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Optimal control for a chemotaxis–haptotaxis model in two space dimensions

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Abstract

This paper deals with a chemotaxis–haptotaxis model which described the process of cancer invasion on the macroscopic scale. We first explore the global-in-time existence and uniqueness of a strong solution. For a class of cost functionals, we prove first-order necessary optimality conditions for the corresponding optimal control problem and establish the existence of Lagrange multipliers. Finally, we derive some extra regularity for the Lagrange multiplier.

MSC: 92C17; 49J20; 49K20; 35K51

Keywords: Chemotaxis–haptotaxis model; Existence; Optimal control; Lagrange multipliers

1 Introduction

In this paper, we investigate the chemotaxis–haptotaxis model with the initial-boundary conditions

$$u_t = \Delta u - \chi\nabla\cdot(u\nabla v) - \xi\nabla\cdot(u\nabla w) + \mu u(1-u-w), \quad \text{in } \Omega\times[0,T], \tag{1.1}$$

$$v_t = \Delta v - v + u + f, \quad \text{in } \Omega\times[0,T], \tag{1.2}$$

$$w_t = -vw, \quad \text{in } \Omega\times[0,T], \tag{1.3}$$

$$\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0, \quad \text{on } \partial\Omega, \tag{1.4}$$

$$u(x,0) = u_0(x), \qquad v(x,0) = v_0(x), \qquad w(x,0) = w_0(x), \quad \text{in } \Omega, \tag{1.5}$$

where $\Omega \subset \mathbb{R}^n (n = 2)$ is a bounded domain with smooth boundary $\partial\Omega$; ν is the outward normal vector to $\partial\Omega$, and χ, μ, ξ are positive constants. The scalar functions $u = u(x,t)$, $v = v(x,t)$, and $w = w(x,t)$ represent the density of cancer cells, the concentration of enzyme, and the density of healthy tissue, respectively. Notice that in the region of Ω where $f \geq 0$ the control acts as a proliferation source of the chemical substance, and inversely, in the region of Ω where $f \leq 0$ the control acts as a degradation source of the chemical substance [23]. In this work, the function $f \geq 0$ lies in a closed convex set $\mathcal{F}$.

Anderson et al. [1] presented the following mathematical model which described the invasion of host tissue by tumour cells:

$$\begin{cases} n_t = d_n \Delta n - \gamma \nabla \cdot (n \nabla f), & x \in \Omega, t > 0, \\ m_t = d_m \Delta m + \alpha n - \beta m, & x \in \Omega, t > 0, \\ f_t = -\eta m f, & x \in \Omega, t > 0. \end{cases}$$

Marciniak-Czochra and Ptashnyk [24] considered the haptotaxis model

$$\begin{cases} u_t = d_u \Delta u - \nabla \cdot (\chi(v) u \nabla v) + \mu_u u(1 - u - v), & x \in \Omega, t > 0, \\ m_t = d_m \Delta m - \rho_m m + \mu_m u v, & x \in \Omega, t > 0, \\ v_t = -\alpha m v, & x \in \Omega, t > 0. \end{cases}$$

They proved the existence of global solutions of the haptotaxis model of cancer invasion for arbitrary non-negative initial conditions. Niño-Celis, Rueda-Gómez and Villamizar-Roa [27] developed two fully discrete schemes for approximating the solutions based on a semi-implicit Euler discretization in time and Finite Element (FE) discretization on space (restricted to triangularization made up of right-angled simplices) of two equivalent systems for the above haptotaxis model.

Chaplain and Lolas [3] first described the process of the cancer invasion on the macroscopic scale by the chemotaxis–haptotaxis system. Tao and Winkler [30] studied the problem

$$\begin{cases} u_t - \Delta u = -\chi \nabla \cdot (u \nabla v) - \xi \nabla \cdot (u \nabla w) + \mu u(1 - u - w), & x \in \Omega, t > 0, \\ \tau v_t - \Delta v + v = u, & x \in \Omega, t > 0, \\ w_t = -v w, & x \in \Omega, t > 0. \end{cases}$$

They discussed the global solvability of classical solutions in a bounded domain $\Omega \subset R^n (n \leq 3)$. Cao [2] proved that for nonnegative and suitably smooth initial data, if χ/μ is sufficiently small, the problem possesses a global classical solution, which is bounded in $\Omega \times (0, \infty)$. The relevant equations have also been studied in [14, 17, 19, 31, 32].

Jin [15] considered the following system:

$$\begin{cases} u_t - \Delta u^m = -\chi \nabla \cdot (u \nabla v) - \xi \nabla \cdot (u \nabla w) + \mu u(1 - u - w), & x \in \Omega, t > 0, \\ v_t - \Delta v + v = u, & x \in \Omega, t > 0, \\ w_t = -v w, & x \in \Omega, t > 0. \end{cases}$$

Under zero-flux boundary conditions, they showed that, for any $m > 0$, the problem admits a global bounded weak solution for any large initial datum if χ/μ is appropriately small.

Mizukami [25] studied the chemotaxis–haptotaxis system with signal-dependent sensitivity

$$\begin{cases} u_t - \Delta u = -\nabla \cdot (\chi(v) u \nabla v) - \xi \nabla \cdot (u \nabla w) + \mu u(1 - u - w), & x \in \Omega, t > 0, \\ v_t - \Delta v + v = u, & x \in \Omega, t > 0, \\ w_t = -v w, & x \in \Omega, t > 0. \end{cases}$$

They established the global existence and boundedness for the above system. The relevant system has also been studied in [33].

During the past years, many authors have been very interested in the optimal control problems governed by the coupled partial differential equations. Colli, Gilardi, Marinoschi and Rocca [8] studied the distributed optimal control problems for a diffuse interface model of tumor growth. Liu and Zhang [21] discussed the optimal distributed control for a new mechanochemical model in biological patterns. Dai and Liu [10] obtained an optimal control problem for a haptotaxis model of solid tumor invasion by considering the multiple treatments of cancer. Recently, Guillén-González, Mallea-Zepeda and Villamizar-Roa [13] studied the following parabolic chemo-repulsion with nonlinear production model in $2D$ domains:

$$\begin{cases} u_t - \Delta u = \nabla \cdot (u \nabla v), & x \in \Omega, t > 0, \\ v_t - \Delta v + v = u^p + fv 1_{\Omega_c}, & x \in \Omega, t > 0. \end{cases}$$

They proved the existence and uniqueness of global-in-time strong state solution for each control, and the existence of global optimum solution. Guillén-González, Mallea-Zepeda and Rodriguez-Bellido [12] considered a bilinear optimal control problem associated to the above $3D$ chemo-repulsion model. Guillén-González et al. [11] studied a bilinear optimal control problem for the chemo-repulsion model with linear production term. The existence, uniqueness and regularity of strong solutions of this model were deduced. They also derived the first-order optimality conditions by using a Lagrange multipliers theorem. López-Ríos and Villamizar-Roa [23] studied an optimal control problem associated to a 3D-chemotaxis-Navier–Stokes model. Some other results can be found in [4–7, 16, 20, 22, 29, 35, 36].

In this paper, we are interested in the optimal control problem for the system (1.1)–(1.5). The main difficulties for treating the problem (1.1)–(1.5) are caused by the nonlinearity of $-\xi \nabla \cdot (u \cdot \nabla w)$ and $\mu u(1-u-w)$. Our method is based on a Lagrange multiplier theorem.

This paper is organized as follows. In Sects. 2 and 3, we show the well-posedness of the state system (1.1)–(1.5). In Sect. 4, the existence of optimal controls is established. Finally, we derive the first-order necessary optimality conditions in Sect. 5.

Notations: $L^p = L^p(\Omega)\,(1 \le p \le \infty)$ denotes the usual Lebesgue space with the usual norm $\|\cdot\|_{L^p}$. The Sobolev space in Ω of order $k, k = 0, 1, 2, \ldots$, is denoted by $H^k(\Omega)$ with norm $\|\cdot\|_{H^k}$, and the space $H^{-k}(\Omega)$ is the dual space of $H^k(\Omega)$. The Sobolev space of fractional order $s > 0$ is denoted by $H^s(\Omega)$ with norm $\|\cdot\|_{H^s}$. $H^s_N(\Omega)$ denotes a closed subspace of $H^s(\Omega)$ such that

$$H^s_N(\Omega) = \left\{ w \in H^s(\Omega) : \frac{\partial w}{\partial n} = 0 \text{ on } \partial\Omega \right\}.$$

2 Local solutions

We first review the existence theorem for local solutions to an abstract equation in a Banach space (see Chap. 4 in [34]). Let Z and $\mathcal{B}$ be two separable Hilbert spaces with dense and compact embedding $Z \subset \mathcal{B}$. Let $\|\cdot\|_Z$ and $\|\cdot\|_{\mathcal{B}}$ be the norms of Z and $\mathcal{B}$, respectively. Let $Z \subset \mathcal{B} \subset Z^*$ be a triplet of spaces. Let $\|\cdot\|_{Z^*}$ be the norm of Z^*. We consider the following Cauchy problem for a semilinear abstract evolution equation:

$$\begin{cases} \frac{dU}{dt} + AU = F(U) + G(t), \quad t > 0, \\ U(0) = U_0. \end{cases} \tag{2.1}$$

Here, A is a sectorial operator of Z^*. The nonlinear operator F is a mapping from Z to Z^*, and for any positive number $\eta > 0$, there exist continuous increasing functions $\varphi(\cdot) \geq 0$ and $\psi(\cdot) \geq 0$ such that the following estimates hold:

$$\|F(U)\|_{Z^*} \leq \eta \|U\|_Z + \varphi(\|U\|_{\mathcal{B}}), \quad U \in Z, \tag{2.2}$$

$$\begin{aligned} &\|F(U) - F(\tilde{U})\|_{Z^*} \\ &\quad \leq \eta \|U - \tilde{U}\|_Z \\ &\qquad + (\|U\|_Z + \|\tilde{U}\|_Z + 1)\psi(\|U\|_{\mathcal{B}} + \|\tilde{U}\|_{\mathcal{B}})\|U - \tilde{U}\|_{\mathcal{B}}, \quad U, \tilde{U} \in Z. \end{aligned} \tag{2.3}$$

Then, we have the existence theorem of the local solutions to (2.1).

Proposition 2.1 ([34, Theorem 4.6]) *Let* (2.2) *and* (2.3) *be satisfied. Then, for* $G \in L^2(0, T; Z^*)$ *and any* $U_0 \in \mathcal{B}$, *there exists a unique local solution* U *to* (2.1) *in the function space*

$$U \in L^2((0, T_{U_0,G}); Z) \cap C([0, T_{U_0,G}]; \mathcal{B}) \cap H^1((0, T_{U_0,G}); Z^*),$$

where $T_{U_0,G} > 0$ *is determined by the norms* $\|U_0\|_{\mathcal{B}}$ *and* $\|G\|_{L^2(0,T;Z^*)}$. *In addition,* U *satisfies the estimate*

$$\|U\|_{L^2((0,T_{U_0,G}];Z)} + \|U\|_{C([0,T_{U_0,G}];\mathcal{B}))} + \|U\|_{H^1((0,T_{U_0,G}];Z^*)} \leq C_{G,U_0},$$

where $C_{G,U_0} > 0$ *is a constant depending on the norm* $\|U_0\|_{\mathcal{B}}$ *and* $\|G\|_{L^2(0,T;Z^*)}$.

Applying Proposition 2.1, we can show the existence of the local-in-time solutions to (1.1)–(1.5).

Theorem 2.1 *For all initial functions* $(u_0, v_0, w_0) \in H^1(\Omega) \times H^2_N(\Omega) \times H^3_N(\Omega)$, $u_0 \geq 0$, $v_0 \geq 0$, $w_0 \geq 0$ *and* $0 \leq f \in L^2(0, T; H^1(\Omega))$, *the problem* (1.1)–(1.5) *admits a unique local-in-time nonnegative solution* (u, v, w) *in the function space*

$$\begin{cases} u \in H^1((0, T); H^{-2}(\Omega)) \cap C([0, T]; H^1(\Omega)) \cap L^2((0, T); H^2_N(\Omega)), \\ v \in H^1((0, T); H^{-3}(\Omega)) \cap C([0, T]; H^2_N(\Omega)) \cap L^2((0, T); H^3_N(\Omega)), \\ w \in H^1((0, T); H^{-3}(\Omega)) \cap C([0, T]; H^3_N(\Omega)) \cap L^2((0, T); H^3_N(\Omega)), \end{cases}$$

with the estimate

$$\begin{aligned} &\|u(t)\|_{H^1} + \|v(t)\|_{H^2} + \|w(t)\|_{H^3} + \|u\|_{H^1((0,T);H^{-2}(\Omega))} \\ &\quad + \|v\|_{H^1((0,T);H^{-3}(\Omega))} + \|w\|_{H^1((0,T);H^{-3}(\Omega))} \leq C, \quad 0 < t \leq T, \end{aligned} \tag{2.4}$$

where T and C are positive constants depending only on the norms $\|u_0\|_{H^1} + \|v_0\|_{H^2} + \|w_0\|_{H^2}$ and $\|h\|_{L^2(0,T;H^1(\Omega))}$.

Proof Let $A_1 = -\Delta + 1$, $A_2 = -\Delta + 1$, and $A_3 = 1$. Then, A_i are three positive definite self-adjoint operators. We define the linear operator A by

$$A = \begin{bmatrix} \Delta + 1 & 0 & 0 \\ 0 & -\Delta + 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad Z = H_N^2(\Omega) \times H_N^3(\Omega) \times H_N^3(\Omega).$$

Problem (1.1)–(1.5) is, then, formulated as an abstract equation,

$$\begin{aligned} &\frac{dU}{dt} + AU = F(U) + G(t), \quad 0 < t \le T, \\ &U(0) = U_0, \end{aligned} \tag{2.5}$$

in a product Banach space $\mathcal{B} = H^1(\Omega) \times H_N^2(\Omega) \times H_N^3(\Omega)$. The nonlinear operator F is defined by

$$F(U) = \begin{bmatrix} -\chi\nabla\cdot(u\cdot\nabla v) - \xi\nabla\cdot(u\cdot\nabla w) + \mu u(1-u-w) + u \\ u \\ w - vw \end{bmatrix}, \qquad G(t) = \begin{bmatrix} 0 \\ f \\ 0 \end{bmatrix},$$

$$U = (u, v, w).$$

The initial value $U_0 = (u_0, v_0, w_0)$ is taken in the function space $H^1(\Omega) \times H_N^2(\Omega) \times H_N^3(\Omega)$. In this setting, we only need to verify conditions (2.2) and (2.3). Let $U = (u, v, w)$ and $\tilde{U} = (\tilde{u}, \tilde{v}, \tilde{w}) \in Z$. Then, using the interpolation of Sobolev spaces ($\|u\|_{H^{3/2}} \le C\|u\|_{H^2}^{1/2}\|u\|_{H^1}^{1/2}$) and the Young inequality, for any positive number $\eta > 0$, we have

$$\begin{aligned} \|F(U)\|_{Z^*} &\le \chi\|\nabla\cdot(u\nabla v)\|_{H^{-2}} + \xi\|\nabla\cdot(u\nabla w)\|_{H^{-2}} \\ &\quad + \mu\|u(1-u-w)\|_{L^2} + 2\|u\|_{L^2} + \|w - vw\|_{L^2} \\ &\le \chi\|u\nabla v\|_{L^2} + \xi\|u\nabla w\|_{L^2} + C\|u\|_{L^2} + C\|w\|_{L^2} + C\|u\|_{L^4}^2 \\ &\quad + C\|v\|_{L^4}^2 + C\|w\|_{L^4}^2 \\ &\le \chi\|u\|_{H^{3/2}}\|v\|_{H^1} + \xi\|u\|_{H^{3/2}}\|w\|_{H^1} + C\|u\|_{L^2} + C\|w\|_{L^2} \\ &\quad + C\big(\|u\|_{H^1}^2 + \|v\|_{H^1}^2 + \|w\|_{H^1}^2\big) \\ &\le \eta\|u\|_{H^2} + C\|v\|_{H^2}^4 + C\|w\|_{H^2}^4 \\ &\quad + C\big(\|u\|_{L^2} + \|w\|_{L^2} + \|u\|_{H^1}^2 + \|v\|_{H^1}^2 + \|w\|_{H^1}^2\big) \\ &\le \eta\|U\|_Z + C\big(\|U\|_{\mathcal{B}}^2 + \|U\|_{\mathcal{B}}^4 + 1\big), \end{aligned}$$

where C is a positive constant depending only on the known quantities.

On the other hand, we derive

$$\begin{aligned}\left\|F(U)-F(\tilde{U})\right\|_{Z^*} &\le \chi\left\|\nabla\cdot(u\nabla v-\tilde{u}\nabla\tilde{v})\right\|_{H^{-2}}+\xi\left\|\nabla\cdot(u\nabla w-\tilde{u}\nabla\tilde{w})\right\|_{H^{-2}}\\&\quad+\mu\left\|\big(u(1-u-w)-\tilde{u}(1-\tilde{u}-\tilde{w})\big)\right\|_{L^2}+2\left\|(u-\tilde{u})\right\|_{L^2}\\&\quad+\left\|(w-vw)-(\tilde{w}-\tilde{v}\tilde{w}\right\|_{L^2}.\end{aligned}$$

For the first term of the right-hand side, we see that

$$\begin{aligned}\chi\left\|\nabla\cdot(u\nabla v-\tilde{u}\nabla\tilde{v})\right\|_{H^{-2}} &\le C\|u\nabla v-\tilde{u}\nabla\tilde{v}\|_{L^2}\\&\le C\|u-\tilde{u}\|_{H^{\frac{3}{2}}}\|\nabla v\|_{L^2}+C\|\tilde{u}\|_{H^{\frac{3}{2}}}\left\|\nabla(v-\tilde{v})\right\|_{L^2}\\&\le \eta\|u-\tilde{u}\|_{H^2}+C\|v\|_{H^2}^2\|u-\tilde{u}\|_{H^1}\\&\quad+C\|\tilde{u}\|_{H^2}\|v-\tilde{v}\|_{H^1}.\end{aligned}$$

Similarly, we deduce that

$$\begin{aligned}\xi\left\|\nabla\cdot(u\nabla w-\tilde{u}\nabla\tilde{w})\right\|_{H^{-2}} &\le \eta\|u-\tilde{u}\|_{H^2}+C\|w\|_{H^2}^2\|u-\tilde{u}\|_{H^1}\\&\quad+C\|\tilde{u}\|_{H^2}\|w-\tilde{w}\|_{H^1}.\end{aligned}$$

For the third term of the right-hand side,

$$\begin{aligned}&\mu\left\|\big(u(1-u-w)-\tilde{u}(1-\tilde{u}-\tilde{w})\big)\right\|_{L^2}+2\|u-\tilde{u}\|_{L^2}\\&\quad=\left\|(u-\tilde{u})+\big(\tilde{u}^2-u^2\big)-u(w-\tilde{w})-w(u-\tilde{u})\right\|_{L^2}+\|u-\tilde{u}\|_{L^2}\\&\quad\le C\Big(\|u-\tilde{u}\|_{L^2}+\big(\|u\|_{L^2}+\|\tilde{u}\|_{L^2}+\|\tilde{w}\|_{L^2}\big)\|u-\tilde{u}\|_{H^{\frac{3}{2}}}+\|u\|_{L^2}\|w-\tilde{w}\|_{H^{\frac{3}{2}}}\Big)\\&\quad\le \eta\big(\|u-\tilde{u}\|_{H^2}+\|w-\tilde{w}\|_{H^2}\big)+C\|u-\tilde{u}\|_{H^1}\big(1+\|u\|_{L^2}^2+\|\tilde{u}\|_{L^2}^2+\|w\|_{L^2}\big)\\&\qquad+C\|w-\tilde{w}\|_{H^1}\|u\|_{L^2}^2.\end{aligned}$$

Similarly, we deduce that

$$\begin{aligned}&\left\|(w-vw)-(\tilde{w}-\tilde{v}\tilde{w}\right\|_{L^2}\\&\quad\le \eta\big(\|v-\tilde{v}\|_{H^2}+\|w-\tilde{w}\|_{H^2}\big)+C\|v-\tilde{v}\|_{H^1}\|\tilde{w}\|_{L^2}^2+C\big(1+\|u\|_{L^2}^2\big)\|w-\tilde{w}\|_{H^1}.\end{aligned}$$

Hence, we can obtain

$$\begin{aligned}&\left\|F(U)-F(\tilde{U})\right\|_{\mathcal{B}}\\&\quad\le \eta\|U-\tilde{U}\|_Z+C\big(\|U\|_Z+\|\tilde{U}\|_Z+1\big)\big(\|U\|_{\mathcal{B}}^2+\|\tilde{U}\|_{\mathcal{B}}^2+1\big)\|U-\tilde{U}\|_{\mathcal{B}}.\end{aligned}\tag{2.6}$$

Thus, we have verified (2.2) and (2.3). Similarly as in the proof of Proposition 2 in [26], we obtain $u\ge 0$. On the other hand, by the comparison principle, we can be sure that v and w are nonnegative. The proof is complete. □

3 Global existence

In this section, we construct several a priori estimates. At first, we introduce the following lemma.

Lemma 3.1 ([28, Lemma 4.3]) *For any nonnegative $u \in H^1(\Omega)$, the estimate*

$$\|u\|_{L^3}^3 \le \delta \|u\|_{H^1}^2 \big\|(u+1)\ln(u+1)\big\|_{L^1} + p\big(\delta^{-1}\big)\|u\|_{L^1}$$

holds for any number $\delta > 0$ and some increasing function $p(\cdot)$.

Lemma 3.2 *Let (u, v, w) be a local solution to (1.1)–(1.5). Then, it holds that*

$$\int_\Omega u\,dx \le \max\left\{\int_\Omega u_0\,dx, |\Omega|\right\} := M_1, \quad \textit{for all } t \in [0,T], \tag{3.1}$$

$$\int_0^t \int_\Omega u^2\,dx \le M_1 T + \frac{M_1}{\mu} := K_1(M_1, T), \quad \textit{for all } t \in [0,T], \tag{3.2}$$

$$\|w\|_{L^\infty} \le \|w_0\|_{L^\infty}, \quad \textit{for all } t \in [0,T]. \tag{3.3}$$

Proof Using the property $u(t) \ge 0$, $v(t) \ge 0$ and $w(t) \ge 0$ for all $t > 0$, and integrating equation (1.1) over Ω, we have

$$\begin{aligned}\frac{d}{dt}\int_\Omega u\,dx &\le \mu\int_\Omega u\,dx - \mu\int_\Omega u^2\,dx \\ &\le \mu\int_\Omega u\,dx - \frac{\mu}{|\Omega|}\left(\int_\Omega u\,dx\right)^2.\end{aligned} \tag{3.4}$$

By the comparison argument of ODE, we derive

$$\int_\Omega u\,dx \le \max\left\{\int_\Omega u_0\,dx, |\Omega|\right\} := M_1. \tag{3.5}$$

Integrating (3.4) over $(0,t)$, it follows from (3.5) that

$$\begin{aligned}\int_0^t \int_\Omega u^2\,dx &\le \int_0^t \int_\Omega u\,dx + \frac{1}{\mu}\left(\int_\Omega u_0\,dx - \int_\Omega u(t)\,dx\right) \\ &\le M_1 T + \frac{M_1}{\mu} := K_1(M_1,T), \quad \text{for all } t \in [0,T].\end{aligned} \tag{3.6}$$

Multiplying equation (1.3) by w^{p-1} and integrating over Ω, we have

$$\frac{1}{p}\frac{d}{dt}\int_\Omega w^p\,dx = -\int_\Omega v w^p\,dx.$$

For all $t \in [0,T]$, due to the fact that v, w are nonnegative, we obtain

$$\frac{d}{dt}\int_\Omega w^p\,dx \le 0, \quad \text{for all } t \in [0,T], \tag{3.7}$$

which yields

$$\|w(\cdot,t_2)\|_{L^p(\Omega)} \le \|w(\cdot,t_1)\|_{L^p(\Omega)} \le \|w(\cdot,0)\|_{L^\infty(\Omega)}, \quad \text{for all } t_2 \ge t_1 \ge 0.$$

Consequently, (3.3) follows by taking the limit $p \to \infty$. Therefore, we complete the proof. □

Lemma 3.3 *Let (u,v,w) be a local solution to (1.1)–(1.5). Then, it holds that*

$$\|u\|_{H^1} + \|v\|_{H^2} + \|w\|_{H^3} + \int_0^t \|u\|_{H^2}^2\,d\tau + \int_0^t \|v\|_{H^3}^2\,d\tau \le C. \tag{3.8}$$

Proof **Step 1.** v is bounded in $L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;H^2(\Omega))$.

Multiplying equation (1.2) by v and integrating over Ω, we have

$$\frac{d}{dt}\int_\Omega v^2\,dx + 2\int_\Omega |\nabla v|^2\,dx + \int_\Omega v^2\,dx \le 2\int_\Omega u^2\,dx + 2\int_\Omega f^2\,dx. \tag{3.9}$$

Integrating (3.9) over $(0,t)$, we derive

$$\|v(t)\|_{L^2}^2 + \int_0^t \|v\|_{H^1}^2\,d\tau \le \|v_0\|_{L^2}^2 + 2\int_0^t \big(\|u\|_{L^2}^2 + \|f\|_{L^2}^2\big)\,d\tau \quad \text{for all } t\in[0,T]. \tag{3.10}$$

Multiplying equation (1.2) by $-\Delta v$ and integrating over Ω, we have

$$\frac{d}{dt}\int_\Omega |\nabla v|^2\,dx + \int_\Omega |\Delta v|^2\,dx + 2\int_\Omega |\nabla v|^2\,dx \le 2\int_\Omega u^2\,dx + 2\int_\Omega f^2\,dx. \tag{3.11}$$

Integrating (3.11) over $(0,t)$, we obtain

$$\begin{aligned}&\|\nabla v(t)\|_{L^2}^2 + \int_0^t \|\Delta v\|_{L^2}^2\,d\tau + \int_0^t \|\nabla v\|_{L^2}^2\,d\tau \\ &\quad\le \|\nabla v_0\|_{L^2}^2 + 2\int_0^t \big(\|u\|_{L^2}^2 + \|f\|_{L^2}^2\big)\,d\tau, \quad \text{for all } t\in[0,T].\end{aligned} \tag{3.12}$$

Moreover, using (3.2) and combining (3.10) and (3.12), we have

$$\begin{aligned}\|v(t)\|_{H^1}^2 + \int_0^t \|v\|_{H^2}^2\,d\tau &\le \|v_0\|_{H^1}^2 + 4\int_0^t \big(\|u\|_{L^2}^2 + \|f\|_{L^2}^2\big)\,d\tau \\ &\le K_2\big(\|v_0\|_{H^1}, M_1, \|f\|_{L^2(Q)}, T\big), \quad \text{for all } t\in[0,T].\end{aligned} \tag{3.13}$$

Step 2. w is bounded in $L^\infty(0,T;H^2(\Omega))$.

Multiplying equation (1.3) by $-\Delta w$ and integrating over Ω, from (3.3) and the negative of v, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla w|^2\,dx &= -\int_\Omega \nabla w\nabla(vw)\,dx \\ &= -\int_\Omega \nabla w(\nabla v w + \nabla w v)\,dx\end{aligned}$$

$$
\begin{aligned}
&\le -\int_\Omega \nabla w \nabla v w \, dx \\
&\le \|w\|_{L^\infty} \|\nabla w\|_{L^2} \|\nabla v\|_{L^2} \\
&\le \frac{1}{2}\|\nabla w\|_{L^2}^2 + \frac{1}{2}\|w\|_{L^\infty}^2 \|\nabla v\|_{L^2}^2.
\end{aligned}
\tag{3.14}
$$

Thanks to (3.3), from (3.13), (3.14) and Gronwall lemma, we have

$$
\begin{aligned}
\|\nabla w\|_{L^2}^2 &\le e^t \left(\|\nabla w_0\|_{L^2}^2 + \int_0^t \|w\|_{L^\infty}^2 \|\nabla v\|_{L^2}^2 \, d\tau \right) \\
&\le e^t \left(\|\nabla w_0\|_{L^2}^2 + \|w_0\|_{L^\infty}^2 \int_0^t \|\nabla v\|_{L^2}^2 \, d\tau \right) \\
&= K_3\big(\|w_0\|_{H^1}, \|w_0\|_{L^\infty}, \|v_0\|_{L^2}, \|f\|_{L^2(Q)} M_1, T\big).
\end{aligned}
\tag{3.15}
$$

Applying Δ to equation (1.3), multiplying by Δw, and integrating the product over Ω, we have

$$
\frac{1}{2}\int_\Omega |\Delta w|^2 \, dx = -\int_\Omega \Delta w \Delta(vw) \, dx. \tag{3.16}
$$

For the term on the right-hand side, using the interpolation of Sobolev spaces,

$$
\begin{aligned}
-\int_\Omega \Delta w \Delta(vw) \, dx &= -\int_\Omega \Delta w(\Delta v w + 2\nabla v \nabla w + \Delta w v) \, dx \\
&\le \|\Delta w\|_{L^2} \|w\|_{L^\infty} \|\Delta v\|_{L^2} + C\|\nabla w\|_{L^4} \|\nabla v\|_{L^4} \|\Delta w\|_{L^2} \\
&\le \|\Delta w\|_{L^2} \|w\|_{L^\infty} \|\Delta v\|_{L^2} + C\|\nabla w\|_{L^2} \|\Delta w\|_{L^2}^{3/2} \|\Delta v\|_{L^2}^{1/2} \|\nabla v\|_{L^2}^{1/2} \\
&\le C\big(\|w\|_{L^\infty} + \|\nabla w\|_{L^2}\big)^2 \|\Delta w\|_{L^2}^2 + \frac{1}{2}\|\Delta v\|_{L^2}^2.
\end{aligned}
\tag{3.17}
$$

Combining (3.16) and (3.17), we can get

$$
\frac{d}{dt}\int_\Omega |\Delta w|^2 \, dx \le C\big(\|w\|_{L^\infty} + \|\nabla w\|_{L^2}\big)^2 \|\Delta w\|_{L^2}^2 + \|\Delta v\|_{L^2}^2. \tag{3.18}
$$

From (3.3), (3.13), (3.15), and Gronwall lemma, we have

$$
\begin{aligned}
\|\Delta w\|_{L^2}^2 &\le e^{C\int_0^t (\|w_0\|_{L^\infty} + \|\nabla w\|_{L^2}) \, d\tau} \left(\|\Delta w_0\|_{L^2}^2 + \int_0^t \|\Delta v\|_{L^2}^2 \, d\tau \right) \\
&\le e^{C\int_0^t (\|w_0\|_{L^\infty} + K_3) \, d\tau} \left(\|\Delta w_0\|_{L^2}^2 + \int_0^t \|\Delta v\|_{L^2}^2 \, d\tau \right) \\
&\le K_4\big(\|w_0\|_{H^1}, \|w_0\|_{L^\infty}, \|\nabla v_0\|_{L^2}, \|f\|_{L^2(Q)}, M_1, T\big).
\end{aligned}
\tag{3.19}
$$

Step 3. u is bounded in $L^\infty(0,T;H^2(\Omega))$.

We observe that, thanks to the positivity of u, we have $0 \le \ln(u+1) \le u$. Then

$$
\int_\Omega \big|\ln(u+1)\big|^2 \, dx \le \int_\Omega |u|^2 \, dx. \tag{3.20}
$$

We also note that

$$\int_\Omega \left|\nabla \ln(u+1)\right|^2 dx = \int_\Omega \left|\frac{\nabla u}{u+1}\right|^2 dx \le \int_\Omega |\nabla u|^2 dx. \tag{3.21}$$

Taking into account that $u \in L^2(0,T;H^1(\Omega))$, from (3.20) and (3.21), we deduce that $\ln(u+1) \in L^2(0,T;H^1(\Omega))$. Note that

$$\frac{d}{dt}\int_\Omega \left\{\left(u(t)+1\right)\ln\left(u(t)+1\right) - u(t)\right\} dx = \left\langle \frac{du}{dt}(t), \ln\left(u(t)+1\right)\right\rangle_{(H^1)'\times H^1}.$$

Testing equation (1.1) with $\ln(u+1) \in L^2(0,T;H^1(\Omega))$, and integrating by parts, we have

$$\begin{aligned} &\frac{d}{dt}\int_\Omega (u+1)\ln(u+1)\,dx + 4\|\nabla\sqrt{u+1}\|_{L^2}^2 \\ &\quad\le -\chi\int_\Omega \frac{u}{u+1}\nabla v\cdot\nabla u - \xi\int_\Omega \frac{u}{u+1}\nabla w\cdot\nabla u + \mu\int_\Omega (u+1)\ln(u+1)\,dx. \end{aligned} \tag{3.22}$$

Applying the Young inequality, we obtain

$$\begin{aligned} \chi\int_\Omega \frac{u}{u+1}\nabla u\cdot\nabla v\,dx &= \chi\int_\Omega \nabla u\left(1-\frac{1}{u+1}\right)\nabla v\,dx \\ &= \chi\int_\Omega \nabla\left(u-\ln(u+1)\right)\nabla v\,dx \\ &= \chi\int_\Omega \left(\ln(u+1)-u\right)\Delta v\,dx \\ &\le \chi\int_\Omega \left|u-\ln(u+1)\right||\Delta v|\,dx \\ &\le \chi\int_\Omega u|\Delta v|\,dx \\ &\le \delta\|u\|_{L^2}^2 + C_\delta\|\Delta v\|_{L^2}^2. \end{aligned} \tag{3.23}$$

Similarly, we have

$$\xi\int_\Omega \frac{u}{u+1}\nabla u\cdot\nabla w \le \delta\|u\|_{L^2}^2 + C_\delta\|\Delta w\|_{L^2}^2. \tag{3.24}$$

Then combining (3.22)–(3.24) and (3.1), we have

$$\begin{aligned} &\frac{d}{dt}\int_\Omega (u+1)\ln(u+1)\,dx + 4\|\nabla\sqrt{u+1}\|_{L^2}^2 \\ &\quad\le 2\delta\|u\|_{L_2}^2 + C\left(\|\Delta v\|_{L^2}^2 + \|\Delta w\|_{L^2}^2\right) + \mu\int_\Omega (u+1)\ln(u+1)\,dx. \end{aligned} \tag{3.25}$$

Then, from (3.2), (3.13) and (3.19), as well as applying Gronwall lemma to (3.25), we deduce

$$\begin{aligned} &\int_\Omega (u+1)\ln(u+1)\,dx \\ &\quad\le e^{\int_0^t \mu\,d\tau}\left(\left\|(u_0+1)\ln(u_0+1)\right\|_{L^1} + C\int_0^t \left(\|u\|_{L^2} + \|\Delta v\|_{L^2}^2 + \|\Delta w\|_{L^2}^2\right) d\tau\right.\end{aligned}$$

$$\leq K_5\big(\|w_0\|_{H^1}, \|w_0\|_{L^\infty}, \|v_0\|_{H^1}, \|f\|_{L^2(Q)}, M_1, T\big). \tag{3.26}$$

Multiplying equation (1.1) by u and integrating over Ω, we see that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega u^2\,dx + \int_\Omega |\nabla u|^2\,dx \\
&\quad\leq \chi\int_\Omega u\nabla u\nabla v\,dx + \xi\int_\Omega u\nabla u\nabla w\,dx + \mu\int_\Omega u^2\,dx \\
&\quad\leq -\frac{\chi}{2}\int_\Omega u^2\Delta v\,dx - \frac{\xi}{2}\int_\Omega u^2\Delta w\,dx + \mu\int_\Omega u^2\,dx.
\end{aligned} \tag{3.27}$$

Here, we note that

$$\begin{aligned}
-\frac{\chi}{2}\int_\Omega u^2\Delta v\,dx &\leq \frac{\chi}{2}\|u\|_{L^3}^2\|\Delta v\|_{L^3} \\
&\leq C\|u\|_{L^3}^2\|v\|_{H^3}^{2/3}\|v\|_{H^1}^{1/3} \\
&\leq C\|u\|_{L^3}^2\|v\|_{H^3}^{2/3},
\end{aligned} \tag{3.28}$$

for some positive constant C. Applying Young's inequality, we further deduce that

$$\begin{aligned}
-\frac{\chi}{2}\int_\Omega u^2\Delta v\,dx &\leq C\big(\delta\|u\|_{H^1}^2\big\|(u+1)\ln(u+1)\big\|_{L^1} + p\big(\delta^{-1}\big)\|u\|_{L^1}\big)^{2/3}\|v\|_{H^3}^{2/3} \\
&\leq C\big(\delta\|u\|_{H^1}^2 + p\big(\delta^{-1}\big)\big)^{2/3}\|v\|_{H^3}^{2/3} \\
&\leq \delta\|v\|_{H^3}^2 + C\delta^{1/2}\|u\|_{H^1}^2 + C_\delta^{-1/2}p\big(\delta^{-1}\big).
\end{aligned} \tag{3.29}$$

So, in the same way, we can derive

$$-\frac{\xi}{2}\int_\Omega u^2\Delta w\,dx \leq \|w\|_{H^3(\Omega)}^2 + C\delta^{1/2}\|u\|_{H^1}^2 + C_\delta^{-1/2}p\big(\delta^{-1}\big). \tag{3.30}$$

Combining (3.27)–(3.30), (3.13) and (3.15), we then deduce

$$\frac{d}{dt}\int_\Omega u^2\,dx + \int_\Omega |\nabla u|^2\,dx \leq C\delta\big(\|v\|_{H^3}^2 + \|w\|_{H^3}^2\big) + C\delta^{-1/2}p\big(\delta^{-1}\big) + \mu\int_\Omega u^2\,dx. \tag{3.31}$$

Next, applying ∇ to the equation of (1.2), multiplying by $\nabla\Delta v$, and integrating the product over Ω, we have

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega |\Delta v|^2\,dx + \int_\Omega |\nabla\Delta v|^2\,dx + 2\int_\Omega |\Delta v|^2\,dx \\
&\quad\leq 2\int_\Omega |\nabla u|^2\,dx + 2\int_\Omega |\nabla f|^2\,dx.
\end{aligned} \tag{3.32}$$

Applying operator $\nabla\Delta$ to equation (1.3), multiplying by $\nabla\Delta w$, and integrating over Ω, we have

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla\Delta w|^2\,dx = -\int_\Omega \nabla\Delta(vw)\nabla\Delta w\,dx. \tag{3.33}$$

Thanks to (3.3), (3.19), we further deduce that

$$\begin{aligned}
&-\int_\Omega \nabla\Delta(vw)\nabla\Delta w\,dx \\
&= -\int_\Omega (\nabla\Delta v w + 3\Delta v\nabla w + 3\nabla v\Delta w + \nabla\Delta w v)\nabla\Delta w\,dx \\
&\le \|\nabla\Delta v\|_{L^2}\|w\|_{L^\infty}\|\nabla\Delta w\|_{L^2} + 3\|\Delta v\|_{L^4}\|\nabla w\|_{L^4}\|\nabla\Delta w\|_{L^2} \\
&\quad + C\|\nabla\Delta w\|_{L^2}\|\Delta w\|_{L^4}\|\nabla v\|_{L^4} \\
&\le \|\nabla\Delta v\|_{L^2}\|w\|_{L^\infty}\|\nabla\Delta w\|_{L^2} + C\|v\|_{H^3}\|w\|_{H^2}\|\nabla\Delta w\|_{L^2} \\
&\quad + C\|w\|_{H^2}^{1/2}\|w\|_{H^3}^{3/2}\|\Delta v\|_{L^2}^{1/2}\|\nabla v\|_{L^2}^{1/2} \\
&\le C\|v\|_{H^3}\|w\|_{H^3} + C\|w\|_{H^3}^{3/2}\|\Delta v\|_{L^2}^{1/2} \le \frac{1}{4}\|v\|_{H^3}^2 + C\|w\|_{H^3}^2.
\end{aligned} \tag{3.34}$$

Replacing (3.34) in (3.33), we have

$$\frac{d}{dt}\int_\Omega |\nabla\Delta w|^2\,dx \le \frac{1}{2}\|v\|_{H^3}^2 + C\|w\|_{H^3}^2. \tag{3.35}$$

Then, choosing δ small enough to absorb $\|v\|_{H^3}$, from (3.7) with $p = 2$, (3.9), (3.11), (3.14), (3.18), (3.31), (3.32), and (3.35), we have

$$\begin{aligned}
&\frac{d}{dt}\big(\|u\|_{L^2}^2 + \|v\|_{H^2}^2 + \|w\|_{H^3}^2\big) + \frac{1}{4}\|v|_{H^3}^2 + \|u\|_{H^1}^2 \\
&\quad \le C\|w\|_{H^3}^2 + C\delta^{-1/2}p\big(\delta^{-1}\big) + 3\mu\|u\|_{L^2}^2 + 2\int_\Omega |\nabla f|^2\,dx.
\end{aligned} \tag{3.36}$$

Then, applying the Gronwall lemma to (3.36), we deduce

$$\|u\|_{L^2} + \|v\|_{H^2} + \|w\|_{H^3} \le K_6\big(\|u_0\|_{L^2}, \|v_0\|_{H^2}, \|w_0\|_{H^3}, \|\nabla f\|_{L^2(Q)}, T\big). \tag{3.37}$$

Integrating (3.36) over $(0,t)$, we obtain

$$\int_0^t \|u\|_{H^1}^2\,d\tau + \int_0^t \|v\|_{H^3}^2\,d\tau \le C. \tag{3.38}$$

Multiplying equation (1.1) by $-\Delta u$ and integrating over Ω, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla u|^2\,dx + \int_\Omega |\Delta u|^2\,dx \\
&= \chi\int_\Omega (\Delta u)(\nabla u\cdot\nabla v + u\Delta v)\,dx + \xi\int_\Omega (\Delta u)(\nabla u\cdot\nabla w + u\Delta w)\,dx \\
&\quad - \int_\Omega (\Delta u)u(1-u-w)\,dx.
\end{aligned} \tag{3.39}$$

The first two terms in the right-hand side can be estimated as follows:

$$\begin{aligned}
\chi\left|\int_\Omega (\Delta u)(\nabla u\cdot\nabla v + u\Delta v)\,dx\right| &\le \chi\|\Delta u\|_{L^2}\big(\|\nabla u\|_{L^3}\|\nabla v\|_{L^6} + \|u\|_{\mathcal{C}}\|\Delta v\|_{L^2}\big) \\
&\le C\|\Delta u\|_{L^2}\big(\|\nabla u\|_{H^{\frac{1}{3}}}\|\nabla v\|_{H^1} + \|u\|_{H^{\frac{4}{3}}}\|\Delta v\|_{L^2}\big)
\end{aligned}$$

$$
\begin{aligned}
&\le C\|u\|_{H^2}\|u\|_{H^{\frac{4}{3}}}\|v\|_{H^2} \\
&\le C\|u\|_{H^2}^{\frac{5}{3}}\|u\|_{L^2}^{\frac{1}{3}}\|v\|_{H^2} \\
&\le \delta\|u\|_{H^2}^2 + C_\delta\|u\|_{L^2}^2\|v\|_{H^2}^6. \qquad (3.40)
\end{aligned}
$$

Through a similar calculation as in obtaining the above inequality, it is easy to get

$$
\xi\left|\int_\Omega (\Delta u)(\nabla u\cdot\nabla w + u\Delta w)\,dx\right| \le \delta\|u\|_{H^2}^2 + C_\delta\|u\|_{L^2}^2\|w\|_{H^2}^6. \qquad (3.41)
$$

For the third term of the right-hand side, thanks to the nonnegativity of u and w, applying the Gagliardo–Nirenberg inequality and (3.19), we have

$$
\begin{aligned}
-\int_\Omega (\Delta u)u(1-u-w)\,dx &= -\int_\Omega (\Delta u)\big(u-u^2-uw\big)\,dx \\
&= \int_\Omega |\nabla u|^2\,dx - 2\int_\Omega |\nabla u|^2 u\,dx - \int_\Omega \nabla u(\nabla uw + u\nabla w)\,dx \\
&\le \int_\Omega |\nabla u|^2\,dx - \int_\Omega \nabla uu\nabla w\,dx \\
&\le \int_\Omega |\nabla u|^2\,dx - \frac{1}{2}\int_\Omega \nabla u^2\nabla w\,dx \\
&= \int_\Omega |\nabla u|^2\,dx + \frac{1}{2}\int_\Omega u^2\Delta w\,dx \\
&\le \|\nabla u\|_{L^2}^2 + \|u\|_{L^4}^2\|\Delta w\|_{L^2} \\
&\le \|\nabla u\|_{L^2}^2 + C\|u\|_{L^2}\|u\|_{H^1}\|\Delta w\|_{L^2} \\
&\le \|\nabla u\|_{L^2}^2 + C\|u\|_{H^1}\|\Delta w\|_{L^2} \\
&\le \|u\|_{H^1}^2 + C. \qquad (3.42)
\end{aligned}
$$

Therefore, we have

$$
\frac{d}{dt}\|u\|_{H^1}^2 + \|u\|_{H^2}^2 \le 2\delta\|u\|_{H^2}^2 + C\|u\|_{L^2}^2\big(\|v\|_{H^2}^6 + \|w\|_{H^2}^6\big) + C\|u\|_{H^1}^2 + C. \qquad (3.43)
$$

Taking $\delta > 0$ small enough, and using (3.39), we can get

$$
\|u\|_{H^1}^2 + \int_0^t \|u\|_{H^2}^2\,d\tau \le K_7\big(\|u_0\|_{H^1}, \|v_0\|_{H^2}, \|w_0\|_{H^3}, \|\nabla f\|_{L^2(Q)}, T\big). \qquad (3.44)
$$

The proof is complete. □

Theorem 3.1 *For all initial functions $(u_0, v_0, w_0) \in H^1(\Omega) \times H_N^2(\Omega) \times H^3(\Omega)$ and $f \in L^2(0,T;H^1(\Omega))$, the problem (1.1)–(1.5) admits a unique global-in-time nonnegative solution (u,v,w) in the function space*

$$
\begin{cases}
u \in H^1((0,T);H^{-2}(\Omega)) \cap C([0,T];H^1(\Omega)) \cap L^2((0,T);H_N^2(\Omega)), \\
v \in H^1((0,T);H^{-3}(\Omega)) \cap C([0,T];H_N^2(\Omega)) \cap L^2((0,T);H_N^3(\Omega)), \\
w \in H^1((0,T);H^{-3}(\Omega)) \cap C([0,T];H_N^3(\Omega)) \cap L^2((0,T);H_N^3(\Omega)).
\end{cases}
$$

In addition, the solution satisfies the uniform estimate involving the norms of initial functions such that

$$\begin{aligned} &\|u\|_{H^1}^2 + \|v\|_{H^2}^2 + \|w\|_{H^3}^2 + \|u\|_{H^1((0,T);H^{-2}(\Omega))} \\ &\quad + \|v\|_{H^1((0,T);H^{-3}(\Omega))} + \|w\|_{H^1((0,T);H^{-3}(\Omega))} \le C, \quad t \ge 0. \end{aligned} \tag{3.45}$$

Proof From Theorem 2.1 and Proposition 2.1, for each triplet of nonnegative initial functions (u_0, v_0, w_0), there exists a unique nonnegative local solution (u, v, w) on an interval $[0, T]$, where the existence time $T > 0$ depends only on the norms of those functions, $\|u_0\|_{H^1} + \|v_0\|_{H^2} + \|w_0\|_{H^3}$. In addition, from Lemma 3.3, the norm $\|u(t)\|_{H^1} + \|v(t)\|_{H^2} + \|w(t)\|_{H^3}$, $0 \le t \le T$, is estimated from above by a uniform constant C, depending only on the norm $\|u_0\|_{H^1} + \|v_0\|_{H^2} + \|w_0\|_{H^3}$. Then, we consider the problem in $[T, 2T]$. Hence, the interval can be extended to $[0, 2T]$, and the norm $\|u(t)\|_{H^1} + \|v(t)\|_{H^2} + \|w(t)\|_{H^3}$, $0 \le t \le 2T$, is estimated again by the same constant C from (3.8). Then, the existence time can be extended to $3T$. Iterating this procedure proves the global-in-time existence of solutions with the estimate (3.8). □

4 Existence of an optimal control

In this section, we will prove the existence of the optimal solution of the control problem. The method we use for treating this problem was inspired by some ideas of Guillén-González et al. [11]. Assume that $\mathcal{F} \subset L^2(0, T; H^1(\Omega_c))$ is a nonempty, closed and convex set, where $\Omega_c \subset \Omega$ is the control domain, and $\Omega_d \subset \Omega$ is the observability domain. We consider data $(u_0, v_0, w_0) \in H^1(\Omega) \times H^2(\Omega) \times H^3(\Omega)$ with $u_0 \ge 0$, $v_0 \ge 0$ and $w_0 \ge 0$ in Ω, and the function $f \in \mathcal{F}$ that describes the control acting on the v-equation.

Now, we consider the optimal control problem for system (1.1)–(1.5) as follows:

$$\begin{cases} \text{Find } (u, v, w, f) \in \mathcal{M} \text{ such that the functional} \\ J(u, v, w, f) = \frac{\beta_1}{2}\|u(x,t) - u_d(x,t)\|_{L^2(Q_d)}^2 + \frac{\beta_2}{2}\|v(x,t) - v_d(x,t)\|_{L^2(Q_d)}^2 \\ \qquad\qquad + \frac{\beta_3}{2}\|w(x,t) - w_d(x,t)\|_{L^2(Q_d)}^2 + \frac{\beta_4}{2}\|f(x,t)\|_{L^2(Q_c)}^2 \\ \text{is minimized, subject to } (u, v, w, f) \text{ satisfying} \\ \text{the system (1.1)–(1.5) a.e. in } Q_d, \end{cases} \tag{4.1}$$

where

$$\begin{aligned} \mathcal{M} &= L^\infty\big(0, T; H^1(\Omega)\big) \cap L^2\big(0, T; H^2(\Omega)\big) \times L^\infty\big(0, T; H^2(\Omega)\big) \\ &\quad \cap L^2\big(0, T; H^3(\Omega)\big) \times L^\infty\big(0, T; H^2(\Omega)\big) \cap L^2\big(0, T; H^3(\Omega)\big) \times \mathcal{F}, \\ Q_d &= [0, T] \times \Omega_d, \qquad Q_c = [0, T] \times \Omega_c. \end{aligned} \tag{4.2}$$

Here $(u_d, v_d, w_d) \in L^2(Q_d) \times L^2(Q_d) \times L^2(Q_d)$ represents the desired states and the $\beta_i (i = 1, 2, 3, 4) > 0$. We will use

$$\mathcal{S}_{\mathrm{ad}} = \big\{s = (u, v, w, f) \in \mathcal{M} : \ s \ \text{ is a solution of (1.1)–(1.5)}\big\}, \tag{4.3}$$

which denotes the set of admissible solutions of (4.1).

First, we will consider the existence of a global optimal solution of problem (4.1). To this end, we start with the definition of optimal solution.

Definition 4.1 An element $(\tilde{u},\tilde{v},\tilde{w},\tilde{f}) \in \mathcal{S}_{\text{ad}}$ will be called a global optimal solution of problem (4.1) if

$$J(\tilde{u},\tilde{v},\tilde{w},\tilde{f}) = \min_{(u,v,w,f)\in\mathcal{S}_{\text{ad}}} J(u,v,w,f). \tag{4.4}$$

Here we state the following result.

Theorem 4.1 *Let $u_0 \in H^1(\Omega)$, $v_0 \in H^2(\Omega)$ and $w_0 \in H^3(\Omega)$ with $u_0 \geq 0$, $v_0 \geq 0$ and $w_0 \geq 0$ in Ω. Then the optimal control problem (4.1) has at least one global optimal solution $(\tilde{u},\tilde{v},\tilde{w},\tilde{f}) \in \mathcal{S}_{\text{ad}}$.*

Proof From Theorem 3.1, recalling that $\mathcal{S}_{\text{ad}}$ is nonempty, there exists a minimizing sequence $\{s_m\}_{m\in\mathbb{N}} \subset \mathcal{S}_{\text{ad}}$ such that $\lim_{m\to+\infty} J(s_m) = \inf_{s\in\mathcal{S}_{\text{ad}}} J(s)$. Then, by the definition of $\mathcal{S}_{\text{ad}}$, we know that for each $m \in \mathbb{N}$, s_m satisfies

$$\begin{cases} u_{mt} = \Delta u_m - \chi\nabla\cdot(u_m\cdot\nabla v_m) - \xi\nabla\cdot(u_m\cdot\nabla w_m) & \\ \qquad + \mu u_m(1-u_m-w_m), & \text{in } Q, \\ v_{mt} = \Delta v_m - v_m + u_m + f_m, & \text{in } Q, \\ w_{mt} = -v_m w_m, & \text{in } Q, \\ \frac{\partial u_m}{\partial \nu} = \frac{\partial v_m}{\partial \nu} = \frac{\partial w_m}{\partial \nu} = 0, & \text{on } \partial\Omega, \\ u_m(x,0) = u_0(x), \qquad v_m(x,0) = v_0(x), \qquad w_m(x,0) = w_0(x), & \text{in } \Omega. \end{cases} \tag{4.5}$$

Hence, it follows that

$$\{f_m\}_{m\in\mathbb{N}} \quad \text{is bounded in } L^2(Q_c). \tag{4.6}$$

By (3.37), (3.38), (3.44), and (3.45), we see that there exists $C > 0$ such that

$$\begin{aligned} &\|u_m\|_{H^1}^2 + \|v_m\|_{H^2}^2 + \|w_m\|_{H^3}^2 + \|u\|_{H^1(0,T;H^{-2}(\Omega))} + \|v\|_{H^1(0,T;H^{-3}(\Omega))} \\ &\quad + \|w\|_{H^1(0,T;H^{-3}(\Omega))} + \int_0^t \|u_m\|_{H^2}^2\,d\tau + \int_0^t \|v_m\|_{H^3}^2\,d\tau \leq C, \quad t \geq 0. \end{aligned} \tag{4.7}$$

Therefore, by (4.6), (4.7) and since $\mathcal{F}$ is a closed convex subset of $L^2(Q_c)$, we deduce that there exist $\tilde{s} = (\tilde{u},\tilde{v},\tilde{w},\tilde{f}) \in \mathcal{M}$ and a subsequence of $\{s_m\}_{m\in\mathbf{N}}$, not relabeled, such that, as $m \to +\infty$,

$$u_m \to \tilde{u}, \quad \text{weakly in } L^2\big(0,T;H^2(\Omega)\big) \cap H^1\big(0,T;H^{-2}(\Omega)\big), \tag{4.8}$$

$$v_m \to \tilde{v}, \quad \text{weakly in } L^2\big(0,T;H^3(\Omega)\big) \cap H^1\big(0,T;H^{-3}(\Omega)\big), \tag{4.9}$$

$$w_m \to \tilde{w}, \quad \text{weakly in } H^1\big(0,T;H^{-3}(\Omega)\big) \tag{4.10}$$

and

$$u_m \to \tilde{u}, \quad \text{weak * in } L^\infty\big(0,T;H^1(\Omega)\big), \tag{4.11}$$

$$v_m \to \tilde{v}, \quad \text{weak * in } L^\infty\big(0,T;H^2(\Omega)\big), \tag{4.12}$$

$$w_m \to \tilde{w}, \quad \text{weak * in } L^\infty\big(0,T;H^3(\Omega)\big), \tag{4.13}$$

$$f_m \to \tilde{f}, \quad \text{weak in } L^2(Q_c), \text{ and } \tilde{f} \in \mathcal{F}. \tag{4.14}$$

From (4.8)–(4.13) and the Aubin–Lions lemma, we have

$$u_m \to \tilde{u}, \quad \text{strongly in } C\big(0,T;L^2(\Omega)\big) \cap L^2\big(0,T;H^1(\Omega)\big), \tag{4.15}$$

$$v_m \to \tilde{v}, \quad \text{strongly in } C\big(0,T;H^1(\Omega)\big) \cap L^2\big(0,T;H^2(\Omega)\big), \tag{4.16}$$

$$w_m \to \tilde{w}, \quad \text{strongly in } C\big(0,T;H^2(\Omega)\big). \tag{4.17}$$

In particular, since $\nabla \cdot (u_m \nabla v_m) = \nabla u_m \cdot \nabla v_m + u_m \Delta v_m$ and $\nabla \cdot (u_m \nabla w_m) = \nabla u_m \cdot \nabla w_m + u_m \Delta w_m$ is bounded in $L^2(0,T;L^2(\Omega))$, one has the weak convergences:

$$\nabla \cdot (u_m \nabla v_m) \to \psi_1, \quad \text{weakly in } L^2\big(0,T;L^2(\Omega)\big),$$

$$\nabla \cdot (u_m \nabla w_m) \to \psi_2, \quad \text{weakly in } L^2\big(0,T;L^2(\Omega)\big).$$

On the other hand, from (4.8)–(4.17), one has

$$u_m \nabla v_m \to \tilde{u} \nabla \tilde{v}, \quad \text{weakly in } L^\infty\big(0,T;L^2(\Omega)\big),$$

$$u_m \nabla w_m \to \tilde{u} \nabla \tilde{w}, \quad \text{weakly in } L^\infty\big(0,T;L^2(\Omega)\big).$$

Therefore, we can identify $\psi_1 = \nabla \cdot (\tilde{u} \nabla \tilde{v})$ and $\psi_2 = \nabla \cdot (\tilde{u} \nabla \tilde{w})$ a.e. in Q, and thus

$$\nabla \cdot (u_m \nabla v_m) \to \nabla \cdot (\tilde{u} \nabla \tilde{v}), \quad \text{weakly in } L^2\big(0,T;L^2(\Omega)\big), \tag{4.18}$$

$$\nabla \cdot (u_m \nabla w_m) \to \nabla \cdot (\tilde{u} \nabla \tilde{w}), \quad \text{weakly in } L^2\big(0,T;L^2(\Omega)\big). \tag{4.19}$$

Moreover, by (4.15)–(4.17), we see $(u_m(0), v_m(0), w_m(0)) \to (\tilde{u}(0), \tilde{v}(0), \tilde{w}(0))$, in $L^2(\Omega) \times H^1(\Omega) \times H^2(\Omega)$. Since $u_m(0) = u_0$, $v_m(0) = v_0$, $w_m(0) = w_0$, we conclude that $\tilde{u}(0) = u_0$, $\tilde{v}(0) = v_0$ and $\tilde{w}(0) = w_0$, thus $\tilde{s}$ satisfies the initial conditions given in (1.1)–(1.5). Therefore, considering the convergences (4.8)–(4.19), we can pass to the limit in (4.5) as $m \to +\infty$, and conclude that $\tilde{s} = (\tilde{u}, \tilde{v}, \tilde{w}, \tilde{f})$ is a solution of the system (1.1)–(1.5), that is, $\tilde{s} \in \mathcal{S}_{\text{ad}}$. Hence,

$$\lim_{m \to +\infty} J(s_m) = \inf_{s \in \mathcal{S}_{\text{ad}}} J(s) \le J(\tilde{s}). \tag{4.20}$$

On the other hand, since J is lower semicontinuous on $\mathcal{S}_{\text{ad}}$, we have $J(\tilde{s}) \le \liminf_{m \to +\infty} J(s_m)$, which, jointly with (4.20), implies (4.4). □

5 First-order necessary optimality condition

Now, we will study the first-order necessary optimality conditions for a local optimal solution $(\tilde{u}, \tilde{v}, \tilde{w}, \tilde{f})$ of problem (4.1). To this end, we will use a result on existence of Lagrange multipliers in Banach spaces [37]. First, we discuss the following problem:

$$\min J(s) \text{ subject to } s \in \mathcal{S} = \big\{ s \in \mathcal{M} : G(s) \in \mathcal{N} \big\}, \tag{5.1}$$

where $J : X \to \mathbb{R}$ is a functional, $G : X \to Y$ is an operator, X and Y are Banach spaces, $\mathcal{M}$ is a nonempty closed convex subset of X, and $\mathcal{N}$ is a nonempty closed convex cone in Y with vertex at the origin.

For a subset A of X (or Y), A^+ denotes its polar cone, that is,

$$A^+ = \left\{\rho \in X' : \langle \rho, a\rangle_{X'} \geq 0, \forall a \in A\right\}.$$

Definition 5.1 A point $\tilde{s} \in \mathcal{S}$ is said to be a local optimal solution of problem (5.1), if there exits $\varepsilon > 0$ such that for all $s \in \mathcal{S}$ satisfying $\|s - \tilde{s}\|_X \leq \varepsilon$ one has $J(\tilde{s}) \leq J(s)$.

Definition 5.2 Let $\tilde{s} \in \mathcal{S}$ be a local optimal solution for problem (5.1) with respect to the X-norm. Suppose that J and G are Fréchet differentiable in $\tilde{s}$, with derivatives $J'(\tilde{s})$ and $G'(\tilde{s})$, respectively. Then, any $\lambda \in Y'$ is called a Lagrange multiplier for (5.1) at the point $\tilde{s}$ if

$$\begin{cases} \lambda \in \mathcal{N}^+, \\ \langle \lambda, G(\tilde{s})\rangle_{Y'} = 0, \\ J'(\tilde{s}) - \lambda \circ G'(\tilde{s}) \in \mathcal{C}(\tilde{s})^+, \end{cases} \tag{5.2}$$

where $\mathcal{C}(\tilde{s}) = \{\theta(s - \tilde{s}) : s \in \mathcal{M}, \theta \geq 0\}$ is the conical hull of $\tilde{s}$ in $\mathcal{M}$.

Definition 5.3 Let $\tilde{s} \in \mathcal{S}$ be a local optimal solution for problem (5.1). We say that $\tilde{s}$ is a regular point if

$$G'(\tilde{s})\big[\mathcal{C}(\tilde{s})\big] - \mathcal{N}\big(G(\tilde{s})\big) = Y, \tag{5.3}$$

where $\mathcal{N}(G(\tilde{s})) = \{(\theta(n - G(\tilde{s})) : n \in \mathcal{N}, \theta \geq 0\}$ is the conical hull of $G(\tilde{s})$ in $\mathcal{N}$.

Theorem 5.1 ([37, Theorem 3.1]) *Let $\tilde{s} \in \mathcal{S}$ be a local optimal solution for problem* (5.1). *Suppose that J is a Fréchet differentiable function and G is continuously Fréchet-differentiable. If $\tilde{s}$ is a regular point, then the set of Lagrange multipliers for* (5.1) *at $\tilde{s}$ is nonempty.*

Remark 5.1 To obtain the existence of first-order necessary optimality conditions, because of the nonlinearity of $-\xi\nabla \cdot (u \cdot \nabla w)$ and $\mu u(1 - u - w)$, the method used in [10] seems not applicable to the present situation. Our method is based on the Lagrange multiplier theorem.

Now, we will reformulate the optimal control problem (4.1) in the abstract setting (5.1). We consider the following Banach spaces:

$$\begin{aligned} &X := \mathcal{W}_u \times \mathcal{W}_v \times \mathcal{W}_w \times L^2(Q_c), \\ &Y := L^2(Q) \times L^2\big(0, T; H^1(\Omega)\big) \times L^\infty\big(0, T; H^3(\Omega)\big) \times H^1(\Omega) \times H^2(\Omega) \times H^3(\Omega), \end{aligned} \tag{5.4}$$

where

$$\mathcal{W}_u = \left\{u \in L^\infty\big(0, T; H^1(\Omega)\big) \cap L^2\big(0, T; H^2(\Omega)\big) : \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega\right\}, \tag{5.5}$$

$$\mathcal{W}_v = \left\{ v \in L^\infty\big(0,T;H^2(\Omega)\big) \cap L^2\big(0,T;H^3(\Omega)\big) : \frac{\partial v}{\partial \nu} = 0 \text{ on } \partial\Omega \right\}, \tag{5.6}$$

$$\mathcal{W}_w = \left\{ w \in L^\infty\big(0,T;H^3(\Omega)\big) : \frac{\partial w}{\partial \nu} = 0 \text{ on } \partial\Omega \right\}. \tag{5.7}$$

By Theorem 3.1, we know that the operator $G = (G_1, G_2, G_3, G_4, G_5, G_6) : X \to Y$, where

$$G_1 : X \to L^2(Q), \qquad G_2 : X \to L^2(Q), \qquad G_3 : X \to L^\infty\big(0,T;H^3(\Omega)\big),$$
$$G_4 : X \to H^1(\Omega), \qquad G_5 : X \to H^2(\Omega), \qquad G_6 : X \to H^3(\Omega)$$

are defined at each point $s = (u, v, w, f) \in X$ by

$$\begin{cases} G_1(s) = \partial_t u - \Delta u + \chi \nabla \cdot (u \nabla v) + \xi \nabla \cdot (u \nabla w) - \mu u(1 - u - w), \\ G_2(s) = \partial_t v - \Delta v + v - u - f, \\ G_3(s) = \partial_t w + vw, \\ G_4(s) = u(0) - u_0, \qquad G_5(s) = v(0) - v_0, \qquad G_6(s) = w(0) - w_0. \end{cases} \tag{5.8}$$

By taking $\mathcal{M} = \mathcal{W}_u \times \mathcal{W}_v \times \mathcal{W}_w \times \mathcal{F}$ a closed convex subset of X and $\mathcal{N} = \{0\}$, the optimal control problem (4.1) is reformulated as follows:

$$\min J(s) \text{ subject to } s \in \mathcal{S}_{\text{ad}} = \big\{ s = (u, v, w, f) \in \mathcal{M} : G(s) = 0 \big\}. \tag{5.9}$$

Similar to [21], by the definition of the Fréchet derivative, using a direct calculation, we have the following results.

Lemma 5.1 *The operator $G : X \to Y$ is continuously Fréchet differentiable and the Fréchet derivative of G in $\tilde{s} = (\tilde{u}, \tilde{v}, \tilde{w}, \tilde{f}) \in X$, in the direction $r = (U, V, W, F) \in X$, is the linear operator*

$$G'(\tilde{s})[r] = \big(G_1'(\tilde{s})[r], G_2'(\tilde{s})[r], G_3'(\tilde{s})[r], G_4'(\tilde{s})[r], G_5'(\tilde{s})[r], G_6'(\tilde{s})[r] \big)$$

defined by

$$\begin{cases} G_1'(\tilde{s})[r] = \partial_t U - \Delta U + \chi \nabla \cdot (U \nabla \tilde{v}) + \chi \nabla \cdot (\tilde{u} \nabla V) + \xi \nabla \cdot (U \nabla \tilde{w}) \\ \qquad\qquad + \xi \nabla \cdot (\tilde{u} \nabla W) - \mu U + 2\mu U \tilde{u} + \mu U \tilde{w} + \mu \tilde{u} W, \\ G_2'(\tilde{s})[r] = \partial_t V - \Delta V + V - U - F, \\ G_3'(\tilde{s})[r] = \partial_t W + V \tilde{w} + \tilde{v} W, \\ G_4'(\tilde{s})[r] = U(0), \qquad G_5'(\tilde{s})[r] = V(0), \qquad G_6'(\tilde{s})[r] = W(0). \end{cases} \tag{5.10}$$

Lemma 5.2 *The functional $J : X \to \mathbb{R}$ is Fréchet differentiable and the Fréchet derivative of J in $\tilde{s} = (\tilde{u}, \tilde{v}, \tilde{w}, \tilde{f}) \in X$ in the direction $r = (U, V, W, F) \in X$ is given by*

$$\begin{aligned} J'(\tilde{s})[r] &= \beta_1 \int_0^T \int_{\Omega_d} (\tilde{u} - u_d) U \,dx\,dt + \beta_2 \int_0^T \int_{\Omega_d} (\tilde{v} - v_d) V \,dx\,dt \\ &\quad + \beta_3 \int_0^T \int_{\Omega_d} (\tilde{w} - w_d) W \,dx\,dt + \beta_4 \int_0^T \int_{\Omega_c} \tilde{f} F \,dx\,dt. \end{aligned} \tag{5.11}$$

We wish to prove the existence of Lagrange multipliers, which is guaranteed if a local optimal solution of problem (5.9) is a regular point of operator G.

Lemma 5.3 *If* $\tilde{s} = (\tilde{u}, \tilde{v}, \tilde{w}, \tilde{f}) \in \mathcal{S}_{\mathrm{ad}}$, *then* $\tilde{s}$ *is a regular point.*

Proof Fix $(\tilde{u}, \tilde{v}, \tilde{w}, \tilde{f}) \in \mathcal{S}_{\mathrm{ad}}$ and let $(g_u, g_v, g_w, U_0, V_0, W_0) \in Y$. Since $0 \in \mathcal{C}(\tilde{f})$, it suffices to show the existence of (U, V, W) such that

$$\begin{cases} \partial_t U - \Delta U + \chi \nabla \cdot (U \nabla \tilde{v}) + \chi \nabla \cdot (\tilde{u} \nabla V) + \xi \nabla \cdot (U \nabla \tilde{w}) + \xi \nabla \cdot (\tilde{u} \nabla W) & \\ \quad - \mu U + 2\mu U \tilde{u} + \mu U \tilde{w} + \mu \tilde{u} W = g_u, & \text{in } Q, \\ \partial_t V - \Delta V + V - U = g_v, & \text{in } Q, \\ \partial_t W + V \tilde{w} + \tilde{v} W = g_w, & \text{in } Q, \\ U(0) = U_0, \qquad V(0) = V_0, \qquad W(0) = W_0, & \text{in } \Omega, \\ \frac{\partial U}{\partial \nu} = 0, \qquad \frac{\partial V}{\partial \nu} = 0, \qquad \frac{\partial W}{\partial \nu} = 0, & \text{on } (0, T) \times \partial\Omega. \end{cases} \tag{5.12}$$

Step 1. Local existence of a solution,

In order to prove the existence of a solution of (5.12), we use Proposition 2.1 to solve the problem

$$\begin{cases} \partial_t U - \Delta U + \chi \nabla \cdot (U \nabla \tilde{v}) + \chi \nabla \cdot (\tilde{u} \nabla V) + \xi \nabla \cdot (U \nabla \tilde{w}) + \xi \nabla \cdot (\tilde{u} \nabla W) & \\ \quad - \mu U + 2\mu U \tilde{u} + \mu U \tilde{w} + \mu \tilde{u} W = g_u, & \text{in } Q, \\ \partial_t V - \Delta V + V - U = g_v, & \text{in } Q, \\ \partial_t W + V \tilde{w} + \tilde{v} W = g_w, & \text{in } Q, \end{cases} \tag{5.13}$$

endowed with the corresponding initial and boundary conditions. On the product Banach space $\mathcal{B} = H^1(\Omega) \times H_N^2(\Omega) \times H_N^3(\Omega)$, we define the linear operator A by

$$A = \begin{bmatrix} -\Delta + 1 & 0 & 0 \\ 0 & -\Delta + 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad Z = H_N^2(\Omega) \times H_N^3(\Omega) \times H_N^3(\Omega).$$

The nonlinear operator F is defined by

$$F(Y) = \begin{bmatrix} -\chi \nabla \cdot (U \nabla \tilde{v}) - \chi \nabla \cdot (\tilde{u} \nabla V) - \xi \nabla \cdot (U \nabla \tilde{w}) \\ -\xi \nabla \cdot (\tilde{u} \nabla W) + (\mu + 1) U - 2\mu U \tilde{u} - \mu U \tilde{w} - \mu \tilde{u} W \\ U \\ W - V \tilde{w} - \tilde{v} W \end{bmatrix}, \qquad G(t) = \begin{bmatrix} g_u \\ g_v \\ g_w \end{bmatrix},$$

$$Y = \begin{bmatrix} U & V & W \end{bmatrix}^T.$$

The remaining part of the proof can be done in the same way as that in the proof of Theorem 2.1, so we omit the details.

Now, we prove the global-in-time solutions in the following part.

Step 2. $(U, V, W) \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)) \times L^\infty(0, T; H^1(\Omega)) \cap L^2(0, T; H^2(\Omega)) \times L^\infty(0, T; H^2(\Omega))$.

By testing the first equation of (5.13) with U, one has

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega U^2\,dx+\int_\Omega|\nabla U|^2\,dx\\
&\quad=\chi\int_\Omega U\nabla U\nabla\tilde{v}\,dx+\chi\int_\Omega\nabla U\tilde{u}\nabla V\,dx+\xi\int_\Omega U\nabla U\nabla\tilde{w}\,dx\\
&\qquad+\xi\int_\Omega\nabla U\tilde{u}\nabla W\,dx+\mu\int_\Omega U^2\,dx-2\mu\int_\Omega U^2\tilde{u}\,dx-\mu\int_\Omega U^2\tilde{w}\,dx\\
&\qquad-\mu\int_\Omega U\tilde{u}W\,dx+\int_\Omega Ug_u\,dx.
\end{aligned}\tag{5.14}$$

Applying the Hölder and Young inequalities as well as (3.8) to the terms on the right-hand side of (5.14),

$$\begin{aligned}
\chi\int_\Omega U\nabla U\nabla\tilde{v}\,dx&=\frac{\chi}{2}\int_\Omega\nabla U^2\nabla\tilde{v}\,dx=-\frac{\chi}{2}\int_\Omega U^2\Delta\tilde{v}\,dx\\
&\le\frac{\chi}{2}\|U\|_{L^4}^2\|\Delta\tilde{v}\|_{L^2}\le C\|\nabla U\|_{L^2}\|U\|_{L^2}\|\Delta\tilde{v}\|_{L^2}\\
&\le\delta\|\nabla U\|_{L^2}^2+C_\delta\|U\|_{L^2}^2,
\end{aligned}\tag{5.15}$$

and

$$\begin{aligned}
\chi\int_\Omega\nabla U\tilde{u}\nabla V\,dx&\le\chi\|\tilde{u}\|_{L^4}\|\nabla V\|_{L^4}\|\nabla U\|_{L^2}\\
&\le C_\delta\|\tilde{u}\|_{L^4}^2\|\nabla V\|_{L^4}^2+\delta\|\nabla U\|_{L^2}^2\\
&\le C_\delta\|\tilde{u}\|_{H^1}^2\|\nabla V\|_{L^2}\|\Delta V\|_{L^2}+\delta\|\nabla U\|_{L^2}^2\\
&\le\delta\big(\|\nabla U\|_{L^2}^2+\|\Delta V\|_{L^2}^2\big)+C_\delta\|\nabla V\|_{L^2}^2.
\end{aligned}\tag{5.16}$$

In the same way, we can get

$$\xi\int_\Omega U\nabla U\nabla\tilde{w}\,dx\le\delta\|\nabla U\|_{L^2}^2+C_\delta\|U\|_{L^2}^2\tag{5.17}$$

and

$$\xi\int_\Omega\nabla U\tilde{u}\nabla W\,dx\le\delta\big(\|\Delta W\|_{L^2}^2+\|\nabla U\|_{L^2}^2\big)+C_\delta\|\nabla W\|_{L^2}^2.\tag{5.18}$$

For the other terms on the right,

$$\begin{aligned}
&-2\mu\int_\Omega U^2\tilde{u}\,dx-\mu\int_\Omega U^2\tilde{w}\,dx-\mu\int_\Omega U\tilde{u}W\,dx+\int_\Omega Ug_u\,dx\\
&\quad\le2\mu\|U\|_{L^4}^2\|\tilde{u}\|_{L^2}+\mu\|U\|_{L^4}^2\|\tilde{w}\|_{L^2}+\mu\|U\|_{L^2}\|\tilde{u}\|_{L^4}\|W\|_{L^4}+\|U\|_{L^2}\|g_u\|_{L^2}\\
&\quad\le C\|U\|_{L^2}\|\nabla U\|_{L^2}\|\tilde{u}\|_{L^2}+C\|U\|_{L^2}\|\nabla U\|_{L^2}\|\tilde{w}\|_{L^2}\\
&\qquad+C\|U\|_{L^2}\|\tilde{u}\|_{H^1}\|\nabla W\|_{L^2}+\|U\|_{L^2}\|g_u\|_{L^2}\\
&\quad\le C\|U\|_{L^2}^2+\delta\|\nabla U\|_{L^2}+C\|\nabla W\|_{L^2}+\|g_u\|_{L^2}.
\end{aligned}\tag{5.19}$$

Replacing (5.15)–(5.19) in (5.14), we have

$$\frac{1}{2}\frac{d}{dt}\int_\Omega U^2\,dx+\int_\Omega|\nabla U|^2\,dx\le 5\delta\|\nabla U\|_{L^2}^2+C\|U\|_{L^2}^2+\delta\bigl(\|\Delta V\|_{L^2}^2+\|\Delta W\|_{L^2}^2\bigr)$$
$$+\|g_u\|_{L^2}^2+C\bigl(\|\nabla V\|_{L^2}^2+\|\nabla W\|_{L^2}^2\bigr). \tag{5.20}$$

By testing the second equation of (5.13) with V, we conclude

$$\frac{d}{dt}\int_\Omega|V|^2\,dx+\int_\Omega|\nabla V|^2\,dx+2\int_\Omega|V|^2\,dx\le\int_\Omega U^2\,dx+\int_\Omega g_v^2\,dx. \tag{5.21}$$

By testing the second equation of (5.13) with ΔV, we get

$$\frac{d}{dt}\int_\Omega|\nabla V|^2\,dx+\int_\Omega|\Delta V|^2\,dx+2\int_\Omega|\nabla V|^2\,dx\le\int_\Omega U^2\,dx+\int_\Omega g_v^2\,dx. \tag{5.22}$$

By testing the third equation of (5.13) with W, we obtain

$$\frac{d}{dt}\int_\Omega|W|^2\,dx\le C\int_\Omega W^2\,dx+C\int_\Omega V^2\,dx+\int_\Omega g_w^2\,dx. \tag{5.23}$$

Next, applying ∇ to the third equation of (5.13), multiplying by ∇W, and integrating the product over Ω, we have

$$\frac{1}{2}\frac{d}{dt}\int_\Omega|\nabla W|^2\,dx=-\int_\Omega\nabla W\nabla(V\tilde{w})\,dx-\int_\Omega\nabla W\nabla(\tilde{v}W)\,dx$$
$$+\int_\Omega\nabla W\nabla g_w\,dx. \tag{5.24}$$

From (3.3) and (3.8), we derive that

$$\begin{aligned}-\int_\Omega\nabla W\nabla(V\tilde{w})\,dx&=-\int_\Omega\nabla W(\nabla V\tilde{w}+V\nabla\tilde{w})\,dx\\&\le\|\nabla W\|_{L^2}\|\nabla V\|_{L^2}\|\tilde{w}\|_{L^\infty}+\|\nabla W\|_{L^2}\|V\|_{L^4}\|\nabla\tilde{w}\|_{L^4}\\&\le\|\nabla W\|_{L^2}\|\nabla V\|_{L^2}\|\tilde{w}\|_{L^\infty}+\|\nabla W\|_{L^2}\|V\|_{H^1}\|\tilde{w}\|_{H^2}\\&\le C\|W\|_{H^1}\|V\|_{H^1}\le\|W\|_{H^1}^2+\frac{1}{2}\|V\|_{H^1}^2,\end{aligned} \tag{5.25}$$

$$\begin{aligned}-\int_\Omega\nabla W\nabla(\tilde{v}W)\,dx&=-\int_\Omega\nabla W(\nabla\tilde{v}W+\tilde{v}\nabla W)\,dx\\&=\frac{1}{2}\int_\Omega W^2\Delta\tilde{v}\,dx-\int_\Omega\tilde{v}|\nabla W|^2\,dx\\&\le\frac{1}{2}\|W\|_{L^4}^2\|\Delta\tilde{v}\|_{L^2}\le C\|W\|_{H^1}^2\|\Delta\tilde{v}\|_{L^2}\le C\|W\|_{H^1}^2,\end{aligned} \tag{5.26}$$

and

$$\int_\Omega\nabla W\nabla g_w\,dx\le C\|\nabla W\|_{L^2}^2+\frac{1}{2}\|\nabla g_w\|_{L^2}^2. \tag{5.27}$$

So, combining (5.23)–(5.27), we can get

$$\frac{d}{dt}\|W\|_{H^1}^2 \leq C\|W\|_{H^1}^2 + \|V\|_{H^1}^2 + \|\nabla g_w\|_{L^2}^2. \tag{5.28}$$

Applying Δ to the third equation of (5.13), multiplying by ΔW, and integrating the product over Ω, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega |\Delta W|^2\,dx = {}& -\int_\Omega \Delta W\Delta(V\tilde{w})\,dx - \int_\Omega \Delta W\Delta(\tilde{v}W)\,dx \\ & + \int_\Omega \Delta W\Delta g_w\,dx.\end{aligned} \tag{5.29}$$

Applying the Hölder and Young inequalities to the terms on the right-hand side of (5.29), we have

$$\begin{aligned}-\int_\Omega \Delta W\Delta(V\tilde{w})\,dx &= -\int_\Omega \Delta W(\Delta V\tilde{w} + 2\nabla V\nabla\tilde{w} + V\Delta\tilde{w}) \\ &\leq \|\Delta W\|_{L^2}\|\Delta V\|_{L^2}\|\tilde{w}\|_{L^\infty} + 2\|\Delta W\|_{L^2}\|\nabla V\|_{L^4}\|\nabla\tilde{w}\|_{L^4} \\ &\quad + \|\Delta W\|_{L^2}\|V\|_{L^4}\|\Delta\tilde{w}\|_{L^4} \\ &\leq \|\Delta W\|_{L^2}\|\Delta V\|_{L^2}\|\tilde{w}\|_{L^\infty} + C\|\Delta W\|_{L^2}\|V\|_{H^2}\|\tilde{w}\|_{H^2} \\ &\quad + \|\Delta W\|_{L^2}\|\nabla V\|_{L^2}\|\tilde{w}\|_{H^3} \\ &\leq C\|W\|_{H^2}\|V\|_{H^2} + C\|W\|_{H^2}\|V\|_{H^1} \\ &\leq C\|W\|_{H^2}^2 + \frac{\delta}{2}\|V\|_{H^2}^2 + C\|V\|_{H^1}^2,\end{aligned} \tag{5.30}$$

$$\begin{aligned}-\int_\Omega \Delta W\Delta(\tilde{v}W)\,dx &= -\int_\Omega \Delta W(\Delta\tilde{v}W + 2\nabla\tilde{v}\nabla W + \tilde{v}\Delta W)\,dx \\ &\leq -\int_\Omega \Delta W(\Delta\tilde{v}W + 2\nabla\tilde{v}\nabla W)\,dx \\ &= -\int_\Omega \Delta W\Delta\tilde{v}W\,dx - \int_\Omega \nabla(\nabla W)^2\nabla\tilde{v}\,dx \\ &= -\int_\Omega \Delta W\Delta\tilde{v}W\,dx + \int_\Omega |\nabla W|^2\Delta\tilde{v}\,dx \\ &\leq \|\Delta W\|_{L^2}\|\Delta\tilde{v}\|_{L^4}\|W\|_{L^4} + \|\nabla W\|_{L^4}^2\|\Delta\tilde{v}\|_{L^2} \\ &\leq C\|W\|_{H^2}^2\|\tilde{v}\|_{H^3},\end{aligned} \tag{5.31}$$

and

$$\int_\Omega \Delta W\Delta g_w\,dx \leq C\|\Delta W\|_{L^2}^2 + \frac{1}{2}\|\Delta g_w\|_{L^2}^2. \tag{5.32}$$

So, we can get

$$\frac{d}{dt}\|W\|_{H^2}^2 \leq C\|W\|_{H^2}^2\left\|\left(\tilde{v}\|_{H^3}^2 + 1\right) + \delta\right\|V\|_{H^2}^2 + C\|V\|_{H^1}^2 + \|g_w\|_{H^2}^2. \tag{5.33}$$

Therefore, we can obtain

$$\begin{aligned}
&\frac{d}{dt}\big(\|U\|_{L^2}^2+\|V\|_{H^1}^2+\|W\|_{H^2}^2\big)+\|U\|_{H^1}^2+\|V\|_{H^2}^2\\
&\quad\le 5\delta\|U\|_{H^1}^2+C\|U\|_{L^2}^2+2\delta\|V\|_{H^2}^2+C\big(\|V\|_{H^1}^2+\|W\|_{H^1}^2\big)\\
&\qquad+C\|W\|_{H^2}^2\big(\|\tilde{v}\|_{H^3}^2+1\big)+\|g_u\|_{L^2}^2+\|g_v\|_{L^2}^2+C\|g_w\|_{H^2}^2. \qquad (5.34)
\end{aligned}$$

By choosing δ small enough, and utilizing the Gronwall inequality, we have

$$\begin{aligned}
&\|U\|_{L^2}+\|V\|_{H^1}+\|W\|_{H^2}+\int_0^t\|U\|_{H^1}^2\,d\tau+\int_0^t\|V\|_{H^2}^2\,d\tau\le C,\\
&\quad\text{for all } t\in[0,T]. \qquad (5.35)
\end{aligned}$$

Step 3. $(U,V,W)\in L^\infty(0,T;H^1(\Omega))\cap L^2(0,T;H^2(\Omega))\times L^\infty(0,T;H^2(\Omega))\cap L^2(0,T;H^3(\Omega))\times L^\infty(0,T;H^2(\Omega))$.

By testing the first equation of (5.13) with $-\Delta U$, one has

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega|\nabla U|^2\,dx+\int_\Omega|\Delta U|^2\,dx\\
&\quad=\chi\int_\Omega\Delta U\nabla\cdot(U\nabla\tilde{v})\,dx+\chi\int_\Omega\Delta U\nabla\cdot(\tilde{u}\nabla V)\,dx+\xi\int_\Omega\Delta U\nabla\cdot(U\nabla\tilde{w})\,dx\\
&\qquad+\xi\int_\Omega\Delta U\nabla\cdot(\tilde{u}\nabla W)\,dx+\mu\int_\Omega|\nabla U|^2\,dx+2\mu\int_\Omega\Delta UU\tilde{u}\,dx\\
&\qquad+\mu\int_\Omega\Delta UU\tilde{w}\,dx+\mu\int_\Omega\Delta U\tilde{u}W\,dx+\int_\Omega\Delta Ug_u\,dx. \qquad (5.36)
\end{aligned}$$

By applying the boundedness of $\|\tilde{v}\|_{H^2}^2$, (5.35), and the Gagliardo–Nirenberg interpolation inequality, we have

$$\begin{aligned}
\chi\int_\Omega\Delta U\nabla\cdot(U\nabla\tilde{v})\,dx&=\chi\int_\Omega\Delta U(\nabla U\nabla\tilde{v}+U\Delta\tilde{v})\,dx\\
&=-\frac{\chi}{2}\int_\Omega|\nabla U|^2\Delta\tilde{v}\,dx+\chi\int_\Omega\Delta UU\Delta\tilde{v}\\
&\le\frac{\chi}{2}\|\nabla U\|_{L^4}^2\|\Delta\tilde{v}\|_{L^2}+\chi\|\Delta U\|_{L^2}\|U\|_{L^4}\|\Delta\tilde{v}\|_{L^4}\\
&\le C\|\nabla U\|_{L^2}\|\Delta U\|_{L^2}+C\|\Delta U\|_{L^2}\|U\|_{L^2}\|U\|_{H^1}\|\Delta\tilde{v}\|_{H^1}\\
&\le C\|U\|_{H^1}^2\big(\|\tilde{v}\|_{H^3}^2+1\big)+\delta\|U\|_{H^2}^2 \qquad (5.37)
\end{aligned}$$

and

$$\begin{aligned}
\chi\int_\Omega\Delta U\nabla\cdot(\tilde{u}\nabla V)\,dx&\le\chi\int_\Omega\Delta U(\nabla\tilde{u}\nabla V+\tilde{u}\Delta V)\\
&\le\|\Delta U\|_{L^2}\|\nabla\tilde{u}\|_{L^4}\|\nabla V\|_{L^4}+\|\Delta U\|_{L^2}\|\tilde{u}\|_{L^\infty}\|\Delta V\|_{L^2}\\
&\le C\|\Delta U\|_{L^2}\|\tilde{u}\|_{H^2}\|V\|_{H^2}\le\delta\|U\|_{H^2}^2+C\|\tilde{u}\|_{H^2}^2\|V\|_{H^2}^2. \qquad (5.38)
\end{aligned}$$

By utilizing the same ideas, we can get

$$\xi \int_\Omega \Delta U \nabla \cdot (U \nabla \tilde{w})\,dx \le C\|U\|_{H^1}^2 \big(\|\tilde{w}\|_{H^3}^2 + 1\big) + \delta \|U\|_{H^2}^2 \tag{5.39}$$

and

$$\xi \int_\Omega \Delta U \nabla \cdot (\tilde{u} \nabla W)\,dx \le \delta \|\Delta U\|_{L^2}^2 + C\|\tilde{u}\|_{H^2}^2. \tag{5.40}$$

With (5.35) and the boundedness of $\|\tilde{w}\|_{H^1}^2$, $\|\tilde{u}\|_{H^1}^2$ in hand, we derive

$$\begin{aligned}
&\mu \int_\Omega \Delta U U \tilde{w}\,dx + \mu \int_\Omega \Delta U \tilde{u} W\,dx + \int_\Omega \Delta U g_u\,dx \\
&\quad \le \mu \|\Delta U\|_{L^2}\|U\|_{L^4}\|\tilde{w}\|_{L^4} + \mu \|\Delta U\|_{L^2}\|\tilde{u}\|_{L^4}\|W\|_{L^4} + \|\Delta U\|_{L^2}\|g_u\|_{L^2} \\
&\quad \le C\|\Delta U\|_{L^2}\|U\|_{H^1}\|\nabla \tilde{w}\|_{L^2} + C\|\Delta U\|_{L^2}\|\tilde{u}\|_{H^1}\|W\|_{H^1} + \|\Delta U\|_{L^2}\|g_u\|_{L^2} \\
&\quad \le \delta \|U\|_{H^2}^2 + C\big(\|U\|_{H^1}^2 + \|g_u\|_{L^2}^2 + 1\big).
\end{aligned} \tag{5.41}$$

Replacing (5.37)–(5.41) in (5.36), and using the fact that $\|\tilde{w}\|_{H^3}^2 \le C$ and (5.35), we have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}(\|U\|_{H^1}^2 + \|U\|_{H^2}^2 &\le C\|U\|_{H^1}^2 \big(\|\tilde{v}\|_{H^3}^2 + \|\tilde{w}\|_{H^3}^2 + 1\big) + 5\delta \|U\|_{H^2}^2 \\
&\quad + C\|\tilde{u}\|_{H^2}^2 \big(\|V\|_{H^2}^2 + 1\big) + C\big(\|g_u\|_{L^2}^2 + 1\big).
\end{aligned} \tag{5.42}$$

Next, applying Δ to the second equation of (5.13), multiplying by ΔV, and integrating the product over Ω, we see

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega |\Delta V|^2\,dx + \int_\Omega |\nabla \Delta V|^2\,dx + \int_\Omega |\Delta V|^2\,dx \\
&\quad = -\int_\Omega \nabla U \nabla \Delta V\,dx - \int_\Omega \nabla g_v \nabla \Delta V\,dx.
\end{aligned} \tag{5.43}$$

Applying the Hölder and Young inequalities, we obtain

$$-\int_\Omega \nabla U \nabla \Delta V\,dx - \int_\Omega \nabla g_v \nabla \Delta V\,dx \le \delta \|\nabla \Delta V\|_{L^2}^2 + C\big(\|\nabla U\|_{L^2}^2 + \|\nabla g_v\|_{L^2}^2\big). \tag{5.44}$$

Therefore, from (5.43) and (5.44), we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega |\Delta V|^2\,dx + \int_\Omega |\nabla \Delta V|^2\,dx + \int_\Omega |\Delta V|^2\,dx \\
&\quad \le \delta \|\nabla \Delta V\|_{L^2}^2 + C\big(\|\nabla U\|_{L^2}^2 + \|\nabla g_v\|_{L^2}^2\big).
\end{aligned} \tag{5.45}$$

Applying $\nabla \Delta$ to the third equation of (5.13), multiplying by $\nabla \Delta W$, and integrating the product over Ω yields

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla \Delta W|^2\,dx \\
&\quad = -\int_\Omega \nabla \Delta W \nabla \Delta (V\tilde{w})\,dx - \int_\Omega \nabla \Delta W \nabla \Delta (\tilde{v} W)\,dx + \int_\Omega \nabla \Delta W \nabla \Delta g_w\,dx.
\end{aligned} \tag{5.46}$$

For the first term on the right,

$$\begin{aligned}
&-\int_\Omega \nabla\Delta W\nabla\Delta(V\tilde{w})\,dx \\
&\quad= -\int_\Omega \nabla\Delta W(\nabla\Delta V\tilde{w} + 3\Delta V\nabla\tilde{w} + 3\nabla V\Delta\tilde{w} + V\nabla\Delta\tilde{w})\,dx \\
&\quad\le \|\nabla\Delta W\|_{L^2}\|\nabla\Delta V\|_{L^2}\|\tilde{w}\|_{L^\infty} + 3\|\nabla\Delta W\|_{L^2}\|\Delta V\|_{L^4}\|\nabla\tilde{w}\|_{L^4} \\
&\qquad + 3\|\nabla\Delta W\|_{L^2}\|\nabla V\|_{L^4}\|\Delta\tilde{w}\|_{L^4} + \|\nabla\Delta W\|_{L^2}\|\nabla\Delta\tilde{w}\|_{L^2}\|V\|_{L^\infty} \\
&\quad\le C\|W\|_{H^3}\|V\|_{H^3}\|\tilde{w}\|_{H^2} + C\|W\|_{H^3}\|V\|_{H^2}\|\tilde{w}\|_{H^3} \\
&\qquad + C\|W\|_{H^3}\|\tilde{w}\|_{H^3}\|W\|_{H^2}.
\end{aligned} \tag{5.47}$$

Thanks to the boundedness of $\|\nabla\Delta\tilde{w}\|_{L^2}$ and (5.35), we can get

$$-\int_\Omega \nabla\Delta W\nabla\Delta(V\tilde{w})\,dx \le C\|W\|_{H^3}^2 + \frac{\delta}{2}\|V\|_{H^3}^2 + C\|V\|_{H^2}^2 + C \tag{5.48}$$

and

$$\begin{aligned}
&-\int_\Omega \nabla\Delta W\nabla\Delta(\tilde{v}W)\,dx \\
&\quad= -\int_\Omega \nabla\Delta W(\nabla\Delta\tilde{v}W + 3\Delta\tilde{v}W + 3\nabla\tilde{v}\Delta W + \tilde{v}\nabla\Delta W) \\
&\quad\le \|\nabla\Delta W\|_{L^2}\|\nabla\Delta\tilde{v}\|_{L^2}\|W\|_{L^\infty} + 3\|\nabla\Delta W\|_{L^2}\|\Delta\tilde{v}\|_{L^4}\|W\|_{L^4} \\
&\qquad + \frac{3}{2}\|\Delta W\|_{L^4}^2\|\Delta\tilde{v}\|_{L^2} + \|\nabla\Delta W\|_{L^2}^2\|\tilde{v}\|_{L^\infty} \\
&\quad\le C\|W\|_{H^3}\|\tilde{v}\|_{H^3}\|W\|_{H^2} + C\|W\|_{H^3}\|\tilde{v}\|_{H^3}\|W\|_{H^1} \\
&\qquad + C\|W\|_{H^2}\|W\|_{H^3}\|\tilde{v}\|_{H^2} + \|W\|_{H^3}^2\|\tilde{v}\|_{H^2}.
\end{aligned} \tag{5.49}$$

Then collecting (5.35) and $\|\tilde{v}\|_{H^2} \le C$, we arrive at

$$-\int_\Omega \nabla\Delta W\nabla\Delta(\tilde{v}W)\,dx \le C\|W\|_{H^3}^2 + C\|\tilde{v}\|_{H^3}^2 + C. \tag{5.50}$$

Applying the Hölder and Young inequalities, we obtain

$$\int_\Omega \nabla\Delta W\nabla\Delta g_w\,dx \le C\|W\|_{H^3}^2 + \frac{1}{2}\|g_w\|_{H^3}^2. \tag{5.51}$$

Thus, in light of (5.46)–(5.51), we arrive at

$$\begin{aligned}
\frac{d}{dt}\int_\Omega |\nabla\Delta W|^2\,dx &\le C\|W\|_{H^3}^2 + \delta\|V\|_{H^3}^2 + C\|\tilde{v}\|_{H^3}^2 \\
&\quad + C\|V\|_{H^2}^2 + \|g_w\|_{H^3}^2 + C.
\end{aligned} \tag{5.52}$$

We invoke (5.34), (5.42), (5.45), and (5.52) to obtain

$$\begin{aligned}
&\frac{d}{dt}\big(\|U\|_{H^1}^2 + |V|_{H^2}^2 + |W|_{H^3}^2\big)\,dx + \|U\|_{H^2}^2 + |V|_{H^3}^2 \\
&\quad \le C\|U\|_{H^1}^2\big(\|\tilde{v}\|_{H^3}^2 + \|\tilde{w}\|_{H^3}^2 + 1\big) + C\big(\|\tilde{u}\|_{H^2}^2 + 1\big)\|V\|_{H^2}^2 + C\|W\|_{H^3}^2 \\
&\qquad + C\|\tilde{u}\|_{H^2}^2 + C\|\tilde{v}\|_{H^3}^2 + C\big(\|g_u\|_{L^2}^2 + \|\nabla g_v\|_{L^2}^2 + \|\nabla\Delta g_w\|_{L^2}^2\big) + C.
\end{aligned} \tag{5.53}$$

By utilizing the Gronwall inequality and $\tilde{v} \in L^2(0,T;H^3(\Omega))$, $\tilde{w} \in L^\infty(0,T;H^3(\Omega))$ $\tilde{u} \in L^2(0,T;H^2(\Omega))$, we have

$$\|U\|_{H^1}^2 + |V|_{H^2}^2 + |W|_{H^3}^2 + \int_0^t \|U\|_{H^2}^2\,d\tau + \int_0^t |V|_{H^3}^2\,d\tau \le C, \quad t\in[0,T]. \tag{5.54}$$

Thus, we conclude the proof. □

Now we show the existence of Lagrange multipliers.

Theorem 5.2 *Let $\tilde{s} = (\tilde{u},\tilde{v},\tilde{w},\tilde{f}) \in \mathcal{S}_{\mathrm{ad}}$ be a local optimal solution for the control problem (5.9). Then, there exist Lagrange multipliers $(\lambda,\eta,\rho,\varphi,\phi,\psi) \in L^2(Q)\times(L^2(0,T;H^1(\Omega)))'\times(L^\infty(0,T;H^3(\Omega)))'\times(H^1(\Omega))'\times(H^2(\Omega))'\times(H^3(\Omega))'$ such that for all $(U,V,W,F)\in\mathcal{W}_u\times\mathcal{W}_v\times\mathcal{W}_w\times\mathcal{C}(\tilde{f})$ one has*

$$\begin{aligned}
&\beta_1\int_0^T\int_{\Omega_d}(\tilde{u}-u_d)U\,dx\,dt + \beta_2\int_0^T\int_{\Omega_d}(\tilde{v}-v_d)V\,dx\,dt \\
&\quad + \beta_3\int_0^T\int_{\Omega_d}(\tilde{w}-w_d)W\,dx\,dt + \beta_4\int_0^T\int_{\Omega_c}\tilde{f}F\,dx\,dt + \int_0^T\int_{\Omega_c}F\eta\,dx\,dt \\
&\quad - \int_0^T\int_\Omega\big(\partial_t U - \Delta U + \chi\nabla\cdot(U\nabla\tilde{v}) + \chi\nabla\cdot(\tilde{u}\nabla V) + \xi\nabla\cdot(U\nabla\tilde{w}) + \xi\nabla\cdot(\tilde{u}\nabla W) \\
&\quad - \mu U + 2\mu U\tilde{u} + \mu U\tilde{w} + \mu\tilde{u}W\big)\lambda\,dx\,dt - \int_0^T\int_\Omega(\partial_t V - \Delta V + V - U)\eta\,dx\,dt \\
&\quad - \int_0^T\int_\Omega(\partial_t W + V\tilde{w} + \tilde{v}W)\rho\,dx\,dt - \int_\Omega U(0)\varphi\,dx - \int_\Omega V(0)\phi\,dx \\
&\quad - \int_\Omega W(0)\psi\,dx \ge 0.
\end{aligned} \tag{5.55}$$

Proof From Lemma 5.3, $\tilde{s}\in\mathcal{S}_{\mathrm{ad}}$ is a regular point, so by Theorem 5.1 there exist Lagrange multipliers

$$\begin{aligned}
(\lambda,\eta,\rho,\varphi,\phi,\psi) \in L^2(Q) &\times \big(L^2\big(0,T;H^1(\Omega)\big)\big)' \times \big(L^\infty\big(0,T;H^3(\Omega)\big)\big)' \\
&\times \big(H^1(\Omega)\big)' \times \big(H^2(\Omega)\big)' \times \big(H^3(\Omega)\big)'
\end{aligned}$$

such that

$$\begin{aligned}
&J'(\bar{s})[r] - \big\langle G_1'(\bar{s})[r],\lambda\big\rangle - \big\langle G_2'(\bar{s})[r],\eta\big\rangle - \big\langle G_3'(\bar{s})[r],\rho\big\rangle \\
&\quad - \big\langle G_4'(\bar{s})[r],\varphi\big\rangle - \big\langle G_5'(\bar{s})[r],\phi\big\rangle - \big\langle G_5'(\bar{s})[r],\psi\big\rangle \ge 0,
\end{aligned} \tag{5.56}$$

for all $r = (U, V, W, F) \in \mathcal{W}_u \times \mathcal{W}_v \times \mathcal{W}_w \times \mathcal{C}(\tilde{f})$. Thus, the proof follows from (5.11) and (5.10). □

From Theorem 5.2, we derive an optimality system for which we consider the following spaces:

$$\mathcal{W}_{u_0} := \{u \in \mathcal{W}_u : u(0) = 0\}, \qquad \mathcal{W}_{v_0} := \{v \in \mathcal{W}_v : v(0) = 0\},$$
$$\mathcal{W}_{w_0} := \{u \in \mathcal{W}_w : w(0) = 0\}. \tag{5.57}$$

Corollary 5.1 *Let $\tilde{s} = (\tilde{u}, \tilde{v}, \tilde{w}, \tilde{f})$ be a local optimal solution for the optimal control problem (5.9). Then the Lagrange multiplier $(\lambda, \eta, \rho) \in L^2(Q) \times (L^2(0,T;H^1(\Omega)))' \times (L^\infty(0,T;H^3(\Omega)))'$, provided by Theorem 5.2, satisfies the system*

$$\int_0^T \int_\Omega \big(\partial_t U - \Delta U + \chi \nabla \cdot (U \nabla \tilde{v}) + \xi \nabla \cdot (U \nabla \tilde{w}) - \mu U + 2\mu U \tilde{u} + \mu U \tilde{w}\big)\lambda \, dx\, dt$$
$$- \int_0^T \int_\Omega U\eta \, dx\, dt = \beta_1 \int_0^T \int_{\Omega_d} (\tilde{u} - u_d) U \, dx\, dt, \quad \forall U \in \mathcal{W}_{u_0}, \tag{5.58}$$
$$\int_0^T \int_\Omega (\partial_t V - \Delta V + V)\eta \, dx\, dt + \chi \int_0^T \int_\Omega \nabla \cdot (\tilde{u} \nabla V)\lambda \, dx\, dt + \int_0^T \int_\Omega V \tilde{w} \rho \, dx\, dt$$
$$= \beta_2 \int_0^T \int_{\Omega_d} (\tilde{v} - v_d) V \, dx\, dt, \quad \forall V \in \mathcal{W}_{v_0}, \tag{5.59}$$
$$\int_0^T \int_\Omega (\partial_t W + \tilde{v} W)\rho \, dx\, dt + \xi \int_0^T \int_\Omega \nabla \cdot (\tilde{u} \nabla W)\lambda \, dx\, dt + \mu \int_0^T \int_\Omega \tilde{u} W \lambda \, dx\, dt$$
$$= \beta_3 \int_0^T \int_{\Omega_d} (\tilde{w} - w_d) W \, dx\, dt, \quad \forall W \in \mathcal{W}_{w_0}, \tag{5.60}$$

which corresponds to the concept of very weak solution of the linear system

$$\begin{cases} \partial_t \lambda + \Delta \lambda + \chi \nabla \lambda \cdot \nabla \tilde{v} + \xi \nabla \lambda \cdot \nabla \tilde{w} + \mu \lambda - 2\mu \tilde{u} \lambda - \mu \lambda \tilde{w} - \eta & \\ \quad = -\beta_1 (\tilde{u} - u_d) \zeta_{\Omega_d}, & \text{in } Q, \\ \partial_t \eta + \Delta \eta - \eta - \chi \nabla \cdot (\nabla \lambda \tilde{u}) - \tilde{w} \rho = -\beta_2 (\tilde{u} - u_d) \zeta_{\Omega_d}, & \text{in } Q, \\ \partial_t \rho - \tilde{v} \rho - \xi \nabla \cdot (\tilde{u} \nabla \lambda) - \mu \tilde{u} \lambda = -\beta_3 (\tilde{v} - v_d) \zeta_{\Omega_d}, & \text{in } Q, \\ \lambda(T) = 0, \quad \eta(T) = 0, \quad \rho(T) = 0, & \text{in } \Omega, \\ \frac{\partial \lambda}{\partial \nu} = 0, \quad \frac{\partial \eta}{\partial \nu} = 0, \quad \frac{\partial \rho}{\partial \nu} = 0, & \text{on } (0,T) \times \partial\Omega, \end{cases} \tag{5.61}$$

and the optimality condition

$$\int_0^T \int_{\Omega_c} (\beta_4 \tilde{f} + \eta)(f - \bar{f}) \, dx\, dt \geq 0, \quad \forall f \in \mathcal{F}. \tag{5.62}$$

Proof From (5.55), taking $(V, W, F) = (0, 0, 0)$, and taking into account that $\mathcal{W}_{u_0}$ is a vector space, we have (5.58). Similarly, taking $(U, W, F) = (0, 0, 0)$ in (5.55), and considering that $\mathcal{W}_{v_0}$ is a vector space, we obtain (5.59). Taking $(U, V, F) = (0, 0, 0)$ in (5.55), and considering that $\mathcal{W}_{w_0}$ is a vector space, we obtain (5.60). Finally, taking $(U, V, W) = (0, 0, 0)$ in (5.55), we have

$$\beta_4 \int_0^T \int_{\Omega_e} \tilde{f} F \, dx \, dt + \int_0^T \int_{\Omega_0} \eta F \, dx \, dt \geq 0, \quad \forall F \in \mathcal{C}(\bar{f}). \tag{5.63}$$

Therefore, choosing $F = f - \bar{f} \in \mathcal{C}(\bar{f})$ for all $f \in \mathcal{F}$ in the last inequality, we derive (5.62). □

In the following result we show that the Lagrange multiplier (λ, η, ρ), provided by Theorem 5.2, has some extra regularity.

Theorem 5.3 *Under conditions of Theorem 5.2, the system (5.61) has a unique strong solution (λ, η, ψ) such that*

$$\lambda \in H^1\big((0,T);L^2(\Omega)\big) \cap L^\infty\big(0,T;H^1(\Omega)\big) \cap L^2\big(0,T;H^2(\Omega)\big), \tag{5.64}$$

$$\eta \in H^1\big((0,T);L^2(\Omega)\big) \cap L^\infty\big(0,T;L^2(\Omega)\big) \cap L^2\big(0,T;H^1(\Omega)\big), \tag{5.65}$$

$$\rho \in H^1\big((0,T);L^2(\Omega)\big) \cap L^\infty\big(0,T;L^2(\Omega)\big). \tag{5.66}$$

Proof Let $s = T - t$, with $t \in (0,T)$ and $\tilde{\lambda}(s) = \lambda(t)$, $\tilde{\eta}(s) = \eta(t)$, $\tilde{\eta}(s) = \psi(t)$. Then system (5.61) is equivalent to

$$\begin{cases} \partial_s \tilde{\lambda} - \Delta\tilde{\lambda} - \chi \nabla\tilde{\lambda} \cdot \nabla\tilde{v} - \xi \nabla\tilde{\lambda} \cdot \nabla\tilde{w} - \mu\tilde{\lambda} + 2\mu\tilde{u}\tilde{\lambda} + \mu\tilde{\lambda}\tilde{w} + \tilde{\eta} & \\ \quad = \beta_1(\tilde{u} - u_d)\zeta_{\Omega_d}, & \text{in } Q, \\ \partial_s \tilde{\eta} - \Delta\tilde{\eta} + \tilde{\eta} + \chi \nabla \cdot (\tilde{u}\nabla\tilde{\lambda}) + \tilde{w}\tilde{\rho} = \beta_2(\tilde{v} - v_d)\zeta_{\Omega_d}, & \text{in } Q, \\ \partial_s \tilde{\rho} + \tilde{v}\tilde{\rho} + \chi \nabla \cdot (\tilde{u}\nabla\tilde{\lambda}) - \mu\tilde{u}\tilde{\lambda} = \beta_3(\tilde{w} - w_d)\zeta_{\Omega_d}, & \text{in } Q, \\ \tilde{\lambda}(0) = 0, \qquad \tilde{\eta}(0) = 0, \qquad \tilde{\psi}(0) = 0, & \text{in } \Omega, \\ \frac{\partial\tilde{\lambda}}{\partial\nu} = 0, \qquad \frac{\partial\tilde{\eta}}{\partial\nu} = 0, \qquad \frac{\partial\tilde{\psi}}{\partial\nu} = 0, & \text{on } (0,T) \times \Omega. \end{cases} \tag{5.67}$$

The proof employs a Galerkin approximation. By testing $(5.67)_1$ with $-\Delta\tilde{\lambda}$, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{ds}\int_\Omega |\nabla\tilde{\lambda}|^2 \, dx + \int_\Omega |\Delta\tilde{\lambda}|^2 \, dx \\ &= -\chi \int_\Omega \Delta\tilde{\lambda}\nabla\tilde{\lambda}\nabla\tilde{v} \, dx - \xi \int_\Omega \Delta\tilde{\lambda}\nabla\tilde{\lambda}\nabla\tilde{w} \, dx + \mu \int_\Omega |\nabla\tilde{\lambda}|^2 \, dx + 2\mu \int_\Omega \tilde{u}\tilde{\lambda}\Delta\tilde{\lambda} \, dx \\ &\quad + \mu \int_\Omega \tilde{w}\tilde{\lambda}\Delta\tilde{\lambda} \, dx + \int_\Omega \Delta\tilde{\lambda}\eta \, dx - \beta_1 \int_\Omega \Delta\tilde{\lambda}(\tilde{u} - u_d) \, dx. \end{aligned} \tag{5.68}$$

Applying the Hölder, Young, and Nirenberg inequalities, as well as the boundedness of $\|\tilde{v}\|_{H^2}^2$, we obtain

$$\begin{aligned} -\chi \int_\Omega \Delta\tilde{\lambda}\nabla\tilde{\lambda}\nabla\tilde{v} \, dx &= \frac{\chi}{2}\int_\Omega |\nabla\tilde{\lambda}|^2 \Delta\tilde{v} \, dx \leq \frac{\chi}{2}\|\nabla\tilde{\lambda}\|_{L^4}^2 \|\Delta\tilde{v}\|_{L^2} \\ &\leq C\|\nabla\tilde{\lambda}\|_{L^2}\|\Delta\tilde{\lambda}\|_{L^2}\|\Delta\tilde{v}\|_{L^2} \leq C\|\nabla\tilde{\lambda}\|_{L^2}^2 + \frac{\delta}{8}\|\Delta\tilde{\lambda}\|_{L^2}^2. \end{aligned} \tag{5.69}$$

Then, utilizing the same procedure gives

$$-\xi \int_\Omega \Delta\tilde{\lambda}\nabla\tilde{\lambda}\nabla\tilde{w} \, dx \leq C\|\nabla\tilde{\lambda}\|_{L^2}^2 + \frac{\delta}{8}\|\Delta\tilde{\lambda}\|_{L^2}^2. \tag{5.70}$$

In light of the boundedness of $\tilde{u}$, $\tilde{w}$, we see that

$$\begin{aligned}
&2\mu\int_\Omega \tilde{u}\tilde{\lambda}\Delta\tilde{\lambda}\,dx+\mu\int_\Omega \tilde{w}\tilde{\lambda}\Delta\tilde{\lambda}\,dx+\int_\Omega \Delta\tilde{\lambda}\tilde{\eta}\,dx-\beta_1\int_\Omega \Delta\tilde{\lambda}(\tilde{u}-u_d)\,dx\\
&\quad\le \mu\big(2\|\tilde{u}\|_{L^4}+\|\tilde{w}\|_{L^4}\big)\|\tilde{\lambda}\|_{L^4}\|\Delta\tilde{\lambda}\|_{L^2}+\|\tilde{\eta}\|_{L^2}\|\Delta\tilde{\lambda}\|_{L^2}-\beta_1\|\Delta\tilde{\lambda}\|_{L^2}\|\tilde{u}-u_d\|_{L^2}\\
&\quad\le C\big(\|\tilde{u}\|_{H^1}+\|\tilde{w}\|_{H^1}\big)\|\nabla\tilde{\lambda}\|_{L^2}\|\Delta\tilde{\lambda}\|_{L^2}+\frac{\delta}{8}\|\Delta\tilde{\lambda}\|_{L^2}^2+C\big(\|\tilde{\eta}\|_{L^2}^2+\|\tilde{u}-u_d\|_{L^2}^2\big)\\
&\quad\le C\|\nabla\tilde{\lambda}\|_{L^2}^2+\frac{\delta}{4}\|\Delta\tilde{\lambda}\|_{L^2}^2+C\big(\|\tilde{\eta}\|_{L^2}^2+\|\tilde{u}-u_d\|_{L^2}^2\big).
\end{aligned}\tag{5.71}$$

In the light of (5.68)–(5.71), we have

$$\begin{aligned}
&\frac{d}{ds}\int_\Omega |\nabla\tilde{\lambda}|^2\,dx+2\int_\Omega |\Delta\tilde{\lambda}|^2\,dx\\
&\quad\le C\|\nabla\tilde{\lambda}\|_{L^2}^2+\delta\|\Delta\tilde{\lambda}\|_{L^2}^2+C\big(\|\tilde{\eta}\|_{L^2}^2+\|\tilde{u}-u_d\|_{L^2}^2\big).
\end{aligned}\tag{5.72}$$

Similarly, testing $(5.67)_1$ with $\tilde{\lambda}$ yields

$$\frac{d}{ds}\int_\Omega |\tilde{\lambda}|^2\,dx+2\int_\Omega |\nabla\tilde{\lambda}|^2\,dx\le C\|\tilde{\lambda}\|_{H^1}^2+C\big(\|\tilde{\eta}\|_{L^2}^2+\|\tilde{u}-u_d\|_{L^2}^2\big).\tag{5.73}$$

By testing $(5.67)_2$ with $\tilde{\eta}$, we conclude

$$\begin{aligned}
&\frac{1}{2}\frac{d}{ds}\int_\Omega \tilde{\eta}^2\,dx+\int_\Omega |\nabla\tilde{\eta}|^2\,dx+\int_\Omega \tilde{\eta}^2\,dx\\
&\quad=\chi\int_\Omega \nabla\tilde{\eta}\tilde{u}\nabla\tilde{\lambda}\,dx-\int_\Omega \tilde{\eta}\tilde{w}\tilde{\rho}\,dx+\beta_2\int_\Omega \tilde{\eta}(\tilde{v}-v_d)\,dx.
\end{aligned}\tag{5.74}$$

Using the Hölder, Young, and Nirenberg inequalities, we obtain

$$\begin{aligned}
\chi\int_\Omega \nabla\tilde{\eta}\tilde{u}\nabla\tilde{\lambda}\,dx&\le\chi\|\nabla\tilde{\eta}\|_{L^2}\|\tilde{u}\|_{L^\infty}\|\nabla\tilde{\lambda}\|_{L^2}\le C\|\nabla\tilde{\eta}\|_{L^2}\|\tilde{u}\|_{H^2}\|\nabla\tilde{\lambda}\|_{L^2}\\
&\le\frac{\delta}{2}\|\nabla\tilde{\eta}\|_{L^2}+C\|\tilde{u}\|_{H^2}^2\|\nabla\tilde{\lambda}\|_{L^2}^2.
\end{aligned}\tag{5.75}$$

Applying the Hölder and Young inequalities, as well as the boundedness of $\tilde{w}$, we get

$$\int_\Omega \tilde{\eta}\tilde{w}\tilde{\rho}\,dx\le\|\tilde{\eta}\|_{L^2}\|\tilde{w}\|_{L^\infty}\|\tilde{\rho}\|_{L^2}\le C\|\tilde{\eta}\|_{L^2}+\frac{1}{2}\|\tilde{\rho}\|_{L^2},\tag{5.76}$$

$$\beta_2\int_\Omega \tilde{\eta}(\tilde{v}-v_d)\,dx\le C\|\tilde{\eta}\|_{L^2}+\frac{1}{2}\|\tilde{v}-v_d\|_{L^2}^2.\tag{5.77}$$

Then collecting (5.74)–(5.77), we see

$$\begin{aligned}
&\frac{d}{ds}\int_\Omega \tilde{\eta}^2\,dx+2\int_\Omega |\nabla\tilde{\eta}|^2\,dx+2\int_\Omega \tilde{\eta}^2\,dx\\
&\quad\le C\|\tilde{\eta}\|_{L^2}+\delta\|\nabla\tilde{\eta}\|_{L^2}+C\|\tilde{u}\|_{H^2}^2\|\nabla\tilde{\lambda}\|_{L^2}^2+\|\tilde{\rho}\|_{L^2}+\|\tilde{v}-v_d\|_{L^2}^2.
\end{aligned}\tag{5.78}$$

Testing $(5.67)_3$ with $\tilde{\rho}$, we have

$$\frac{1}{2}\frac{d}{ds}\int_\Omega \tilde{\rho}^2\,dx = -\int_\Omega \tilde{v}\tilde{\rho}^2\,dx - \chi\int_\Omega \tilde{\rho}\nabla(\tilde{u}\nabla\tilde{\lambda})\,dx + \mu\int_\Omega \tilde{u}\tilde{\lambda}\tilde{\rho}\,dx + \beta_3\int_\Omega \tilde{\rho}(\tilde{w}-w_d)\,dx. \tag{5.79}$$

First, we see that

$$-\int_\Omega \tilde{v}\tilde{\rho}^2\,dx \le \|\tilde{v}\|_{L^\infty}\|\tilde{\rho}\|_{L^2}^2. \tag{5.80}$$

Similarly, we know that

$$\begin{aligned} -\chi\int_\Omega \tilde{\rho}\nabla(\tilde{u}\nabla\tilde{\lambda})\,dx &= -\chi\int_\Omega \tilde{\rho}(\nabla\tilde{u}\nabla\tilde{\lambda} + \tilde{u}\Delta\tilde{\lambda})\,dx \\ &\le \chi\|\tilde{\rho}\|_{L^2}\|\nabla\tilde{u}\|_{L^4}\|\nabla\tilde{\lambda}\|_{L^4} + \chi\|\tilde{\rho}\|_{L^2}\|\tilde{u}\|_{L^\infty}\|\Delta\tilde{\lambda}\|_{L^2} \\ &\le C\|\tilde{\rho}\|_{L^2}\|\tilde{u}\|_{H^2}\|\tilde{\lambda}\|_{H^2} \le C\|\tilde{\rho}\|_{L^2}^2\|\tilde{u}\|_{H^2}^2 + \delta\|\tilde{\lambda}\|_{H^2}^2, \end{aligned} \tag{5.81}$$

$$\begin{aligned} \mu\int_\Omega \tilde{u}\tilde{\lambda}\tilde{\rho}\,dx &\le \mu\|\tilde{\lambda}\|_{L^4}\|\tilde{u}\|_{L^4}\|\tilde{\rho}\|_{L^2} \le C\|\tilde{\lambda}\|_{H^1}\|\tilde{u}\|_{H^1}\|\tilde{\rho}\|_{L^2} \\ &\le C\|\tilde{\lambda}\|_{H^1}^2 + \|\tilde{\rho}\|_{L^2}^2, \end{aligned} \tag{5.82}$$

and

$$\beta_3\int_\Omega \tilde{\rho}(\tilde{w}-w_d)\,dx \le \|\tilde{\rho}\|_{L^2}^2 + C\|\tilde{w}-w_d\|_{L^2}^2. \tag{5.83}$$

Then collecting (5.79)–(5.83), we get

$$\frac{d}{ds}\int_\Omega \tilde{\rho}^2\,dx \le C\|\tilde{\rho}\|_{L^2}^2\big(\|\tilde{u}\|_{H^2}^2+1\big) + C\|\tilde{\lambda}\|_{H^1}^2 + \delta\|\tilde{\lambda}\|_{H^2}^2 + C\|\tilde{w}-w_d\|_{L^2}^2. \tag{5.84}$$

Accordingly, we invoke (5.72), (5.73), (5.78), and (5.84) to obtain

$$\begin{aligned} &\frac{d}{ds}\big(\|\tilde{\lambda}\|_{H^1}^2 + \|\tilde{\eta}\|_{L^2}^2 + \|\tilde{\rho}\|_{L^2}^2\big)\,dx + \|\tilde{\lambda}\|_{H^2}^2 + \|\tilde{\eta}\|_{H^1}^2 \\ &\quad\le 2\delta\|\tilde{\lambda}\|_{H^2}^2 + C\big(\|\tilde{u}\|_{H^2}^2+1\big)\|\tilde{\lambda}\|_{H^1}^2 + C\|\tilde{\eta}\|_{L^2}^2 + \delta\|\tilde{\eta}\|_{H^1}^2 \\ &\qquad + C\|\tilde{\rho}\|_{L^2}^2\big(\|\tilde{u}\|_{H^2}^2+1\big) + C\big(\|\tilde{v}-v_d\|_{L^2}^2 + \|\tilde{w}-w_d\|_{L^2}^2 + \|\tilde{u}-u_d\|_{L^2}^2\big). \end{aligned} \tag{5.85}$$

By taking δ small enough, a Gronwall argument leads to

$$\|\tilde{\lambda}\|_{H^1}^2 + \|\tilde{\eta}\|_{L^2}^2 + \|\tilde{\rho}\|_{L^2}^2 + \int_0^s \|\tilde{\lambda}\|_{H^2}^2\,d\tau + \int_0^s \|\tilde{\eta}\|_{H^1}^2\,d\tau \le C. \tag{5.86}$$

Testing the first equation of (5.67) with $\tilde{\lambda}_s$, integrating over $\Omega\times(0,T)$, and using the above inequality, we have

$$\int_0^T\int_\Omega \|\tilde{\lambda}_s\|_{L^2}^2\,dt \le \frac{1}{2}\int_0^T\int_\Omega \|\tilde{\lambda}_s\|_{L^2}^2\,dt + C.$$

Similarly, we obtain

$$\int_0^T \int_\Omega \|\tilde{\eta}_s\|_{L^2}^2 \, dt \le C, \qquad \int_0^T \int_\Omega \|\tilde{\rho}_s\|_{L^2}^2 \, dt \le C.$$

The proof is complete. □

Corollary 5.2 (Optimality System) *Let* $\tilde{s} = (\tilde{u}, \tilde{v}, \tilde{w}, \tilde{f}) \in \mathcal{S}_{\text{ad}}$ *be a local optimal solution for the control problem* (5.9). *Then, the Lagrange multiplier* (λ, η, ρ) *has the regularity as in* (5.67) *and satisfies the following optimality system*:

$$\begin{aligned}
&\partial_t \lambda + \Delta\lambda + \chi \nabla\lambda \cdot \nabla\tilde{v} + \xi \nabla\lambda \cdot \nabla\tilde{w} + \mu\lambda - 2\mu\tilde{u}\lambda - \mu\lambda\tilde{w} - \eta \\
&\quad = -\beta_1(\tilde{u} - u_d)\zeta_{\Omega_d}, \quad \text{in } Q, \\
&\partial_t \eta + \Delta\eta - \eta - \chi\nabla\cdot(\nabla\lambda\tilde{u}) - \tilde{w}\rho = -\beta_2(\tilde{u} - u_d)\zeta_{\Omega_d}, \quad \text{in } Q, \\
&\partial_t \rho - \tilde{v}\rho - \xi\nabla\cdot(\tilde{u}\nabla\lambda) - \mu\tilde{u}\lambda = -\beta_3(\tilde{v} - v_d)\zeta_{\Omega_d}, \quad \text{in } Q, \\
&\lambda(T) = 0, \qquad \eta(T) = 0, \qquad \rho(T) = 0, \quad \text{in } \Omega, \\
&\frac{\partial\lambda}{\partial\nu} = 0, \qquad \frac{\partial\eta}{\partial\nu} = 0, \qquad \frac{\partial\rho}{\partial\nu} = 0, \quad \text{on } (0,T)\times\partial\Omega, \\
&\int_0^T \int_{\Omega_c} (\beta_4\tilde{f} + \eta)(f - \bar{f}) \, dx \, dt \ge 0, \quad \forall f \in \mathcal{F}.
\end{aligned}$$

Remark 5.2 The first-order necessary optimality conditions for chemotaxis models have been intensively studied [10, 12, 23]. Recently, Colli, Signori and Sprekels [9] established both first-order necessary and second-order sufficient conditions for a tumor growth model of Cahn–Hilliard type, including chemotaxis with possibly singular potentials.

Remark 5.3 For the numerical analysis of the optimal control problem, the ringlike diffusion and aggregation patterns and the dynamics of tumor invasion, as well as the optimal control strategies, are presented numerically in Dai and Liu [10] for a haptotaxis model. Khajanchi and Ghosh [18] studied the numerical aspect of the optimal control problem for the immunogenic tumors model. They demonstrated the numerical illustrations that the optimal regimens reduce the tumor burden under different scenarios.

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Competing interests
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