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Solvability of mixed Hilfer fractional functional boundary value problems with p-Laplacian at resonance

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Abstract

This article investigates the existence of solutions of mixed Hilfer fractional differential equations with p-Laplacian under the functional boundary conditions at resonance. By defining Banach spaces with appropriate norms, constructing suitable operators, and using the extension of the continuity theorem, some of the current results are extended to the nonlinear situation, and some new existence results of the problem are obtained. Finally, an example is given to verify our main results.

MSC: Primary 34A08; secondary 34B15

Keywords: Hilfer fractional derivative; Functional boundary conditions; Continuation theorem; p-Laplacian; Resonance

1 Introduction

The fractional differential equations have become an important research field because of the in-depth development of fractional calculus theory and its wide applications in many sciences such as physics, engineering, biology and so on [1–5].

There are various definitions of fractional derivatives, such as Riemann–Liouville and Caputo fractional derivatives [6, 7]. On this basis, a more generalized fractional derivative "Hilfer" derivative has been studied [8]. The Hilfer fractional derivative is an extension of the Riemann–Liouville and Caputo fractional derivatives. Hilfer fractional differential equations are very suitable for describing processes with memory and hereditary properties. They have the advantages of simple modeling and accurate description of complex systems, and have become one of the important tools for mathematical modeling of mechanical and physical processes. Therefore, fractional differential equations with Hilfer derivative have gradually become a research hotspot [9–11].

Ri et al. [11] considered the following multi-point boundary value problems of the Hilfer fractional differential equations at resonance:

$$\begin{cases} D_{0+}^{\alpha,\beta}x(t) = f(t,x(t)), \quad 0 < t \le T, \\ I_{0+}^{1-\gamma}u(0) = \sum_{i=1}^{m} c_i x(\tau_i), \end{cases}$$

where $0 < \alpha < 1$, $0 \leq \beta \leq 1$, $\tau_i \in (0, T]$, $D_{0+}^{\alpha,\beta}$ is Hilfer fractional derivative of order α and type β.

In the past, the boundary conditions of boundary value problems were generally specific. In recent years, some scholars have changed the boundary value conditions into abstract conditions, which contains many specific boundary conditions. And many achievements have been made in the study of functional boundary value problems [12–17].

Zhao and Liang [15] first used Mawhin's coincidence degree theory to discuss the solvability of functional boundary value problems:

$$\begin{cases} x''(t) = f(t, x(t), x'(t)), \quad 0 < t < 1, \\ \Gamma_1(x) = 0, \qquad \Gamma_2(x) = 0, \end{cases}$$

where $\Gamma_1, \Gamma_2 : C^1[0,1] \to R$ are continuous linear functionals. It was discussed according to the six situations of non-resonance and resonance, and some existence results of the solution of the functional boundary value problems were obtained.

However, the existence of solutions under the condition of $\Gamma_1(t)\Gamma_2(1) = \Gamma_1(1)\Gamma_2(t)$ was not discussed in [15]. Furthermore, Kosmatov and Jiang [16] considered the solvability of functional boundary value problems under the condition $\Gamma_1(t)\Gamma_2(1) = \Gamma_1(1)\Gamma_2(t)$:

$$\begin{cases} x''(t) = f(t, x(t), x'(t)), \quad t \in (0, 1), \\ \Gamma_1(x) = 0, \qquad \Gamma_2(x) = 0, \end{cases}$$

where Γ_1, Γ_2 are linear functionals. The conditions in [15] were supplemented here, and the solvability of functional boundary value problems was analyzed more comprehensively.

The p-Laplacian operator originated from the research of turbulence in porous media. Leibenson [18] first considered the following p-Laplacian equation:

$$\left(\phi_p\left(x'(t)\right)\right)' = f\left(t, x(t), x'(t)\right).$$

Later, many scholars conducted more in-depth research on the p-Laplacian operator and obtained some excellent results [19–21].

Jiang [22] considered the solvability of fractional differential equations with p-Laplacian by the extended continuous theorem:

$$\begin{cases} D_{0+}^{\beta}(\varphi_p(D_{0+}^{\alpha} u))(t) + f(t, u(t), D_{0+}^{\alpha-1} u(t), D_{0+}^{\alpha} u(t)) = 0, \\ u(0) = D_{0+}^{\alpha} u(0) = 0, \qquad u(1) = \int_0^1 h(t) u(t)\, dt, \end{cases}$$

where $0 < \beta \leq 1$, $1 < \alpha \leq 2$, $\varphi_p(s) = |s|^{p-2}s$, $p > 1$, $\int_0^1 h(t) t^{\alpha-1}\, dt = 1$, D_{0+}^{α} is the Riemann–Liouville fractional derivative.

Based on the above literature, this paper studies the solvability of mixed Hilfer fractional functional boundary value problems with p-Laplacian operator at resonance:

$$\begin{cases} D_{1-}^{\alpha_1,\beta_1} \varphi_p(D_{0+}^{\alpha_2,\beta_2} u(t)) = f(t, u(t), D_{0+}^{\alpha_2-2,\beta_2} u(t), D_{0+}^{\alpha_2-1,\beta_2} u(t), D_{0+}^{\alpha_2,\beta_2} u(t)), \\ u(0) = 0, \qquad D_{0+}^{\alpha_2,\beta_2} u(1) = 0, \qquad T_1(u) = T_2(u) = 0, \quad t \in [0, 1], \end{cases} \tag{1.1}$$

where $0 < \alpha_1 < 1$, $2 < \alpha_2 < 3$, $0 \le \beta_1, \beta_2 \le 1$, $\gamma_1 = \alpha_1 + \beta_1 - \alpha_1\beta_1$, $\gamma_2 = \alpha_2 + 3\beta_2 - \alpha_2\beta_2$, $\varphi_p(s) = |s|^{p-2}s$, $p > 1$, $\varphi_p(0) = 0$, $D_{a\pm}^{\alpha,\beta}$ is Hilfer right-/left-sided fractional derivative of order α and type β, $f \in C([0,1] \times \mathbb{R}^4, \mathbb{R})$ and $T_1, T_2 : C[0,1] \to \mathbb{R}$ are linear bounded functionals.

2 Preliminaries

Definition 2.1 ([23]) Let X and Y be two Banach spaces with norms $\|\cdot\|_X$, $\|\cdot\|_Y$, respectively. A continuous operator $L: X \cap \operatorname{dom} L \to Y$ is said to be quasilinear if

(i) $\operatorname{Im} L := L(X \cap \operatorname{dom} L)$ is a closed subset of Y,

(ii) $\operatorname{Ker} L := \{x \in X \cap \operatorname{dom} L : Lx = 0\}$ is linearly homeomorphic to $\mathbb{R}^n$, $n < \infty$,

where $\operatorname{dom} L$ denotes the domain of the operator L.

Let $X_1 = \operatorname{Ker} L$ and X_2 be the complement space of X_1 in X, then $X = X_1 \oplus X_2$. Let $P : X \to X_1$ be the projector and $\Omega \subset X$ be an open and bounded set with the origin $\theta \in \Omega$.

Definition 2.2 ([22]) Suppose that $N_\lambda : \overline{\Omega} \to Y$, $\lambda \in [0,1]$ is a continuous and bounded operator. Denote N_1 by N. Let $\Sigma_\lambda = \{x \in \overline{\Omega} : Lx = N_\lambda x\}$. N_λ is said to be L-quasicompact in $\overline{\Omega}$ if there exists a vector subspace Y_1 of Y satisfying $\dim Y_1 = \dim X_1$ and two operators Q and R such that for $\lambda \in [0,1]$,

(a) $\operatorname{Ker} Q = \operatorname{Im} L$,

(b) $QN_\lambda x = \theta$, $\lambda \in (0,1) \Leftrightarrow QNx = \theta$,

(c) $R(\cdot,0)$ is the zero operator and $R(\cdot,\lambda)\,|_{\Sigma_\lambda} = (I-P)\,|_{\Sigma_\lambda}$,

(d) $L[P + R(\cdot,\lambda)] = (I-Q)N_\lambda$,

where $Q : Y \to Y_1$, $QY = Y_1$ is continuous, bounded and satisfies $Q(I-Q) = 0$ and $R : \overline{\Omega} \times [0,1] \to X_2$ is continuous and compact.

Lemma 2.3 ([22]) *Let X and Y be two Banach spaces with the norms $\|\cdot\|_X$, $\|\cdot\|_Y$, respectively, and let $\Omega \subset X$ be an open and bounded nonempty set. Suppose that $L : \operatorname{dom} L \cap X \to Y$ is a quasilinear operator and that $N_\lambda : \overline{\Omega} \to Y$, $\lambda \in [0,1]$ is L-quasicompact. In addition, if the following conditions hold*:

(a) $Lx \ne N_\lambda x$, $\forall x \in \partial\Omega \cap \operatorname{dom} L$, $\lambda \in (0,1)$,

(b) $\deg\{JQN, \Omega \cap \operatorname{Ker} L, 0\} \ne 0$,

then the abstract equation $Lx = Nx$ has at least one solution in $\operatorname{dom} L \cap \overline{\Omega}$, where $N = N_1$, $J : \operatorname{Im} Q \to \operatorname{Ker} L$ is a homeomorphism with $J(\theta) = \theta$.

Definition 2.4 ([6]) The left-sided and right-sided Riemann–Liouville fractional integrals of order $\alpha > 0$ of a function $y : (0,+\infty) \to R$ are given by

$$I_{0+}^{\alpha} y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)\,ds, \qquad I_{1-}^{\alpha} y(t) = \frac{1}{\Gamma(\alpha)} \int_t^1 (s-t)^{\alpha-1} y(s)\,ds.$$

Definition 2.5 ([6]) The left-sided and right-sided Riemann–Liouville fractional derivatives of order $\alpha > 0$ of a function $y : (0,+\infty) \to R$ are given by

$$D_{0^+}^{\alpha} y(t) = \frac{d^n}{dt^n}\bigl(I_{0^+}^{n-\alpha} y\bigr)(t), \qquad D_{1^-}^{\alpha} y(t) = (-1)^n \frac{d^n}{dt^n}\bigl(I_{1^-}^{n-\alpha} y\bigr)(t),$$

where $n = [\alpha] + 1$.

Definition 2.6 ([8]) The right-/left-sided Hilfer fractional derivative of order α and type β for a function $y:(0,+\infty)\to R$ is given by

$$D_{a\pm}^{\alpha,\beta}y(t)=(\pm)^n I_{a\pm}^{\beta(n-\alpha)}\frac{d^n}{dt^n}\big(I_{a\pm}^{(1-\beta)(n-\alpha)}y\big)(t),\quad n-1<\alpha<n,0\le\beta\le 1.$$

Remark

(1) The operator $D_{a\pm}^{\alpha,\beta}$ can also be written as $D_{a\pm}^{\alpha,\beta}=I_{a\pm}^{\beta(n-\alpha)}D_{a\pm}^{\gamma}$, $\gamma=\alpha+n\beta-\alpha\beta$.

(2) If $\beta=0$, then the Riemann–Liouville fractional derivative can be presented as $D_{a\pm}^{\alpha}=D_{a\pm}^{\alpha,0}$.

(3) If $\beta=1$, then the Caputo fractional derivative can be presented as ${}^{C}D_{a\pm}^{\alpha}=D_{a\pm}^{\alpha,1}$.

Lemma 2.7 ([6]) *For $n-1<\alpha\le n$, $n\in N$, the general solution of the fractional differential equation $D_{1-}^{\alpha}u(t)=0$ is given by*

$$u(t)=c_1(1-t)^{\alpha-1}+c_2(1-t)^{\alpha-2}+\cdots+c_n(1-t)^{\alpha-n},$$

where $c_i\in\mathbb{R}$, $i=1,2,\ldots,n$, $n=[\alpha]+1$.

Lemma 2.8 ([6]) *Let $\alpha>0$, $n=[\alpha]+1$, if $y\in L_1(0,1)$ and $I_{0+}^{n-\alpha}y\in AC^n[0,1]$, then the following holds*:

$$I_{0+}^{\alpha}D_{0+}^{\alpha}y(t)=y(t)-\sum_{j=1}^{n}\frac{(I_{0+}^{n-\alpha}y(t))^{(n-j)}|_{t=0}}{\Gamma(\alpha-j+1)}t^{\alpha-j}.$$

Lemma 2.9 ([6]) *For $n-1<\alpha\le n$, $n\in N$, the general solution of the fractional differential equation $D_{0+}^{\alpha}u(t)=0$ is given by*

$$u(t)=c_1t^{\alpha-1}+c_2t^{\alpha-2}+\cdots+c_nt^{\alpha-n},$$

where $c_i\in\mathbb{R}$, $i=1,2,\ldots,n$, $n=[\alpha]+1$.

Lemma 2.10 ([6]) *If $\alpha>0$, $\beta>-1$, and $\beta\ne\alpha-i$, $i=1,2,\ldots,[\alpha]+1$, then*

$$D_{0+}^{\alpha}t^{\beta}=\frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}t^{\beta-\alpha},\qquad D_{0+}^{\alpha}t^{\alpha-i}=0.$$

Lemma 2.11 ([6]) *If $\alpha>\beta>0$, and $y\in L_1(\mathbb{R}^+)$, then*

$$D_{0+}^{\beta}I_{0+}^{\alpha}y(t)=I_{0+}^{\alpha-\beta}y(t),\qquad D_{0+}^{\alpha}I_{0+}^{\beta}y(t)=D_{0+}^{\alpha-\beta}y(t).$$

In particular, when $\beta=k\in\mathbb{N}$ and $\alpha>k$, then

$$\frac{d^k}{dt^k}I_{0+}^{\alpha}y(t)=I_{0+}^{\alpha-k}y(t).$$

Lemma 2.12 ([24]) *For any $u,v\ge 0$, then*

(1) $\varphi_p(u+v)\le\varphi_p(u)+\varphi_p(v)$, $1<p\le 2$,

(2) $\varphi_p(u+v)\le 2^{p-2}(\varphi_p(u)+\varphi_p(v))$, $p\ge 2$,

where $\varphi_p(s)=|s|^{p-2}s=s^{p-1}$, $s\ge 0$.

3 Main results

Take

$$X = \{u \mid u(t), D_{0+}^{\alpha_2-2,\beta_2}u(t), D_{0+}^{\alpha_2-1,\beta_2}u(t), D_{0+}^{\alpha_2,\beta_2}u(t) \in C[0,1]\}, \qquad Y = C[0,1],$$

with norms

$$\|u\|_X = \max_{t\in[0,1]}\{\|u\|_\infty, \|D_{0+}^{\alpha_2-2,\beta_2}u\|_\infty, \|D_{0+}^{\alpha_2-1,\beta_2}u\|_\infty, \|D_{0+}^{\alpha_2,\beta_2}u\|_\infty\}, \qquad \|y\|_Y = \|y\|_\infty,$$

where $\|y\|_\infty = \max_{t\in[0,1]}|y(t)|$.

Lemma 3.1 $(X, \|\cdot\|)$, $(Y, \|\cdot\|)$ *are Banach spaces.*

Proof It is easy to see that $(Y, \|\cdot\|)$ is a Banach space. Next, we prove that $(X, \|\cdot\|)$ is also a Banach space. Suppose that $\{u_n\}$ is a Cauchy sequence of X, then $\{u_n\}$, $\{D_{0+}^{\alpha_2-2,\beta_2}u_n\}$, $\{D_{0+}^{\alpha_2-1,\beta_2}u_n\}$, $\{D_{0+}^{\alpha_2,\beta_2}u_n\}$ are Cauchy sequences of $C[0,1]$. So, there exist functions $u, v, w, g \in C[0,1]$ such that u_n, $D_{0+}^{\alpha_2-2,\beta_2}u_n$, $D_{0+}^{\alpha_2-1,\beta_2}u_n$, $D_{0+}^{\alpha_2,\beta_2}u_n$ converge uniformly to u, v, w, g on $[0,1]$, respectively. We need to prove that $D_{0+}^{\alpha_2-2,\beta_2}u = v$, $D_{0+}^{\alpha_2-1,\beta_2}u = w$, $D_{0+}^{\alpha_2,\beta_2}u = g$. By Lemma 2.8, we get

$$I_{0+}^{\alpha_2-2}D_{0+}^{\alpha_2-2,\beta_2}u_n = I_{0+}^{\alpha_2-2}I_{0+}^{\beta_2(3-\alpha_2)}D_{0+}^{\gamma_2-2}u_n = I_{0+}^{\gamma_2-2}D_{0+}^{\gamma_2-2}u_n = u_n + ct^{\gamma_2-3}.$$

So, we have

$$\frac{1}{\Gamma(\alpha_2-2)}\int_0^t (t-s)^{\alpha_2-3}D_{0+}^{\alpha_2-2,\beta_2}u_n(s)\,ds = u_n + ct^{\gamma_2-3}.$$

Let $n \to \infty$, we get

$$\frac{1}{\Gamma(\alpha_2-2)}\int_0^t (t-s)^{\alpha_2-3}v(s)\,ds = u + ct^{\gamma_2-3}. \tag{3.1}$$

Applying $D_{0+}^{\gamma_2-2}$ and $I_{0+}^{\beta_2(3-\alpha_2)}$ to the both sides of (3.1), we obtain

$$I_{0+}^{\beta_2(3-\alpha_2)}D_{0+}^{\gamma_2-2}I_{0+}^{\alpha_2-2}v(t) = I_{0+}^{\beta_2(3-\alpha_2)}D_{0+}^{\gamma_2-2}u = D_{0+}^{\alpha_2-2,\beta_2}u.$$

Therefore, from Lemma 2.11 and Lemma 2.8, we get $v = D_{0+}^{\alpha_2-2,\beta_2}u$.

Since $I_{0+}^{\alpha_2-1}D_{0+}^{\alpha_2-1,\beta_2}u_n = u_n + c_1t^{\gamma_2-2} + c_2t^{\gamma_2-3}$ and $I_{0+}^{\alpha_2}D_{0+}^{\alpha_2,\beta_2}u_n = u_n + c_1t^{\gamma_2-1} + c_2t^{\gamma_2-2} + c_3t^{\gamma_2-3}$, similar to the above proof we can get $w = D_{0+}^{\alpha_2-1,\beta_2}u$ and $g = D_{0+}^{\alpha_2,\beta_2}u$. So, $(X, \|\cdot\|)$ is a Banach space. The proof is completed. □

In order to obtain our main results, we always suppose that the following conditions hold:

(H_1) $T_1(t^{\gamma_2-1})T_2(t^{\gamma_2-2}) = T_1(t^{\gamma_2-2})T_2(t^{\gamma_2-1})$.

(H_2) Functionals $T_i : X \to \mathbb{R}$ are linear bounded with the respective norms $\|T_i\|$, $i = 1,2$. And the functionals T_1, T_2 satisfy the relations $T_1(t^{\gamma_2-1}) = \delta_2$, $T_1(t^{\gamma_2-2}) = \delta_1$, $T_2(t^{\gamma_2-1}) = k\delta_2$, $T_2(t^{\gamma_2-2}) = k\delta_1$, where $\delta_1, \delta_2, k \in \mathbb{R}$, $\delta_1^2 + \delta_2^2 \neq 0$.

(H_3) Functional $G(y) = (T_2 - kT_1)(I_{0+}^{\alpha_2}\varphi_q(I_{1-}^{\alpha_1}y))$, $\frac{1}{p} + \frac{1}{q} = 1$ is increasing.

Define operators $L : \operatorname{dom} L \cap X \to Y$ and $N_\lambda : X \to Y$ as follows

$$\begin{aligned} &Lu(t) = D_{1-}^{\alpha_1,\beta_1}\varphi_p\big(D_{0+}^{\alpha_2,\beta_2}u(t)\big), \\ &N_\lambda u(t) = \lambda f\big(t, u(t), D_{0+}^{\alpha_2-2,\beta_2}u(t), D_{0+}^{\alpha_2-1,\beta_2}u(t), D_{0+}^{\alpha_2,\beta_2}u(t)\big), \quad t \in [0,1], \lambda \in [0,1], \end{aligned}$$

where

$$\begin{aligned} \operatorname{dom} L = \big\{&u(t) \mid u(t) \in X, D_{1-}^{\alpha_1,\beta_1}\varphi_p\big(D_{0+}^{\alpha_2,\beta_2}u(t)\big) \in Y, u(0) = 0, \\ &D_{0+}^{\alpha_2,\beta_2}u(1) = 0, T_1(u) = T_2(u) = 0\big\}. \end{aligned}$$

Lemma 3.2 *Suppose that* (H_1) *holds, then* L *is a quasilinear operator.*

Proof It is easy to get that $\operatorname{Ker} L = \{u \in \operatorname{dom} L \mid u(t) = c(\delta_2 t^{\gamma_2-2} - \delta_1 t^{\gamma_2-1}), c \in \mathbb{R}\}$.

For $y \in \operatorname{Im} L$, there exists $u \in \operatorname{dom} L$ such that $D_{1-}^{\alpha_1,\beta_1}\varphi_p(D_{0+}^{\alpha_2,\beta_2}u(t)) = y(t)$. According to *Remark*, we get

$$I_{1-}^{\beta_1(1-\alpha_1)}D_{1-}^{\gamma_1}\varphi_p\big(D_{0+}^{\alpha_2,\beta_2}u(t)\big) = y(t). \tag{3.2}$$

Thus, applying $D_{1-}^{\beta_1(1-\alpha_1)}$ to the both sides of (3.2), and by Lemma 2.7, we have

$$D_{0+}^{\alpha_2,\beta_2}u(t) = \varphi_q\big(I_{1-}^{\alpha_1}y(t) + c_1(1-t)^{\gamma_1-1}\big).$$

Since $D_{0+}^{\alpha_2,\beta_2}u(1) = 0$, we can get

$$D_{0+}^{\alpha_2,\beta_2}u(t) = \varphi_q\big(I_{1-}^{\alpha_1}y(t)\big). \tag{3.3}$$

Applying $D_{0+}^{\beta_2(3-\alpha_2)}$ to the both sides of (3.3), and because of $u(0) = 0$, we obtain

$$u(t) = I_{0+}^{\alpha_2}\varphi_q\big(I_{1-}^{\alpha_1}y(t)\big) + c_2 t^{\gamma_2-1} + c_3 t^{\gamma_2-2}.$$

The functional boundary condition $T_1(u) = T_2(u) = 0$ implies that

$$\begin{aligned} &T_1(u) = T_1\big(I_{0+}^{\alpha_2}\varphi_q\big(I_{1-}^{\alpha_1}y(t)\big)\big) + c_2\delta_2 + c_3\delta_1 = 0, \\ &T_2(u) = T_2\big(I_{0+}^{\alpha_2}\varphi_q\big(I_{1-}^{\alpha_1}y(t)\big)\big) + kc_2\delta_2 + kc_3\delta_1 = 0. \end{aligned}$$

Obviously,

$$(T_2 - kT_1)\big(I_{0+}^{\alpha_2}\varphi_q\big(I_{1-}^{\alpha_1}y(t)\big)\big) = 0. \tag{3.4}$$

Hence, $\operatorname{Im} L \subseteq \{y \in Y | (T_2 - kT_1)(I_{0+}^{\alpha_2}\varphi_q(I_{1-}^{\alpha_1}y(t))) = 0\}$.

Conversely, if $y \in Y$ and satisfies (3.4), let

$$u(t) = I_{0+}^{\alpha_2}\varphi_q\big(I_{1-}^{\alpha_1}y(t)\big) + \frac{T_1(I_{0+}^{\alpha_2}\varphi_q(I_{1-}^{\alpha_1}y(t)))}{\delta_1^2 + \delta_2^2}\big(\delta_2 t^{\gamma_2-1} + \delta_1 t^{\gamma_2-2}\big).$$

It is easy to prove that $u(t)$ satisfies the boundary conditions of problem (1.1), and we have

$$\begin{aligned}Lu(t) &= I_{1-}^{\beta_1(1-\alpha_1)} D_{1-}^{\gamma_1} \varphi_p\big(I_{0+}^{\beta_2(3-\alpha_2)} D_{0+}^{\gamma_2} I_{0+}^{\alpha_2} \varphi_q\big(I_{1-}^{\alpha_1} y(t)\big)\big) \\ &= I_{1-}^{\beta_1(1-\alpha_1)} D_{1-}^{\beta_1(1-\alpha_1)} y(t) = y(t).\end{aligned}$$

Therefore,

$$\operatorname{Im} L \supseteq \big\{ y \in Y \mid (T_2 - kT_1)\big(I_{0+}^{\alpha_2} \varphi_q\big(I_{1-}^{\alpha_1} y(t)\big)\big) = 0 \big\}.$$

In summary, we get

$$\operatorname{Im} L = \big\{ y \in Y \mid (T_2 - kT_1)\big(I_{0+}^{\alpha_2} \varphi_q\big(I_{1-}^{\alpha_1} y(t)\big)\big) = 0 \big\}.$$

Clearly, $\operatorname{Im} L \subset Y$ is closed. So, L is a quasilinear operator. □

Define the operator $P : X \to \operatorname{Ker} L$ by

$$Pu(t) = \frac{\delta_2 D_{0+}^{\gamma_2-2} u(0) - \delta_1 D_{0+}^{\gamma_2-1} u(0)}{\delta_2^2 \Gamma(\gamma_2 - 1) + \delta_1^2 \Gamma(\gamma_2)} \big(\delta_2 t^{\gamma_2-2} - \delta_1 t^{\gamma_2-1}\big).$$

It is clear that $P^2 u = Pu$ and $\operatorname{Im} P = \operatorname{Ker} L$, $X = \operatorname{Ker} L \oplus \operatorname{Ker} P$. So, $P : X \to \operatorname{Ker} L$ is a projector.

Define the operator $Q : Y \to R$ by

$$Qy(t) = c,$$

where c satisfies

$$(T_2 - kT_1)\big(I_{0+}^{\alpha_2} \varphi_q\big(I_{1-}^{\alpha_1}\big(y(t) - c\big)\big)\big) = 0. \tag{3.5}$$

Next, we will prove that c is the unique constant satisfying (3.5). For $y \in Y$, let

$$F(c) = (T_2 - kT_1)\big(I_{0+}^{\alpha_2} \varphi_q\big(I_{1-}^{\alpha_1}\big(y(t) - c\big)\big)\big).$$

Obviously, $F(c)$ is continuous and strictly decreasing in $\mathbb{R}$. We make $c_1 = \min_{t\in[0,1]} y(t)$, $c_2 = \max_{t\in[0,1]} y(t)$. It is easy to see that $F(c_1) \geq 0$, $F(c_2) \leq 0$, then, there exists a unique constant $c \in [c_1, c_2]$ such that $F(c) = 0$.

Lemma 3.3 *$Q : Y \to Y_1$ is continuous, bounded and $Q(I - Q)y = Q(y - Qy) = 0$, $y \in Y$, $QY = Y_1$, where $Y_1 = \mathbb{R}$.*

Proof For $y_1, y_2 \in Y$, assume $Qy_1 = c_1$, $Qy_2 = c_2$. Since φ_q is strictly increasing, if $c_2 - c_1 > \max_{t\in[0,1]}(y_2(t) - y_1(t))$, then

$$\begin{aligned}0 &= (T_2 - kT_1)\big(I_{0+}^{\alpha_2} \varphi_q\big(I_{1-}^{\alpha_1}\big(y_2(t) - c_2\big)\big)\big) \\ &= (T_2 - kT_1)\big(I_{0+}^{\alpha_2} \varphi_q\big(I_{1-}^{\alpha_1}\big(y_1(t) - c_1 + y_2(t) - y_1(t) - (c_2 - c_1)\big)\big)\big) \\ &< (T_2 - kT_1)\big(I_{0+}^{\alpha_2} \varphi_q\big(I_{1-}^{\alpha_1}\big(y_1(t) - c_1\big)\big)\big) = 0.\end{aligned}$$

A contradiction. On the other hand, if $c_2 - c_1 < \min_{t\in[0,1]}(y_2(t) - y_1(t))$, then

$$\begin{aligned}0 &= (T_2 - kT_1)\big(I_{0+}^{\alpha_2}\varphi_q\big(I_{1-}^{\alpha_1}\big(y_2(t) - c_2\big)\big)\big)\\ &= (T_2 - kT_1)\big(I_{0+}^{\alpha_2}\varphi_q\big(I_{1-}^{\alpha_1}\big(y_1(t) - c_1 + y_2(t) - y_1(t) - (c_2 - c_1)\big)\big)\big)\\ &> (T_2 - kT_1)\big(I_{0+}^{\alpha_2}\varphi_q\big(I_{1-}^{\alpha_1}\big(y_1(t) - c_1\big)\big)\big) = 0.\end{aligned}$$

A contradiction, too. So, we can get

$$\min_{t\in[0,1]}\big(y_2(t) - y_1(t)\big) \le c_2 - c_1 \le \max_{t\in[0,1]}\big(y_2(t) - y_1(t)\big), \quad \text{i.e.} \quad |c_2 - c_1| \le \|y_2 - y_1\|_\infty.$$

Therefore, Q is continuous. In addition, if $\Omega \subset Y$ is bounded, then $Q(\Omega)$ is bounded, *i.e.*, Q is bounded. According to the definition of Q, we can easily know that Q is not a projector but satisfies $Q(I - Q)Y = Q(Y - QY) = 0$, $y \in Y$ and $QY = Y_1$. □

Lemma 3.4 *Define an operator* $R : X \times [0,1] \to X_2$ *as*

$$\begin{aligned}R(u,\lambda)(t) &= I_{0+}^{\alpha_2}\varphi_q\big(I_{1-}^{\alpha_1}(I - Q)N_\lambda u(t)\big)\\ &\quad - \frac{T_1(I_{0+}^{\alpha_2}\varphi_q(I_{1-}^{\alpha_1}(I - Q)N_\lambda u(t)))}{\delta_2^2\Gamma(\gamma_2 - 1) + \delta_1^2\Gamma(\gamma_2)}\big(\delta_2\Gamma(\gamma_2 - 1)t^{\gamma_2-1} + \delta_1\Gamma(\gamma_2)t^{\gamma_2-2}\big),\end{aligned}$$

where $\operatorname{Ker} L \oplus X_2 = X$.

Then $R : \overline{\Omega} \times [0,1] \to X_2$ *is continuous and compact, where* $\Omega \subset X$ *is an open bounded set.*

Proof Obviously, R is continuous. Let A be any bounded set in X, for $\forall u \in A$, $D_{0+}^{\alpha_2-1,\beta_2}u \in A$, $D_{0+}^{\alpha_2-2,\beta_2}u \in A$, $\lambda \in [0,1]$. By the continuity of f and the boundedness of Q, we can get that there exist constants $k_1 > 0$, $k_2 > 0$ such that $|f(t, u(t), D_{0+}^{\alpha_2-2,\beta_2}u(t), D_{0+}^{\alpha_2-1,\beta_2}u(t), D_{0+}^{\alpha_2,\beta_2}u(t))| \le k_1$, $|Qf| \le k_2$ for $u \in \overline{\Omega}$. Note that

$$\begin{aligned}&\big|I_{0+}^{\alpha_2}\varphi_q\big(I_{1-}^{\alpha_1}(I - Q)N_\lambda u(t)\big)\big|\\ &\quad\le \frac{1}{\Gamma(\alpha_2)}\int_0^t (t - s)^{\alpha_2-1}\varphi_q\left(\frac{1}{\Gamma(\alpha_1)}\int_s^1 (x - s)^{\alpha_1-1}\big|(I - Q)N_\lambda u(x)\big|\,dx\right)ds\\ &\quad\le \frac{1}{\Gamma(\alpha_2)}\int_0^t (t - s)^{\alpha_2-1}\varphi_q\left(\frac{k_1 + k_2}{\Gamma(\alpha_1 + 1)}\right)ds\\ &\quad\le \frac{1}{\Gamma(\alpha_2 + 1)}\varphi_q\left(\frac{k_1 + k_2}{\Gamma(\alpha_1 + 1)}\right),\\ &\big|D_{0+}^{\alpha_2-2,\beta_2}I_{0+}^{\alpha_2}\varphi_q\big(I_{1-}^{\alpha_1}(I - Q)N_\lambda u(t)\big)\big|\\ &\quad\le \int_0^t (t - s)\varphi_q\left(\frac{1}{\Gamma(\alpha_1)}\int_s^1 (x - s)^{\alpha_1-1}\big|(I - Q)N_\lambda u(x)\big|\,dx\right)ds\\ &\quad\le \frac{1}{2}\varphi_q\left(\frac{k_1 + k_2}{\Gamma(\alpha_1 + 1)}\right),\end{aligned}$$

$$\begin{aligned}
&\left|D_{0+}^{\alpha_2-1,\beta_2} I_{0+}^{\alpha_2}\varphi_q\left(I_{1-}^{\alpha_1}(I-Q)N_\lambda u(t)\right)\right| \\
&\quad\le \int_0^t \varphi_q\left(\frac{1}{\Gamma(\alpha_1)}\int_s^1 (x-s)^{\alpha_1-1}\left|(I-Q)N_\lambda u(x)\right|dx\right)ds \\
&\quad\le \varphi_q\left(\frac{k_1+k_2}{\Gamma(\alpha_1+1)}\right),
\end{aligned}$$

and

$$\left|D_{0+}^{\alpha_2,\beta_2} I_{0+}^{\alpha_2}\varphi_q\left(I_{1-}^{\alpha_1}(I-Q)N_\lambda u(t)\right)\right| \le \varphi_q\left(\frac{k_1+k_2}{\Gamma(\alpha_1+1)}\right).$$

Therefore,

$$\begin{aligned}
&\left\|I_{0+}^{\alpha_2}\varphi_q\left(I_{1-}^{\alpha_1}(I-Q)N_\lambda u\right)\right\|_X \\
&\quad\le \max\left\{\frac{1}{\Gamma(\alpha_2+1)}\varphi_q\left(\frac{k_1+k_2}{\Gamma(\alpha_1+1)}\right), \frac{1}{2}\varphi_q\left(\frac{k_1+k_2}{\Gamma(\alpha_1+1)}\right), \varphi_q\left(\frac{k_1+k_2}{\Gamma(\alpha_1+1)}\right)\right\} \\
&\quad= \varphi_q\left(\frac{k_1+k_2}{\Gamma(\alpha_1+1)}\right),
\end{aligned}$$

then we have

$$\begin{aligned}
\left|R(u,\lambda)(t)\right| &\le \left\|I_{0+}^{\alpha_2}\varphi_q\left(I_{1-}^{\alpha_1}(I-Q)N_\lambda u\right)\right\|_X \\
&\quad + \frac{\|T_1\|_\infty(|\delta_2|\Gamma(\gamma_2-1)+|\delta_1|\Gamma(\gamma_2))}{\delta_2^2\Gamma(\gamma_2-1)+\delta_1^2\Gamma(\gamma_2)}\left\|I_{0+}^{\alpha_2}\varphi_q\left(I_{1-}^{\alpha_1}(I-Q)N_\lambda u\right)\right\|_X \\
&\le \left[1+\frac{\|T_1\|_\infty(|\delta_2|\Gamma(\gamma_2-1)+|\delta_1|\Gamma(\gamma_2))}{\delta_2^2\Gamma(\gamma_2-1)+\delta_1^2\Gamma(\gamma_2)}\right]\left\|I_{0+}^{\alpha_2}\varphi_q\left(I_{1-}^{\alpha_1}(I-Q)N_\lambda u\right)\right\|_X \\
&\le \left[1+\frac{\|T_1\|_\infty(|\delta_2|\Gamma(\gamma_2-1)+|\delta_1|\Gamma(\gamma_2))}{\delta_2^2\Gamma(\gamma_2-1)+\delta_1^2\Gamma(\gamma_2)}\right]\varphi_q\left(\frac{k_1+k_2}{\Gamma(\alpha_1+1)}\right),
\end{aligned}$$

$$\begin{aligned}
&\left|D_{0+}^{\alpha_2-2,\beta_2}R(u,\lambda)(t)\right| \\
&\quad\le \int_0^t (t-s)\varphi_q\left(\frac{1}{\Gamma(\alpha_1)}\int_s^1 (x-s)^{\alpha_1-1}\left|(I-Q)N_\lambda u(x)\right|dx\right)ds \\
&\qquad + \frac{\|T_1\|_\infty\|I_{0+}^{\alpha_2}\varphi_q(I_{1-}^{\alpha_1}(I-Q)N_\lambda u)\|_X}{\delta_2^2\Gamma(\gamma_2-1)+\delta_1^2\Gamma(\gamma_2)} \\
&\qquad \times \left(|\delta_2|\Gamma(\gamma_2-1)I_{0+}^{\beta_2(3-\alpha_2)}\Gamma(\gamma_2)t + |\delta_1|\Gamma(\gamma_2)I_{0+}^{\beta_2(3-\alpha_2)}\Gamma(\gamma_2-1)\right) \\
&\quad\le \int_0^t (t-s)\varphi_q\left(\frac{k_1+k_2}{\Gamma(\alpha_1+1)}\right)ds \\
&\qquad + \frac{\|T_1\|_\infty\varphi_q(\frac{k_1+k_2}{\Gamma(\alpha_1+1)})}{\delta_2^2\Gamma(\gamma_2-1)+\delta_1^2\Gamma(\gamma_2)}\left(\frac{|\delta_2|\Gamma(\gamma_2-1)\Gamma(\gamma_2)}{\Gamma(\beta_2(3-\alpha_2)+2)} + \frac{|\delta_1|\Gamma(\gamma_2-1)\Gamma(\gamma_2)}{\Gamma(\beta_2(3-\alpha_2)+1)}\right) \\
&\quad\le \left[\frac{1}{2}+\frac{\|T_1\|_\infty(|\delta_2|\Gamma(\gamma_2)+(\beta_2(3-\alpha_2)+1)|\delta_1|\Gamma(\gamma_2))}{(\delta_2^2+\delta_1^2(\gamma_2-1))\Gamma(\beta_2(3-\alpha_2)+2)}\right]\varphi_q\left(\frac{k_1+k_2}{\Gamma(\alpha_1+1)}\right),
\end{aligned}$$

$$\begin{aligned}
&\left|D_{0+}^{\alpha_2-1,\beta_2}R(u,\lambda)(t)\right| \\
&\quad\le \int_0^t \varphi_q\left(\frac{1}{\Gamma(\alpha_1)}\int_s^1 (x-s)^{\alpha_1-1}\left|(I-Q)N_\lambda u(x)\right|dx\right)ds
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{\|T_1\|_\infty \|I_{0+}^{\alpha_2}\varphi_q(I_{1-}^{\alpha_1}(I-Q)N_\lambda u)\|_X}{\delta_2^2\Gamma(\gamma_2-1)+\delta_1^2\Gamma(\gamma_2)} \times |\delta_2|\Gamma(\gamma_2-1) I_{0+}^{\beta_2(3-\alpha_2)}\Gamma(\gamma_2) \\
&\leq \int_0^t \varphi_q\left(\frac{k_1+k_2}{\Gamma(\alpha_1+1)}\right) ds + \frac{\|T_1\|_\infty \varphi_q(\frac{k_1+k_2}{\Gamma(\alpha_1+1)})}{\delta_2^2\Gamma(\gamma_2-1)+\delta_1^2\Gamma(\gamma_2)} \times \frac{|\delta_2|\Gamma(\gamma_2-1)\Gamma(\gamma_2)}{\Gamma(\beta_2(3-\alpha_2)+1)} \\
&\leq \left[1 + \frac{\|T_1\|_\infty |\delta_2|\Gamma(\gamma_2)}{(\delta_2^2+\delta_1^2(\gamma_2-1))\Gamma(\beta_2(3-\alpha_2)+1)}\right]\varphi_q\left(\frac{k_1+k_2}{\Gamma(\alpha_1+1)}\right),
\end{aligned}
$$

$$
\begin{aligned}
\left|D_{0+}^{\alpha_2,\beta_2}R(u,\lambda)(t)\right| &\leq \varphi_q\left(\frac{1}{\Gamma(\alpha_1)}\int_t^1 (s-t)^{\alpha_1-1}\left|(I-Q)N_\lambda u(s)\right| ds\right) \\
&\leq \varphi_q\left(\frac{k_1+k_2}{\Gamma(\alpha_1+1)}\right).
\end{aligned}
$$

So, R is bounded in $\overline{\Omega}\times[0,1]$.

For $(u,\lambda)\in\overline{\Omega}\times[0,1]$, $0\leq t_1<t_2\leq 1$, we have

$$
\begin{aligned}
&\left|R(u,\lambda)(t_2)-R(u,\lambda)(t_1)\right| \\
&\leq \left|\frac{1}{\Gamma(\alpha_2)}\int_0^{t_2}(t_2-s)^{\alpha_2-1}\varphi_q\left(\frac{1}{\Gamma(\alpha_1)}\int_s^1 (x-s)^{\alpha_1-1}(I-Q)N_\lambda u(x)\,dx\right) ds\right. \\
&\quad \left. - \frac{1}{\Gamma(\alpha_2)}\int_0^{t_1}(t_1-s)^{\alpha_2-1}\varphi_q\left(\frac{1}{\Gamma(\alpha_1)}\int_s^1 (x-s)^{\alpha_1-1}(I-Q)N_\lambda u(x)\,dx\right) ds\right| \\
&\quad + \frac{\|T_1\|_\infty \|I_{0+}^{\alpha_2}\varphi_q(I_{1-}^{\alpha_1}(I-Q)N_\lambda u)\|_X}{\delta_2^2\Gamma(\gamma_2-1)+\delta_1^2\Gamma(\gamma_2)} \\
&\quad \times \left(|\delta_2|\Gamma(\gamma_2-1)\left(t_2^{\gamma_2-1}-t_1^{\gamma_2-1}\right) + |\delta_1|\Gamma(\gamma_2)\left(t_2^{\gamma_2-2}-t_1^{\gamma_2-2}\right)\right) \\
&\leq \frac{1}{\Gamma(\alpha_2)}\int_0^{t_1}\left[(t_2-s)^{\alpha_2-1}-(t_1-s)^{\alpha_2-1}\right] \\
&\quad \times \varphi_q\left(\frac{1}{\Gamma(\alpha_1)}\int_s^1 (x-s)^{\alpha_1-1}\left|(I-Q)N_\lambda u(x)\right| dx\right) ds \\
&\quad + \frac{1}{\Gamma(\alpha_2)}\int_{t_1}^{t_2}(t_2-s)^{\alpha_2-1}\varphi_q\left(\frac{1}{\Gamma(\alpha_1)}\int_s^1 (x-s)^{\alpha_1-1}\left|(I-Q)N_\lambda u(x)\right| dx\right) ds \\
&\quad + \frac{\|T_1\|_\infty \varphi_q(\frac{k_1+k_2}{\Gamma(\alpha_1+1)})}{\delta_2^2+\delta_1^2(\gamma_2-1)}\left(|\delta_2|\left(t_2^{\gamma_2-1}-t_1^{\gamma_2-1}\right)+|\delta_1|(\gamma_2-1)\left(t_2^{\gamma_2-2}-t_1^{\gamma_2-2}\right)\right) \\
&\leq \frac{\varphi_q(\frac{k_1+k_2}{\Gamma(\alpha_1+1)})}{\Gamma(\alpha_2)}\left[\int_0^{t_1}\left[(t_2-s)^{\alpha_2-1}-(t_1-s)^{\alpha_2-1}\right] ds + \int_{t_1}^{t_2}(t_2-s)^{\alpha_2-1}\,ds\right] \\
&\quad + \frac{\|T_1\|_\infty \varphi_q(\frac{k_1+k_2}{\Gamma(\alpha_1+1)})}{\delta_2^2+\delta_1^2(\gamma_2-1)}\left(|\delta_2|\left(t_2^{\gamma_2-1}-t_1^{\gamma_2-1}\right)+|\delta_1|(\gamma_2-1)\left(t_2^{\gamma_2-2}-t_1^{\gamma_2-2}\right)\right) \\
&\leq \varphi_q\left(\frac{k_1+k_2}{\Gamma(\alpha_1+1)}\right)\left[\frac{(t_2^{\alpha_2}-t_1^{\alpha_2})}{\Gamma(\alpha_2+1)} + \frac{\|T_1\|_\infty|\delta_2|}{\delta_2^2+\delta_1^2(\gamma_2-1)}\left(t_2^{\gamma_2-1}-t_1^{\gamma_2-1}\right)\right. \\
&\quad \left. + \frac{\|T_1\|_\infty|\delta_1|(\gamma_2-1)}{\delta_2^2+\delta_1^2(\gamma_2-1)}\left(t_2^{\gamma_2-2}-t_1^{\gamma_2-2}\right)\right],
\end{aligned}
$$

$$
\begin{aligned}
&\left|D_{0+}^{\alpha_2-2,\beta_2}R(u,\lambda)(t_2)-D_{0+}^{\alpha_2-2,\beta_2}R(u,\lambda)(t_1)\right| \\
&\leq \left|\int_0^{t_2}(t_2-s)\varphi_q\left(\frac{1}{\Gamma(\alpha_1)}\int_s^1 (x-s)^{\alpha_1-1}(I-Q)N_\lambda u(x)\,dx\right) ds\right.
\end{aligned}
$$

$$
\begin{aligned}
&\quad - \int_0^{t_1} (t_1 - s)\varphi_q\left(\frac{1}{\Gamma(\alpha_1)}\int_s^1 (x-s)^{\alpha_1-1}(I-Q)N_\lambda u(x)\,dx\right)ds\Bigg| \\
&\quad + \frac{\|T_1\|_\infty \varphi_q(\frac{k_1+k_2}{\Gamma(\alpha_1+1)})}{\delta_2^2\Gamma(\gamma_2-1)+\delta_1^2\Gamma(\gamma_2)} \\
&\quad \times \left(\frac{|\delta_2|\Gamma(\gamma_2-1)\Gamma(\gamma_2)}{\Gamma(\beta_2(3-\alpha_2)+2)} t_2^{\beta_2(3-\alpha_2)+1} - \frac{|\delta_2|\Gamma(\gamma_2-1)\Gamma(\gamma_2)}{\Gamma(\beta_2(3-\alpha_2)+2)} t_1^{\beta_2(3-\alpha_2)+1}\right) \\
&\le \int_0^{t_1} \big[(t_2-s)-(t_1-s)\big]\varphi_q\left(\frac{1}{\Gamma(\alpha_1)}\int_s^1 (x-s)^{\alpha_1-1}\big|(I-Q)N_\lambda u(x)\big|\,dx\right)ds \\
&\quad + \int_{t_1}^{t_2} (t_2-s)\varphi_q\left(\frac{1}{\Gamma(\alpha_1)}\int_s^1 (x-s)^{\alpha_1-1}\big|(I-Q)N_\lambda u(x)\big|\,dx\right)ds \\
&\quad + \frac{\|T_1\|_\infty \varphi_q(\frac{k_1+k_2}{\Gamma(\alpha_1+1)})|\delta_2|\Gamma(\gamma_2)}{\Gamma(\beta_2(3-\alpha_2)+2)(\delta_2^2+\delta_1^2(\gamma_2-1))}\big(t_2^{\beta_2(3-\alpha_2)+1} - t_1^{\beta_2(3-\alpha_2)+1}\big) \\
&\le \varphi_q\left(\frac{k_1+k_2}{\Gamma(\alpha_1+1)}\right)\left[\int_0^{t_1}\big[(t_2-s)-(t_1-s)\big]\,ds + \int_{t_1}^{t_2}(t_2-s)\,ds\right] \\
&\quad + \frac{\|T_1\|_\infty \varphi_q(\frac{k_1+k_2}{\Gamma(\alpha_1+1)})|\delta_2|\Gamma(\gamma_2)}{\Gamma(\beta_2(3-\alpha_2)+2)(\delta_2^2+\delta_1^2(\gamma_2-1))}\big(t_2^{\beta_2(3-\alpha_2)+1} - t_1^{\beta_2(3-\alpha_2)+1}\big) \\
&\le \varphi_q\left(\frac{k_1+k_2}{\Gamma(\alpha_1+1)}\right)\left[\frac{(t_2^2-t_1^2)}{2}\right. \\
&\quad \left. + \frac{\|T_1\|_\infty |\delta_2|\Gamma(\gamma_2)}{\Gamma(\beta_2(3-\alpha_2)+2)(\delta_2^2+\delta_1^2(\gamma_2-1))}\big(t_2^{\beta_2(3-\alpha_2)+1} - t_1^{\beta_2(3-\alpha_2)+1}\big)\right],
\end{aligned}
$$

$$
\begin{aligned}
&\big|D_{0+}^{\alpha_2-1,\beta_2}R(u,\lambda)(t_2) - D_{0+}^{\alpha_2-1,\beta_2}R(u,\lambda)(t_1)\big| \\
&\le \Bigg|\int_0^{t_2} \varphi_q\left(\frac{1}{\Gamma(\alpha_1)}\int_s^1 (x-s)^{\alpha_1-1}(I-Q)N_\lambda u(x)\,dx\right)ds \\
&\quad - \int_0^{t_1} \varphi_q\left(\frac{1}{\Gamma(\alpha_1)}\int_s^1 (x-s)^{\alpha_1-1}(I-Q)N_\lambda u(x)\,dx\right)ds\Bigg| \\
&\le \int_{t_1}^{t_2} \varphi_q\left(\frac{1}{\Gamma(\alpha_1)}\int_s^1 (x-s)^{\alpha_1-1}\big|(I-Q)N_\lambda u(x)\big|\,dx\right)ds \\
&\le \varphi_q\left(\frac{k_1+k_2}{\Gamma(\alpha_1+1)}\right)(t_2-t_1).
\end{aligned}
$$

So, $\{R(u,\lambda) \mid (u,\lambda) \in \overline{\Omega} \times [0,1]\}$, $\{D_{0+}^{\alpha_2-2,\beta_2}R(u,\lambda) \mid (u,\lambda) \in \overline{\Omega} \times [0,1]\}$ and $\{D_{0+}^{\alpha_2-1,\beta_2}R(u,\lambda) \mid (u,\lambda) \in \overline{\Omega} \times [0,1]\}$ are equicontinuous. Next, we prove that $\{D_{0+}^{\alpha_2,\beta_2}R(u,\lambda) \mid (u,\lambda) \in \overline{\Omega} \times [0,1]\}$ is also equicontinuous.

For $(u,\lambda) \in \overline{\Omega} \times [0,1]$, $0 \le t_1 < t_2 \le 1$, then

$$
\begin{aligned}
&\big|D_{0+}^{\alpha_2,\beta_2}R(u,\lambda)(t_2) - D_{0+}^{\alpha_2,\beta_2}R(u,\lambda)(t_1)\big| \\
&= \Bigg|\varphi_q\left(\frac{1}{\Gamma(\alpha_1)}\int_{t_2}^1 (s-t_2)^{\alpha_1-1}(I-Q)N_\lambda u(s)\,ds\right) \\
&\quad - \varphi_q\left(\frac{1}{\Gamma(\alpha_1)}\int_{t_1}^1 (s-t_1)^{\alpha_1-1}(I-Q)N_\lambda u(s)\,ds\right)\Bigg|.
\end{aligned}
$$

Since

$$\begin{aligned}
&\left|\frac{1}{\Gamma(\alpha_1)}\int_{t_2}^{1}(s-t_2)^{\alpha_1-1}(I-Q)N_\lambda u(s)\,ds-\frac{1}{\Gamma(\alpha_1)}\int_{t_1}^{1}(s-t_1)^{\alpha_1-1}(I-Q)N_\lambda u(s)\,ds\right|\\
&\quad\le\frac{1}{\Gamma(\alpha_1)}\left(\int_{t_2}^{1}\left[(s-t_2)^{\alpha_1-1}-(s-t_1)^{\alpha_1-1}\right]\left|(I-Q)N_\lambda u(s)\right|ds\right.\\
&\qquad\left.+\int_{t_1}^{t_2}(s-t_1)^{\alpha_1-1}\left|(I-Q)N_\lambda u(s)\right|ds\right)\\
&\quad\le\frac{2(k_1+k_2)}{\Gamma(\alpha_1+1)}(t_2-t_1)^{\alpha_1},
\end{aligned}$$

and

$$\left|\frac{1}{\Gamma(\alpha_1)}\int_{t}^{1}(s-t)^{\alpha_1-1}(I-Q)N_\lambda u(s)\,ds\right|\le\frac{k_1+k_2}{\Gamma(\alpha_1+1)},\quad (u,\lambda)\in\overline{\Omega}\times[0,1],$$

and taking into account that φ_q is uniformly continuous in $[-\frac{k_1+k_2}{\Gamma(\alpha_1+1)},\frac{k_1+k_2}{\Gamma(\alpha_1+1)}]$, we can obtain $\{D_{0+}^{\alpha_2,\beta_2}R(u,\lambda)\mid(u,\lambda)\in\overline{\Omega}\times[0,1]\}$ is also equicontinuous. By the Arzela–Ascoli theorem, we get that $R:\Omega\times[0,1]\to X_2$ is compact. □

Lemma 3.5 *Assume that $\Omega\subset X$ is an open and bounded set. Then N_λ is L-quasicompact in $\overline{\Omega}$.*

Proof It is obvious that $\operatorname{Im}P=\operatorname{Ker}L$, $\dim\operatorname{Ker}L=\dim\operatorname{Im}Q$, $Q(I-Q)=0$, $\operatorname{Ker}Q=\operatorname{Im}L$, $R(\cdot,0)=0$ and that Definition 2.2(b) holds.

For $u\in\Sigma_\lambda=\{u\in\overline{\Omega}\mid Lu=N_\lambda u\}$, we can get $N_\lambda u\in\operatorname{Im}L=\operatorname{Ker}Q$. Thus, we have $QN_\lambda u=0$ and $N_\lambda u=Lu=D_{1-}^{\alpha_1,\beta_1}\varphi_p(D_{0^+}^{\alpha_2,\beta_2}u)$, then

$$\begin{aligned}
I_{0+}^{\alpha_2}\varphi_q\big(I_{1-}^{\alpha_1}N_\lambda u(t)\big)&=I_{0+}^{\alpha_2}\varphi_q\big(I_{1-}^{\alpha_1}I_{1-}^{\beta_1(1-\alpha_1)}D_{1-}^{\gamma_1}\varphi_p\big(I_{0+}^{\beta_2(3-\alpha_2)}D_{0+}^{\gamma_2}u(t)\big)\big)=I_{0+}^{\gamma_2}D_{0+}^{\gamma_2}u(t)\\
&=u(t)-\frac{D_{0+}^{\gamma_2-1}u(0)}{\Gamma(\gamma_2)}t^{\gamma_2-1}-\frac{D_{0+}^{\gamma_2-2}u(0)}{\Gamma(\gamma_2-1)}t^{\gamma_2-2}-\frac{D_{0+}^{\gamma_2-3}u(0)}{\Gamma(\gamma_2-2)}t^{\gamma_2-3}.
\end{aligned}$$

Since $u(0)=0$, we obtain $D_{0+}^{\gamma_2-3}u(0)=0$. It follows from $D_{0+}^{\alpha_2,\beta_2}u(1)=u(0)=D_{0+}^{\alpha_2,\beta_2}R(u,\lambda)(1)=R(u,\lambda)(0)=D_{0+}^{\gamma_2-3}u(0)=T_1(u)=0$ that

$$\begin{aligned}
R(u,\lambda)&=I_{0+}^{\alpha_2}\varphi_q\big(I_{1-}^{\alpha_1}\big(N_\lambda u(t)-QN_\lambda u(t)\big)\big)\\
&\quad-\frac{T_1(I_{0+}^{\alpha_2}\varphi_q(I_{1-}^{\alpha_1}(N_\lambda u(t)-QN_\lambda u(t))))}{\delta_2^2\Gamma(\gamma_2-1)+\delta_1^2\Gamma(\gamma_2)}\big(\delta_2\Gamma(\gamma_2-1)t^{\gamma_2-1}+\delta_1\Gamma(\gamma_2)t^{\gamma_2-2}\big)\\
&=u(t)-\frac{D_{0+}^{\gamma_2-1}u(0)}{\Gamma(\gamma_2)}t^{\gamma_2-1}-\frac{D_{0+}^{\gamma_2-2}u(0)}{\Gamma(\gamma_2-1)}t^{\gamma_2-2}\\
&\quad+\frac{T_1(\frac{D_{0+}^{\gamma_2-1}u(0)}{\Gamma(\gamma_2)}t^{\gamma_2-1})+T_1(\frac{D_{0+}^{\gamma_2-2}u(0)}{\Gamma(\gamma_2-1)}t^{\gamma_2-2})}{\delta_2^2\Gamma(\gamma_2-1)+\delta_1^2\Gamma(\gamma_2)}\big(\delta_2\Gamma(\gamma_2-1)t^{\gamma_2-1}+\delta_1\Gamma(\gamma_2)t^{\gamma_2-2}\big)\\
&=u(t)-\frac{D_{0+}^{\gamma_2-1}u(0)}{\Gamma(\gamma_2)}t^{\gamma_2-1}+\frac{\delta_2^2\Gamma(\gamma_2-1)D_{0+}^{\gamma_2-1}u(0)+\delta_1\delta_2\Gamma(\gamma_2)D_{0+}^{\gamma_2-2}u(0)}{\Gamma(\gamma_2)(\delta_2^2\Gamma(\gamma_2-1)+\delta_1^2\Gamma(\gamma_2))}t^{\gamma_2-1}
\end{aligned}$$

$$
\begin{aligned}
&-\frac{D_{0+}^{\gamma_2-2}u(0)}{\Gamma(\gamma_2-1)}t^{\gamma_2-2}+\frac{\delta_1\delta_2\Gamma(\gamma_2-1)D_{0+}^{\gamma_2-1}u(0)+\delta_1^2\Gamma(\gamma_2)D_{0+}^{\gamma_2-2}u(0)}{\Gamma(\gamma_2-1)(\delta_2^2\Gamma(\gamma_2-1)+\delta_1^2\Gamma(\gamma_2))}t^{\gamma_2-2}\\
&=u(t)-\frac{\delta_2D_{0+}^{\gamma_2-2}u(0)-\delta_1D_{0+}^{\gamma_2-1}u(0)}{\delta_2^2\Gamma(\gamma_2-1)+\delta_1^2\Gamma(\gamma_2)}\big(\delta_2t^{\gamma_2-2}-\delta_1t^{\gamma_2-1}\big)\\
&=u(t)-Pu(t)=(I-P)u,
\end{aligned}
$$

i.e., Definition 2.2(c) holds.

For $u\in\overline{\Omega}$, we have

$$
\begin{aligned}
L\big[Pu(t)+R(u,\lambda)(t)\big]&=I_{1-}^{\beta_1(1-\alpha_1)}D_{1-}^{\gamma_1}\varphi_p\big(I_{0+}^{\beta_2(3-\alpha_2)}D_{0+}^{\gamma_2}\big(Pu(t)+R(u,\lambda)(t)\big)\big)\\
&=I_{1-}^{\beta_1(1-\alpha_1)}D_{1-}^{\beta_1(1-\alpha_1)}(I-Q)N_\lambda u(t)\\
&=(I-Q)N_\lambda u(t),
\end{aligned}
$$

i.e., Definition 2.2(d) holds. Therefore, N_λ is L-quasicompact in $\overline{\Omega}$. □

Theorem 3.6 *Suppose that (H_1)–(H_3) and the following conditions hold:*

(H_4) *There exists a constant $M_0>0$ such that if $|t^{-\beta_2(3-\alpha_2)}D_{0+}^{\alpha_2-2,\beta_2}u(t)|+|t^{-\beta_2(3-\alpha_2)}\times D_{0+}^{\alpha_2-1,\beta_2}u(t)|>M_0$, then $(T_2-kT_1)(I_{0+}^{\alpha_2}\varphi_q(I_{1-}^{\alpha_1}Nu(t)))\neq 0$.*

(H_5) *There exist nonnegative functions $a(t),b(t),c(t),d(t),e(t)\in C[0,1]$, such that*

$$
\begin{aligned}
\big|f(t,x,y,z,w)\big|\le{}& a(t)+b(t)\varphi_p\big(|x|\big)+c(t)\varphi_p\big(|y|\big)\\
&+d(t)\varphi_p\big(|z|\big)+e(t)\varphi_p\big(|w|\big),\quad x,y,z,w\in\mathbb{R},
\end{aligned}
$$

where $\Gamma(\alpha_1+1)>A(\frac{2\Gamma(\gamma_2-1)+5\Gamma(\beta_2(3-\alpha_2)+1)\Gamma(\alpha_2+1)}{2\Gamma(\alpha_2+1)\Gamma(\gamma_2-1)})^{p-1}\|b\|_\infty+A3^{p-1}\|c\|_\infty+A2^{p-1}\|d\|_\infty+\|e\|_\infty$, $A=\max_{p\in(1,+\infty)}\{1,2^{p-2}\}$.

(H_6) *There exists $B_1>0$ such that one of the following inequalities holds:*

$$
(1)\quad cQN\big(c\big(\delta_2t^{\gamma_2-2}-\delta_1t^{\gamma_2-1}\big)\big)>0,\qquad (2)\quad cQN\big(c\big(\delta_2t^{\gamma_2-2}-\delta_1t^{\gamma_2-1}\big)\big)<0.
$$

Then problem (1.1) has at least one solution in X.

Lemma 3.7 *Suppose that (H_4) and (H_5) hold, then $\Omega_1=\{u\mid u\in\operatorname{dom}L\setminus\operatorname{Ker}L, Lu=N_\lambda u,\lambda\in(0,1)\}$ is bounded in X.*

Proof For $u\in\operatorname{dom}L$, according to Lemma 2.8, we obtain

$$
u(t)=I_{0+}^{\gamma_2}D_{0+}^{\gamma_2}u(t)+c_1t^{\gamma_2-1}+c_2t^{\gamma_2-2}.\tag{3.6}
$$

Applying $D_{0+}^{\alpha_2-1,\beta_2}$ and $D_{0+}^{\alpha_2-2,\beta_2}$ to both sides of (3.6) respectively, we can get

$$
\begin{aligned}
D_{0+}^{\alpha_2-1,\beta_2}u(t)&=\int_0^tD_{0+}^{\alpha_2,\beta_2}u(s)\,ds+\frac{c_1\Gamma(\gamma_2)}{\Gamma(\beta_2(3-\alpha_2)+1)}t^{\beta_2(3-\alpha_2)},\\
D_{0+}^{\alpha_2-2,\beta_2}u(t)&=\int_0^t(t-s)D_{0+}^{\alpha_2,\beta_2}u(s)\,ds+\frac{c_1\Gamma(\gamma_2)}{\Gamma(\beta_2(3-\alpha_2)+2)}t^{\beta_2(3-\alpha_2)+1}\\
&\quad+\frac{c_2\Gamma(\gamma_2-1)}{\Gamma(\beta_2(3-\alpha_2)+1)}t^{\beta_2(3-\alpha_2)}.
\end{aligned}
$$

Therefore,

$$c_1 = \frac{\Gamma(\beta_2(3-\alpha_2)+1)}{\Gamma(\gamma_2)}\left(t^{-\beta_2(3-\alpha_2)}D_{0+}^{\alpha_2-1,\beta_2}u(t) - t^{-\beta_2(3-\alpha_2)}\int_0^t D_{0+}^{\alpha_2,\beta_2}u(s)\,ds\right), \tag{3.7}$$

$$c_2 = \frac{\Gamma(\beta_2(3-\alpha_2)+1)}{\Gamma(\gamma_2-1)}\left(t^{-\beta_2(3-\alpha_2)}D_{0+}^{\alpha_2-2,\beta_2}u(t) - t^{-\beta_2(3-\alpha_2)}\int_0^t (t-s)D_{0+}^{\alpha_2,\beta_2}u(s)\,ds - \frac{c_1\Gamma(\gamma_2)t}{\Gamma(\beta_2(3-\alpha_2)+2)}\right). \tag{3.8}$$

For $u \in \Omega_1$, we have $Lu = N_\lambda u$, $N_\lambda u \in \operatorname{Im} L = \operatorname{Ker} Q$, we get $QN_\lambda u(t) = 0$. It follows from (H_4) that there exists $t_0 \in (0,1]$, such that $|t_0^{-\beta_2(3-\alpha_2)}D_{0+}^{\alpha_2-2,\beta_2}u(t_0)| + |t_0^{-\beta_2(3-\alpha_2)}D_{0+}^{\alpha_2-1,\beta_2}u(t_0)| \le M_0$, then $|t_0^{-\beta_2(3-\alpha_2)}D_{0+}^{\alpha_2-2,\beta_2}u(t_0)| \le M_0$ and $|t_0^{-\beta_2(3-\alpha_2)}D_{0+}^{\alpha_2-1,\beta_2}u(t_0)| \le M_0$. Taking $t = t_0$ into equations (3.7) and (3.8), we have

$$|c_1| \le \frac{\Gamma(\beta_2(3-\alpha_2)+1)}{\Gamma(\gamma_2)}\left(M_0 + \left\|D_{0+}^{\alpha_2,\beta_2}u\right\|_\infty\right),$$

$$\begin{aligned}|c_2| &\le \frac{\Gamma(\beta_2(3-\alpha_2)+1)}{\Gamma(\gamma_2-1)}\left(M_0 + \frac{1}{2}\left\|D_{0+}^{\alpha_2,\beta_2}u\right\|_\infty + \frac{|c_1|\Gamma(\gamma_2)}{\Gamma(\beta_2(3-\alpha_2)+2)}\right)\\ &\le \frac{\Gamma(\beta_2(3-\alpha_2)+1)}{\Gamma(\gamma_2-1)}\left(2M_0 + \frac{3}{2}\left\|D_{0+}^{\alpha_2,\beta_2}u\right\|_\infty\right).\end{aligned}$$

Thus,

$$\left\|D_{0+}^{\alpha_2-1,\beta_2}u\right\|_\infty \le M_0 + 2\left\|D_{0+}^{\alpha_2,\beta_2}u\right\|_\infty, \qquad \left\|D_{0+}^{\alpha_2-2,\beta_2}u\right\|_\infty \le 3\left(M_0 + \left\|D_{0+}^{\alpha_2,\beta_2}u\right\|_\infty\right).$$

Since $u(t)$ in (3.6) can also be written as $u(t) = I_{0+}^{\alpha_2}D_{0+}^{\alpha_2,\beta_2}u(t) + c_1t^{\gamma_2-1} + c_2t^{\gamma_2-2}$, then

$$\|u\|_\infty \le \frac{3\Gamma(\beta_2(3-\alpha_2)+1)}{\Gamma(\gamma_2-1)}M_0 + \frac{2\Gamma(\gamma_2-1) + 5\Gamma(\beta_2(3-\alpha_2)+1)\Gamma(\alpha_2+1)}{2\Gamma(\alpha_2+1)\Gamma(\gamma_2-1)}\left\|D_{0+}^{\alpha_2,\beta_2}u\right\|_\infty.$$

According to $Lu(t) = N_\lambda u(t)$ and boundary conditions, we can get

$$u(t) = I_{0+}^{\alpha_2}\varphi_q\left(I_{1-}^{\alpha_1}N_\lambda u(t)\right) + c_3t^{\gamma_2-1} + c_4t^{\gamma_2-2}.$$

Therefore,

$$\begin{aligned}&\left|\varphi_p\left(D_{0+}^{\alpha_2,\beta_2}u(t)\right)\right|\\ &\quad= \left|\varphi_p\left(I_{0+}^{\beta_2(3-\alpha_2)}D_{0+}^{\gamma_2}I_{0+}^{\alpha_2}\varphi_q\left(I_{1-}^{\alpha_1}N_\lambda u(t)\right)\right)\right| = \left|I_{1-}^{\alpha_1}N_\lambda u(t)\right|\\ &\quad\le \frac{\lambda}{\Gamma(\alpha_1)}\int_t^1 (s-t)^{\alpha_1-1}\left|f\left(s,u(s),D_{0+}^{\alpha_2-2,\beta_2}u(s),D_{0+}^{\alpha_2-1,\beta_2}u(s),D_{0+}^{\alpha_2,\beta_2}u(s)\right)\right|\,ds\\ &\quad\le \frac{\lambda}{\Gamma(\alpha_1)}\int_t^1 (s-t)^{\alpha_1-1}\left(a(t) + b(t)\varphi_p\left(|u(s)|\right) + c(t)\varphi_p\left(\left|D_{0+}^{\alpha_2-2,\beta_2}u(t)\right|\right)\right.\\ &\qquad\left. + d(t)\varphi_p\left(\left|D_{0+}^{\alpha_2-1,\beta_2}u(t)\right|\right) + e(t)\varphi_p\left(\left|D_{0+}^{\alpha_2,\beta_2}u(t)\right|\right)\right)\,ds\end{aligned}$$

$$
\begin{aligned}
&\le \frac{1}{\Gamma(\alpha_1+1)}\big(\|a\|_\infty + \|b\|_\infty\|u\|_\infty^{p-1} + \|c\|_\infty\big\|D_{0+}^{\alpha_2-2,\beta_2}u\big\|_\infty^{p-1} + \|d\|_\infty\big\|D_{0+}^{\alpha_2-1,\beta_2}u\big\|_\infty^{p-1} \\
&\quad + \|e\|_\infty\big\|D_{0+}^{\alpha_2,\beta_2}u\big\|_\infty^{p-1}\big) \\
&\le \frac{1}{\Gamma(\alpha_1+1)}\bigg[\|a\|_\infty + \|b\|_\infty\bigg(\frac{2\Gamma(\gamma_2-1)+5\Gamma(\beta_2(3-\alpha_2)+1)\Gamma(\alpha_2+1)}{2\Gamma(\alpha_2+1)\Gamma(\gamma_2-1)}\big\|D_{0+}^{\alpha_2,\beta_2}u\big\|_\infty \\
&\quad + \frac{3M_0\Gamma(\beta_2(3-\alpha_2)+1)}{\Gamma(\gamma_2-1)}\bigg)^{p-1} + \|c\|_\infty\big(3M_0 + 3\big\|D_{0+}^{\alpha_2,\beta_2}u\big\|_\infty\big)^{p-1} \\
&\quad + \|d\|_\infty\big(M_0 + 2\big\|D_{0+}^{\alpha_2,\beta_2}u\big\|_\infty\big)^{p-1} + \|e\|_\infty\big\|D_{0+}^{\alpha_2,\beta_2}u\big\|_\infty^{p-1}\bigg].
\end{aligned}
$$

It is known that $|\varphi_p(D_{0+}^{\alpha_2,\beta_2}u(t))| = |D_{0+}^{\alpha_2,\beta_2}u(t)|^{p-1}$, then

$$
\begin{aligned}
&\big|D_{0+}^{\alpha_2,\beta_2}u(t)\big|^{p-1} \\
&\le \frac{1}{\Gamma(\alpha_1+1)}\bigg[\|a\|_\infty + \|b\|_\infty\bigg(\frac{2\Gamma(\gamma_2-1)+5\Gamma(\beta_2(3-\alpha_2)+1)\Gamma(\alpha_2+1)}{2\Gamma(\alpha_2+1)\Gamma(\gamma_2-1)}\big\|D_{0+}^{\alpha_2,\beta_2}u\big\|_\infty \\
&\quad + \frac{3M_0\Gamma(\beta_2(3-\alpha_2)+1)}{\Gamma(\gamma_2-1)}\bigg)^{p-1} + \|c\|_\infty\big(3M_0 + 3\big\|D_{0+}^{\alpha_2,\beta_2}u\big\|_\infty\big)^{p-1} \\
&\quad + \|d\|_\infty\big(M_0 + 2\big\|D_{0+}^{\alpha_2,\beta_2}u\big\|_\infty\big)^{p-1} + \|e\|_\infty\big\|D_{0+}^{\alpha_2,\beta_2}u\big\|_\infty^{p-1}\bigg].
\end{aligned}
$$

If $1 < p \le 2$, then

$$
\begin{aligned}
&\big\|D_{0+}^{\alpha_2,\beta_2}u\big\|_\infty^{p-1} \\
&\le \frac{1}{\Gamma(\alpha_1+1)}\bigg[\|a\|_\infty + \|b\|_\infty\bigg(\frac{2\Gamma(\gamma_2-1)+5\Gamma(\beta_2(3-\alpha_2)+1)\Gamma(\alpha_2+1)}{2\Gamma(\alpha_2+1)\Gamma(\gamma_2-1)}\bigg)^{p-1} \\
&\quad \times \big\|D_{0+}^{\alpha_2,\beta_2}u\big\|_\infty^{p-1} \\
&\quad + \|b\|_\infty\bigg(\frac{3M_0\Gamma(\beta_2(3-\alpha_2)+1)}{\Gamma(\gamma_2-1)}\bigg)^{p-1} + \|c\|_\infty(3M_0)^{p-1} + \|c\|_\infty 3^{p-1}\big\|D_{0+}^{\alpha_2,\beta_2}u\big\|_\infty^{p-1} \\
&\quad + \|d\|_\infty M_0^{p-1} + \|d\|_\infty 2^{p-1}\big\|D_{0+}^{\alpha_2,\beta_2}u\big\|_\infty^{p-1} + \|e\|_\infty\big\|D_{0+}^{\alpha_2,\beta_2}u\big\|_\infty^{p-1}\bigg].
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
&\big\|D_{0+}^{\alpha_2,\beta_2}\big\|_\infty \\
&\le \bigg(\frac{\|a\|_\infty + \|b\|_\infty(\frac{3M_0\Gamma(\beta_2(3-\alpha_2)+1)}{\Gamma(\gamma_2-1)})^{p-1} + \|c\|_\infty(3M_0)^{p-1} + \|d\|_\infty M_0^{p-1}}{\Gamma(\alpha_1+1) - [\|b\|_\infty(\frac{2\Gamma(\gamma_2-1)+5\Gamma(\beta_2(3-\alpha_2)+1)\Gamma(\alpha_2+1)}{2\Gamma(\alpha_2+1)\Gamma(\gamma_2-1)})^{p-1} + \|c\|_\infty 3^{p-1} + \|d\|_\infty 2^{p-1} + \|e\|_\infty]}\bigg)^{p-1}.
\end{aligned}
$$

If $p > 2$, then

$$
\begin{aligned}
&\big\|D_{0+}^{\alpha_2,\beta_2}\big\|_\infty \\
&\le \bigg(\frac{\|a\|_\infty + 2^{p-1}\|b\|_\infty(\frac{3M_0\Gamma(\beta_2(3-\alpha_2)+1)}{\Gamma(\gamma_2-1)})^{p-1} + 2^{p-1}\|c\|_\infty(3M_0)^{p-1} + 2^{p-1}\|d\|_\infty M_0^{p-1}}{\Gamma(\alpha_1+1) - 2^{p-1}[\|b\|_\infty(\frac{2\Gamma(\gamma_2-1)+5\Gamma(\beta_2(3-\alpha_2)+1)\Gamma(\alpha_2+1)}{2\Gamma(\alpha_2+1)\Gamma(\gamma_2-1)})^{p-1} + \|c\|_\infty 3^{p-1} + \|d\|_\infty 2^{p-1} + \frac{\|e\|_\infty}{2^{p-1}}]}\bigg)^{p-1}.
\end{aligned}
$$

Set $A = \max_{p\in(1,+\infty)}\{1,2^{p-2}\}$, then the above inequality is equivalent to

$$\begin{aligned}&\|D_{0+}^{\alpha_2,\beta_2}\|_\infty\\&\le\left(\frac{\|a\|_\infty + A\|b\|_\infty(\frac{3M_0\Gamma(\beta_2(3-\alpha_2)+1)}{\Gamma(\gamma_2-1)})^{p-1} + A\|c\|_\infty(3M_0)^{p-1} + A\|d\|_\infty M_0^{p-1}}{\Gamma(\alpha_1+1) - [A\|b\|_\infty(\frac{2\Gamma(\gamma_2-1)+5\Gamma(\beta_2(3-\alpha_2)+1)\Gamma(\alpha_2+1)}{2\Gamma(\alpha_2+1)\Gamma(\gamma_2-1)})^{p-1} + A\|c\|_\infty 3^{p-1} + A\|d\|_\infty 2^{p-1} + \|e\|_\infty]}\right)^{p-1}\\&:= M_1.\end{aligned}$$

Therefore,

$$\begin{aligned}&\|D_{0+}^{\alpha_2,\beta_2}\|_\infty \le M_1,\qquad \|D_{0+}^{\alpha_2-1,\beta_2}\|_\infty \le M_0 + 2M_1,\qquad \|D_{0+}^{\alpha_2-2,\beta_2}\|_\infty \le 3(M_0+M_1),\\&\|u\|_\infty \le \frac{3\Gamma(\beta_2(3-\alpha_2)+1)}{\Gamma(\gamma_2-1)}M_0 + \frac{2\Gamma(\gamma_2-1)+5\Gamma(\beta_2(3-\alpha_2)+1)\Gamma(\alpha_2+1)}{2\Gamma(\alpha_2+1)\Gamma(\gamma_2-1)}M_1 := M_2,\end{aligned}$$

we can get

$$\begin{aligned}\|u\|_X &= \max\left\{\|u\|_\infty, \left\|D_{0+}^{\alpha_2-2,\beta_2}u\right\|_\infty, \left\|D_{0+}^{\alpha_2-1,\beta_2}u\right\|_\infty, \left\|D_{0+}^{\alpha_2,\beta_2}u\right\|_\infty\right\}\\&\le \max\left\{M_2, 3(M_0+M_1), M_0+2M_1, M_1\right\} := M_3.\end{aligned}$$

Hence, we can conclude that Ω_1 is bounded in X. □

Lemma 3.8 *Suppose that* (H_1)*–*(H_3) *and* (H_6) *hold, then* $\Omega_2 = \{u|u \in \operatorname{Ker} L, QNu = 0\}$ *is bounded in* X.

Proof Let $u \in \Omega_2$, we have $u(t) = c(\delta_2 t^{\gamma_2-2} - \delta_1 t^{\gamma_2-1})$, $c \in \mathbb{R}$.

Since $QNu(t) = 0$, according to (H_6), there exists a constant $B_1 > 0$ such that $|c| \le B_1$, then

$$\begin{aligned}&\|u\|_\infty \le B_1\left(|\delta_2| + |\delta_1|\right),\\&\left\|D_{0+}^{\alpha_2-2,\beta_2}u\right\|_\infty \le B_1\left(\frac{|\delta_2|(\beta_2(3-\alpha_2)+1)\Gamma(\gamma_2-1) + |\delta_1|\Gamma(\gamma_2)}{\Gamma(\beta_2(3-\alpha_2)+2)}\right),\\&\left\|D_{0+}^{\alpha_2-1,\beta_2}u\right\|_\infty \le \frac{B_1|\delta_1|\Gamma(\gamma_2)}{\Gamma(\beta_2(3-\alpha_2)+1)}.\end{aligned}$$

Thus,

$$\begin{aligned}\|u\|_X \le \max&\left\{B_1\left(|\delta_2| + |\delta_1|\right), B_1\left(\frac{|\delta_2|(\beta_2(3-\alpha_2)+1)\Gamma(\gamma_2-1) + |\delta_1|\Gamma(\gamma_2)}{\Gamma(\beta_2(3-\alpha_2)+2)}\right),\right.\\&\left.\frac{B_1|\delta_1|\Gamma(\gamma_2)}{\Gamma(\beta_2(3-\alpha_2)+1)}\right\}\\&:= M_4,\end{aligned}$$

we can conclude that Ω_2 is bounded in X. □

Proof of Theorem 3.6 Let $\Omega \supset \overline{\Omega_1} \cup \overline{\Omega_2} \cup \{u \mid u \in X, \|u\|_X \le \max\{M_3, M_4\} + 1\}$ be an open and bounded set of X. By Lemma 3.7 and Lemma 3.8, we can get $Lu \ne N_\lambda u$, $u \in \operatorname{dom} L \cap \partial\Omega$ and $QNu \ne 0$, $u \in \operatorname{Ker} L \cap \partial\Omega$.

Let $H(u,\xi) = \rho\xi u + (1-\xi)JQNu$, $\xi \in [0,1]$, $u \in \operatorname{Ker} L \cap \overline{\Omega}$, where $J : \operatorname{Im} Q \to \operatorname{Ker} L$ is a homeomorphism with $Jc = c(\delta_2 t^{\gamma_2-2} - \delta_1 t^{\gamma_2-1})$,

$$\rho = \begin{cases} 1, & \text{if } (H_6)\ (1) \text{ holds}, \\ -1, & \text{if } (H_6)\ (2) \text{ holds}. \end{cases}$$

For $u \in \operatorname{Ker} L \cap \partial\Omega$, we have $u(t) = c(\delta_2 t^{\gamma_2-2} - \delta_1 t^{\gamma_2-1})$. Therefore

$$H(u,\xi) = \rho\xi c\bigl(\delta_2 t^{\gamma_2-2} - \delta_1 t^{\gamma_2-1}\bigr) + (1-\xi)QN\bigl(c\bigl(\delta_2 t^{\gamma_2-2} - \delta_1 t^{\gamma_2-1}\bigr)\bigr)\bigl(\delta_2 t^{\gamma_2-2} - \delta_1 t^{\gamma_2-1}\bigr).$$

If $\xi = 1$, then $H(u,1) = \rho c(\delta_2 t^{\gamma_2-2} - \delta_1 t^{\gamma_2-1}) \neq 0$. If $\xi = 0$, then $H(u,0) = QN(c(\delta_2 t^{\gamma_2-2} - \delta_1 t^{\gamma_2-1}))(\delta_2 t^{\gamma_2-2} - \delta_1 t^{\gamma_2-1}) \neq 0$. If $0 < \xi < 1$, suppose $H(u,\xi) = 0$, then $\rho\xi c(\delta_2 t^{\gamma_2-2} - \delta_1 t^{\gamma_2-1}) = -(1-\xi)QN(c(\delta_2 t^{\gamma_2-2} - \delta_1 t^{\gamma_2-1}))(\delta_2 t^{\gamma_2-2} - \delta_1 t^{\gamma_2-1})$. So, $c = -(\frac{1-\xi}{\rho\xi})QN(c(\delta_2 t^{\gamma_2-2} - \delta_1 t^{\gamma_2-1}))$. By (H_6), we get

$$c^2 = -\left(\frac{1-\xi}{\rho\xi}\right)cQN\bigl(c\bigl(\delta_2 t^{\gamma_2-2} - \delta_1 t^{\gamma_2-1}\bigr)\bigr) < 0.$$

A contradiction. That is, $H(u,\xi) \neq 0$, $u \in \operatorname{Ker} L \cap \partial\Omega$, $\xi \in [0,1]$.

Therefore, via the homotopy property of degree, we obtain

$$\begin{aligned} \deg(JQN, \Omega \cap \operatorname{Ker} L, 0) &= \deg\bigl(H(\cdot,0), \Omega \cap \operatorname{Ker} L, 0\bigr) \\ &= \deg\bigl(H(\cdot,1), \Omega \cap \operatorname{Ker} L, 0\bigr) \\ &= \deg(\rho I, \Omega \cap \operatorname{Ker} L, 0) \neq 0. \end{aligned}$$

Applying Lemma 2.3, we conclude that boundary value problem (1.1) has at least one solution in X. □

For another result of problem (1.1), suppose that the inequality $|t^{-\beta_2(3-\alpha_2)} D_{0+}^{\alpha_2-2,\beta_2} u(t)| + |t^{-\beta_2(3-\alpha_2)} D_{0+}^{\alpha_2-1,\beta_2} u(t)| > M_0$ in condition (H_4) is replaced by $|t^{-\beta_2(3-\alpha_2)} D_{0+}^{\alpha_2-1,\beta_2} u(t)| > M_0'$ or $|t^{-\beta_2(3-\alpha_2)} D_{0+}^{\alpha_2-2,\beta_2} u(t)| > M_0''$, which will cause the proof of Lemma 3.7 to change, but the result of Theorem 3.6 can still be obtained, as shown below.

Theorem 3.9 *Suppose that* (H_1)–(H_3), (H_6) *and the following conditions hold*:

(H_7) *There exists a constant* $M_0' > 0$ *such that if* $|t^{-\beta_2(3-\alpha_2)} D_{0+}^{\alpha_2-1,\beta_2} u(t)| > M_0'$, *then* $(T_2 - kT_1)(I_{0+}^{\alpha_2} \varphi_q(I_{1-}^{\alpha_1} Nu(t))) \neq 0$.

(H_8) *There exist nonnegative functions* $a(t), b(t), c(t), d(t), e(t) \in C[0,1]$, *such that*

$$\begin{aligned} \bigl|f(t,x,y,z,w)\bigr| \leq{}& a(t) + b(t)\varphi_p\bigl(|x|\bigr) + c(t)\varphi_p\bigl(|y|\bigr) \\ &+ d(t)\varphi_p\bigl(|z|\bigr) + e(t)\varphi_p\bigl(|w|\bigr), \quad x,y,z,w \in R, \end{aligned}$$

where $L(C_1 M + C_2)(\|b\|_\infty + \|c\|_\infty + \|d\|_\infty + \|e\|_\infty)^{q-1} < 1$, $L = \max_{q\in(1,+\infty)}\{1, 2^{q-2}\}$,

$$\begin{aligned} C_1 = \max\biggl\{&|\delta_2| + |\delta_1|, \frac{|\delta_2|(\beta_2(3-\alpha_2)+1)\Gamma(\gamma_2-1) + |\delta_1|\Gamma(\gamma_2)}{\Gamma(\beta_2(3-\alpha_2)+2)}, \\ &\frac{|\delta_1|\Gamma(\gamma_2)}{\Gamma(\beta_2(3-\alpha_2)+1)}\biggr\}, \end{aligned}$$

$$\begin{aligned} C_2 = \max\bigg\{&1+\frac{\|T_1\|_\infty(|\delta_2|\Gamma(\gamma_2-1)+|\delta_1|\Gamma(\gamma_2))}{\delta_2^2\Gamma(\gamma_2-1)+\delta_1^2\Gamma(\gamma_2)},\\ &\frac{1}{2}+\frac{\|T_1\|_\infty(|\delta_2|\Gamma(\gamma_2)+(\beta_2(3-\alpha_2)+1)|\delta_1|\Gamma(\gamma_2))}{(\delta_2^2+\delta_1^2(\gamma_2-1))\Gamma(\beta_2(3-\alpha_2)+2)},\\ &1+\frac{\|T_1\|_\infty|\delta_2|\Gamma(\gamma_2)}{(\delta_2^2+\delta_1^2(\gamma_2-1))\Gamma(\beta_2(3-\alpha_2)+1)}\bigg\}\times\varphi_q\bigg(\frac{1}{\Gamma(\alpha_1+1)}\bigg),\\ M = {}&\frac{\Gamma(\beta_2(3-\alpha_2)+1)\varphi_q(\frac{1}{\Gamma(\alpha_1+1)})}{|\delta_1|\Gamma(\gamma_2)}+\frac{\|T_1\|_\infty|\delta_2|\Gamma(\gamma_2-1)\varphi_q(\frac{1}{\Gamma(\alpha_1+1)})}{|\delta_1|(\delta_2^2\Gamma(\gamma_2-1)+\delta_1^2\Gamma(\gamma_2))}. \end{aligned}$$

Then problem (1.1) *has at least one solution in* X.

Proof For $u\in\Omega_1$, we have $QN_\lambda u(t)=0$. It follows from (H_7) that there exists $t_0\in(0,1]$, such that $|t_0^{-\beta_2(3-\alpha_2)}D_{0+}^{\alpha_2-1,\beta_2}u(t_0)|\le M_0'$. By Lemma 3.5, we obtain $R(u,\lambda)(t)=(I-P)u(t)=u(t)-Pu(t)$. So $D_{0+}^{\alpha_2-1,\beta_2}Pu(t)=D_{0+}^{\alpha_2-1,\beta_2}u(t)-D_{0+}^{\alpha_2-1,\beta_2}R(u,\lambda)(t)$. According to the definition of P, we can get

$$\begin{aligned} &\bigg|\frac{\delta_2D_{0+}^{\gamma_2-2}u(0)-\delta_1D_{0+}^{\gamma_2-1}u(0)}{\delta_2^2\Gamma(\gamma_2-1)+\delta_1^2\Gamma(\gamma_2)}\bigg|\\ &\quad\le\frac{\Gamma(\beta_2(3-\alpha_2)+1)}{|\delta_1|\Gamma(\gamma_2)}\big(\big|t^{-\beta_2(3-\alpha_2)}D_{0+}^{\alpha_2-1,\beta_2}u(t)\big|+\big|t^{-\beta_2(3-\alpha_2)}D_{0+}^{\alpha_2-1,\beta_2}R(u,\lambda)(t)\big|\big). \end{aligned}\tag{3.9}$$

Taking $t=t_0$ into equation (3.9), we have

$$\begin{aligned} &\bigg|\frac{\delta_2D_{0+}^{\gamma_2-2}u(0)-\delta_1D_{0+}^{\gamma_2-1}u(0)}{\delta_2^2\Gamma(\gamma_2-1)+\delta_1^2\Gamma(\gamma_2)}\bigg|\\ &\quad\le\frac{\Gamma(\beta_2(3-\alpha_2)+1)}{|\delta_1|\Gamma(\gamma_2)}\big(M_0'+\big|t_0^{-\beta_2(3-\alpha_2)}D_{0+}^{\alpha_2-1,\beta_2}R(u,\lambda)(t_0)\big|\big). \end{aligned}$$

Since

$$\begin{aligned} &\big|t_0^{-\beta_2(3-\alpha_2)}D_{0+}^{\alpha_2-1,\beta_2}R(u,\lambda)(t_0)\big|\\ &\quad\le t_0^{-\beta_2(3-\alpha_2)}\int_0^{t_0}\varphi_q\bigg(\int_s^1\frac{(x-s)^{\alpha_1-1}}{\Gamma(\alpha_1)}\big|N_\lambda u(x)\big|\,dx\bigg)\,ds\\ &\qquad+\frac{\|T_1\|_\infty\|I_{0+}^{\alpha_2}\varphi_q(I_{1-}^{\alpha_1}N_\lambda u)\|_X|\delta_2|\Gamma(\gamma_2-1)\Gamma(\gamma_2)}{(\delta_2^2\Gamma(\gamma_2-1)+\delta_1^2\Gamma(\gamma_2))\Gamma(\beta_2(3-\alpha_2)+1)}\\ &\quad\le t_0^{-\beta_2(3-\alpha_2)}\int_0^{t_0}\varphi_q\bigg(\frac{\|N_\lambda u\|_\infty}{\Gamma(\alpha_1+1)}\bigg)\,ds+\frac{\|T_1\|_\infty|\delta_2|\Gamma(\gamma_2-1)\Gamma(\gamma_2)\varphi_q(\frac{\|N_\lambda u\|_\infty}{\Gamma(\alpha_1+1)})}{(\delta_2^2\Gamma(\gamma_2-1)+\delta_1^2\Gamma(\gamma_2))\Gamma(\beta_2(3-\alpha_2)+1)}\\ &\quad\le\bigg[\varphi_q\bigg(\frac{1}{\Gamma(\alpha_1+1)}\bigg)+\frac{\|T_1\|_\infty|\delta_2|\Gamma(\gamma_2-1)\Gamma(\gamma_2)\varphi_q(\frac{1}{\Gamma(\alpha_1+1)})}{(\delta_2^2\Gamma(\gamma_2-1)+\delta_1^2\Gamma(\gamma_2))\Gamma(\beta_2(3-\alpha_2)+1)}\bigg]\\ &\qquad\times\|N_\lambda u\|_\infty^{q-1}, \end{aligned}$$

we can obtain

$$\begin{aligned}
&\left|\frac{\delta_2 D_{0+}^{\gamma_2-2}u(0)-\delta_1 D_{0+}^{\gamma_2-1}u(0)}{\delta_2^2\Gamma(\gamma_2-1)+\delta_1^2\Gamma(\gamma_2)}\right| \\
&\quad\le\left[\frac{\Gamma(\beta_2(3-\alpha_2)+1)\varphi_q(\frac{1}{\Gamma(\alpha_1+1)})}{|\delta_1|\Gamma(\gamma_2)}+\frac{\|T_1\|_\infty|\delta_2|\Gamma(\gamma_2-1)\varphi_q(\frac{1}{\Gamma(\alpha_1+1)})}{|\delta_1|(\delta_2^2\Gamma(\gamma_2-1)+\delta_1^2\Gamma(\gamma_2))}\right]\|N_\lambda u\|_\infty^{q-1} \\
&\qquad+\frac{M_0'\Gamma(\beta_2(3-\alpha_2)+1)}{|\delta_1|\Gamma(\gamma_2)}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|Pu\|_X&\le\max\left\{|\delta_2|+|\delta_1|,\frac{|\delta_2|(\beta_2(3-\alpha_2)+1)\Gamma(\gamma_2-1)+|\delta_1|\Gamma(\gamma_2)}{\Gamma(\beta_2(3-\alpha_2)+2)},\frac{|\delta_1|\Gamma(\gamma_2)}{\Gamma(\beta_2(3-\alpha_2)+1)}\right\} \\
&\quad\times\left(\left[\frac{\Gamma(\beta_2(3-\alpha_2)+1)\varphi_q(\frac{1}{\Gamma(\alpha_1+1)})}{|\delta_1|\Gamma(\gamma_2)}+\frac{\|T_1\|_\infty|\delta_2|\Gamma(\gamma_2-1)\varphi_q(\frac{1}{\Gamma(\alpha_1+1)})}{|\delta_1|(\delta_2^2\Gamma(\gamma_2-1)+\delta_1^2\Gamma(\gamma_2))}\right]\right. \\
&\quad\left.\times\|N_\lambda u\|_\infty^{q-1}+\frac{M_0'\Gamma(\beta_2(3-\alpha_2)+1)}{|\delta_1|\Gamma(\gamma_2)}\right) \\
&:=C_1\left(M\|N_\lambda u\|_\infty^{q-1}+\frac{M_0'\Gamma(\beta_2(3-\alpha_2)+1)}{|\delta_1|\Gamma(\gamma_2)}\right),
\end{aligned}$$

where

$$\begin{aligned}
&C_1=\max\left\{|\delta_2|+|\delta_1|,\frac{|\delta_2|(\beta_2(3-\alpha_2)+1)\Gamma(\gamma_2-1)+|\delta_1|\Gamma(\gamma_2)}{\Gamma(\beta_2(3-\alpha_2)+2)},\frac{|\delta_1|\Gamma(\gamma_2)}{\Gamma(\beta_2(3-\alpha_2)+1)}\right\}, \\
&M=\frac{\Gamma(\beta_2(3-\alpha_2)+1)\varphi_q(\frac{1}{\Gamma(\alpha_1+1)})}{|\delta_1|\Gamma(\gamma_2)}+\frac{\|T_1\|_\infty|\delta_2|\Gamma(\gamma_2-1)\varphi_q(\frac{1}{\Gamma(\alpha_1+1)})}{|\delta_1|(\delta_2^2\Gamma(\gamma_2-1)+\delta_1^2\Gamma(\gamma_2))}.
\end{aligned}$$

According to Lemma 3.4, we can get

$$\begin{aligned}
\big\|R(u,\lambda)\big\|_X\le{}&\left[\max\left\{1+\frac{\|T_1\|_\infty(|\delta_2|\Gamma(\gamma_2-1)+|\delta_1|\Gamma(\gamma_2))}{\delta_2^2\Gamma(\gamma_2-1)+\delta_1^2\Gamma(\gamma_2)},\right.\right. \\
&1+\frac{\|T_1\|_\infty|\delta_2|\Gamma(\gamma_2)}{(\delta_2^2+\delta_1^2(\gamma_2-1))\Gamma(\beta_2(3-\alpha_2)+1)}, \\
&\left.\left.\frac{1}{2}+\frac{\|T_1\|_\infty(|\delta_2|\Gamma(\gamma_2)+(\beta_2(3-\alpha_2)+1)|\delta_1|\Gamma(\gamma_2))}{(\delta_2^2+\delta_1^2(\gamma_2-1))\Gamma(\beta_2(3-\alpha_2)+2)}\right\}\times\varphi_q\left(\frac{1}{\Gamma(\alpha_1+1)}\right)\right] \\
&\times\|N_\lambda u\|_\infty^{q-1} \\
:={}&C_2\|N_\lambda u\|_\infty^{q-1},
\end{aligned}$$

where

$$\begin{aligned}
C_2={}&\max\left\{1+\frac{\|T_1\|_\infty(|\delta_2|\Gamma(\gamma_2-1)+|\delta_1|\Gamma(\gamma_2))}{\delta_2^2\Gamma(\gamma_2-1)+\delta_1^2\Gamma(\gamma_2)},\right. \\
&1+\frac{\|T_1\|_\infty|\delta_2|\Gamma(\gamma_2)}{(\delta_2^2+\delta_1^2(\gamma_2-1))\Gamma(\beta_2(3-\alpha_2)+1)}, \\
&\left.\frac{1}{2}+\frac{\|T_1\|_\infty(|\delta_2|\Gamma(\gamma_2)+(\beta_2(3-\alpha_2)+1)|\delta_1|\Gamma(\gamma_2))}{(\delta_2^2+\delta_1^2(\gamma_2-1))\Gamma(\beta_2(3-\alpha_2)+2)}\right\}\times\varphi_q(\frac{1}{\Gamma(\alpha_1+1)}).
\end{aligned}$$

Using Lemma 3.5 and hypothetical condition (H_8), we have

$$\begin{aligned}\|u\|_X &\le \|Pu\|_X + \big\|R(u,\lambda)\big\|_X \\ &\le (C_1M + C_2)\|N_\lambda u\|_\infty^{q-1} + \frac{C_1M_0'\Gamma(\beta_2(3-\alpha_2)+1)}{|\delta_1|\Gamma(\gamma_2)} \\ &\le (C_1M + C_2)\big(\|a\|_\infty + \|b\|_\infty\|u\|_\infty^{p-1} + \|c\|_\infty\big\|D_{0+}^{\alpha_2-2,\beta_2}u\big\|_\infty^{p-1} \\ &\quad + \|d\|_\infty\big\|D_{0+}^{\alpha_2-1,\beta_2}u\big\|_\infty^{p-1} + \|e\|_\infty\big\|D_{0+}^{\alpha_2,\beta_2}u\big\|_\infty^{p-1}\big)^{q-1} + \frac{C_1M_0'\Gamma(\beta_2(3-\alpha_2)+1)}{|\delta_1|\Gamma(\gamma_2)}.\end{aligned}$$

If $1 < q \le 2$, then

$$\begin{aligned}\|u\|_X &\le (C_1M + C_2)\|a\|_\infty^{q-1} + (C_1M + C_2)\big(\|b\|_\infty + \|c\|_\infty + \|d\|_\infty + \|e\|_\infty\big)^{q-1}\|u\|_X \\ &\quad + \frac{C_1M_0'\Gamma(\beta_2(3-\alpha_2)+1)}{|\delta_1|\Gamma(\gamma_2)}.\end{aligned}$$

Thus,

$$\|u\|_X \le \frac{(C_1M + C_2)\|a\|_\infty^{q-1} + \frac{C_1M_0'\Gamma(\beta_2(3-\alpha_2)+1)}{|\delta_1|\Gamma(\gamma_2)}}{1-(C_1M + C_2)(\|b\|_\infty + \|c\|_\infty + \|d\|_\infty + \|e\|_\infty)^{q-1}}.$$

If $q > 2$, then

$$\begin{aligned}\|u\|_X &\le 2^{q-2}(C_1M + C_2)\|a\|_\infty^{q-1} \\ &\quad + 2^{q-2}(C_1M + C_2)\big(\|b\|_\infty + \|c\|_\infty + \|d\|_\infty + \|e\|_\infty\big)^{q-1}\|u\|_X \\ &\quad + \frac{C_1M_0'\Gamma(\beta_2(3-\alpha_2)+1)}{|\delta_1|\Gamma(\gamma_2)}.\end{aligned}$$

Therefore,

$$\|u\|_X \le \frac{2^{q-2}(C_1M + C_2)\|a\|_\infty^{q-1} + \frac{C_1M_0'\Gamma(\beta_2(3-\alpha_2)+1)}{|\delta_1|\Gamma(\gamma_2)}}{1-2^{q-2}(C_1M + C_2)(\|b\|_\infty + \|c\|_\infty + \|d\|_\infty + \|e\|_\infty)^{q-1}}.$$

Set $L = \max_{q\in(1,+\infty)}\{1, 2^{q-2}\}$, then the above inequality is equivalent to

$$\|u\|_X \le \frac{L(C_1M + C_2)\|a\|_\infty^{q-1} + \frac{C_1M_0'\Gamma(\beta_2(3-\alpha_2)+1)}{|\delta_1|\Gamma(\gamma_2)}}{1-L(C_1M + C_2)(\|b\|_\infty + \|c\|_\infty + \|d\|_\infty + \|e\|_\infty)^{q-1}}.$$

This means that Ω_1 is bounded. The remaining proof is similar to Theorem 3.6 and is omitted here. Finally, we can get that boundary value problem (1.1) has at least one solution in X. □

Remark When the inequality $|t^{-\beta_2(3-\alpha_2)}D_{0+}^{\alpha_2-1,\beta_2}u(t)| > M_0'$ in assumption condition (H_7) is replaced by $|t^{-\beta_2(3-\alpha_2)}D_{0+}^{\alpha_2-2,\beta_2}u(t)| > M_0''$, the method of proving the existence of the solution of boundary value problem (1.1) is similar to Theorem 3.9. There is no detailed explanation here.

4 Example

Consider the following boundary value problem at resonance:

$$\begin{cases} D_{1-}^{\frac{1}{2},\frac{1}{3}}\varphi_{\frac{3}{2}}(D_{0+}^{\frac{5}{2},\frac{1}{2}}u(t)) = \frac{1}{45}[5+\sin(\sqrt{|u(t)|})+\sin(\sqrt{|D_{0+}^{\frac{1}{2},\frac{1}{2}}u(t)|})+\sin(\sqrt{|D_{0+}^{\frac{3}{2},\frac{1}{2}}u(t)|}) \\ \qquad + \sin(\sqrt{|D_{0+}^{\frac{5}{2},\frac{1}{2}}u(t)|})], \quad t\in[0,1], \\ u(0) = D_{0+}^{\frac{5}{2},\frac{1}{2}}u(1) = 0, \qquad T_1(u) = D_{0+}^{\frac{7}{4}}u(1) + D_{0+}^{\frac{3}{4}}u(1) = 0, \\ T_2(u) = 2D_{0+}^{\frac{7}{4}}u(1) + \int_0^2 D_{0+}^{\frac{3}{4}}u(t)\,dt = 0. \end{cases} \tag{4.1}$$

Corresponding to boundary value problem (1.1), we have $\alpha_1 = \frac{1}{2}$, $\alpha_2 = \frac{5}{2}$, $\beta_1 = \frac{1}{3}$, $\beta_2 = \frac{1}{2}$, $\gamma_1 = \frac{2}{3}$, $\gamma_2 = \frac{11}{4}$, $p = \frac{3}{2}$, $k = 2$ and

$$f\big(t,u(t),D_{0+}^{\frac{1}{2},\frac{1}{2}}u(t),D_{0+}^{\frac{3}{2},\frac{1}{2}}u(t),D_{0+}^{\frac{5}{2},\frac{1}{2}}u(t)\big) = \frac{1}{45}\big[5+\sin\big(\sqrt{|u(t)|}\big)+\sin\big(\sqrt{|D_{0+}^{\frac{1}{2},\frac{1}{2}}u(t)|}\big) \\ + \sin\big(\sqrt{|D_{0+}^{\frac{3}{2},\frac{1}{2}}u(t)|}\big)+\sin\big(\sqrt{|D_{0+}^{\frac{5}{2},\frac{1}{2}}u(t)|}\big)\big].$$

Boundary value problem (4.1) is at resonance with

$$\operatorname{Ker} L = \left\{c\left(\frac{2t^{\frac{3}{4}}}{\Gamma(\frac{7}{4})} - \frac{t^{\frac{7}{4}}}{\Gamma(\frac{11}{4})}\right), c\in R\right\}, \qquad D_{0+}^{\frac{7}{4}}u(t) = -c, \qquad D_{0+}^{\frac{3}{4}}u(t) = c(2-t).$$

Thus,

$$T_1\big(t^{\frac{3}{4}}\big) = \delta_1 = \frac{1}{\Gamma(\frac{11}{4})} \neq 0, \qquad T_1\big(t^{\frac{7}{4}}\big) = \delta_2 = \frac{2}{\Gamma(\frac{7}{4})} \neq 0.$$

Take $a(t) = \frac{1}{9}$, $b(t) = c(t) = d(t) = e(t) = \frac{1}{45}$, and $q = 3$, then

$$C_1 = \max\{1.715, 3.097, 1.098\} = 3.097, \qquad C_2 = \max\{2.714, 1.148, 2.219\} = 2.714,$$

$$M = \left(\frac{1}{\Gamma(1.5)}\right)^2\Gamma(0.25) + \frac{2(\frac{1}{\Gamma(1.5)})^2}{\frac{1}{4\Gamma(2.75)}(\frac{4}{\Gamma(1.75)}+\frac{1}{\Gamma(2.75)})} = 8.00025, \qquad L = \max\big\{1, 2^2\big\} = 4.$$

Therefore,

$$L(C_1M + C_2)\big(\|b\|_\infty + \|c\|_\infty + \|d\|_\infty + \|e\|_\infty\big)^{q-1} = 0.8688 < 1,$$

$$\big|f(t,x,y,z,w)\big| \le \frac{1}{9} + \frac{1}{45}\varphi_p\big(|x|\big) + \frac{1}{45}\varphi_p\big(|y|\big) + \frac{1}{45}\varphi_p\big(|z|\big) + \frac{1}{45}\varphi_p\big(|w|\big).$$

That means condition (H_8) holds.

Let $M_0' = 5$, if $|t^{-\frac{1}{4}}D_{0+}^{\frac{3}{2},\frac{1}{2}}u(t)| > M_0'$ holds for any $t \in (0,1]$, then

$$\begin{aligned} &f\big(t,u(t),D_{0+}^{\frac{1}{2},\frac{1}{2}}u(t),D_{0+}^{\frac{3}{2},\frac{1}{2}}u(t),D_{0+}^{\frac{5}{2},\frac{1}{2}}u(t)\big) \\ &\quad = \frac{1}{45}\big[5+\sin\big(\sqrt{|u(t)|}\big)+\sin\big(\sqrt{|D_{0+}^{\frac{1}{2},\frac{1}{2}}u(t)|}\big)+\sin\big(\sqrt{|D_{0+}^{\frac{3}{2},\frac{1}{2}}u(t)|}\big)+\sin\big(\sqrt{|D_{0+}^{\frac{5}{2},\frac{1}{2}}u(t)|}\big)\big] \\ &\quad > 0 \end{aligned}$$

and

$$(T_2 - kT_1)\big(I_{0+}^{\alpha_2}\varphi_q\big(I_{1-}^{\alpha_1}Nu(t)\big)\big)$$
$$= \frac{1}{\Gamma(\frac{7}{4})}\int_0^2\left[\int_0^t (t-s)^{\frac{3}{4}}\varphi_3\big(I_{1-}^{\frac{1}{2}}Nu(s)\big)\,ds - \int_0^1 (1-s)^{\frac{3}{4}}\varphi_3\big(I_{1-}^{\frac{1}{2}}Nu(s)\big)\,ds\right]dt < 0.$$

Hence, condition (H_7) holds.

Similarly, let $B_1 = 24$, $u(t) = c(\frac{2t^{\frac{3}{4}}}{\Gamma(\frac{7}{4})} - \frac{t^{\frac{7}{4}}}{\Gamma(\frac{11}{4})})$, $c \in R$, if $|c| > B_1$, then $|t^{-\frac{1}{4}}D_{0+}^{\frac{3}{2},\frac{1}{2}}u(t)| = \frac{1}{4}|c| > M_0'$. Therefore, $(T_2 - kT_1)(I_{0+}^{\alpha_2}\varphi_q(I_{1-}^{\alpha_1}Nc(\frac{2t^{\frac{3}{4}}}{\Gamma(\frac{7}{4})} - \frac{t^{\frac{7}{4}}}{\Gamma(\frac{11}{4})}))) \neq 0$. Clearly, condition (H_6) holds. Through the application of Theorem 3.9, we obtain that boundary value problem (4.1) has at least one solution.

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Declarations

Competing interests
The authors declare no competing interests.

Author contributions
M, J and Z studied the solvability of mixed Hilfer fractional boundary value problems under functional boundary value conditions and were major contributors to the writing of the manuscript. G gives an example of this article. All authors read and approved the final manuscript.

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