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Global dynamics for an SVEIR epidemic model with diffusion and nonlinear incidence rate

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Abstract

In this paper, we investigate an SVEIR epidemic model with reaction–diffusion and nonlinear incidence. We establish the well-posedness of the solutions and the basic reproduction number $\mathfrak{R}_0$. Moreover, we show that the disease-free steady state is globally asymptotically stable when $\mathfrak{R}_0 < 1$, whereas the disease will be persistent when $\mathfrak{R}_0 > 1$. Furthermore, using the method of Lyapunov functional, we prove the global stability of the positive steady state for the spatial homogeneous model.

Keywords: Reaction–diffusion; Nonlinear incidence rate; Persistence; Lyapunov function; Global stability

1 Introduction

In the recent years, much effort has been paid on epidemic models by many researchers due to their important role in describing the dynamical evolution of infectious diseases. For better understanding of epidemiological scheme and intervening spreading of infectious diseases, see [1–8] and references therein.

Vaccination is one of the effective control measures to prevent and weaken the transmission of infectious diseases. Currently, various modeling studies have been made to explain the effect of vaccination on the spread of diseases [3, 9–11]. In particular, Liu et al. [3] proposed and studied the following model:

$$\begin{cases} \frac{dS}{dt} = \mu - (\mu + \xi)S - \beta SI, \\ \frac{dV}{dt} = \xi S - \beta_1 VI - \mu V - \alpha V, \\ \frac{dI}{dt} = \beta SI + \beta_1 VI - (\mu + \gamma)I, \\ \frac{dR}{dt} = \alpha V + \gamma I - \mu R, \end{cases} \tag{1.1}$$

where S, I, R, and V denote the numbers of compartments of susceptible, infected, recovered, and vaccinated individuals, respectively. The global stability of the equilibrium of model (1.1) has been studied by constructing Lyapunov functions. For more biological background of model (1.1), we refer to [3].

Note that the latent period was not considered in model (1.1). However, many diseases have a latent period before the hosts become infectious, and the length of latent period differs from disease to disease [12]. Besides, it well known that the spatial structure has also been considered as an important factor that affects the spatial spreading of disease due to the carrier hosts of infectious sources randomly moving in space. There are many mathematical studies of the influence of the spatial aspect and mobility of host populations on the dynamics of diseases; see [13–18] and the references therein. Taking into consideration the latent period and individuals movements, we consider the diffusive version of model (1.1) with latency, which is a more realistic biological model. Thus we consider the diffusive SVEIR model

$$\begin{cases} \frac{\partial S}{\partial t} = \nabla \cdot (d_1(x)\nabla S) + \Lambda(x) - (\mu_1(x) + \xi(x))S - \beta_1(x)f_1(S,I), \\ \quad t > 0, x \in \Omega, \\ \frac{\partial V}{\partial t} = \nabla \cdot (d_2(x)\nabla V) + \xi(x)S - \beta_2(x)f_2(V,I) - \mu_2(x) - \alpha(x)V, \\ \quad t > 0, x \in \Omega, \\ \frac{\partial E}{\partial t} = \nabla \cdot (d_3(x)\nabla E) + \beta_1(x)f_1(S,I) + \beta_2(x)f_2(V,I) - (\mu_3(x) + \sigma(x))E, \\ \quad t > 0, x \in \Omega, \\ \frac{\partial I}{\partial t} = \nabla \cdot (d_4(x)\nabla I) + \sigma(x)E - (\mu_4(x) + \delta(x) + \gamma(x))I, \\ \quad t > 0, x \in \Omega, \\ \frac{\partial R}{\partial t} = \nabla \cdot (d_5(x)\nabla R) + \alpha(x)V + \gamma(x)I - \mu_5(x)R, \\ \quad t > 0, x \in \Omega, \end{cases} \tag{1.2}$$

with e homogeneous Neumann boundary conditions

$$\frac{\partial S}{\partial \nu} = \frac{\partial V}{\partial \nu} = \frac{\partial E}{\partial \nu} = \frac{\partial I}{\partial \nu} = \frac{\partial R}{\partial \nu} = 0, \quad t > 0, x \in \partial\Omega, \tag{1.3}$$

and the initial conditions

$$\begin{aligned} &S(0,x) = S_0(x) \geq 0, \qquad V(0,x) = V_0(x) \geq 0, \qquad E(0,x) = E_0(x) \geq 0, \\ &I(0,x) = I_0(x) \geq 0, \qquad R(0,x) = R_0(x) \geq 0, \quad x \in \Omega, \end{aligned} \tag{1.4}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, and ν is the outward normal vector to $\partial\Omega$; $S = S(t,x)$, $V = V(t,x)$, $E = E(t,x)$, $I = I(t,x)$, and $R = R(t,x)$ stand for the densities of the susceptible, vaccinated, latent, infective, and recovered individuals at time t and spatial location x, respectively; $\Lambda(x)$ is the input rate of S in spatial location x; $\mu_k(x)$ ($k = 1,2,3,4,5$) denote the natural death rates of S, V, E, I, and R in spatial location x, respectively; $\delta(x)$ is the death rate induced by the disease in spatial location x; $\xi(x)$ is the vaccination rate of S in spatial location x; $\gamma(x)$ is the rate of recovery from infection in spatial location x; $\sigma(x)$ represents the transition rate from E to I; $\alpha(x)$ is the rate of obtaining immunity by vaccinees; $\beta_1(x)$ and $\beta_2(x)$ are the infection rates of S and V infected by I in spatial location x, respectively; $d_k(x)$ ($k = 1,2,3,4,5$) are the diffusion rates of S, V, E, I, and R in spatial location x, respectively. All location-dependent parameters are continuous and strictly positive d on $\overline{\Omega}$; $f_j(u,I)$ ($j = 1,2$; $u \in \{S,V\}$) denotes the force of infection.

We assume that the function $f_j(u,I)$ satisfies the following properties:

$$\begin{aligned}
&f_j : \mathbb{R}_+^2 \to \mathbb{R}_+ \text{ are differentiable}, f_j(u,I) \le uI, f_j(0,I) = f_j(u,0) = 0, \\
&\frac{\partial f_j(u,I)}{\partial u} > 0, \frac{\partial f_j(u,I)}{\partial I} > 0, \frac{\partial^2 f_j(u,I)}{\partial I^2} \le 0 \text{ for all } u, I > 0, \text{ and} \\
&\frac{\partial f_j(u,I)}{\partial I} \text{ is increasing with respect to } u.
\end{aligned} \tag{1.5}$$

Obviously, $f_j(u,I) = uI$, $f_j(u,I) = \frac{uI}{1+I}$, and $f_j(u,I) = \frac{uI}{(1+u)(1+I)}$ satisfy these assumptions. For convenience, let $f_{jI}(u,I) = \frac{\partial f_j(u,I)}{\partial I}$.

Since the last equation of model (1.1) is decoupled from other equations, we indeed need to study the following subsystem of model (1.2):

$$\begin{cases}
\frac{\partial S}{\partial t} = \nabla \cdot (d_1(x)\nabla S) + \Lambda(x) - (\mu_1(x) + \xi(x))S - \beta_1(x) f_1(S,I), \\
\quad t > 0, x \in \Omega, \\
\frac{\partial V}{\partial t} = \nabla \cdot (d_2(x)\nabla V) + \xi(x)S - \beta_2(x) f_2(V,I) - \mu_2(x)V - \alpha(x)V, \\
\quad t > 0, x \in \Omega, \\
\frac{\partial E}{\partial t} = \nabla \cdot (d_3(x)\nabla E) + \beta_1(x) f_1(S,I) + \beta_2(x) f_2(V,I) - (\mu_3(x) + \sigma(x))E, \\
\quad t > 0, x \in \Omega, \\
\frac{\partial I}{\partial t} = \nabla \cdot (d_4(x)\nabla I) + \sigma(x)E - (\mu_4(x) + \delta(x) + \gamma(x))I, \\
\quad t > 0, x \in \Omega.
\end{cases} \tag{1.6}$$

The rest of this paper is organized as follows. In Sect. 2, we introduce some preliminaries for the well-posedness of the model. In Sect. 3, we define the basic reproduction number $\mathfrak{R}_0$ and establish the threshold dynamics in terms of $\mathfrak{R}_0$. A particular case is performed as a supplementary to the theoretical results in Sect. 4. A brief conclusion ends the paper.

2 Well-posedness

For convenience, denote $\overline{k} = \max_{x\in\overline{\Omega}} k(x)$ and $\underline{k} = \min_{x\in\overline{\Omega}} k(x)$. Let $\mathbb{X} = C(\overline{\Omega}, \mathbb{R}^4)$ with norm $\|\cdot\|_{\mathbb{X}}$, and let $\mathbb{X}^+ = C(\overline{\Omega}, \mathbb{R}_+^4)$. Denote by

$$T_i(t) : C(\overline{\Omega}, \mathbb{R}) \to C(\overline{\Omega}, \mathbb{R}), \quad i = 1,2,3,4,$$

the C_0 semigroups associated with $\nabla \cdot (d_i(\cdot)\nabla) - \rho_i(\cdot)$ subject to (1.3), where $\rho_1(x) = \mu_1(x) + \xi(x)$, $\rho_2(x) = \mu_2(x) + \alpha(x)$, $\rho_3(x) = \mu_3(x) + \sigma(x)$, and $\rho_4(x) = \mu_4(x) + \delta(x) + \gamma(x)$. Thus we have

$$\big(T_i(t)\psi\big)(x) = \int_\Omega \Gamma_i(t,x,y)\psi(y)\,dy, \quad t > 0, \psi \in C(\overline{\Omega}, \mathbb{R}), \tag{2.1}$$

where $\Gamma_i(t,x,y)$ are the Green functions associated with $\nabla \cdot (d_i(\cdot)\nabla) - \rho_i(\cdot)$ $(i = 1,2,3,4)$ subject to the Neumann boundary condition. It follows from [19] that $T_i(t)$ is strongly positive and compact for all $t > 0$. Thus there exists $M > 0$ such that $\|T_i(t)\| \le Me^{\omega_i t}$ for all $t \ge 0$, where $\omega_i < 0$ denotes the principal eigenvalue of $\nabla \cdot (d_i(\cdot)\nabla) - \rho_i(\cdot)$ $(i = 1,2,3,4)$ subject to Neumann boundary condition (1.3).

Define $F = (F_1, F_2, F_3, F_4) : \mathbb{X} \to \mathbb{X}$ by

$$\begin{aligned}
&F_1(\psi)(x) = \Lambda(x) - \beta_1(x)f_1\big(\psi_1(x), \psi_4(x)\big),\\
&F_2(\psi)(x) = \xi(x)\psi_1(x) - \beta_2(x)f_2\big(\psi_2(x), \psi_4(x)\big),\\
&F_3(\psi)(x) = \beta_1(x)f_1\big(\psi_1(x), \psi_4(x)\big) + \beta_2(x)f_2\big(\psi_2(x), \psi_4(x)\big),\\
&F_3(\psi)(x) = \sigma(x)\psi_3(x)
\end{aligned} \tag{2.2}$$

for $t \geq 0$, $x \in \overline{\Omega}$, and $\psi = (\psi_1, \psi_2, \psi_3, \psi_4) \in \mathbb{X}^+$. Then model (1.6) can be written as

$$u(t,x) = \big(T(t)\psi\big)(x) + \int_0^t T(t-s)F\big(u(s,x)\big)\,ds, \tag{2.3}$$

where $u(t,x) = (S(t,x), V(t,x), E(t,x), I(t,x))$ and $T(t) = diag(T_1(t), T_2(t), T_3(t), T_4(t))$.

It is easy to check that

$$\lim_{\rho\to 0^+} \frac{1}{\rho} dist\big(\psi + \rho F(\psi), \mathbb{X}^+\big) = 0, \quad (x,\psi) \in \bar{\Omega} \times \mathbb{X}^+. \tag{2.4}$$

For any $\psi \in \mathbb{X}^+$, it follows from expressions (2.2) that

$$\begin{aligned}
\psi(x) + \rho F(\psi)(x) &= \begin{pmatrix} \psi_1(x) + \rho(\Lambda(x) - \beta_1(x)f_1(\psi_1(x), \psi_4(x))) \\ \psi_2(x) + \rho(\xi(x)\psi_1(x) - \beta_2(x)f_2(\psi_2(x), \psi_4(x))) \\ \psi_3(x) + \rho(\beta_1(x)f_1(\psi_1(x), \psi_4(x)) + \beta_2(x)f_2(\psi_2(x), \psi_4(x))) \\ \psi_4(x) + \rho\sigma(x)\psi_3(x) \end{pmatrix}\\
&\geq \begin{pmatrix} \psi_1(x)(1 - \rho\bar{\beta}_1\psi_4(x)) \\ \psi_2(x)(1 - \rho\bar{\beta}_2\psi_4(x)) \\ \psi_3(x) \\ \psi_4(x) \end{pmatrix}.
\end{aligned}$$

Firstly, if $\psi_i > 0$, then $\psi + \rho F(\psi) > 0$ for all $\rho > 0$ sufficiently small, so that (2.4) holds. Secondly, if $\psi_i = 0$, then as $F_i(\psi)|_{\psi_i=0} \geq 0$ by (2.2), we have $\psi_i + \rho F(\psi) \geq 0$ for all $\rho > 0$. Thus $\psi + \rho F(\psi) \in \mathbb{X}^+$ when ρ is sufficiently small. Then it follows from [20] that (2.4) holds.

By Corollary 4 in [20] we have have the following:

Lemma 2.1 *For $\psi = (\psi_1, \psi_2, \psi_3, \psi_4) \in \mathbb{X}^+$, model (1.6) has a unique nonnegative mild solution $u(t, \cdot, \psi) = (S(t, \cdot, \psi), V(t, \cdot, \psi), E(t, \cdot, \psi), I(t, \cdot, \psi)) \in \mathbb{X}^+$ on its maximal existence interval $[0, \tau_\psi)$, where $\tau_\psi \leq \infty$. Moreover, this solution is a classical solution.*

Next, we will show the existence of solutions of model (1.6).

Theorem 2.1 *Model (1.6) has a unique solution $u(t, x, \psi)$ on $[0, \infty)$ with $\psi = (\psi_1, \psi_2, \psi_3, \psi_4) \in \mathbb{X}^+$. Furthermore, the solution semiflow $\Phi(t) = u(t, \cdot) : \mathbb{X}^+ \to \mathbb{X}^+$ of model (1.6) is defined by*

$$\Phi(t)\psi = u(t, \cdot, \psi), \quad t \geq 0,$$

admits a global compact attractor.

Proof Suppose to the contrary that $\tau_\psi < \infty$. Then $\|u(t,x,\psi)\| \to +\infty$ as $t \to \tau_\psi$ by Theorem 2 in [20]. Recalling the first equation of model (1.6), we have

$$\frac{\partial S}{\partial t} \leq \nabla \cdot \big(d_1(x)\nabla S\big) + \bar{\Lambda} - (\underline{\mu}_1 + \underline{\xi})S, \quad t \in [0,\tau_\psi), x \in \Omega. \tag{2.5}$$

By the comparison principle and Lemma 2.2 in [21] there exists a constant $\mathcal{Q}_1 > 0$ such that $S(t,x) \leq \mathcal{Q}_1$ for $t \in [0,\tau_\psi)$, $x \in \overline{\Omega}$. Furthermore, a similar procedure can be applied to the second equation of model (1.6). Then there exists a constant $\mathcal{Q}_2 > 0$ such that $V(t,x) \leq \mathcal{Q}_2$ for $t \in [0,\tau_\psi)$, $x \in \overline{\Omega}$. Then from the last two equations of model (1.6) we have

$$\begin{aligned}
&\frac{\partial E}{\partial t} \leq \nabla \cdot \big(d_3(x)\nabla E\big) + \big(\overline{\beta}_1 f_{1I}(\mathcal{Q}_1,0) + \overline{\beta}_2 f_{2I}(\mathcal{Q}_2,0)\big)I - (\underline{\mu}_3 + \underline{\sigma})E, \quad t \in (0,\tau_\psi), x \in \Omega,\\
&\frac{\partial I}{\partial t} \leq \nabla \cdot \big(d_4(x)\nabla I\big) + \overline{\sigma}E - (\underline{\mu}_4 + \underline{\delta} + \underline{\gamma})I, \quad t \in (0,\tau_\psi), x \in \Omega,\\
&\frac{\partial E}{\partial \nu} = \frac{\partial I}{\partial \nu} = 0, \quad t \in (0,\tau_\psi), x \in \partial\Omega,
\end{aligned}$$

where $f_{1I}(\mathcal{Q}_1,0) = \frac{\partial f_1(\mathcal{Q}_1,0)}{\partial I}$ and $f_{2I}(\mathcal{Q}_2,0) = \frac{\partial f_2(\mathcal{Q}_2,0)}{\partial I}$.

Consider the linear system

$$\begin{aligned}
&\frac{\partial u_1}{\partial t} = \nabla \cdot \big(d_3(x)\nabla u_1\big) + \big(\overline{\beta}_1 f_{1I}(\mathcal{Q}_1,0) + \overline{\beta}_2 f_{2I}(\mathcal{Q}_2,0)\big)u_2 - (\underline{\mu}_3 + \underline{\sigma})u_1,\\
&\quad t \in (0,\tau_\psi), x \in \Omega,\\
&\frac{\partial u_2}{\partial t} = \nabla \cdot \big(d_4(x)\nabla u_2\big) + \overline{\sigma}u_1 - (\underline{\mu}_4 + \underline{\delta} + \underline{\gamma})u_2, \quad t \in (0,\tau_\psi), x \in \Omega,\\
&\frac{\partial u_1}{\partial \nu} = \frac{\partial u_2}{\partial \nu} = 0, \quad t \in (0,\tau_\psi), x \in \partial\Omega.
\end{aligned} \tag{2.6}$$

The standard Krein–Rutman theorem (see [22]) implies that the eigenvalue problem of model (2.6) admits a principal eigenvalue λ with strongly positive eigenfunction $\varphi = (\varphi_1,\varphi_2)$. Thus model (2.6) has a solution $\varsigma e^{\lambda t}\varphi(x)$ for $t \geq 0$, where ς is a positive constant such that $\varsigma\varphi = (u_1(0,x), u_2(0,x)) \geq (E(0,x), I(0,x))$ for $x \in \overline{\Omega}$. Then from the comparison principle it follows that

$$\big(E(t,x), I(t,x)\big) \leq \varsigma e^{\lambda t}\varphi(x), \quad t \in (0,\tau_\psi), x \in \overline{\Omega}.$$

Therefore there exists a constant κ such that $E(t,x) \leq \kappa$, $I(t,x) \leq \kappa$, $x \in \overline{\Omega}$, which leads to a contradiction. Hence the global existence of $u(t,\cdot,\psi)$ is derived.

Furthermore, it follows from the comparison principle and Lemma 2.2 in [21] that there exist $t_1 > 0$ and $\mathcal{L}_1 > 0$, $\mathcal{L}_2 > 0$ such that $S(t,x) \leq \mathcal{L}_1$ and $V(t,x) \leq \mathcal{L}_2$ for all $t \geq t_1$ and $x \in \overline{\Omega}$. Set

$$P(t) = \int_\Omega \big(S(t,x) + V(t,x) + E(t,x) + I(t,x)\big)\,dx.$$

Then we have

$$\frac{dP(t)}{dt} = \int_\Omega \Big(\Lambda(x) - \mu_1(x)S(t,x) - \big(\mu_2(x) + \alpha(x)\big)V(t,x) - \mu_3(x)E(t,x)$$

$$
\begin{aligned}
&\quad - \bigl(\mu_4(x) + \delta(x) + \gamma(x)\bigr)I(t,x)\bigr)\,dx \\
&\le \int_\Omega \Lambda(x)\,dx - \min_{x\in\overline{\Omega}}\{\underline{\mu}_1, \underline{\mu}_2 + \underline{\alpha}, \underline{\mu}_3, \underline{\mu}_4 + \underline{\delta} + \underline{\gamma}\}P(t).
\end{aligned}
$$

Thus there exist $t_2 > 0$ and $\mathcal{L}_3 > 0$ such that $P(t) \le \mathcal{L}_3$ for all $t \ge t_2$. It follows from [23] that

$$
\Gamma_3(t,x,y) = \sum_{n\ge 1} e^{\pi_n t}\varphi_n(x)\varphi_n(y), \quad t > 0,
$$

where π_n are the eigenvalues of $\nabla\cdot(d_3(x)\nabla) - (\mu_3(x) + \sigma(x))$ subject to the Neumann boundary condition with eigenfunction $\varphi_n(x)$ and satisfy $\pi_1 > \pi_2 \ge \pi_2 \ge \cdots \ge \pi_n \ge \cdots$. Then for some $\kappa_1 > 0$, we have $\Gamma_3(t,x,y) \le \kappa_1 \sum_{n\ge 1} e^{\pi_n t}$ for $t > 0$. Moreover, we assume that τ_n are the eigenvalues of $\nabla\cdot(\underline{d}_3\nabla) - (\underline{\mu}_3 + \underline{\sigma})$ subject to the Neumann boundary condition, which satisfy $\tau_1 = -(\underline{\mu}_3 + \underline{\sigma}) > \tau_2 \ge \tau_3 \ge \cdots \ge \tau_n \ge \cdots$. Following Theorem 2.4.7 in Wang [24], we get $\tau_i \ge \pi_i$ for all $i \in \mathbb{N}_+$. Then for some $\kappa_2 > 0$, we have

$$
\Gamma_3 \le \kappa_1 \sum_{n\ge 1} e^{\tau_n t} \le \kappa_2 e^{\tau_1 t} = \kappa_2 e^{-(\underline{\mu}_3 + \underline{\sigma})t}, \quad t > 0.
$$

Let $t_3 = \max\{t_1, t_2\}$. For all $t \ge t_3$, we obtain

$$
\begin{aligned}
E(t,x) &= T_3(t)E(t_3,x) + \int_{t_3}^t T_3(t-s)\bigl[\beta_1(x)f_1\bigl(S(s,x), I(s,x)\bigr) + \beta_2(x)f_2\bigl(V(s,x), I(s,x)\bigr)\bigr]\,ds \\
&\le M\bigl\|E(t_3,x)\bigr\|e^{\omega_3(t-t_3)} + \int_{t_3}^t \int_\Omega \Gamma_3(t-s,x,y)\bigl[\beta_1(y)f_1\bigl(S(s,y), I(s,y)\bigr) \\
&\quad + \beta_2(y)f_2\bigl(V(s,y), I(s,y)\bigr)\bigr]\,dy\,ds \\
&\le M\bigl\|E(t_3,x)\bigr\|e^{\omega_3(t-t_3)} \\
&\quad + \int_{t_3}^t \int_\Omega \kappa_2 e^{-(\mu_3+\sigma)(t-s)}\bigl[\overline{\beta}_1 f_{1I}(\mathcal{L}_1, 0) + \overline{\beta}_2 f_{2I}(\mathcal{L}_2, 0)\bigr]I(s,y)\,dy\,ds \\
&\le M\bigl\|E(t_3,x)\bigr\|e^{\omega_3(t-t_3)} + \int_{t_3}^t \kappa_2 e^{-(\mu_3+\sigma)(t-s)}\bigl[\overline{\beta}_1 f_{1I}(\mathcal{L}_1, 0) + \overline{\beta}_2 f_{2I}(\mathcal{L}_2, 0)\bigr]\mathcal{L}_3\,ds \\
&\le M\bigl\|E(t_3,x)\bigr\|e^{\omega_3(t-t_3)} + \frac{\kappa_2\mathcal{L}_3}{\mu_3 + \sigma}\bigl[\overline{\beta}_1 f_{1I}(\mathcal{L}_1, 0) + \overline{\beta}_2 f_{2I}(\mathcal{L}_2, 0)\bigr],
\end{aligned}
$$

which yields that

$$
\limsup_{t\to\infty}\bigl\|E(t,x)\bigr\| \le \frac{\kappa_2\mathcal{L}_3}{\mu_3 + \sigma}\bigl[\overline{\beta}_1 f_{1I}(\mathcal{L}_1, 0) + \overline{\beta}_2 f_{2I}(\mathcal{L}_2, 0)\bigr].
$$

Similarly, we can also obtain that there exists a positive constant $\mathcal{L}_4 > 0$ such that $\limsup_{t\to\infty}\|I(t,x)\| \le \mathcal{L}_4$. Thus the above discussions imply that the system is point dissipative. Furthermore, by Theorem 2.2.6 in [25] the solution semiflow $\Phi(t)$ is compact for all $t > 0$. Therefore it follows from Theorem 3.4.8 in [26] that $\Phi(t)$ has a global compact attractor. □

3 Threshold dynamics

3.1 Basic reproduction number

Considering the following subsystem of model (1.6):

$$\begin{cases} \frac{\partial S}{\partial t} = \nabla \cdot (d_1(x)\nabla S) + \Lambda(x) - (\mu_1(x) + \xi(x))S, & t > 0, x \in \Omega, \\ \frac{\partial V}{\partial t} = \nabla \cdot (d_2(x)\nabla V) + \xi(x)S - (\mu_2(x) + \alpha(x))V, & t > 0, x \in \Omega, \\ \frac{\partial S}{\partial \nu} = \frac{\partial V}{\partial \nu} = 0, & t > 0, x \in \partial\Omega. \end{cases} \tag{3.1}$$

It follows from the Lemma 2.2 in [21] that the system

$$\begin{cases} \frac{\partial S}{\partial t} = \nabla \cdot (d_1(x)\nabla S) + \Lambda(x) - (\mu_1(x) + \xi(x))S, & t > 0, x \in \Omega, \\ \frac{\partial S}{\partial \nu} = 0, & t > 0, x \in \partial\Omega, \end{cases} \tag{3.2}$$

admits a unique positive steady state $S^0(x)$ that satisfies the equation

$$\nabla \cdot \big(d_1(x)\nabla S^0(x)\big) + \Lambda(x) - \big(\mu_1(x) + \xi(x)\big)S^0(x) = 0$$

with $\frac{\partial S^0(x)}{\partial \nu} = 0$ for $x \in \partial\Omega$, which is globally asymptotically stable in $C(\overline{\Omega}, \mathbb{R}_+)$. Then the second equation of model (3.1) is asymptotic to

$$\frac{\partial V}{\partial t} = \nabla \cdot \big(d_2(x)\nabla V\big) + \xi(x)S^0(x) - \big(\mu_2(x) + \alpha(x)\big)V. \tag{3.3}$$

By Lemma 2.2 in [21] and Corollary 4.3 in [27] we easily obtain that model (1.6) admits a unique disease-free steady state $P_0(x) = (S^0(x), V^0(x), 0, 0)$. Furthermore, if all the parameters of model (1.6) are positive constants, then we have $S^0(x) = \frac{\Lambda}{\mu_1+\xi}$ and $V^0(x) = \frac{\Lambda\xi}{(\mu_2+\alpha)(\mu_1+\xi)}$. Linearizing model (1.6) at $P_0(x)$, we obtain the linearized subsystem

$$\begin{cases} \frac{\partial E}{\partial t} = \nabla \cdot (d_3(x)\nabla E) + (\beta_1(x)f_{1I}(S^0(x), 0) + \beta_2(x)f_{2I}(V^0(x), 0))I \\ \qquad - (\mu_3(x) + \sigma(x))E, & t > 0, x \in \Omega, \\ \frac{\partial I}{\partial t} = \nabla \cdot (d_4(x)\nabla I) + \sigma(x)E - (\mu_4(x) + \delta(x) + \gamma(x))I, & t > 0, x \in \Omega, \\ \frac{\partial E}{\partial \nu} = \frac{\partial I}{\partial \nu} = 0, & t > 0, x \in \partial\Omega. \end{cases} \tag{3.4}$$

Letting $(E(t,x), I(t,x)) = e^{\lambda t}(\psi_3(x), \psi_4(x))$, it follows from model (3.4) that

$$\begin{cases} \lambda\psi_3(x) = \nabla \cdot (d_3(x)\nabla\psi_3(x)) + (\beta_1(x)f_{1I}(S^0(x), 0) \\ \qquad + \beta_2(x)f_{2I}(V^0(x), 0))\psi_4(x) - (\mu_3(x) + \sigma(x))\psi_3(x), & t > 0, x \in \Omega, \\ \lambda\psi_4(x) = \nabla \cdot (d_4(x)\nabla\psi_4(x)) + \sigma(x)\psi_3(x) \\ \qquad - (\mu_4(x) + \delta(x) + \gamma(x))\psi_4(x), & t > 0, x \in \Omega, \\ \frac{\partial \psi_3}{\partial \nu} = \frac{\partial \psi_4}{\partial \nu} = 0, & t > 0, x \in \partial\Omega. \end{cases} \tag{3.5}$$

It follows from the Krein–Rutman theorem that system (3.5) admits a unique principal eigenvalue $\lambda_0(S^0(x), V^0(x))$ with strongly positive eigenfunction $\psi(x) = (\psi_3(x), \psi_4(x))$.

Let $\Psi(t) : C(\bar{\Omega}, \mathbb{R}^2) \to C(\bar{\Omega}, \mathbb{R}^2)$ be the semigroup associated with the following system:

$$\begin{cases} \frac{\partial E}{\partial t} = \nabla \cdot (d_3(x)\nabla E) - (\mu_3(x) + \sigma(x))E(x,t), & t > 0, x \in \Omega, \\ \frac{\partial I}{\partial t} = \nabla \cdot (d_4(x)\nabla I) + \sigma(x)E(x,t) - (\mu_4(x) + \delta(x) + \gamma(x))I(x,t), & t > 0, x \in \Omega, \\ \frac{\partial E}{\partial \nu} = \frac{\partial I}{\partial \nu} = 0, & t > 0, x \in \partial\Omega. \end{cases}$$

Define

$$\Gamma(x) = \begin{pmatrix} 0 & \beta_1(x)f_{1I}(S^0(x), 0) + \beta_2(x)f_{2I}(V^0(x), 0) \\ 0 & 0 \end{pmatrix}.$$

Assuming that the distribution of initial infection is $\psi(x) = (\psi_3(x), \psi_4(x))$, the distribution of totally new infective numbers is given by

$$\mathcal{L}(\psi)(x) = \int_0^\infty \Gamma(x)\Psi(t)\psi(x)\,dt.$$

According to [28], the basic reproduction number is defined by $\Re_0 = r(\mathcal{L})$, where $r(\mathcal{L})$ is the spectral radius of the operator $\mathcal{L}$. Furthermore, by Theorem 3.1 in [28] we have

Lemma 3.1 *The principal eigenvalue λ_0 and $\Re_0 - 1$ have the same sign, and the disease-free steady state $P_0(x)$ is locally asymptotically stable if $\Re_0 < 1$ and unstable if $\Re_0 > 1$.*

3.2 Persistence analysis

In this section, we investigate the extinction and persistence of the disease in terms of $\Re_0$.

Theorem 3.1 *If $\Re_0 < 1$, then the disease-free steady state $P_0(x)$ is globally asymptotically stable.*

Proof It follows from Lemma 3.1 that $\lambda_0(S^0(x), V^0(x)) < 0$ for $\Re_0 < 1$. By the continuity there is $\varepsilon > 0$ such that $\lambda_0(S^0(x) + \varepsilon, V^0(x) + \varepsilon) < 0$. Furthermore, according to the first two equation of model (1.6), there exists $t_1 > 0$ such that $S(t,x) \le S^0(x) + \varepsilon$ and $V(t,x) \le V^0(x) + \varepsilon$ for $t \ge t_1$ and $x \in \Omega$. Then we have

$$\begin{cases} \frac{\partial E}{\partial t} \le \nabla \cdot (d_3(x)\nabla E) + [\beta_1(x)f_{1I}(S^0(x) + \varepsilon, 0) + \beta_2(x)f_{2I}(V^0(x) + \varepsilon, 0)]I \\ \qquad - (\mu_3(x) + \sigma(x))E, & x \in \Omega, t \ge t_1, \\ \frac{\partial I}{\partial t} \le \nabla \cdot (d_4(x)\nabla I) + \sigma(x)E - (\mu_4(x) + \delta(x) + \gamma(x))I, & x \in \Omega, t \ge t_1, \\ \frac{\partial E}{\partial \nu} = \frac{\partial I}{\partial \nu} = 0, & x \in \partial\Omega, t \ge t_1. \end{cases}$$

Assume that $\zeta(\psi_3(x), \psi_4(x)) \ge (E(t_1,x), I(t_1,x))$ with $\zeta > 0$ and the eigenfunction $\psi = (\psi_3(x), \psi_4(x))$ corresponding to the principal eigenvalue λ_0. Since $\lambda_0(S^0(x) + \varepsilon, V^0(x) + \varepsilon) < 0$, according to the comparison principle, we obtain

$$\big(E(t,x), I(t,x)\big) \le \zeta\big(\psi_3(x), \psi_4(x)\big)e^{\lambda_0(S^0(x)+\varepsilon, V^0(x)+\varepsilon)(t-t_1)}, t \ge t_1,$$

which yields $\lim_{t\to\infty}(E(t,x,)I(t,x)) = 0$ uniformly for $x \in \overline{\Omega}$. Thus model (1.6) is asymptotic to (3.1). Consequently, we have $\lim_{t\to\infty} S(t,x) = S^0(x)$ and $\lim_{t\to\infty} V(t,x) = V^0(x)$. Therefore the disease-free steady state $P_0(x)$ is globally asymptotically stable. □

Theorem 3.2 *If $\Re_0 > 1$, then there exists a constant $\rho > 0$ such that for any $\psi \in \mathbb{X}^+$ with $\psi_3 \not\equiv 0$ and $\psi_4 \not\equiv 0$,*

$$\liminf_{t\to\infty} S(t,x,\psi) \geq \rho, \qquad \liminf_{t\to\infty} V(t,x,\psi) \geq \rho,$$
$$\liminf_{t\to\infty} E(t,x,\psi) \geq \rho, \qquad \liminf_{t\to\infty} I(t,x,\psi) \geq \rho,$$

uniformly for all $x \in \overline{\Omega}$.

Proof Let

$$\mathbb{X}_0 = \left\{(S,V,E,I) \in \mathbb{X}^+ : E(\cdot) \not\equiv 0, I(\cdot) \not\equiv 0\right\}.$$

Then we have

$$\partial\mathbb{X}_0 = \mathbb{X}^+ \backslash \mathbb{X}_0 = \left\{(S,V,E,I) \in \mathbb{X}^+ : E(\cdot) \equiv 0 \text{ or } I(\cdot) \equiv 0\right\}.$$

Thus $\mathbb{X}_0$ is positively invariant for $\Phi(t)$. Set

$$\mathcal{M}_\partial = \left\{\psi \in \mathbb{X}^+ : \Phi(t)\psi \in \partial\mathbb{X}_0, \forall t \geq 0\right\},$$

and let $\omega(\psi)$ be the omega limit set of the orbit $\mathcal{O}^+(\psi) = \{\Phi(t)\psi : t \geq 0\}$. We first prove the following claim.

Claim 1. $\bigcup_{\psi\in\mathcal{M}_\partial} \omega(\psi) = P_0(x)$.

For $\psi \in \mathcal{M}_\partial$, $u_t(\psi) \in \partial X_0$, since $u_t(\psi) = u(t,\cdot,\psi)$, for $t \geq 0$, there are two possible cases, either $E(t,\cdot,\psi) \equiv 0$ or $I(t,\cdot,\psi) \equiv 0$. If $E(t,\cdot,\psi) \equiv 0$ for $t \geq 0$, then it follows from the last equation of model (1.6) that $\lim_{t\to+\infty} I(t,\cdot,\psi) = 0$ uniformly for $x \in \Omega$. Consequently, we have $\lim_{t\to+\infty} S(t,\cdot,\psi) = S^0(x)$ and $\lim_{t\to+\infty} V(t,\cdot,\psi) = V^0(x)$ for $x \in \Omega$. If $E(t,\cdot,\psi) \not\equiv 0$ for some $t_0 > 0$, then $E(t,\cdot,\psi) > 0$ for $t > t_0$, $x \in \Omega$, and thus $I(t,\cdot,\psi) \equiv 0$ for $t \geq t_0$, which implies that $E(t,\cdot,\psi) = 0$, a contradiction. Hence the proof of the claim is completed.

Claim 2. $P_0(x)$ is a uniform weak repeller for $\mathbb{X}_0$ in the sense that

$$\limsup_{t\to\infty} \left\| \Phi(t)\phi - P_0(x) \right\| \geq \eta \quad \forall\psi \in \mathbb{X}_0.$$

Suppose this is not true. Then there exists $\psi_0 \in \mathbb{X}_0$ such that

$$\limsup_{t\to\infty} \left\| \Phi(t)\psi - P_0(x) \right\| < \eta.$$

Then there exists $t_1 > 0$ such that

$$S^0(x) - \eta < S(t,x,\psi) < S^0(x) + \eta, \qquad V^0(x) - \eta < V(t,x,\psi) < V^0(x) + \eta,$$
$$0 < E(t,x,\psi) < \eta, \qquad 0 < I(t,x,\psi) < \eta, \quad \forall t \geq t_1, x \in \overline{\Omega}.$$

Therefore from model (1.6) we have

$$\begin{cases} \frac{\partial E}{\partial t} \geq \nabla \cdot (d_3(x)\nabla E) + \beta_1(x) f_{1I}(S^0(x) - \eta, \eta) I + \beta_2(x) f_{2I}(V^0(x) - \eta, \eta) I \\ \qquad - (\mu_3(x) + \sigma(x))E, & x \in \Omega, t \geq t_1, \\ \frac{\partial I}{\partial t} \geq \nabla \cdot (d_4(x)\nabla I) + \sigma(x)E - (\mu_4(x) + \delta(x) + \gamma(x))I, & x \in \Omega, t \geq t_1, \\ \frac{\partial E}{\partial \nu} = \frac{\partial I}{\partial \nu} = 0, & x \in \partial\Omega, t \geq t_1. \end{cases}$$

Let $(\psi_3(x), \psi_4(x))$ be the eigenfunction associated with principle eigenvalue $\lambda_0(S^0(x) - \eta, V^0(x) - \eta) > 0$. Suppose that $(E(t_1, x), I(t_1, x)) \geq \xi(\psi_3(x), \psi_4(x))$ for some $\xi > 0$. Then we obtain that

$$\big(E(t,x), I(t,x)\big) \geq \xi e^{\lambda_0(S^0(x)-\eta, V^0(x)-\eta)(t-t_1)} \big(\psi_3(x), \psi_4(x)\big), \quad t \geq t_1,$$

which implies that $\limsup_{t\to\infty} E(t,x) = \infty$ and $\limsup_{t\to\infty} I(t,x) = \infty$. This gives a contradiction.

Define the continuous function $p : \mathbb{X}^+ \to [0, +\infty)$ by

$$p(\psi) = \min\Big\{\min_{x\in\overline{\Omega}} \psi_3(x), \min_{x\in\overline{\Omega}} \psi_4(x)\Big\} \forall \psi \in \mathbb{X}^+.$$

Clearly, $p^{-1}(0, +\infty) \subseteq \mathbb{X}_0$, and p has the property that if either $p(\psi) > 0$ or $p(\psi) = 0$ and $\psi \in \mathbb{X}_0$, then $p(\Phi(t)\psi) > 0$. Moreover, we can show that $P_0(x)$ is isolated in $\mathbb{X}^+$ and $W^s(P_0(x)) \cap \mathbb{X}_0 = \emptyset$, where $W^s(P_0(x))$ is the stable set of $P_0(x)$. It then follows from Theorem 3 in [29] that there exists a constant $\varrho > 0$ such that $\liminf_{t\to\infty} p(\Phi(t)\psi) \geq \varrho$ for all $\psi \in \mathbb{X}_0$. The proof is complete. $\square$

4 A particular case

In this section, we consider a particular case of model (1.6) to study global stabilities of steady states. Select $f_1(S,I) = Sf_1(I)$ and $f_2(V,I) = Vf_2(I)$ and consider the model

$$\begin{cases} \frac{\partial S}{\partial t} = d_1 \Delta S + \Lambda - (\mu_1 + \xi)S - \beta_1 S f_1(I), & t > 0, x \in \Omega, \\ \frac{\partial V}{\partial t} = d_2 \Delta V + \xi S - \beta_2 V f_2(I) - (\mu_2 + \alpha)V, & t > 0, x \in \Omega, \\ \frac{\partial E}{\partial t} = d_3 \Delta E + \beta_1 S f_1(I) + \beta_2 V f_2(I) - (\mu_3 + \sigma)E, & t > 0, x \in \Omega, \\ \frac{\partial I}{\partial t} = d_4 \Delta I + \sigma E - (\mu_4 + \delta + \gamma)I, & t > 0, x \in \Omega, \\ \frac{\partial S}{\partial \nu} = \frac{\partial V}{\partial \nu} = \frac{\partial E}{\partial \nu} = \frac{\partial I}{\partial \nu} = 0, & t > 0, x \in \partial\Omega. \end{cases} \tag{4.1}$$

It follows that

$$\mathfrak{R}_0 = \frac{(\beta_1 S_0 f_1'(0) + \beta_2 V_0 f_2'(0))\sigma}{(\mu_3 + \sigma)(\mu_4 + \delta + \gamma)},$$

where $S_0 = \frac{\Lambda}{\mu_1+\xi}$ and $V_0 = \frac{\xi\Lambda}{(\mu_2+\alpha)(\mu_1+\xi)}$. If $\mathfrak{R}_0 > 1$, then the positive steady state $P_* = (S_*, V_*, E_*, I_*)$ of model (4.1) satisfies the following equations:

$$\begin{aligned}
&\Lambda = (\mu_1 + \xi)S_* + \beta_1 S_* f_1(I_*),\\
&\xi S_* = (\mu_2 + \alpha)V_* + \beta_2 V_* f_2(I_*),\\
&\beta_1 S_* f_1(I_*) + \beta_2 V_* f_2(I_*) = (\mu_3 + \sigma)E_*,\\
&\sigma E_* = (\mu_4 + \delta + \gamma)I_*.
\end{aligned} \tag{4.2}$$

Then we have

$$S_* = \frac{\Lambda}{\mu_1 + \xi + \beta_1 f_1(I_*)}, \qquad V_* = \frac{\xi S_*}{\mu_1 + \alpha + \beta_2 f_2(I_*)}, \qquad E_* = \frac{(\mu_3+\sigma)(\mu_4+\delta+\gamma)}{\sigma} I_*,$$

and I_* is the positive root of the equation

$$\begin{aligned}
F(I) &= \frac{\Lambda\beta_1 f_1(I)}{I(\mu_1 + \xi + \beta_1 f_1(I))} + \frac{\Lambda\xi\beta_2 f_2(I)}{I(\mu_1 + \xi + \beta_1 f_1(I))(\mu_2 + \alpha + \beta_2 f_2(I))}\\
&\quad - \frac{(\mu_3+\sigma)(\mu_4+\delta+\gamma)}{\sigma}\\
&= 0.
\end{aligned}$$

It is easy to show that $\lim_{I\to 0^+} F(I) = \frac{(\mu_3+\sigma)(\mu_4+\delta+\gamma)}{\sigma}(\mathfrak{R}_0 - 1) > 0$, $\lim_{I\to+\infty} F(I) < 0$, and

$$F'(I) = \frac{\Theta_1}{[I(\mu_1 + \xi + \beta_1 f_1(I))]^2} + \frac{\Theta_2}{[I(\mu_1 + \xi + \beta_1 f_1(I))(\mu_2 + \alpha + \beta_2 f_2(I))]^2},$$

where

$$\begin{aligned}
\Theta_1 &= \bigl(\mu_1 + \xi + \beta_1 f_1(I)\bigr)\bigl(f_1'(I)I - f_1(I)\bigr) - \beta_1 I f_1'(I) f_1(I),\\
\Theta_2 &= \bigl(I f_2'(I) - f_2(I)\bigr)\bigl(\mu_1 + \xi + \beta_1 f_1(I)\bigr)\bigl(\mu_2 + \alpha + \beta_2 f_2(I)\bigr)\\
&\quad - f_2(I)\bigl[\beta_1 I f_1'(I)\bigl(\mu_2 + \alpha + \beta_2 f_2(I)\bigr) + \beta_2 I f_2(I)\bigl(\mu_1 + \xi + \beta_1 f_1(I)\bigr)\bigr].
\end{aligned}$$

It follows from the assumption on the functions $f_j(u, I)$ $(j = 1, 2)$ that $\Theta_1 < 0$, $\Theta_2 < 0$. Then we have $F'(I) < 0$. Thus there exists a positive steady state P_*.

Theorem 4.1 *If $\mathfrak{R}_0 > 1$, then the unique positive steady state P_* of model (4.1) is globally asymptotically stable.*

Proof Define

$$\begin{aligned}
L(t) = \int_\Omega \biggl\{ & S - S_* - S_* \ln\frac{S}{S_*} + V - V_* - V_* \ln\frac{V}{V_*} + E - E_* - E_* \ln\frac{E}{E_*}\\
& + \frac{\mu_3+\sigma}{\sigma}\left(I - I_* - I_* \ln\frac{I}{I_*}\right)\biggr\}\, dx.
\end{aligned}$$

Using (4.2) and the function $\varphi(z) = 1 + \ln z - z$ with global maximum $\varphi(1) = 0$. Then we have

$$\begin{aligned}
L' = {} & \int_\Omega \left\{ \left(1 - \frac{S}{S_*}\right)\left[d_1 \Delta S + \Lambda - \beta_1 S f_1(I) - (\mu_1 + \xi)S\right] + \left(1 - \frac{V}{V_*}\right)\left[d_2 \Delta V + \xi S \right.\right. \\
& \left. - \beta_2 S f_2(I) - (\mu_2 + \alpha)V\right] + \left(1 - \frac{E}{E_*}\right)\left[d_3 \Delta E + \beta_1 S f_1(I) + \beta_2 V f_2(I) - (\mu_3 + \sigma)E\right] \\
& \left. + \left(1 - \frac{I}{I_*}\right)\left[d_4 \Delta I + \sigma E - (\mu_4 + \delta + \gamma)E\right] \right\} dx \\
= {} & \int_\Omega \left\{ -d_1 S_* \frac{\|\nabla S\|^2}{S^2} - d_2 V_* \frac{\|\nabla V\|^2}{V^2} - d_3 E_* \frac{\|\nabla E\|^2}{E^2} - d_4 I_* \frac{\|\nabla I\|^2}{I^2} \right. \\
& + \mu_1 S_* \left(2 - \frac{S}{S_*} - \frac{S_*}{S}\right) + (\mu_2 + \alpha) V_* \left(3 - \frac{V}{V_*} - \frac{S_*}{S} - \frac{S V_*}{S_* V}\right) \\
& + \beta_1 S_* f_1(I_*) \left[\varphi\left(\frac{S_*}{S}\right) + \varphi\left(\frac{E I_*}{E_* I}\right) + \varphi\left(\frac{E_* S f_1(I)}{E S_* f_1(I_*)}\right) + \varphi\left(\frac{I f_1(I_*)}{I_* f_1(I)}\right) \right. \\
& \left. + \left(1 - \frac{f_1(I_*)}{f_1(I)}\right)\left(\frac{f_1(I)}{f_1(I_*)} - \frac{I}{I_*}\right)\right] + \beta_2 V_* f_2(I_*) \left[\varphi\left(\frac{S_*}{S}\right) + \varphi\left(\frac{S V_*}{S_* V}\right) + \varphi\left(\frac{I_* E}{I E_*}\right) \right. \\
& \left.\left. + \varphi\left(\frac{E_* V f_2(I)}{E V_* f_2(I_*)}\right) + \varphi\left(\frac{I f_2(I_*)}{I_* f_2(I)}\right) + \left(1 - \frac{f_2(I_*)}{f_2(I)}\right)\left(\frac{f_2(I)}{f_2(I_*)} - \frac{I}{I_*}\right)\right]\right\} dx \\
\leq {} & 0.
\end{aligned}$$

Thus by the LaSalle invariance principle it is clear that P_* is globally asymptotically stable. □

5 Conclusions

In this paper, we investigated an SEVIR model with diffusion and nonlinear incidence rate. The basic reproduction number $\mathfrak{R}_0$, which serves as a threshold index, is defined. By applying the comparison principle we prove that the disease-free steady state $P_0(x)$ is globally asymptotically stable when $\mathfrak{R}_0 < 1$. If $\mathfrak{R}_0 > 1$, then the disease will persist. Furthermore, for the spatially homogeneous model, we establish the global stability of the positive steady state in terms of the corresponding basic reproduction number. Some existing global dynamical results can be covered and improved (see [18]). Though the well-posedness, the basic reproduction number and persistence of model (1.6) are established. The uniqueness and stability of the positive steady state remain an open problem. In addition, the global stability of the positive steady state of model (4.1) of general forms $f_1(S,I)$ and $f_2(V,I)$ is still an open problem. We leave these problems for future investigation.

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Declarations

Competing interests
The authors declare no competing interests.

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