} - p^3q\frac{[3]_{p,q}-[2]_{p,q}}{[2]_{p,q}[3]_{p,q}}\right)\left|{}_{\nu}\mathfrak{D}_{p,q}\hbar(\nu)\right|\right. \\
&\quad + \left(\frac{p^3+p^2q+p^2}{[2]_{p,q}} - p^3q\frac{[3]_{p,q}+[2]_{p,q}}{[2]_{p,q}[3]_{p,q}}\right)\left|{}_{\nu}\mathfrak{D}_{p,q}\hbar(\omega)\right| \\
&\quad + \left(\frac{p^3+p^2q+p^2}{[2]_{p,q}} - p^3q\frac{[3]_{p,q}+[2]_{p,q}}{[2]_{p,q}[3]_{p,q}}\right)\left|{}^{\omega}\mathfrak{D}_{p,q}\hbar(\nu)\right|
\end{aligned}$$

$$+\left(\frac{p^3+p^2q-p^2}{[2]_{p,q}}-p^3q\frac{[3]_{p,q}-[2]_{p,q}}{[2]_{p,q}[3]_{p,q}}\right)\left|{}^{\omega}\mathfrak{D}_{p,q}\hbar(\omega)\right|\Bigg]$$
$$=\frac{27}{64}\left[\left(\frac{8}{9}-\frac{128}{567}\right)\frac{7}{27}+\left(\frac{8}{9}-\frac{128}{567}\right)\frac{7}{3}+\left(0+\frac{16}{567}\right)9\right]$$
$$=\frac{472}{567}.$$

It is clear that

$$\frac{9}{42}<\frac{472}{567}.$$

Theorem 4.6 *If Lemma 4.1 holds and $|_{\nu}\mathfrak{D}_{p,q}\hbar|^s$ and $|^{\omega}\mathfrak{D}_{p,q}\hbar|^s$ are convex for $s\geq 1$, then*

$$\left|\frac{1}{p(\omega-\nu)}\left[\int_{p\nu+(1-p)\frac{\nu+\omega}{2}}^{\frac{\nu+\omega}{2}}\hbar(x)\,{}^{\frac{\nu+\omega}{2}}d_{p,q}x+\int_{\frac{\nu+\omega}{2}}^{p\omega+(1-p)\frac{\nu+\omega}{2}}\hbar(x)\,{}_{\frac{\nu+\omega}{2}}d_{p,q}x\right]-\hbar\left(\frac{\nu+\omega}{2}\right)\right|$$
$$\leq\frac{\omega-\nu}{4p^2}\left(\frac{p^2+pq-qp^2}{[2]_{p,q}}\right)^{1-\frac{1}{s}}$$
$$\times\Bigg[\Bigg(\frac{1}{2p}\left(\frac{p^3+p^2q-p^2}{[2]_{p,q}}-p^3q\frac{[3]_{p,q}-[2]_{p,q}}{[2]_{p,q}[3]_{p,q}}\right)\left|{}_{\nu}\mathfrak{D}_{p,q}\hbar(\nu)\right|^s$$
$$+\frac{1}{2p}\left(\frac{p^3+p^2q+p^2}{[2]_{p,q}}-p^3q\frac{[3]_{p,q}+[2]_{p,q}}{[2]_{p,q}[3]_{p,q}}\right)\left|{}_{\nu}\mathfrak{D}_{p,q}\hbar(\omega)\right|^s\Bigg)^{\frac{1}{s}}$$
$$+\Bigg(\frac{1}{2p}\left(\frac{p^3+p^2q+p^2}{[2]_{p,q}}-p^3q\frac{[3]_{p,q}+[2]_{p,q}}{[2]_{p,q}[3]_{p,q}}\right)\left|{}^{\omega}\mathfrak{D}_{p,q}\hbar(\nu)\right|^s$$
$$+\frac{1}{2p}\left(\frac{p^3+p^2q-p^2}{[2]_{p,q}}-p^3q\frac{[3]_{p,q}-[2]_{p,q}}{[2]_{p,q}[3]_{p,q}}\right)\left|{}^{\omega}\mathfrak{D}_{p,q}\hbar(\omega)\right|^s\Bigg)^{\frac{1}{s}}\Bigg].$$

Proof Taking the modulus in (14), by the power mean inequality we have

$$\left|\frac{1}{p(\omega-\nu)}\left[\int_{p\nu+(1-p)\frac{\nu+\omega}{2}}^{\frac{\nu+\omega}{2}}\hbar(x)\,{}^{\frac{\nu+\omega}{2}}d_{p,q}x+\int_{\frac{\nu+\omega}{2}}^{p\omega+(1-p)\frac{\nu+\omega}{2}}\hbar(x)\,{}_{\frac{\nu+\omega}{2}}d_{p,q}x\right]-\hbar\left(\frac{\nu+\omega}{2}\right)\right|$$
$$\leq\frac{\omega-\nu}{4p^2}\Bigg[\int_0^p(1-qt)\Bigg(\left|{}_{\nu}\mathfrak{D}_{p,q}\hbar\left(\frac{p-t}{2p}\nu+\frac{p+t}{2p}\omega\right)\right|$$
$$+\left|{}^{\omega}\mathfrak{D}_{p,q}\hbar\left(\frac{p+t}{2p}\nu+\frac{p-t}{2p}\omega\right)\right|\Bigg)\,d_{p,q}t\Bigg]$$
$$\leq\frac{\omega-\nu}{4p^2}\left(\int_0^p(1-qt)\,d_{p,q}t\right)^{1-\frac{1}{s}}\Bigg[\left(\int_0^p(1-qt)\left|{}_{\nu}\mathfrak{D}_{p,q}\hbar\left(\frac{p-t}{2p}\nu+\frac{p+t}{2p}\omega\right)\right|^s d_{p,q}t\right)^{\frac{1}{s}}$$
$$+\left(\int_0^p(1-qt)\left|{}^{\omega}\mathfrak{D}_{p,q}\hbar\left(\frac{p+t}{2p}\nu+\frac{p-t}{2p}\omega\right)\right|^s d_{p,q}t\right)^{\frac{1}{s}}\Bigg].$$

By the convexity of $|_{\nu}\mathfrak{D}_{p,q}\hbar|^s$ and $|^{\omega}\mathfrak{D}_{p,q}\hbar|^s$ we have

$$\left|\frac{1}{p(\omega-\nu)}\left[\int_{p\nu+(1-p)\frac{\nu+\omega}{2}}^{\frac{\nu+\omega}{2}}\hbar(x)\,{}^{\frac{\nu+\omega}{2}}d_{p,q}x+\int_{\frac{\nu+\omega}{2}}^{p\omega+(1-p)\frac{\nu+\omega}{2}}\hbar(x)\,{}_{\frac{\nu+\omega}{2}}d_{p,q}x\right]-\hbar\left(\frac{\nu+\omega}{2}\right)\right|$$
$$\leq\frac{\omega-\nu}{4p^2}\left(\frac{p^2+pq-qp^2}{[2]_{p,q}}\right)^{1-\frac{1}{s}}$$

$$\times \Bigg[\Bigg(\frac{1}{2p}\bigg(\frac{p^3+p^2q-p^2}{[2]_{p,q}}-p^3q\frac{[3]_{p,q}-[2]_{p,q}}{[2]_{p,q}[3]_{p,q}}\bigg)\big|{}_{\nu}\mathfrak{D}_{p,q}\hbar(\nu)\big|^s$$

$$+\frac{1}{2p}\bigg(\frac{p^3+p^2q+p^2}{[2]_{p,q}}-p^3q\frac{[3]_{p,q}+[2]_{p,q}}{[2]_{p,q}[3]_{p,q}}\bigg)\big|{}_{\nu}\mathfrak{D}_{p,q}\hbar(\omega)\big|^s\Bigg)^{\frac{1}{s}}$$

$$+\Bigg(\frac{1}{2p}\bigg(\frac{p^3+p^2q+p^2}{[2]_{p,q}}-p^3q\frac{[3]_{p,q}+[2]_{p,q}}{[2]_{p,q}[3]_{p,q}}\bigg)\big|{}^{\omega}\mathfrak{D}_{p,q}\hbar(\nu)\big|^s$$

$$+\frac{1}{2p}\bigg(\frac{p^3+p^2q-p^2}{[2]_{p,q}}-p^3q\frac{[3]_{p,q}-[2]_{p,q}}{[2]_{p,q}[3]_{p,q}}\bigg)\big|{}^{\omega}\mathfrak{D}_{p,q}\hbar(\omega)\big|^s\Bigg)^{\frac{1}{s}}\Bigg].$$

The proof is completed. □

Remark 4.7 In Theorem 4.6, for $p = 1$, we have the inequality

$$\left|\frac{1}{\omega-\nu}\left[\int_{\nu}^{\frac{\nu+\omega}{2}}\hbar(x)\,{}^{\frac{\nu+\omega}{2}}d_qx+\int_{\frac{\nu+\omega}{2}}^{\omega}\hbar(x)\,{}_{\frac{\nu+\omega}{2}}d_qx\right]-\hbar\left(\frac{\nu+\omega}{2}\right)\right|$$

$$\leq\frac{\omega-\nu}{4}\left(\frac{1}{[2]_q}\right)^{1-\frac{1}{s}}\Bigg[\Bigg(\frac{1}{2}\bigg(\frac{q}{[2]_q}-\frac{q^3}{[2]_q[3]_q}\bigg)\big|{}_{\nu}\mathfrak{D}_q\hbar(\nu)\big|^s$$

$$+\frac{1}{2}\bigg(\frac{2+q}{[2]_q}-q\frac{[3]_q+[2]_q}{[2]_q[3]_q}\bigg)\big|{}_{\nu}\mathfrak{D}_q\hbar(\omega)\big|^s\Bigg)^{\frac{1}{s}}$$

$$+\Bigg(\frac{1}{2}\bigg(\frac{2+q}{[2]_q}-q\frac{[3]_q+[2]_q}{[2]_q[3]_q}\bigg)\big|{}^{\omega}\mathfrak{D}_q\hbar(\nu)\big|^s+\frac{1}{2}\bigg(\frac{q}{[2]_q}-\frac{q^3}{[2]_q[3]_q}\bigg)\big|{}^{\omega}\mathfrak{D}_q\hbar(\omega)\big|^s\Bigg)^{\frac{1}{s}}\Bigg],$$

which was obtained by Sitthiwirattham et al. [36].

Theorem 4.8 *If Lemma 4.1 holds and $|{}_{\nu}\mathfrak{D}_{p,q}\hbar|^s$ and $|{}^{\omega}\mathfrak{D}_{p,q}\hbar|^s$ are convex for $s > 1$, then*

$$\left|\frac{1}{p(\omega-\nu)}\left[\int_{p\nu+(1-p)\frac{\nu+\omega}{2}}^{\frac{\nu+\omega}{2}}\hbar(x)\,{}^{\frac{\nu+\omega}{2}}d_{p,q}x+\int_{\frac{\nu+\omega}{2}}^{p\omega+(1-p)\frac{\nu+\omega}{2}}\hbar(x)\,{}_{\frac{\nu+\omega}{2}}d_{p,q}x\right]-\hbar\left(\frac{\nu+\omega}{2}\right)\right|$$

$$\leq\frac{\omega-\nu}{4p^2}\left(\frac{1-(1-pq)^{r+1}}{q[r+1]_{p,q}}\right)^{\frac{1}{r}}$$

$$\times\Bigg[\bigg(\frac{p^3+p^2q-p^2}{2p[2]_{p,q}}\big|{}_{\nu}\mathfrak{D}_{p,q}\hbar(\nu)\big|^s+\frac{p^3+p^2q+p^2}{2p[2]_{p,q}}\big|{}_{\nu}\mathfrak{D}_{p,q}\hbar(\omega)\big|^s\bigg)^{\frac{1}{s}}$$

$$+\bigg(\frac{p^3+p^2q+p^2}{2p[2]_{p,q}}\big|{}^{\omega}\mathfrak{D}_{p,q}\hbar(\nu)\big|^s+\frac{p^3+p^2q-p^2}{2p[2]_{p,q}}\big|{}^{\omega}\mathfrak{D}_{p,q}\hbar(\omega)\big|^s\bigg)^{\frac{1}{s}}\Bigg],$$

where $s^{-1}+r^{-1}=1$.

Proof Taking the modulus in (14), by the Hölder inequality we have

$$\left|\frac{1}{p(\omega-\nu)}\left[\int_{p\nu+(1-p)\frac{\nu+\omega}{2}}^{\frac{\nu+\omega}{2}}\hbar(x)\,{}^{\frac{\nu+\omega}{2}}d_{p,q}x+\int_{\frac{\nu+\omega}{2}}^{p\omega+(1-p)\frac{\nu+\omega}{2}}\hbar(x)\,{}_{\frac{\nu+\omega}{2}}d_{p,q}x\right]-\hbar\left(\frac{\nu+\omega}{2}\right)\right|$$

$$\leq\frac{\omega-\nu}{4p^2}\Bigg[\int_0^p(1-qt)\bigg(\bigg|{}_{\nu}\mathfrak{D}_{p,q}\hbar\bigg(\frac{p-t}{2p}\nu+\frac{p+t}{2p}\omega\bigg)\bigg|$$

$$
\begin{aligned}
&+\left|{}^{\omega}\mathfrak{D}_{p,q}\hbar\left(\frac{p+t}{2p}\nu+\frac{p-t}{2p}\omega\right)\right|\Big)\,d_{p,q}t\Bigg]\\
&\le\frac{\omega-\nu}{4p^2}\left(\int_0^p(1-qt)^r\,d_{p,q}t\right)^{\frac{1}{r}}\left[\left(\int_0^p\left|{}_{\nu}\mathfrak{D}_{p,q}\hbar\left(\frac{p-t}{2p}\nu+\frac{p+t}{2p}\omega\right)\right|^s d_{p,q}t\right)^{\frac{1}{s}}\right.\\
&\left.+\left(\int_0^p\left|{}^{\omega}\mathfrak{D}_{p,q}\hbar\left(\frac{p+t}{2p}\nu+\frac{p-t}{2p}\omega\right)\right|^s d_{p,q}t\right)^{\frac{1}{s}}\right].
\end{aligned}
$$

By the convexity of $|{}_{\nu}\mathfrak{D}_{p,q}\hbar|^s$ and $|{}^{\omega}\mathfrak{D}_{p,q}\hbar|^s$ we have

$$
\begin{aligned}
&\left|\frac{1}{p(\omega-\nu)}\left[\int_{p\nu+(1-p)\frac{\nu+\omega}{2}}^{\frac{\nu+\omega}{2}}\hbar(x)\,{}^{\frac{\nu+\omega}{2}}d_{p,q}x+\int_{\frac{\nu+\omega}{2}}^{p\omega+(1-p)\frac{\nu+\omega}{2}}\hbar(x)\,{}_{\frac{\nu+\omega}{2}}d_{p,q}x\right]-\hbar\left(\frac{\nu+\omega}{2}\right)\right|\\
&\le\frac{\omega-\nu}{4p^2}\left(\frac{1-(1-pq)^{r+1}}{q[r+1]_{p,q}}\right)^{\frac{1}{r}}\\
&\times\left[\left(\frac{p^3+p^2q-p^2}{2p[2]_{p,q}}\left|{}_{\nu}\mathfrak{D}_{p,q}\hbar(\nu)\right|^s+\frac{p^3+p^2q+p^2}{2p[2]_{p,q}}\left|{}_{\nu}\mathfrak{D}_{p,q}\hbar(\omega)\right|^s\right)^{\frac{1}{s}}\right.\\
&\left.+\left(\frac{p^3+p^2q+p^2}{2p[2]_{p,q}}\left|{}^{\omega}\mathfrak{D}_{p,q}\hbar(\nu)\right|^s+\frac{p^3+p^2q-p^2}{2p[2]_{p,q}}\left|{}^{\omega}\mathfrak{D}_{p,q}\hbar(\omega)\right|^s\right)^{\frac{1}{s}}\right].
\end{aligned}
$$

□

Remark 4.9 In Theorem 4.8, for $p = 1$, we have the inequality

$$
\begin{aligned}
&\left|\frac{1}{\omega-\nu}\left[\int_{\nu}^{\frac{\nu+\omega}{2}}\hbar(x)\,{}^{\frac{\nu+\omega}{2}}d_qx+\int_{\frac{\nu+\omega}{2}}^{\omega}\hbar(x)\,{}_{\frac{\nu+\omega}{2}}d_qx\right]-\hbar\left(\frac{\nu+\omega}{2}\right)\right|\\
&\le\frac{\omega-\nu}{4}\left(\frac{1-(1-q)^{r+1}}{q[r+1]_q}\right)^{\frac{1}{r}}\left[\left(\frac{q}{2(1+q)}\left|{}_{\nu}\mathfrak{D}_q\hbar(\nu)\right|^s+\frac{2+q}{2(1+q)}\left|{}_{\nu}\mathfrak{D}_q\hbar(\omega)\right|^s\right)^{\frac{1}{s}}\right.\\
&\left.+\left(\frac{2+q}{2(1+q)}\left|{}^{\omega}\mathfrak{D}_q\hbar(\nu)\right|^s+\frac{q}{2(1+q)}\left|{}^{\omega}\mathfrak{D}_q\hbar(\omega)\right|^s\right)^{\frac{1}{s}}\right],
\end{aligned}
$$

which was obtained by Sitthiwirattham et al. [36].

5 (p,q)-Trapezoid inequalities

Now we obtain some (p,q)-trapezoidal inequalities. Let us begin by the following important equality.

Lemma 5.1 *For $\hbar:[\nu,\omega]\subset\mathbb{R}\to\mathbb{R}$, if ${}_{\nu}\mathfrak{D}_{p,q}\hbar$ and ${}^{\omega}\mathfrak{D}_{p,q}\hbar$ are continuous and integrable mappings over $[\nu,\omega]$, then we have the identity*

$$
\begin{aligned}
&\frac{\hbar(\nu)+\hbar(\omega)}{2}-\frac{1}{p(\omega-\nu)}\left[\int_{p\nu+(1-p)\frac{\nu+\omega}{2}}^{\frac{\nu+\omega}{2}}\hbar(x)\,{}^{\frac{\nu+\omega}{2}}d_{p,q}x+\int_{\frac{\nu+\omega}{2}}^{p\omega+(1-p)\frac{\nu+\omega}{2}}\hbar(x)\,{}_{\frac{\nu+\omega}{2}}d_{p,q}x\right]\\
&=\frac{\omega-\nu}{4p^2}\left[\int_0^p(qt)\left({}_{\nu}\mathfrak{D}_{p,q}\hbar\left(\frac{p-t}{2p}\nu+\frac{p+t}{2p}\omega\right)\right.\right.\\
&\left.\left.-{}^{\omega}\mathfrak{D}_{p,q}\hbar\left(\frac{p+t}{2p}\nu+\frac{p-t}{2p}\omega\right)\right)d_{p,q}t\right].
\end{aligned}\tag{20}
$$

Proof By using (15) and Definition 2.8 we get

$$\begin{aligned}
&\int_0^p (qt)^{\omega}\mathfrak{D}_{p,q}\hbar\left(\frac{p+t}{2p}\nu+\frac{p-t}{2p}\omega\right)d_{p,q}t \\
&= \int_0^p (qt)p\frac{\hbar(\frac{q}{p}t\nu+(1-\frac{q}{p}t)\frac{\nu+\omega}{2})-\hbar(t\nu+(1-t)\frac{\nu+\omega}{2})}{(p-q)(\frac{\omega-\nu}{2})t}\,d_{p,q}t \\
&= \frac{2p^2}{\omega-\nu}\left[\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\hbar\left(\frac{q^{n+1}}{p^{n+1}}\nu+\left(1-\frac{q^{n+1}}{p^{n+1}}\right)\frac{\nu+\omega}{2}\right)\right. \\
&\quad \left.-\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\hbar\left(\frac{q^n}{p^n}\nu+\left(1-\frac{q^n}{p^n}\right)\frac{\nu+\omega}{2}\right)\right] \\
&= \frac{2p^2q}{\omega-\nu}\left[\frac{1}{q}\sum_{n=0}^{\infty}\frac{q^n}{p^n}\hbar\left(\frac{q^n}{p^n}\nu+\left(1-\frac{q^n}{p^n}\right)\frac{\nu+\omega}{2}\right)-\frac{1}{q}\hbar(\nu)\right. \\
&\quad \left.-\frac{1}{p}\sum_{n=0}^{\infty}\frac{q^n}{p^n}\hbar\left(\frac{q^n}{p^n}\nu+\left(1-\frac{q^n}{p^n}\right)\frac{\nu+\omega}{2}\right)\right] \\
&= \frac{2p^2}{\omega-\nu}\hbar(\nu)+\frac{2p^2q}{\omega-\nu}\left[\frac{p-q}{pq}\sum_{n=0}^{\infty}\frac{q^n}{p^n}\hbar\left(\frac{q^n}{p^n}\nu+\left(1-\frac{q^n}{p^n}\right)\frac{\nu+\omega}{2}\right)\right] \\
&= -\frac{2p^2}{\omega-\nu}\hbar(\nu)+\frac{4p}{(\omega-\nu)^2}\int_{p\nu+(1-p)\frac{\nu+\omega}{2}}^{\frac{\nu+\omega}{2}}\hbar(x)\,^{\frac{\nu+\omega}{2}}d_{p,q}x.
\end{aligned}$$

This gives

$$\begin{aligned}
&\frac{\omega-\nu}{4p^2}\int_0^p (qt)^{\omega}\mathfrak{D}_{p,q}\hbar\left(\frac{p+t}{2p}\nu+\frac{p-t}{2p}\omega\right)d_{p,q}t \\
&= \frac{1}{p(\omega-\nu)}\int_{p\nu+(1-p)\frac{\nu+\omega}{2}}^{\frac{\nu+\omega}{2}}\hbar(x)\,^{\frac{\nu+\omega}{2}}d_{p,q}x-\frac{1}{2}\hbar(\nu).
\end{aligned} \tag{21}$$

Similarly, by (16) and Definition 2.7 it becomes

$$\begin{aligned}
&\frac{\omega-\nu}{4p^2}\int_0^p (qt)\,_{\nu}\mathfrak{D}_{p,q}\hbar\left(\frac{p-t}{2p}\nu+\frac{p+t}{2p}\omega\right)d_{p,q}t \\
&= \frac{1}{2}\hbar(\omega)-\frac{1}{p(\omega-\nu)}\int_{\frac{\nu+\omega}{2}}^{p\omega+(1-p)\frac{\nu+\omega}{2}}\hbar(x)\,_{\frac{\nu+\omega}{2}}d_{p,q}x.
\end{aligned} \tag{22}$$

Therefore we establish the required identity (20) by equalities (21) and (22). □

Corollary 5.2 *In Lemma 5.1, for $p = 1$, we obtain the new identity*

$$\begin{aligned}
&\frac{\hbar(\nu)+\hbar(\omega)}{2}-\frac{1}{\omega-\nu}\left[\int_{\nu}^{\frac{\nu+\omega}{2}}\hbar(x)\,^{\frac{\nu+\omega}{2}}d_qx+\int_{\frac{\nu+\omega}{2}}^{\omega}\hbar(x)\,_{\frac{\nu+\omega}{2}}d_qx\right] \\
&= \frac{\omega-\nu}{4}\left[\int_0^1 (qt)\left(_{\nu}\mathfrak{D}_q\hbar\left(\frac{1-t}{2}\nu+\frac{1+t}{2}\omega\right)-{}^{\omega}\mathfrak{D}_q\hbar\left(\frac{1+t}{2}\nu+\frac{1-t}{2}\omega\right)\right)d_qt\right].
\end{aligned}$$

This identity helps us to find some estimates of q-trapezoidal inequalities.

Theorem 5.3 *If Lemma 5.1 holds and* $|{}_{\nu}\mathfrak{D}_{p,q}\hbar|$ *and* $|{}^{\omega}\mathfrak{D}_{p,q}\hbar|$ *are convex, then*

$$\left|\frac{\hbar(\nu)+\hbar(\omega)}{2}-\frac{1}{p(\omega-\nu)}\left[\int_{p\nu+(1-p)\frac{\nu+\omega}{2}}^{\frac{\nu+\omega}{2}}\hbar(x)\,{}^{\frac{\nu+\omega}{2}}d_{p,q}x+\int_{\frac{\nu+\omega}{2}}^{p\omega+(1-p)\frac{\nu+\omega}{2}}\hbar(x)\,{}_{\frac{\nu+\omega}{2}}d_{p,q}x\right]\right|$$
$$\leq \frac{q(\omega-\nu)}{8[2]_{p,q}[3]_{p,q}}\Big[\big([3]_{p,q}-[2]_{p,q}\big)\big|{}_{\nu}\mathfrak{D}_{p,q}\hbar(\nu)\big|+\big([3]_{p,q}+[2]_{p,q}\big)\big|{}_{\nu}\mathfrak{D}_{p,q}\hbar(\omega)\big|$$
$$+\big([3]_{p,q}+[2]_{p,q}\big)\big|{}^{\omega}\mathfrak{D}_{p,q}\hbar(\nu)\big|+\big([3]_{p,q}-[2]_{p,q}\big)\big|{}^{\omega}\mathfrak{D}_{p,q}\hbar(\omega)\big|\Big]. \tag{23}$$

Proof By taking the modulus in equality (20) we may write

$$\left|\frac{\hbar(\nu)+\hbar(\omega)}{2}-\frac{1}{p(\omega-\nu)}\left[\int_{p\nu+(1-p)\frac{\nu+\omega}{2}}^{\frac{\nu+\omega}{2}}\hbar(x)\,{}^{\frac{\nu+\omega}{2}}d_{p,q}x+\int_{\frac{\nu+\omega}{2}}^{p\omega+(1-p)\frac{\nu+\omega}{2}}\hbar(x)\,{}_{\frac{\nu+\omega}{2}}d_{p,q}x\right]\right|$$
$$\leq \frac{\omega-\nu}{4p^2}\left[\int_0^p(qt)\left(\left|{}_{\nu}\mathfrak{D}_{p,q}\hbar\left(\frac{p-t}{2p}\nu+\frac{p+t}{2p}\omega\right)\right|\right.\right.$$
$$\left.\left.+\left|{}^{\omega}\mathfrak{D}_{p,q}\hbar\left(\frac{p+t}{2p}\nu+\frac{p-t}{2p}\omega\right)\right|\right)d_{p,q}t\right]. \tag{24}$$

Since the functions $|{}_{\nu}\mathfrak{D}_{p,q}\hbar|$ and $|{}^{\omega}\mathfrak{D}_{p,q}\hbar|$ are convex, we have

$$\left|\frac{\hbar(\nu)+\hbar(\omega)}{2}-\frac{1}{p(\omega-\nu)}\left[\int_{p\nu+(1-p)\frac{\nu+\omega}{2}}^{\frac{\nu+\omega}{2}}\hbar(x)\,{}^{\frac{\nu+\omega}{2}}d_{p,q}x+\int_{\frac{\nu+\omega}{2}}^{p\omega+(1-p)\frac{\nu+\omega}{2}}\hbar(x)\,{}_{\frac{\nu+\omega}{2}}d_{p,q}x\right]\right|$$
$$\leq \frac{\omega-\nu}{4p^2}\left[\int_0^p(qt)\left(\frac{p-t}{2p}\big|{}_{\nu}\mathfrak{D}_{p,q}\hbar(\nu)\big|+\frac{p+t}{2p}\big|{}_{\nu}\mathfrak{D}_{p,q}\hbar(\omega)\big|\right)d_{p,q}t\right.$$
$$\left.+\int_0^p(qt)\left(\frac{p+t}{2p}\big|{}^{\omega}\mathfrak{D}_{p,q}\hbar(\nu)\big|+\frac{p-t}{2p}\big|{}^{\omega}\mathfrak{D}_{p,q}\hbar(\omega)\big|\right)d_{p,q}t\right]$$
$$=\frac{\omega-\nu}{8p^3}\left[\left(p^3q\frac{[3]_{p,q}-[2]_{p,q}}{[2]_{p,q}[3]_{p,q}}\right)\big|{}_{\nu}\mathfrak{D}_{p,q}\hbar(\nu)\big|+\left(p^3q\frac{[3]_{p,q}+[2]_{p,q}}{[2]_{p,q}[3]_{p,q}}\right)\big|{}_{\nu}\mathfrak{D}_{p,q}\hbar(\omega)\big|\right.$$
$$\left.+\left(p^3q\frac{[3]_{p,q}+[2]_{p,q}}{[2]_{p,q}[3]_{p,q}}\right)\big|{}^{\omega}\mathfrak{D}_{p,q}\hbar(\nu)\big|+\left(p^3q\frac{[3]_{p,q}-[2]_{p,q}}{[2]_{p,q}[3]_{p,q}}\right)\big|{}^{\omega}\mathfrak{D}_{p,q}\hbar(\omega)\big|\right].$$

Thus the proof is completed. □

Corollary 5.4 *In Theorem 5.3, for* $p=1$, *we derive the new q-trapezoidal inequality*

$$\left|\frac{\hbar(\nu)+\hbar(\omega)}{2}-\frac{1}{\omega-\nu}\left[\int_{\nu}^{\frac{\nu+\omega}{2}}\hbar(x)\,{}^{\frac{\nu+\omega}{2}}d_qx+\int_{\frac{\nu+\omega}{2}}^{\omega}\hbar(x)\,{}_{\frac{\nu+\omega}{2}}d_qx\right]\right|$$
$$\leq\frac{q(\omega-\nu)}{8[2]_q[3]_q}\Big[q^2\big|{}_{\nu}\mathfrak{D}_q\hbar(\nu)\big|+\big([3]_q+[2]_q\big)\big|{}_{\nu}\mathfrak{D}_q\hbar(\omega)\big|$$
$$+\big([3]_q+[2]_q\big)\big|{}^{\omega}\mathfrak{D}_q\hbar(\nu)\big|+q^2\big|{}^{\omega}\mathfrak{D}_q\hbar(\omega)\big|\Big].$$

Example 5.5 Consider $\hbar:[0,1]\to\mathbb{R}$ defined by $\hbar(x)=x^3$. Let also $q=\frac{1}{3}$ and $p=\frac{2}{3}$. Then we have the convex functions ${}_{\nu}\mathfrak{D}_{p,q}\hbar(x)=\frac{7x^2}{27}$ and ${}^{\omega}\mathfrak{D}_{p,q}\hbar(x)=\frac{1}{3}(7x^2+13x+7)$. So the left-hand side of (23) can be written as

$$\left|\frac{\hbar(\nu)+\hbar(\omega)}{2}-\frac{1}{p(\omega-\nu)}\left[\int_{p\nu+(1-p)\frac{\nu+\omega}{2}}^{\frac{\nu+\omega}{2}}\hbar(x)\,{}^{\frac{\nu+\omega}{2}}d_{p,q}x+\int_{\frac{\nu+\omega}{2}}^{p\omega+(1-p)\frac{\nu+\omega}{2}}\hbar(x)\,{}_{\frac{\nu+\omega}{2}}d_{p,q}x\right]\right|$$

$$= \left|\frac{1}{2} - \frac{3}{2}\left[\frac{18}{2520} + \frac{551}{2520}\right]\right|$$
$$= \frac{271}{1680}.$$

On the other hand, since ${}_{\nu}\mathfrak{D}_{p,q}\hbar(\nu) = 0$, $|{}_{\nu}\mathfrak{D}_{p,q}\hbar(\omega)| = \frac{7}{27}$, $|{}^{\omega}\mathfrak{D}_{p,q}\hbar(\nu)| = \frac{7}{3}$, and $|{}^{\omega}\mathfrak{D}_{p,q}\hbar(\omega)| = 9$, the right-hand side of inequality (23) becomes

$$\frac{q(\omega-\nu)}{8[2]_{p,q}[3]_{p,q}}\Big[\big([3]_{p,q}-[2]_{p,q}\big)\big|{}_{\nu}\mathfrak{D}_{p,q}\hbar(\nu)\big| + \big([3]_{p,q}+[2]_{p,q}\big)\big|{}_{\nu}\mathfrak{D}_{p,q}\hbar(\omega)\big|$$
$$+ \big([3]_{p,q}+[2]_{p,q}\big)\big|{}^{\omega}\mathfrak{D}_{p,q}\hbar(\nu)\big| + \big([3]_{p,q}-[2]_{p,q}\big)\big|{}^{\omega}\mathfrak{D}_{p,q}\hbar(\omega)\big|\Big]$$
$$= \frac{3}{56}\left[\left(\frac{7}{9}+1\right)\frac{7}{27} + \left(\frac{7}{9}+1\right)\frac{7}{3} + \left(\frac{7}{9}-1\right)9\right]$$
$$= \frac{3}{56}\left[\frac{16}{9}\frac{7}{27} + \frac{16}{9}\frac{7}{3} - 2\right]$$
$$= \frac{3}{56}\left[\frac{16}{9}\frac{7}{27} + \frac{16}{9}\frac{7}{3} - 2\right]$$
$$= \frac{877}{4536}.$$

It is clear that

$$\frac{271}{1680} < \frac{877}{4536}.$$

Theorem 5.6 *If Lemma 5.1 holds and $|{}_{\nu}\mathfrak{D}_{p,q}\hbar|^s$ and $|{}^{\omega}\mathfrak{D}_{p,q}\hbar|^s$ are convex for $s \geq 1$, then*

$$\left|\frac{\hbar(\nu)+\hbar(\omega)}{2} - \frac{1}{p(\omega-\nu)}\left[\int_{p\nu+(1-p)\frac{\nu+\omega}{2}}^{\frac{\nu+\omega}{2}} \hbar(x)\ {}^{\frac{\nu+\omega}{2}}d_{p,q}x + \int_{\frac{\nu+\omega}{2}}^{p\omega+(1-p)\frac{\nu+\omega}{2}} \hbar(x)\ {}_{\frac{\nu+\omega}{2}}d_{p,q}x\right]\right|$$
$$\leq \frac{q(\omega-\nu)}{4p^2[2]_{p,q}}$$
$$\times\left[\left(\left(\frac{[3]_{p,q}-[2]_{p,q}}{2[3]_{p,q}}\right)\big|{}_{\nu}\mathfrak{D}_{p,q}\hbar(\nu)\big|^s + \left(\frac{[3]_{p,q}+[2]_{p,q}}{2[3]_{p,q}}\right)\big|{}_{\nu}\mathfrak{D}_{p,q}\hbar(\omega)\big|^s\right)^{\frac{1}{s}}\right.$$
$$\left.+\left(\left(\frac{[3]_{p,q}+[2]_{p,q}}{2[3]_{p,q}}\right)\big|{}^{\omega}\mathfrak{D}_{p,q}\hbar(\nu)\big|^s + \left(\frac{[3]_{p,q}-[2]_{p,q}}{2[3]_{p,q}}\right)\big|{}^{\omega}\mathfrak{D}_{p,q}\hbar(\omega)\big|^s\right)^{\frac{1}{s}}\right].$$

Proof In (24), by the power mean inequality we get

$$\left|\frac{\hbar(\nu)+\hbar(\omega)}{2} - \frac{1}{p(\omega-\nu)}\left[\int_{p\nu+(1-p)\frac{\nu+\omega}{2}}^{\frac{\nu+\omega}{2}} \hbar(x)\ {}^{\frac{\nu+\omega}{2}}d_{p,q}x + \int_{\frac{\nu+\omega}{2}}^{p\omega+(1-p)\frac{\nu+\omega}{2}} \hbar(x)\ {}_{\frac{\nu+\omega}{2}}d_{p,q}x\right]\right|$$
$$\leq \frac{\omega-\nu}{4p^2}\left(\int_0^p (qt)\,d_{p,q}t\right)^{1-\frac{1}{s}}\left[\left(\int_0^p (qt)\left|{}_{\nu}\mathfrak{D}_{p,q}\hbar\left(\frac{p-t}{2p}\nu + \frac{p+t}{2p}\omega\right)\right|^s d_{p,q}t\right)^{\frac{1}{s}}\right.$$
$$\left.+\left(\int_0^p (qt)\left|{}^{\omega}\mathfrak{D}_{p,q}\hbar\left(\frac{p+t}{2p}\nu + \frac{p-t}{2p}\omega\right)\right|^s d_{p,q}t\right)^{\frac{1}{s}}\right].$$

By the convexity of the functions $|_{\nu}\mathfrak{D}_{p,q}\hbar|^s$ and $|^{\omega}\mathfrak{D}_{p,q}\hbar|^s$ we obtain

$$\begin{aligned}&\left|\frac{\hbar(\nu)+\hbar(\omega)}{2}-\frac{1}{p(\omega-\nu)}\left[\int_{p\nu+(1-p)\frac{\nu+\omega}{2}}^{\frac{\nu+\omega}{2}}\hbar(x)\,{}^{\frac{\nu+\omega}{2}}d_{p,q}x+\int_{\frac{\nu+\omega}{2}}^{p\omega+(1-p)\frac{\nu+\omega}{2}}\hbar(x)\,{}_{\frac{\nu+\omega}{2}}d_{p,q}x\right]\right|\\&\leq\frac{\omega-\nu}{4p^2}\left(\frac{qp^2}{[2]_{p,q}}\right)^{1-\frac{1}{s}}\\&\quad\times\left[\left(\frac{1}{2p}\left(p^3q\frac{[3]_{p,q}-[2]_{p,q}}{[2]_{p,q}[3]_{p,q}}\right)\left|_{\nu}\mathfrak{D}_{p,q}\hbar(\nu)\right|^s\right.\right.\\&\quad\left.+\frac{1}{2p}\left(p^3q\frac{[3]_{p,q}+[2]_{p,q}}{[2]_{p,q}[3]_{p,q}}\right)\left|_{\nu}\mathfrak{D}_{p,q}\hbar(\omega)\right|^s\right)^{\frac{1}{s}}\\&\quad+\left(\frac{1}{2p}\left(p^3q\frac{[3]_{p,q}+[2]_{p,q}}{[2]_{p,q}[3]_{p,q}}\right)\left|^{\omega}\mathfrak{D}_{p,q}\hbar(\nu)\right|^s\right.\\&\quad\left.\left.+\frac{1}{2p}\left(p^3q\frac{[3]_{p,q}-[2]_{p,q}}{[2]_{p,q}[3]_{p,q}}\right)\left|^{\omega}\mathfrak{D}_{p,q}\hbar(\omega)\right|^s\right)^{\frac{1}{s}}\right],\end{aligned}$$

which completes the proof. □

Corollary 5.7 *For $p = 1$ in Theorem 5.6, we derive the new q-trapezoidal inequality*

$$\begin{aligned}&\left|\frac{\hbar(\nu)+\hbar(\omega)}{2}-\frac{1}{\omega-\nu}\left[\int_{\nu}^{\frac{\nu+\omega}{2}}\hbar(x)\,{}^{\frac{\nu+\omega}{2}}d_qx+\int_{\frac{\nu+\omega}{2}}^{\omega}\hbar(x)\,{}_{\frac{\nu+\omega}{2}}d_qx\right]\right|\\&\leq\frac{q(\omega-\nu)}{4[2]_q}\left[\left(\frac{q^2|_{\nu}\mathfrak{D}_q\hbar(\nu)|^s+([3]_q+[2]_q)|_{\nu}\mathfrak{D}_q\hbar(\omega)|^s}{2[3]_q}\right)^{\frac{1}{s}}\right.\\&\quad\left.+\left(\frac{([3]_q+[2]_q)|^{\omega}\mathfrak{D}_q\hbar(\nu)|^s+q^2|^{\omega}\mathfrak{D}_q\hbar(\omega)|^s}{2[3]_q}\right)^{\frac{1}{s}}\right].\end{aligned}$$

Theorem 5.8 *If Lemma 5.1 holds and $|_{\nu}\mathfrak{D}_{p,q}\hbar|^s$ and $|^{\omega}\mathfrak{D}_{p,q}\hbar|^s$ are convex for $s > 1$, then*

$$\begin{aligned}&\left|\frac{\hbar(\nu)+\hbar(\omega)}{2}-\frac{1}{p(\omega-\nu)}\left[\int_{p\nu+(1-p)\frac{\nu+\omega}{2}}^{\frac{\nu+\omega}{2}}\hbar(x)\,{}^{\frac{\nu+\omega}{2}}d_{p,q}x+\int_{\frac{\nu+\omega}{2}}^{p\omega+(1-p)\frac{\nu+\omega}{2}}\hbar(x)\,{}_{\frac{\nu+\omega}{2}}d_{p,q}x\right]\right|\\&\leq\frac{\omega-\nu}{4p^2}\left(\frac{(qp)^{r+1}}{q[r+1]_{p,q}}\right)^{\frac{1}{r}}\\&\quad\times\left[\left(\frac{p^3+p^2q-p^2}{2p[2]_{p,q}}\left|_{\nu}\mathfrak{D}_{p,q}\hbar(\nu)\right|^s+\frac{p^3+p^2q+p^2}{2p[2]_{p,q}}\left|_{\nu}\mathfrak{D}_{p,q}\hbar(\omega)\right|^s\right)^{\frac{1}{s}}\right.\\&\quad\left.+\left(\frac{p^3+p^2q+p^2}{2p[2]_{p,q}}\left|^{\omega}\mathfrak{D}_{p,q}\hbar(\nu)\right|^s+\frac{p^3+p^2q-p^2}{2p[2]_{p,q}}\left|^{\omega}\mathfrak{D}_{p,q}\hbar(\omega)\right|^s\right)^{\frac{1}{s}}\right],\end{aligned}$$

where $s^{-1} + r^{-1} = 1$.

Proof Applying the Hölder inequality to (24), we establish

$$\left|\frac{\hbar(\nu)+\hbar(\omega)}{2}-\frac{1}{p(\omega-\nu)}\left[\int_{p\nu+(1-p)\frac{\nu+\omega}{2}}^{\frac{\nu+\omega}{2}}\hbar(x)\,{}^{\frac{\nu+\omega}{2}}d_{p,q}x+\int_{\frac{\nu+\omega}{2}}^{p\omega+(1-p)\frac{\nu+\omega}{2}}\hbar(x)\,{}_{\frac{\nu+\omega}{2}}d_{p,q}x\right]\right|$$

$$\leq \frac{\omega-\nu}{4p^2}\left(\int_0^p (qt)^r\,d_{p,q}t\right)^{\frac{1}{r}}\left[\left(\int_0^p \left|{}_\nu\mathfrak{D}_{p,q}\hbar\left(\frac{p-t}{2p}\nu+\frac{p+t}{2p}\omega\right)\right|^s d_{p,q}t\right)^{\frac{1}{s}}\right.$$
$$\left.+\left(\int_0^p \left|{}^\omega\mathfrak{D}_{p,q}\hbar\left(\frac{p+t}{2p}\nu+\frac{p-t}{2p}\omega\right)\right|^s d_{p,q}t\right)^{\frac{1}{s}}\right].$$

Since the functions $|_\nu\mathfrak{D}_{p,q}\hbar|^s$ and $|^\omega\mathfrak{D}_{p,q}\hbar|^s$ are convex, we have

$$\left|\frac{\hbar(\nu)+\hbar(\omega)}{2}-\frac{1}{p(\omega-\nu)}\left[\int_{p\nu+(1-p)\frac{\nu+\omega}{2}}^{\frac{\nu+\omega}{2}}\hbar(x)\,{}^{\frac{\nu+\omega}{2}}d_{p,q}x+\int_{\frac{\nu+\omega}{2}}^{p\omega+(1-p)\frac{\nu+\omega}{2}}\hbar(x)\,{}_{\frac{\nu+\omega}{2}}d_{p,q}x\right]\right|$$
$$\leq \frac{\omega-\nu}{4p^2}\left(\frac{(qp)^{r+1}}{q[r+1]_{p,q}}\right)^{\frac{1}{r}}$$
$$\times\left[\left(\frac{p^3+p^2q-p^2}{2p[2]_{p,q}}\left|{}_\nu\mathfrak{D}_{p,q}\hbar(\nu)\right|^s+\frac{p^3+p^2q+p^2}{2p[2]_{p,q}}\left|{}_\nu\mathfrak{D}_{p,q}\hbar(\omega)\right|^s\right)^{\frac{1}{s}}\right.$$
$$\left.+\left(\frac{p^3+p^2q+p^2}{2p[2]_{p,q}}\left|{}^\omega\mathfrak{D}_{p,q}\hbar(\nu)\right|^s+\frac{p^3+p^2q-p^2}{2p[2]_{p,q}}\left|{}^\omega\mathfrak{D}_{p,q}\hbar(\omega)\right|^s\right)^{\frac{1}{s}}\right].$$

The proof is ended. □

Corollary 5.9 *For $p = 1$ in Theorem 5.8, we derive the new q-trapezoidal inequality*

$$\left|\frac{\hbar(\nu)+\hbar(\omega)}{2}-\frac{1}{\omega-\nu}\left[\int_\nu^{\frac{\nu+\omega}{2}}\hbar(x)\,{}^{\frac{\nu+\omega}{2}}d_qx+\int_{\frac{\nu+\omega}{2}}^{\omega}\hbar(x)\,{}_{\frac{\nu+\omega}{2}}d_qx\right]\right|$$
$$\leq \frac{\omega-\nu}{4}\left(\frac{q^r}{[r+1]_q}\right)^{\frac{1}{r}}\left[\left(\frac{q}{2(1+q)}\left|{}_\nu\mathfrak{D}_q\hbar(\nu)\right|^s+\frac{2+q}{2(1+q)}\left|{}_\nu\mathfrak{D}_q\hbar(\omega)\right|^s\right)^{\frac{1}{s}}\right.$$
$$\left.+\left(\frac{2+q}{2(1+q)}\left|{}^\omega\mathfrak{D}_q\hbar(\nu)\right|^s+\frac{q}{2(1+q)}\left|{}^\omega\mathfrak{D}_q\hbar(\omega)\right|^s\right)^{\frac{1}{s}}\right].$$

6 Conclusions

In the present research work, we analyzed a new variant of Hermite–Hadamard inequality in relation to convex functions in the framework of (p,q)-calculus. Moreover, we derived some new estimates for (p,q)-midpoint and (p,q)-trapezoidal inequalities for (p,q)-differentiable convex functions using the left and right (p,q)-integrals. The upcoming researchers can obtain similar inequalities for different kinds of convexity and coordinated convexity in the context of (p,q)-calculus theory in their future research works.

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Author contributions
T.S. and M.A.A. and H.B. and S.E. dealt with the conceptualization, supervision, methodology, investigation, and writing-original draft preparation. T.S. and M.A.A. and H.B. and S.E. and S.R. made the formal analysis, writing-review, editing. All authors read and approved the final manuscript.

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References

1. Kirmaci, U.S.: Inequalities for differentiable mappings and applications to special means of real numbers and to midpoint formula. Appl. Math. Comput. **147**(1), 137–146 (2004). https://doi.org/10.1016/S0096-3003(02)00657-4
2. Dragomir, S.S., Agarwal, R.P.: Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula. Appl. Math. Lett. **11**(5), 91–95 (1998). https://doi.org/10.1016/S0893-9659(98)00086-X
3. Dragomir, S.S.: Inequalities of Hermite–Hadamard type for *h*-convex functions on linear spaces. Proyecciones J. Math. **34**(4), 323–341 (2015). https://doi.org/10.4067/S0716-09172015000400002
4. Alomari, M.W., Darus, M., Kirmaci, U.S.: Some inequalities of Hermite–Hadamard type for *s*-convex functions. Acta Math. Sci. **31**(4), 1643–1652 (2011). https://doi.org/10.1016/S0252-9602(11)60350-0
5. Samet, B.: A convexity concept with respect to a pair of functions. Numer. Funct. Anal. Optim. **43**(5), 522–540 (2022). https://doi.org/10.1080/01630563.2022.2050753
6. Tunc, M.: On new inequalities for *h*-convex functions via Riemann–Liouville fractional integration. Filomat **27**(4), 559–565 (2013). https://doi.org/10.2298/FIL1304559T
7. Sarikaya, M.Z., Yildrim, H.: On Hermite–Hadamard type inequalities for Riemann–Liouville fractional integrals. Miskolc Math. Notes **17**(2), 1049–1059 (2016)
8. Matar, M.M., Abbas, M.I., Alzabut, J., Kaabar, M.K.A., Etemad, S., Rezapour, S.: Investigation of the p-Laplacian nonperiodic nonlinear boundary value problem via generalized Caputo fractional derivatives. Adv. Differ. Equ. **2021**, 68 (2021). https://doi.org/10.1186/s13662-021-03228-9
9. Omame, A., Abbas, M., Abdel-Aty, A.: Assessing the impact of SARS-CoV-2 infection on the dynamics of Dengue and HIV via fractional derivatives. Chaos Solitons Fractals **162**, 112427 (2022). https://doi.org/10.1016/j.chaos.2022.112427
10. Omame, A., Abbas, M., Onyenegecha, C.P.: Backward bifurcation and optimal control in a co-infection model for SARS-CoV-2 and ZIKV. Results Phys. **37**, 105481 (2022). https://doi.org/10.1016/j.rinp.2022.105481
11. Mohammadi, H., Kumar, S., Rezapour, S., Etemad, S.: A theoretical study of the Caputo–Fabrizio fractional modeling for hearing loss due to Mumps virus with optimal control. Chaos Solitons Fractals **144**, 110668 (2021). https://doi.org/10.1016/j.chaos.2021.110668
12. Etemad, S., Avci, I., Kumar, P., Baleanu, D., Rezapour, S.: Some novel mathematical analysis on the fractal–fractional model of the AH1N1/09 virus and its generalized Caputo-type version. Chaos Solitons Fractals **162**, 112511 (2022). https://doi.org/10.1016/j.chaos.2022.112511
13. Kumar, P., Erturk, V.S., Abboubakar, H., Nisar, K.S.: Prediction studies of the epidemic peak of coronavirus disease in Brazil via new generalised Caputo type fractional derivatives. Alex. Eng. J. **60**(3), 3189–3204 (2021). https://doi.org/10.1016/j.aej.2021.01.032
14. Baleanu, D., Etemad, S., Rezapour, S.: On a fractional hybrid integro-differential equation with mixed hybrid integral boundary value conditions by using three operators. Alex. Eng. J. **59**(5), 3019–3027 (2020). https://doi.org/10.1016/j.aej.2020.04.053
15. Nabi, K.N., Abboubakar, H., Kumar, P.: Forecasting of COVID-19 pandemic: from integer derivatives to fractional derivatives. Chaos Solitons Fractals **141**, 110283 (2020). https://doi.org/10.1016/j.chaos.2020.110283
16. Rezapour, S., Etemad, S., Mohammadi, H.: A mathematical analysis of a system of Caputo–Fabrizio fractional differential equations for the anthrax disease model in animals. Adv. Differ. Equ. **2020**, 481 (2020). https://doi.org/10.1186/s13662-020-02937-x
17. Wang, Y., Wang, X.: The evolution of immersed locally convex plane curves driven by anisotropic curvature flow. Adv. Nonlinear Anal. **12**(1), 117–131 (2023). https://doi.org/10.1515/anona-2022-0245

18. Eiter, T., Hopf, K., Lasarzik, R.: Weak-strong uniqueness and energy-variational solutions for a class of viscoelastoplastic fluid models. Adv. Nonlinear Anal. **12**(1), 20220274 (2023). https://doi.org/10.1515/anona-2022-0274
19. Thabet, S.T.M., Etemad, S., Rezapour, S.: On a coupled Caputo conformable system of pantograph problems. Turk. J. Math. **45**(1), 496–519 (2021). https://doi.org/10.3906/mat-2010-70
20. Baleanu, D., Etemad, S., Rezapour, S.: A hybrid Caputo fractional modeling for thermostat with hybrid boundary value conditions. Bound. Value Probl. **2020**, 64 (2020). https://doi.org/10.1186/s13661-020-01361-0
21. Baleanu, D., Mohammadi, H., Rezapour, S.: Analysis of the model of HIV-1 infection of $CD4^+$ T-cell with a new approach of fractional derivative. Adv. Differ. Equ. **2020**, 71 (2020). https://doi.org/10.1186/s13662-020-02544-w
22. Baleanu, D., Mohammadi, H., Rezapour, S.: Mathematical theoretical study of a particular system of Caputo–Fabrizio fractional differential equations for the Rubella disease model. Adv. Differ. Equ. **2020**, 184 (2020). https://doi.org/10.1186/s13662-020-02614-z
23. Baleanu, D., Jajarmi, A., Mohammadi, H., Rezapour, S.: A new study on the mathematical modelling of human liver with Caputo–Fabrizio fractional derivative. Chaos Solitons Fractals **134**, 109705 (2020). https://doi.org/10.1016/j.chaos.2020.109705
24. Rezapour, S., Kumar, S., Iqbal, M.Q., Hussain, A., Etemad, S.: On two abstract Caputo multi-term sequential fractional boundary value problems under the integral conditions. Math. Comput. Simul. **194**, 365–382 (2022). https://doi.org/10.1016/j.matcom.2021.11.018
25. Ahmad, B., Nieto, J.J., Alsaedi, A., Al-Hutami, H.: Boundary value problems of nonlinear fractional q-difference (integral) equations with two fractional orders and four-point nonlocal integral boundary conditions. Filomat **28**(8), 1719–1736 (2014). https://doi.org/10.2298/FIL1408719A
26. Ren, J., Zhai, C.: A fractional q-difference equation with integral boundary conditions and comparison theorem. Int. J. Nonlinear Sci. Numer. Simul. **18**(7–8), 575–583 (2017). https://doi.org/10.1515/ijnsns-2017-0056
27. Rezapour, S., Imran, A., Hussain, A., Martinez, F., Etemad, S., Kaabar, M.K.A.: Condensing functions and approximate endpoint criterion for the existence analysis of quantum integro-difference FBVPs. Symmetry **13**(3), 469 (2021). https://doi.org/10.3390/sym13030469
28. Abdeljawad, T., Alzabut, J.: On Riemann–Liouville fractional q-difference equations and their application to retarded logistic type model. Math. Methods Appl. Sci. **41**(18), 8953–8962 (2018). https://doi.org/10.1002/mma.4743
29. Etemad, S., Ntouyas, S.K., Ahmad, B.: Existence theory for a fractional q-integro-difference equation with q-integral boundary conditions of different orders. Mathematics **7**(8), 659 (2019). https://doi.org/10.3390/math7080659
30. Zhang, L., Sun, S.: Existence and uniqueness of solutions for mixed fractional q-difference boundary value problems. Bound. Value Probl. **2019**, 100 (2019). https://doi.org/10.1186/s13661-019-1215-z
31. Neang, P., Nonlaopon, K., Tariboon, J., Ntouyas, S.K., Ahmad, B.: Existence and uniqueness results for fractional (p,q)-difference equations with separated boundary conditions. Mathematics **10**(5), 767 (2022). https://doi.org/10.3390/math10050767
32. Kamsrisuk, N., Promsakon, C., Ntouyas, S.K., Tariboon, J.: Nonlocal boundary value problems for (p,q)-difference equations. Differ. Equ. Appl. **10**(2), 183–195 (2018). https://doi.org/10.7153/dea-2018-10-11
33. Alp, N., Sarikaya, M.Z., Kunt, M., İşcan, I.: q-Hermite Hadamard inequalities and quantum estimates for midpoint type inequalities via convex and quasi-convex functions. J. King Saud Univ., Sci. **30**(2), 193–203 (2018). https://doi.org/10.1016/j.jksus.2016.09.007
34. Bermudo, S., Kórus, P., Valdés, J.N.: On q-Hermite–Hadamard inequalities for general convex functions. Acta Math. Hung. **162**, 364–374 (2020). https://doi.org/10.1007/s10474-020-01025-6
35. Ali, M.A., Budak, M.F.H., Khan, S.: A new version of q-Hermite–Hadamard's midpoint and trapezoid type inequalities for convex functions. Math. Slovaca (2022, in press)
36. Sitthiwirattham, T., Ali, M.A., Ali, A., Budak, H.: A new q-Hermite–Hadamard's inequality and estimates for midpoint type inequalities for convex functions. Miskolc Math. Notes (2022, in press)
37. Kunt, M., İşcan, I., Alp, N., Sarikaya, M.Z.: (p,q)-Hermite–Hadamard inequalities and (p,q)-estimates for midpoint inequalities via convex quasi-convex functions. Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat. **112**, 969–992 (2018)
38. Vivas-Cortez, M., Ali, M.A., Budak, H., Kalsoom, H., Agarwal, P.: Some new Hermite–Hadamard and related inequalities for convex functions via (p,q)-integral. Entropy **23**(7), 828 (2021). https://doi.org/10.3390/e23070828
39. Budak, H.: Some trapezoid and midpoint type inequalities for newly defined quantum integrals. Proyecciones **40**(1), 199–215 (2021). https://doi.org/10.22199/issn.0717-6279-2021-01-0013
40. Latif, M.A., Kunt, M., Dragomir, S.S., İşcan, I.: Post-quantum trapezoid type inequalities. AIMS Math. **5**(4), 4011–4026 (2020). https://doi.org/10.3934/math.2020258
41. Ali, M.A., Budak, H., Zhang, Z., Yildrim, H.: Some new Simpson's type inequalities for co-ordinated convex functions in quantum calculus. Math. Methods Appl. Sci. **44**(6), 4515–4540 (2021). https://doi.org/10.1002/mma.7048
42. Ali, M.A., Abbas, M., Budak, H., Agarwal, P., Murtaza, G., Chu, Y.M.: New quantum boundaries for quantum Simpson's and quantum Newton's type inequalities for preinvex functions. Adv. Differ. Equ. **2021**, 64 (2021). https://doi.org/10.1186/s13662-021-03226-x
43. Budak, H., Erden, S., Ali, M.A.: Simpson and Newton type inequalities for convex functions via newly defined quantum integrals. Math. Methods Appl. Sci. **44**(1), 378–390 (2021). https://doi.org/10.1002/mma.6742
44. Sial, I.B., Mei, S., Ali, M.A., Nonlaopon, K.: On some generalized Simpson's and Newton's inequalities for (α,m)-convex functions in q-calculus. Mathematics **9**(24), 3266 (2021). https://doi.org/10.3390/math9243266
45. Soontharanon, J., Ali, M.A., Budak, H., Nonlaopon, K., Abdullah, Z.: Simpson's and Newton's type inequalities for (α,m)-convex functions via quantum calculus. Symmetry **14**(4), 736 (2022). https://doi.org/10.3390/sym14040736
46. Luangboon, W., Nonlaopon, K., Tariboon, J., Ntouyas, S.K.: Simpson- and Newton-type inequalities for convex functions via (p,q)-calculus. Mathematics **9**(12), 1338 (2021). https://doi.org/10.3390/math9121338
47. Vivas-Cortez, M.J., Ali, M.A., Qaisar, S., Sial, I.B., Jansem, S., Mateen, A.: On some new Simpson's formula type inequalities for convex functions in post-quantum calculus. Symmetry **13**(12), 2419 (2021). https://doi.org/10.3390/sym13122419
48. Luangboon, W., Nonlaopon, K., Tariboon, J., Ntouyas, S.K.: On Simpson type inequalities for generalized strongly preinvex functions via (p,q)-calculus and applications. AIMS Math. **6**(9), 9236–9261 (2021). https://doi.org/10.3934/math.2021537
49. Sudsutad, W., Ntouyas, S.K., Tariboon, J.: Quantum integral inequalities for convex functions. J. Math. Inequal. **9**(3), 781–793 (2015). https://doi.org/10.7153/jmi-09-64

50. Zhuang, H., Liu, W., Park, J.: Some quantum estimates of Hermite–Hadamard inequalities for quasi-convex functions. Mathematics **7**(2), 152 (2019). https://doi.org/10.3390/math7020152
51. Gauchman, H.: Integral inequalities in q-calculus. Comput. Math. Appl. **47**(2–3), 281–300 (2004). https://doi.org/10.1016/S0898-1221(04)90025-9
52. Zhang, Y., Du, T.S., Wang, H., Shen, Y.J.: Different types of quantum integral inequalities via (α, m)-convexity. J. Inequal. Appl. **2018**, 264 (2018). https://doi.org/10.1186/s13660-018-1860-2
53. Nwaeze, E.R., Tameru, A.M.: New parameterized quantum integral inequalities via η-quasiconvexity. Adv. Differ. Equ. **2019**, 425 (2019). https://doi.org/10.1186/s13662-019-2358-z
54. Awan, M.U., Talib, S., Noor, M.A., Noor, K.I., Chu, Y.M.: On post quantum integral inequalities. J. Math. Inequal. **15**(2), 629–654 (2021). https://doi.org/10.7153/jmi-2021-15-46
55. Kac, V., Cheung, P.: Quantum Calculus. Springer, New York (2001)
56. Tariboon, J., Ntouyas, S.K.: Quantum calculus on finite intervals and applications to impulsive difference equations. Adv. Differ. Equ. **2013**, 282 (2013). https://doi.org/10.1186/1687-1847-2013-282
57. Tunc, M., Gov, E.: Some integral inequalities via (p,q)-calculus on finite intervals. Filomat **35**(5), 1421–1430 (2021). https://doi.org/10.2298/FIL2105421T
58. Jackson, F.H.: On a q-definite integrals. Q. J. Pure Appl. Math. **41**(15), 193–203 (1910)