Sun and He *Boundary Value Problems* (2022) 2022:82
https://doi.org/10.1186/s13661-022-01663-5

Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

Scattering for 2D quasilinear wave equations with null conditions

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Abstract

We study the scattering theory of solutions to the quasilinear wave equations with null conditions and small initial data in two dimensions. Based on the scattering profile that was described precisely in He–Liu–Wang (J. Differ. Equ. 269(4):3067–3088, 2020), we establish precise estimates for the difference of the solution and the scattering profile. Therefore, collecting the results in this paper and those in He–Liu–Wang (J. Differ. Equ. 269(4):3067–3088, 2020), we have given a basically systematic study on the long-time behavior of a small data solution to the 2D wave equation with null conditions.

MSC: 35L05; 35L15; 35L72

Keywords: Quasilinear wave equation; Null conditions; Two dimensions; Long-time behavior; Scattering

1 Introduction and main results

1.1 Introduction

We consider the Cauchy problem for the two-dimensional quasilinear wave equations with null conditions:

$$\begin{cases} \partial_t^2 u - \Delta u = g^{jk}(\partial u)\partial_{jk}^2 u, & (t,x) \in \mathbb{R}_+ \times \mathbb{R}^2, \\ u(0,x) = \varepsilon u_0(x), & \partial_t u(0,x) = \varepsilon u_1(x), \end{cases} \tag{1.1}$$

where we used the convention that repeated upper and lower indices are summed over $j,k = 0,1,2$ and $\partial_0 = \partial_t$, $\partial_i = \partial_{x_i}$, $i = 1,2$. Without loss of generality, we impose the symmetry constraint on the coefficients g^{jk} as follows:

$$g^{jk}(\eta) = g^{kj}(\eta), \quad \forall j,k = 0,1,2,\ \forall \eta \in \mathbb{R}^3. \tag{1.2}$$

Furthermore, we restrict g^{jk} to be smooth real functions vanishing at the origin, and hence

$$g^{jk}(\eta) = g^{jk,l}\eta_l + h^{jk,lm}\eta_l\eta_m + r^{jk}(\eta), \qquad r^{jk}(\eta) = O\big(|\eta|^3\big), \tag{1.3}$$

where $g^{jk,l}$ and $h^{jk,lm}$ are constants, and r^{jk} are smooth functions. Since our work is based on the global well-posedness theory of the system (1.1) by Alinhac [2], we assume that

(u_0, u_1) are smooth functions with compact supports. To proceed further, we define $g(\omega) = g^{jk,l}\omega_j\omega_k\omega_l$, $h(\omega) = h^{jk,lm}\omega_j\omega_k\omega_l\omega_m$, where $\omega = (\omega_0, w_1, w_2) \in \mathbb{R}^{1+2}$ and $-\omega_0^2 + \omega_1^2 + \omega_2^2 = 0$, then we introduce both null conditions from [2]

$$g(\omega) \equiv 0, \qquad h(\omega) \equiv 0. \tag{1.4}$$

For simplicity of presentation, we assume that

$$r^{jk} \equiv 0, \qquad \operatorname{supp}(u_0, u_1) \subseteq B_1 = \{x \in \mathbb{R}^2 : |x| \le 1\}. \tag{1.5}$$

Let us denote

$$\sigma = r - t, \qquad x = r(\cos\varphi, \sin\varphi), \qquad r = |x| = \sqrt{x_1^2 + x_2^2},$$

and it is well known that for the solution u_L of 2D linear wave equations with smooth and compactly supported initial data (u_0, u_1), there exists a smooth function $F = F(\sigma, \varphi)$ such that for the arbitrary multiindex I

$$\left|\partial^\alpha \Gamma^I \left(u_L(t,x) - \frac{1}{\sqrt{r}} F(\sigma,\varphi)\right)\right| \le C_{\alpha,I}(1+t)^{-\frac{3}{2}}\left(1+|\sigma|\right)^{\frac{1}{2}}, \quad \text{if } r \ge \frac{t}{2} > 1, \tag{1.6}$$

where Γ^I is a product of order $|I|$ of the Klainerman vector fields:

$$S = t\partial_t + r\partial_r, \qquad \Omega = x_1\partial_{x_2} - x_2\partial_{x_1}, \qquad \partial_{x_j}, \qquad \partial_t, \quad j = 1, 2, \tag{1.7}$$

and F is the so-called Friedlander radiation field, see Theorem 6.2.1 of [9]. In the case $n = 2$ and the null condition $g(\omega) \equiv 0$ is satisfied, Alinhac [1] introduced the slow time variable $s = \varepsilon^2 \ln(1 + t)$ and made the ansatz (see [1, 2]), which indicates that the solution admits the asymptotic expansion

$$u(t,x) \approx \frac{\varepsilon}{\sqrt{r}} U(s, \sigma, \varphi), \tag{1.8}$$

for some function U and σ, φ defined as above. Moreover, if both null conditions (1.4) are satisfied, then the global existence was established by Alinhac in [2], one can also refer to [3] and [10]. More precisely, for $\varepsilon > 0$ small enough and any integer $N \ge 7$, the global solution u of (1.1) satisfies

$$\begin{aligned} E_N(t) &= \sum_{|I|\le N} \left\|\partial\Gamma^I u(t,\cdot)\right\|_{L^2(\mathbb{R}^2)} \le C\varepsilon(1+t)^{c\varepsilon^2}, \\ E_{N-2}(t) &= \sum_{|I|\le N-2} \left\|\partial\Gamma^I u(t,\cdot)\right\|_{L^2(\mathbb{R}^2)} \le C\varepsilon. \end{aligned} \tag{1.9}$$

Furthermore, for $|I| \le \frac{N}{2}$, one has the linear decay estimate as follows:

$$\left|\partial\Gamma^I u(t,x)\right| \le \varepsilon(1+t)^{-\frac{1}{2}}\left(1+|r-t|\right)^{-\frac{1}{2}}. \tag{1.10}$$

Recently, He, Liu and Wang [8] obtained a result about the asymptotic behavior similar to (1.6):

Proposition 1.1 *Suppose that the conditions* (1.3)–(1.5) *are satisfied. Let u be the solution of* (1.1) *satisfying* (1.9) *and* (1.10), *then there exists a function* $U = U(s,\sigma,\varphi)$ *satisfying*

$$\left\| \partial_\sigma U(s,\sigma,\varphi) \right\|_{L^\infty_\sigma} \lesssim \varepsilon, \qquad \partial_s \partial_\sigma U(s,\sigma,\varphi) = O\left(\varepsilon^2 e^{-\frac{1}{2}s}\right), \tag{1.11}$$

such that u satisfies the asymptotic behavior

$$\sup_{|I| \le \frac{N}{2}} \left| \partial \partial^I \left(u(t,x) - \frac{1}{\sqrt{r}} U(\ln t, r-t, \varphi) \right) \right| \lesssim \varepsilon t^{-\frac{3}{2}+}, \tag{1.12}$$

for $r \ge \frac{t}{2} > 1$, *where* $r = |x|$ *and* $x/|x| = (\cos\varphi, \sin\varphi)$.

1.2 Main results

In this paper, we give a complete description of the asymptotic behavior of u. More specifically, we establish estimates of $\partial(u - U/\sqrt{r})$ under the action of general Klainerman vector fields. The main result is the following:

Theorem 1.2 *Suppose that the conditions* (1.3)–(1.5) *are satisfied. Let u be the solution of* (1.1) *satisfying* (1.9) *and* (1.10). *Then, for the function* $U = U(s,\sigma,\varphi)$ *in Proposition* 1.1 *and* $\Gamma \in \{\Omega, S\}$, *u satisfies the asymptotic behavior*

$$\sup_{|I| \le \frac{N}{2}} \left| \partial \Gamma^I \left(u(t,x) - \frac{1}{\sqrt{r}} U(\ln t, r-t, \varphi) \right) \right| \lesssim \varepsilon t^{-\frac{3}{2}+}, \tag{1.13}$$

for $r \ge \frac{t}{2} > 1$, *where* $r = |x|$ *and* $x/|x| = (\cos\varphi, \sin\varphi)$.

Remark 1.3 By delicate analysis, we show that $\partial\Gamma^I u$ admits representation similar to that of ∂u (see (3.18) in [8]). This observation allows us to control $\partial\Gamma^I(u - U/\sqrt{r})$ by the estimates of the corresponding profile function $f(\Gamma^I u)$. Now, the key point here is how to understand the relationship between the profile $f(\Gamma^I u)$ and the action of Γ^I on $f(u)$. Fortunately, we are able to write $f(\Gamma^I u)$ in terms of $\Gamma^J(f(u))$, $|J| \le |I|$ with a delicate inductive argument. Finally, we establish the improvement from (1.12) to (1.13) by some delicate computations, which generalizes the conclusion in Proposition 1.1.

This paper is organized as follows: In the next section we state the main idea of the proof and list some necessary a priori estimates. In Sect. 3 we utilize the a priori estimates together with the asymptotic expansion of the oscillatory integral to establish the expected estimates.

2 Preliminaries

We first give a sketch of the proof. For the real-valued solution u of (1.1), we set

$$f := e^{it|\nabla|}\left(\partial_t - i|\nabla|\right)u, \tag{2.1}$$

where f is called the profile function for u and satisfies the following equality

$$\partial_t f = e^{it|\nabla|}\left(\partial_t + i|\nabla|\right)\left(\partial_t - i|\nabla|\right)u = e^{it|\nabla|}(\Box u). \tag{2.2}$$

By (2.1), we can write the solution as

$$u = -\frac{\mathrm{Im}(e^{-it|\nabla|}f)}{|\nabla|} = -\frac{1}{2\pi}\,\mathrm{Im}\int_{\mathbb{R}^2} e^{ix\cdot\xi}\,\frac{e^{-it|\xi|}\widehat{f}(t,\xi)}{|\xi|}\,\mathrm{d}\xi. \tag{2.3}$$

Then, using the polar coordinate on the phase plane, for fixed $x \in \mathbb{R}^2$, we denote $\frac{x}{|x|} = (\cos\varphi, \sin\varphi)$, $\rho = |\xi|$ and $\frac{\xi}{|\xi|} = (\cos(\theta+\varphi), \sin(\theta+\varphi))$, thus

$$u(t,x) = -\frac{1}{2\pi}\,\mathrm{Im}\int_0^\infty e^{i\rho(r-t)}\,\mathrm{d}\rho \int_{-\pi}^{\pi} e^{ir\rho(\cos\theta-1)}\widehat{f}\big(t, \rho\cos(\theta+\varphi), \rho\sin(\theta+\varphi)\big)\,\mathrm{d}\theta. \tag{2.4}$$

This kind of representation in (2.4) appeared in [6], one can also refer to [4], [5], [7], and [8]. For every fixed ρ, the inner integral with respect to θ in (2.4) is an oscillatory one. By the advantage of the property of this oscillatory integral, the authors of [8] constructed an approximate profile function U as the following:

$$U(s,\sigma,\varphi) = -\,\mathrm{Im}\int_0^\infty e^{i\rho\sigma}\frac{\widehat{f}(e^s, \rho\cos\varphi, \rho\sin\varphi)}{\sqrt{2\pi i\rho}}\,\mathrm{d}\rho, \tag{2.5}$$

and the approximate estimate (1.12) was obtained. In this paper, we apply delicate computation to show that $\partial\Gamma^I u$ admits similar representation, we shall explain how our analysis relies on the explicit properties of the approximate profile together with the inductive method. Then, by some crucial observations of the structure of the representation formula, we manage to control the decay speed of the error term in $\partial\Gamma^I(u - U/\sqrt{r})$, and (1.13) follows immediately. For this aim, we first recall some necessary a priori estimates from [8].

Lemma 2.1 *If f is the profile function defined by* (2.1), *then for* $\Gamma \in \{\partial, \Omega, S\}$ *and each* I, $|I| \le \frac{N}{2}$

$$\big\|\widehat{\Gamma^I\partial_t f}(t,\cdot)\big\|_{L^2_\xi} \lesssim \varepsilon^2(1+t)^{-\frac{3}{2}}, \tag{2.6}$$

and

$$\big|\widehat{\Gamma^I\partial_t f}(t,\xi)\big| \lesssim \varepsilon^2\langle\xi\rangle^{-15}(1+t)^{-1}\ln(1+t). \tag{2.7}$$

Furthermore,

$$\big\|\widehat{\Gamma^I f}(t,\cdot)\big\|_{L^2_\xi} \lesssim \varepsilon, \tag{2.8}$$

and

$$\big|\widehat{\Gamma^I f}(t,\xi)\big| \lesssim \varepsilon\langle\xi\rangle^{-15}\big(\ln(1+t)\big)^2. \tag{2.9}$$

3 Proof of Theorem 1.2

In this part we prove (1.13). Based on the results in [8], we utilize the idea of formula (2.4) and some delicate computations to show that for general Klainerman vector fields, the estimate in (1.12) could be generalized to (1.13).

In the proof of (1.13) we only deal with the case $\partial = \partial_t$, for that the case $\partial = \partial_x$ can be treated completely analogously.

3.1 $\Gamma = \Omega$

For a function $v = v(t,x)$, we can define a map:

$$v \longmapsto f(v) = e^{it|\nabla|}\big(\partial_t - i|\nabla|\big)v.$$

Hence,

$$\partial_t v(t,x) = \frac{1}{2\pi}\operatorname{Re}\int_{\mathbb{R}^2} e^{ix\cdot\xi - it|\xi|}\widehat{f(v)}(t,\xi)\,\mathrm{d}\xi. \tag{3.1}$$

On the other hand, a direct computation shows that for any index I,

$$\Omega^I f = \Omega^I\big(e^{it|\nabla|}\big(\partial_t - i|\nabla|\big)u\big) = e^{it|\nabla|}\big(\partial_t - i|\nabla|\big)\Omega^I u = f\big(\Omega^I u\big). \tag{3.2}$$

Thus,

$$\begin{aligned}
&\partial_t\big(\Omega^I u\big)(t,x)\\
&\quad= \operatorname{Re}\big(e^{-it|\nabla|} f\big(\Omega^I u\big)\big)(t,x)\\
&\quad= \frac{1}{2\pi}\operatorname{Re}\int_{\mathbb{R}^2} e^{i(x\cdot\xi - t|\xi|)}\widehat{\Omega^I f}(t,\xi)\,\mathrm{d}\xi\\
&\quad= \frac{1}{2\pi}\operatorname{Re}\int_0^\infty e^{i\rho(r-t)}\rho\,\mathrm{d}\rho\int_{-\pi}^{\pi} e^{ir\rho(\cos\theta-1)}\widehat{\Omega^I f}\big(t,\rho\cos(\theta+\varphi)\rho\sin(\theta+\varphi)\big)\,\mathrm{d}\theta.
\end{aligned} \tag{3.3}$$

Recall that in Lemma 2.1, $\widehat{f}$ and $\widehat{\Omega^I f}$ admit the same estimate, thus the last line of (3.3) has the same property as the formula (3.18) in [8]. In fact,

$$\begin{aligned}
&\left|\int_0^{t^{-1+\delta}} e^{i\rho(r-t)}\rho\,\mathrm{d}\rho\int_{-\pi}^{\pi} e^{ir\rho(\cos\theta-1)}\widehat{\Omega^I f}\big(t,\rho\cos(\theta+\varphi),\rho\sin(\theta+\varphi)\big)\,\mathrm{d}\theta\right|\\
&\quad\le \int_0^{t^{-1+\delta}}\big|\widehat{\Omega^I f}\big(t,\rho\cos(\theta+\varphi),\rho\sin(\theta+\varphi)\big)\big|\rho\,\mathrm{d}\rho\\
&\quad\lesssim \int_0^{t^{-1+\delta}} \varepsilon\langle\rho\rangle^{-15}\big(\ln(1+t)\big)^2\rho\,\mathrm{d}\rho \lesssim \varepsilon(1+t)^{-2+2\delta},
\end{aligned} \tag{3.4}$$

where δ is a positive constant that could be chosen arbitrarily small, hence we have

$$\begin{aligned}
&\partial_t\big(\Omega^I u\big)(t,x)\\
&\quad= \frac{1}{2\pi}\operatorname{Re}\int_{t^{-1+\delta}}^\infty e^{i\rho(r-t)}\rho\,\mathrm{d}\rho\int_{-\pi}^{\pi} e^{ir\rho(\cos\theta-1)}\widehat{\Omega^I f}\big(t,\rho\cos(\theta+\varphi),\rho\sin(\theta+\varphi)\big)\,\mathrm{d}\theta\\
&\qquad+ O\big(\varepsilon(1+t)^{-2+2\delta}\big).
\end{aligned} \tag{3.5}$$

For the remaining part, where $\rho \ge t^{-1+\delta}$, we have $\lambda = r\rho \ge t^\delta \to \infty$ as $t \to \infty$. Consequently, we will apply the asymptotic expansion in [11] to the inner integral

$$\int_{-\pi}^{\pi} e^{ir\rho(\cos\theta-1)}\widehat{\Omega^I f}\big(t,\rho\cos(\theta+\varphi),\rho\sin(\theta+\varphi)\big)\,\mathrm{d}\theta.$$

For this aim, we use the cut-off function $\chi \in C^\infty(\mathbb{R})$ defined as follows

$$\chi(y) = \begin{cases} 1, & -1 \le y \le 1, \\ 0, & |y| \ge 2, \end{cases} \tag{3.6}$$

and rewrite (3.5) as

$$\begin{aligned}
&\partial_t\big(\Omega^I u\big)(t,x) \\
&\quad = \frac{1}{2\pi} \operatorname{Re} \int_{t^{-1+\delta}}^{\infty} e^{i\rho(r-t)} \rho \,\mathrm{d}\rho \int_{-\pi}^{\pi} e^{ir\rho(\cos\theta-1)} \chi\left(\frac{10\theta}{\pi}\right) \\
&\qquad \times \widehat{\Omega^I f}\big(t, \rho\cos(\theta+\varphi), \rho\sin(\theta+\varphi)\big) \,\mathrm{d}\theta \\
&\qquad + \frac{1}{2\pi} \operatorname{Re} \int_{t^{-1+\delta}}^{\infty} e^{i\rho(r-t)} \rho \,\mathrm{d}\rho \int_{-\pi}^{\pi} e^{ir\rho(\cos\theta-1)} \left(1 - \chi\left(\frac{10\theta}{\pi}\right)\right) \\
&\qquad \times \widehat{\Omega^I f}\big(t, \rho\cos(\theta+\varphi), \rho\sin(\theta+\varphi)\big) \,\mathrm{d}\theta \\
&\qquad + O\big(\varepsilon(1+t)^{-2+2\delta}\big) \\
&\quad =: I + II + O\big(\varepsilon(1+t)^{-2+2\delta}\big).
\end{aligned} \tag{3.7}$$

As for the principal term I, we apply Stein's lemma of an oscillatory integral (see Sect. 8.1.3 of [11], one can also refer to Lemma 3.1 of [8]) to obtain

$$I = \frac{1}{2\pi} \operatorname{Re} \int_{t^{-1+\delta}}^{\infty} e^{i\rho(r-t)} \left(\sqrt{\frac{2\pi}{ir\rho}} \widehat{\Omega^I f}(t, \rho\cos\varphi, \rho\sin\varphi) + O\big(A(r\rho)^{-\frac{3}{2}}\big) \right) \rho \,\mathrm{d}\rho,$$

where

$$|A| \lesssim \sum_{|J|\le 3} \big\| \widehat{\Omega^J \Omega^I f}(t, \rho\cos\varphi, \rho\sin\varphi) \big\|_{L^\infty_\varphi} \lesssim \varepsilon\big(\ln(1+t)\big)^2,$$

hence,

$$\begin{aligned}
I &= \frac{1}{\sqrt{r}} \operatorname{Re} \int_{t^{-1+\delta}}^{\infty} e^{i\rho(r-t)} \sqrt{\frac{\rho}{2\pi i}} \widehat{\Omega^I f}(t, \rho\cos\varphi, \rho\sin\varphi) \,\mathrm{d}\rho + O\big(\varepsilon(1+t)^{-\frac{3}{2}+2\delta}\big) \\
&= \frac{1}{\sqrt{r}} \operatorname{Re} \int_{0}^{\infty} e^{i\rho(r-t)} \sqrt{\frac{\rho}{2\pi i}} \widehat{\Omega^I f}(t, \rho\cos\varphi, \rho\sin\varphi) \,\mathrm{d}\rho + O\big(\varepsilon(1+t)^{-\frac{3}{2}+2\delta}\big).
\end{aligned}$$

For the remaining term II in (3.7), one can integrate with respect to ρ, then by the relation $\rho\partial_\rho \widehat{f} = t\partial_t \widehat{f} - \widehat{Sf} - 2\widehat{f}$,

$$\begin{aligned}
II &= \frac{1}{2\pi} \operatorname{Re} \int_{-\pi}^{\pi} \left(1 - \chi\left(\frac{10\theta}{\pi}\right)\right) \mathrm{d}\theta \int_{t^{-1+\delta}}^{\infty} \frac{\partial_\rho e^{i\rho(r\cos\theta - t)}}{i(r\cos\theta - r)} \\
&\quad \times \widehat{\Omega^I f}\big(t, \rho\cos(\theta+\varphi), \rho\sin(\theta+\varphi)\big) \rho \,\mathrm{d}\rho \\
&= \frac{1}{2\pi} \operatorname{Re} \int_{t^{-1+\delta}}^{\infty} e^{-it\rho} \,\mathrm{d}\rho \int_{-\pi}^{\pi} \frac{e^{ir\rho\cos\theta}}{i(t - r\cos\theta)} \left(1 - \chi\left(\frac{10\theta}{\pi}\right)\right) \big(\widehat{t\partial_t \Omega^I f} - \widehat{\Omega^I f} - \widehat{S\Omega^I f}\big) \,\mathrm{d}\theta \\
&= \frac{1}{2\pi} \operatorname{Re} \int_{t^{-1+\delta}}^{\infty} e^{-i\rho(t+r)} \,\mathrm{d}\rho \int_{-\pi}^{\pi} \frac{e^{ir\rho(\cos\theta+1)}}{i(t - r\cos\theta)} \left(1 - \chi\left(\frac{10\theta}{\pi}\right)\right) \chi\left(\frac{10(\pi-\theta)}{\pi}\right)
\end{aligned}$$

$$
\begin{aligned}
&\times \big(\widehat{t\partial_t \Omega^I f} - \widehat{\Omega^I f} - \widehat{S\Omega^I f}\big)\,\mathrm{d}\theta \\
&+ \frac{1}{2\pi}\operatorname{Re}\int_{t^{-1+\delta}}^{\infty} e^{-it\rho}\,\mathrm{d}\rho \int_{-\pi}^{\pi} \frac{e^{ir\rho\cos\theta}}{i(t-r\cos\theta)}\left(1-\chi\left(\frac{10\theta}{\pi}\right)\right)\left(1-\chi\left(\frac{10(\pi-\theta)}{\pi}\right)\right) \\
&\times \big(\widehat{t\partial_t \Omega^I f} - \widehat{\Omega^I f} - \widehat{S\Omega^I f}\big)\,\mathrm{d}\theta \\
=:&\, II_1 + II_2.
\end{aligned}
$$

For II_1 we can apply Lemma 3.1 in [8] for $\phi(\theta) = \cos\theta + 1$ and $V = \{\theta : |\theta - \pi| \le \frac{\pi}{5}\}$ to derive

$$
\begin{aligned}
II_1 = \frac{1}{2\pi}\operatorname{Re}\int_{t^{-1+\delta}}^{\infty} e^{-i\rho(t+r)}\Bigg(&\sqrt{\frac{2\pi}{ir\rho}}\frac{1}{i(t+r)}\big(\widehat{t\partial_t\Omega^I f} - \widehat{\Omega^I f} - \widehat{S\Omega^I f}\big)(t,\rho\cos\varphi,\rho\sin\varphi) \\
&+ O\big(B(r\rho)^{-\frac{3}{2}}\big)\Bigg)\,\mathrm{d}\rho,
\end{aligned}
$$

where

$$
|B| \lesssim \sum_{|J|\le 3}\left\|\Omega^J\left(\frac{\widehat{t\partial_t\Omega^I f} - \widehat{\Omega^I f} - \widehat{S\Omega^I f}}{t - r\cos\theta}\right)\right\|_{L^\infty_\theta} \lesssim \varepsilon\frac{(\ln(1+t))^2}{1+t},
$$

thus a direct computation derives $II_1 = O(\varepsilon(1+t)^{-\frac{3}{2}+2\delta})$. For II_2, if the integrand of II_2 does not vanish, then $|\sin\theta| \ge \sin\frac{\pi}{10} > 0$, thus one can integrate by parts with respect to θ to obtain

$$
|II_2| \lesssim \int_{t^{-1+\delta}}^{\infty} \frac{1}{\rho r(t - r\cos\frac{\pi}{10})}\sum_{|J|\le 1}\big\|\Omega^J\big(\widehat{t\partial_t\Omega^I f} - \widehat{\Omega^I f} - \widehat{S\Omega^I f}\big)\big\|_{L^\infty_\varphi}\,\mathrm{d}\rho \lesssim O\big(\varepsilon(1+t)^{-2+2\delta}\big).
$$

Therefore, we obtain that

$$
\begin{aligned}
&\partial_t\big(\Omega^I u\big)(t,x) \\
&\quad= \frac{1}{\sqrt{r}}\operatorname{Re}\int_0^\infty e^{i\rho(r-t)}\sqrt{\frac{\rho}{2\pi i}}\widehat{\Omega^I f}(t,\rho\cos\varphi,\rho\sin\varphi)\,\mathrm{d}\rho + O\big(\varepsilon(1+t)^{-\frac{3}{2}+2\delta}\big).
\end{aligned} \tag{3.8}
$$

On the other hand, by (2.5) we have,

$$
\begin{aligned}
\partial_t\Omega^I\left(\frac{1}{\sqrt{r}}U(\ln t, r-t,\varphi)\right) &= \partial_t\left(-\frac{1}{\sqrt{r}}\operatorname{Im}\int_0^\infty e^{i\rho(r-t)}\frac{\widehat{\Omega^I f}(t,\rho\cos\varphi,\rho\sin\varphi)}{\sqrt{2\pi i\rho}}\,\mathrm{d}\rho\right) \\
&= \frac{1}{\sqrt{r}}\operatorname{Re}\int_0^\infty e^{i\rho(r-t)}\sqrt{\frac{\rho}{2\pi i}}\widehat{\Omega^I f}(t,\rho\cos\varphi,\rho\sin\varphi)\,\mathrm{d}\rho \\
&\quad - \frac{1}{\sqrt{r}}\operatorname{Im}\int_0^\infty e^{i\rho(r-t)}\frac{\widehat{\Omega^I\partial_t f}(t,\rho\cos\varphi,\rho\sin\varphi)}{\sqrt{2\pi i\rho}}\,\mathrm{d}\rho.
\end{aligned} \tag{3.9}
$$

Thus, we obtain

$$
\begin{aligned}
&\left|\partial_t\Omega^I\left(u(t,x) - \frac{1}{\sqrt{r}}U(\ln t, r-t,\varphi)\right)\right| \\
&\quad\lesssim \varepsilon t^{-\frac{3}{2}+2\delta} + \left|\frac{1}{\sqrt{r}}\operatorname{Im}\int_0^\infty e^{i\rho(r-t)}\frac{\widehat{\Omega^I\partial_t f}(t,\rho\cos\varphi,\rho\sin\varphi)}{\sqrt{2\pi i\rho}}\,\mathrm{d}\rho\right|.
\end{aligned}
$$

By (2.7),

$$\begin{aligned} &\left|\int_0^\infty e^{i\rho(r-t)} \frac{\widehat{\Omega^I \partial_t f}(t,\rho\cos\varphi,\rho\sin\varphi)}{\sqrt{2\pi i\rho}}\,\mathrm{d}\rho\right| \\ &\quad\lesssim \int_0^\infty \varepsilon\langle\rho\rangle^{-15}\rho^{-\frac{1}{2}}\frac{\ln(1+t)}{1+t}\,\mathrm{d}\rho \lesssim \varepsilon(1+t)^{-1+2\delta}, \end{aligned} \tag{3.10}$$

therefore we have proved

$$\left|\partial_t\Omega^I\left(u(t,x)-\frac{1}{\sqrt{r}}U(\ln t, r-t,\varphi)\right)\right| \lesssim \varepsilon(1+t)^{-\frac{3}{2}+2\delta} \quad \text{if } r \geq \frac{t}{2}. \tag{3.11}$$

3.2 $\Gamma = S$

This case is a little involved. First, note that

$$\begin{aligned} &S\big(f(v)\big) = Se^{it|\nabla|}\big(\partial_t - i|\nabla|\big)v = e^{it|\nabla|}S\big(\partial_t - i|\nabla|\big)v = e^{it|\nabla|}\big(\partial_t - i|\nabla|\big)Sv - e^{it|\nabla|}\big(\partial_t - i|\nabla|\big)v \\ &\quad\Longrightarrow\quad f(Sv) = S\big(f(v)\big) + f(v) \quad\Longrightarrow\quad f\big(S^I v\big) = (S+Id)^I f(v). \end{aligned} \tag{3.12}$$

Then, by (3.1), the solution u of (1.1) satisfies

$$\begin{aligned} &\partial_t\big(S^I u\big)(t,x) \\ &\quad= \operatorname{Re}\big(e^{-it|\nabla|}f\big(S^I u\big)\big)(t,x) \\ &\quad= \frac{1}{2\pi}\operatorname{Re}\int_{\mathbb{R}^2} e^{i(x\cdot\xi - t|\xi|)}\widehat{f\big(S^I u\big)}(t,\xi)\,\mathrm{d}\xi \\ &\quad= \frac{1}{2\pi}\operatorname{Re}\int_{\mathbb{R}^2} e^{ix\cdot\xi - it|\xi|}\widehat{(S+Id)^I f}(u)(t,\xi)\,\mathrm{d}\xi \\ &\quad= \frac{1}{2\pi}\operatorname{Re}\int_0^\infty e^{i\rho(r-t)}\rho\,\mathrm{d}\rho\int_{-\pi}^{\pi} e^{ir\rho(\cos\theta-1)}\sum_{|J|\le|I|}\widehat{c_J S^J f}(u)\big(t,\rho\cos(\theta+\varphi),\rho\sin(\theta+\varphi)\big)\,\mathrm{d}\theta, \end{aligned}$$

where c_J are constants. By (2.7) and (2.9), $\widehat{S^J f(u)}$ and $\widehat{f(u)} = \widehat{f}$ satisfy the same estimate, thus by an analysis similar to Sect. 3.1,

$$\begin{aligned} &\partial_t\big(S^I u\big)(t,x) \\ &\quad= \frac{1}{\sqrt{r}}\operatorname{Re}\int_0^\infty e^{i\rho(r-t)}\sqrt{\frac{\rho}{2\pi i}}\sum_{|J|\le|I|}\widehat{c_J S^J f}(u)(t,\rho\cos\varphi,\rho\sin\varphi)\,\mathrm{d}\rho \\ &\qquad + O\big(\varepsilon(1+t)^{-\frac{3}{2}+2\delta}\big) \\ &\quad= \partial_t\left(-\frac{1}{\sqrt{r}}\operatorname{Im}\int_0^\infty e^{i\rho(r-t)}\frac{\sum_{|J|\le|I|}\widehat{c_J S^J f}(u)(t,\rho\cos\varphi,\rho\sin\varphi)}{\sqrt{2\pi i\rho}}\,\mathrm{d}\rho\right) \\ &\qquad + \frac{1}{\sqrt{r}}\operatorname{Im}\int_0^\infty e^{i\rho(r-t)}\frac{1}{\sqrt{2\pi i\rho}}\sum_{|J|\le|I|}\widehat{c_J\partial_t S^J f}(u)(t,\rho\cos\varphi,\rho\sin\varphi)\,\mathrm{d}\rho \\ &\qquad + O\big(\varepsilon(1+t)^{-\frac{3}{2}+2\delta}\big) \\ &\quad=: A + B + O\big(\varepsilon(1+t)^{-\frac{3}{2}+2\delta}\big). \end{aligned} \tag{3.13}$$

We shall handle the terms A and B, respectively. It is clear that

$$\begin{aligned} A &= \partial_t\left(-\frac{1}{\sqrt{r}}\operatorname{Im}\int_0^\infty e^{i\rho(r-t)}\frac{\widehat{(S+Id)^I f(u)}(t,\rho\cos\varphi,\rho\sin\varphi)}{\sqrt{2\pi i\rho}}\,\mathrm{d}\rho\right) \\ &= \partial_t\left(-\frac{1}{\sqrt{r}}\operatorname{Im}\int_0^\infty e^{i\rho(r-t)}\frac{\widehat{f(S^I u)}(t,\rho\cos\varphi,\rho\sin\varphi)}{\sqrt{2\pi i\rho}}\,\mathrm{d}\rho\right). \end{aligned} \tag{3.14}$$

Note that for any function v,

$$\widehat{Sf(v)} = t\partial_t\widehat{f(v)} - \rho\partial_\rho\widehat{f(v)} - 2\widehat{f(v)}, \tag{3.15}$$

therefore

$$\begin{aligned} &-\frac{1}{\sqrt{r}}\operatorname{Im}\int_0^\infty e^{i\rho(r-t)}\frac{\widehat{f(Sv)}(t,\rho\cos\varphi,\rho\sin\varphi)}{\sqrt{2\pi i\rho}}\,\mathrm{d}\rho \\ &\quad= -\frac{1}{\sqrt{r}}\operatorname{Im}\left(\int_0^\infty e^{i\rho(r-t)}\frac{1}{\sqrt{2\pi i\rho}}t\partial_t\widehat{f(v)}(t,\rho\cos\varphi,\rho\sin\varphi)\,\mathrm{d}\rho\right. \\ &\qquad - \int_0^\infty e^{i\rho(r-t)}\frac{1}{\sqrt{2\pi i\rho}}\widehat{f(v)}(t,\rho\cos\varphi,\rho\sin\varphi)\,\mathrm{d}\rho \\ &\qquad \left. - \int_0^\infty e^{i\rho(r-t)}\sqrt{\frac{\rho}{2\pi i}}\partial_\rho\widehat{f(v)}(t,\rho\cos\varphi,\rho\sin\varphi)\,\mathrm{d}\rho\right). \end{aligned} \tag{3.16}$$

For the last integral in (3.16), integrating by parts w.r.t. ρ, we obtain

$$\begin{aligned} &-\frac{1}{\sqrt{r}}\operatorname{Im}\int_0^\infty e^{i\rho(r-t)}\frac{\widehat{f(Sv)}(t,\rho\cos\varphi,\rho\sin\varphi)}{\sqrt{2\pi i\rho}}\,\mathrm{d}\rho \\ &\quad= -\frac{1}{\sqrt{r}}\operatorname{Im}\int_0^\infty e^{i\rho(r-t)}\sqrt{\frac{\rho}{2\pi i}}\big(t\partial_t\widehat{f(v)}(t,\rho\cos\varphi,\rho\sin\varphi) \\ &\qquad + i(r-t)\rho\widehat{f(v)}(t,\rho\cos\varphi,\rho\sin\varphi)\big)\,\mathrm{d}\rho \\ &\qquad + \frac{1}{2\sqrt{r}}\operatorname{Im}\int_0^\infty e^{i\rho(r-t)}\frac{1}{\sqrt{2\pi i\rho}}\widehat{f(v)}(t,\rho\cos\varphi,\rho\sin\varphi)\,\mathrm{d}\rho \\ &\quad= S\left(-\frac{1}{\sqrt{r}}\operatorname{Im}\int_0^\infty e^{i\rho(r-t)}\frac{\widehat{f(v)}(t,\rho\cos\varphi,\rho\sin\varphi)}{\sqrt{2\pi i\rho}}\,\mathrm{d}\rho\right). \end{aligned} \tag{3.17}$$

Thus,

$$\begin{aligned} A &= \partial_t S\left(-\frac{1}{\sqrt{r}}\operatorname{Im}\int_0^\infty e^{i\rho(r-t)}\frac{\widehat{f(S^{I-1}u)}(t,\rho\cos\varphi,\rho\sin\varphi)}{\sqrt{2\pi i\rho}}\,\mathrm{d}\rho\right) \\ &= \cdots \\ &= \partial_t S^I\left(-\frac{1}{\sqrt{r}}\operatorname{Im}\int_0^\infty e^{i\rho(r-t)}\frac{\widehat{f(u)}(t,\rho\cos\varphi,\rho\sin\varphi)}{\sqrt{2\pi i\rho}}\,\mathrm{d}\rho\right) \\ &= \partial_t S^I\left(\frac{U(\ln t, r-t,\varphi)}{\sqrt{r}}\right). \end{aligned} \tag{3.18}$$

Hence, the remaining task is to control B. Note that

$$\partial_t(S + Id) = (S + 2Id)\partial_t,$$

then

$$\begin{aligned}B &= \frac{1}{\sqrt{r}}\,\mathrm{Im}\int_0^\infty e^{i\rho(r-t)}\frac{1}{\sqrt{2\pi i\rho}}\partial_t\widehat{(S+Id)^I f}(u)(t,\rho\cos\varphi,\rho\sin\varphi)\,\mathrm{d}\rho\\ &= \frac{1}{\sqrt{r}}\,\mathrm{Im}\int_0^\infty e^{i\rho(r-t)}\frac{1}{\sqrt{2\pi i\rho}}\widehat{(S+2Id)^I\partial_t f}(u)(t,\rho\cos\varphi,\rho\sin\varphi)\,\mathrm{d}\rho\\ &= \frac{1}{\sqrt{r}}\,\mathrm{Im}\int_0^\infty e^{i\rho(r-t)}\frac{1}{\sqrt{2\pi i\rho}}\sum_{|J|\le|I|}\widehat{d_J S^J\partial_t f}(u)(t,\rho\cos\varphi,\rho\sin\varphi)\,\mathrm{d}\rho,\end{aligned} \tag{3.19}$$

where d_J are constants. For each J, by (2.7),

$$\begin{aligned}&\left|\int_0^\infty e^{i\rho(r-t)}\frac{1}{\sqrt{2\pi i\rho}}\sum_{|J|\le|I|}\widehat{d_J S^J\partial_t f}(u)(t,\rho\cos\varphi,\rho\sin\varphi)\,\mathrm{d}\rho\right|\\ &\quad\lesssim\int_0^\infty \varepsilon\langle\rho\rangle^{-15}\rho^{-\frac{1}{2}}\frac{\ln(1+t)}{1+t}\,\mathrm{d}\rho\lesssim\varepsilon(1+t)^{-1+2\delta}.\end{aligned}$$

Therefore, $|B|\lesssim\varepsilon(1+t)^{-\frac{3}{2}+2\delta}$ if $r\ge t/2$, combining this with (3.13) and (3.18), we see that the estimate of $\partial_t S^I(u - U/\sqrt{r})$ has been established. The case of $\partial = \partial_x$ can be handled similarly, therefore the proof of (1.13) is completed.

Acknowledgements
The authors would like to thank Professor Z. Lei and Professor H. Gao for many helpful discussions.

Funding
Yaqing Sun was supported by the National Natural Science Foundation of China (grant No. 11531006). Daoyin He was supported by the National Natural Science Foundation of China (grant No. 11901103).

Availability of data and materials
All the data supporting our results can be found in Sects. 2 and 3.

Declarations

Competing interests
The authors declare no competing interests.

Author contribution
DH produced the idea of the proof, YS completed the computation in detail. Both of the authors made efforts in the writing of the manuscript. All authors read and approved the final manuscript.

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Publisher's Note

Received: 30 July 2020 Accepted: 24 October 2022 Published online: 04 November 2022

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