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Oscillatory and spectral properties of fourth-order differential operator and weighted differential inequality with boundary conditions

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Abstract

In the paper, we study the oscillatory and spectral properties of a fourth-order differential operator. These properties are established based on the validity of some weighted second-order differential inequality, where the inequality's weights are the coefficients of the operator. The inequality, in turn, is established for functions satisfying certain boundary conditions that depend on the boundary behavior of its weights at infinity and at zero.

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1 Introduction

Let $I = (0,\infty)$ and $1 < p,q < \infty$. Let r, υ, and u be a.e. positive functions such that r is continuously differentiable, u and υ are locally integrable on the interval I. Moreover, $r^{-1} \equiv \frac{1}{r} \in L_1^{\mathrm{loc}}(I)$ and $\upsilon^{-p'} \in L_1^{\mathrm{loc}}(I)$, where $p' = \frac{p}{p-1}$.

We consider the following inequality:

$$\left(\int_0^\infty \left|u(t)f(t)\right|^q dt\right)^{\frac{1}{q}} \le C\left(\int_0^\infty \left|\upsilon(t)D_r^2 f(t)\right|^p dt\right)^{\frac{1}{p}}, \quad f \in C_0^\infty(I), \tag{1}$$

where $D_r^2 f(t) = \frac{d}{dt} r(t)\frac{df(t)}{dt}$ and $C_0^\infty(I)$ is the set of compactly supported functions infinitely time continuously differentiable on I. Moreover, assume that $D_r^1 f(t) = r(t)\frac{df(t)}{dt}$.

For $r = 1$, inequality (1) has the form

$$\left(\int_0^\infty \left|u(t)f(t)\right|^q dt\right)^{\frac{1}{q}} \le C\left(\int_0^\infty \left|\upsilon(t)f''(t)\right|^p dt\right)^{\frac{1}{p}}. \tag{2}$$

At the end of the 70s of the last century, the Czech mathematician A. Kufner set the problem of studying a weighted differential inequality in the form

$$\left(\int_0^\infty |u(t)f(t)|^q\,dt\right)^{\frac{1}{q}} \le C\left(\int_0^\infty |\upsilon(t)f^{(n)}(t)|^p\,dt\right)^{\frac{1}{p}}, \quad f \in C_0^\infty(I), \tag{3}$$

where $n \ge 1$. For $n = 1$, the validity of inequality (3) was first established by P. Gurka, and this result was presented in [24, Chap. 8]. Improvements to this result are given in [1] and [11]. A survey of results for $n \ge 2$ with comments and some proofs are given in [17, Chap. 4] and [18].

Let $W^2_{p,\upsilon}(r) \equiv W^2_{p,\upsilon}(r,I)$ be a set of functions $f : I \to \mathbb{R}$, having generalized derivatives together with functions $D^1_r f(t)$ on the interval I, with the finite norm

$$\|f\|_{W^2_{p,\upsilon}(r)} = \|\upsilon D^2_r f\|_p + |D^1_r f(1)| + |f(1)|, \tag{4}$$

where $\|\cdot\|_p$ is the standard norm of the space $L_p(I)$.

By the conditions on the functions r and υ, we have that $C_0^\infty(I) \subset W^2_{p,\upsilon}(r)$. Denote by $\mathring{W}^2_{p,\upsilon}(r) \equiv \mathring{W}^2_{p,\upsilon}(r,I)$ the closure of the set $C_0^\infty(I)$ with respect to norm (4). Then inequality (1) is equivalent to the inequality

$$\left(\int_0^\infty |u(t)f(t)|^q\,dt\right)^{\frac{1}{q}} \le C\left(\int_0^\infty |\upsilon(t)D^2_r f(t)|^p\,dt\right)^{\frac{1}{p}}, \quad f \in \mathring{W}^2_{p,\upsilon}(r); \tag{5}$$

in addition, the least constants in (1) and (5) coincide.

Criteria for the fulfillment of inequality (2) under various boundary conditions on the function f are given in [21] and [22]. In this paper, following the ideas of [22], we give criteria for the fulfillment of inequality (5) and two-sided estimates for its least constant, suitable for establishing the oscillatory properties of the differential equation

$$D^2_r\big(\upsilon(t)D^2_r y(t)\big) - u(t)y(t) = 0, \quad t > 0, \tag{6}$$

and the spectral properties of the operator L generated by the differential expression

$$Ly(t) = \frac{1}{u(t)}D^2_r\big(\upsilon(t)D^2 y(t)\big). \tag{7}$$

For $r = 1$, relations (6) and (7) have the forms

$$\big(\upsilon(t)y''(t)\big)'' - u(t)y(t) = 0, \tag{8}$$

$$Ly(t) = \frac{1}{u(t)}\big(\upsilon(t)y''(t)\big)''. \tag{9}$$

The recent paper [14] presents a relationship between inequality (5), the oscillatory properties of equation (6), and the spectral properties of the operator L generated by expression (7). This relationship shows that oscillation and nonoscillation of equation (6) depend on the value of the least constant in inequality (5), while the spectral properties of the operator L depend on the strong nonoscillation of equation (6). Thus, the study of inequality (5) plays a leading role. In turn, the study of inequality (5) depends on the degree of singularity of the functions υ^{-1} and r^{-1} at the endpoints of the interval I. The concept of "degree of singularity" follows from the results of the work [13] (see Theorems C^+ and C^- below). For $f \in W^2_{p,\nu}(r,I)$, we assume that $\lim_{t\to 0+} f(t) = f(0)$, $\lim_{t\to 0+} D^1_r f(t) = D^1_r f(0)$, $\lim_{t\to\infty} f(t) = f(\infty)$, and $\lim_{t\to\infty} D^1_r f(t) = D^1_r f(\infty)$ regardless of whether they are finite or infinite. Let $W^2_{p,\nu}(r,I_0)$ and $W^2_{p,\nu}(r,I_\infty)$ be the contraction sets of functions from $W^2_{p,\nu}(r,I)$ on $(0,1]$ and $[1,\infty)$, respectively.

Theorem C⁺ *Let* $1 < p < \infty$.

(i) *If* $\nu^{-1} \notin L_{p'}(I_\infty)$, $r^{-1} \notin L_1(I_\infty)$ *or* $r^{-1} \in L_1(I_\infty)$, $\int_1^\infty \nu^{-p'}(t)(\int_t^\infty r^{-1}(x)\,dx)^{p'}\,dx = \infty$, *then* $\mathring{W}^2_{p,\nu}(r,I_\infty) = W^2_{p,\nu}(r,I_\infty)$. *(In this case, for all* $f \in W^2_{p,\nu}(r,I)$, *there do not exist* $f(\infty)$ *and* $D^1_r f(\infty)$.*)*

(ii) *If* $\nu^{-1} \notin L_{p'}(I_\infty)$, $r^{-1} \in L_1(I_\infty)$ *and* $\int_1^\infty \nu^{-p'}(t)(\int_t^\infty r^{-1}(x)\,dx)^{p'}\,dt < \infty$, *then*

$$\mathring{W}^2_{p,\nu}(r,I_\infty) = \{f \in W^2_{p,\nu}(r,I_\infty) : f(\infty) = 0\}.$$

(In this case, for all $f \in W^2_{p,\nu}(r,I)$, *there exists only* $f(\infty)$.*)*

(iii) *If* $\nu^{-1} \in L_{p'}(I_\infty)$, $r^{-1} \notin L_1(I_\infty)$, *and* $\int_1^\infty \nu^{-p'}(t)(\int_1^t r^{-1}(x)\,dx)^{p'}\,dt = \infty$, *then*

$$\mathring{W}^2_{p,\nu}(r,I_\infty) = \{f \in W^2_{p,\nu}(r,I_\infty) : D^1_r f(\infty) = 0\}.$$

(In this case, for all $f \in W^2_{p,\nu}(r,I)$, *there exists only* $D^1_r f(\infty)$.*)*

(iv) *If* $\nu^{-1} \in L_{p'}(I_\infty)$ *and* $r^{-1} \in L_1(I_\infty)$, *then*

$$\mathring{W}^2_{p,\nu}(r,I_\infty) = \{f \in W^2_{p,\nu}(r,I_\infty) : f(\infty) = D^1_r f(\infty) = 0\}.$$

(In this case, for all $f \in W^2_{p,\nu}(r,I)$, *there exist both* $f(\infty)$ *and* $D^1_r f(\infty)$.*)*

Theorem C⁻ *Let* $1 < p < \infty$.

(i) *If* $\nu^{-1} \notin L_{p'}(I_0)$, $r^{-1} \notin L_1(I_0)$ *or* $r^{-1} \in L_1(I_0)$, $\int_0^1 \nu^{-p'}(t)(\int_0^t r^{-1}(x)\,dx)^{p'}\,dx = \infty$, *then* $\mathring{W}^2_{p,\nu}(r,I_0) = W^2_{p,\nu}(r,I_0)$. *(In this case, for all* $f \in W^2_{p,\nu}(r,I)$, *there do not exist* $f(0)$ *and* $D^1_r f(0)$.*)*

(ii) *If* $\nu^{-1} \notin L_{p'}(I_0)$, $r^{-1} \in L_1(I_0)$, *and* $\int_0^1 \nu^{-p'}(t)(\int_0^t r^{-1}(x)\,dx)^{p'}\,dt < \infty$, *then*

$$\mathring{W}^2_{p,\nu}(r,I_0) = \{f \in W^2_{p,\nu}(r,I_0) : f(0) = 0\}.$$

(In this case, for all $f \in W^2_{p,\nu}(r,I)$, *there exists only* $f(0)$.*)*

(iii) *If* $\nu^{-1} \in L_{p'}(I_0)$, $r^{-1} \notin L_1(I_0)$, *and* $\int_0^1 \nu^{-p'}(t)(\int_t^1 r^{-1}(x)\,dx)^{p'}\,dt = \infty$, *then*

$$\mathring{W}^2_{p,\nu}(r,I_0) = \{f \in W^2_{p,\nu}(r,I_0) : D^1_r f(0) = 0\}.$$

(In this case, for all $f \in W^2_{p,\nu}(r,I)$, *there exists only* $D^1_r f(0)$.*)*

(iv) *If* $\nu^{-1} \in L_{p'}(I_0)$ *and* $r^{-1} \in L_1(I_0)$, *then*

$$\mathring{W}^2_{p,\nu}(r,I_0) = \{f \in W^2_{p,\nu}(r,I_0) : f(0) = D^1_r f(0) = 0\}.$$

(In this case, for all $f \in W^2_{p,\nu}(r,I)$, *there exist both* $f(0)$ *and* $D^1_r f(0)$.*)*

If neither $f(\infty)$ nor $D_r^1 f(\infty)$ exist, the functions v^{-1} and r^{-1} are said to be strongly singular at infinity (see item (i), Theorem C⁺). If only $f(\infty)$ or $D_r^1 f(\infty)$ exists, the functions v^{-1} and r^{-1} are said to be weakly singular at infinity (see item (ii) or (iii), Theorem C⁺). If both $f(\infty)$ and $D_r^1 f(\infty)$ exist, the functions v^{-1} and r^{-1} are said to be regular at infinity (see item (iv), Theorem C⁺). Similarly, Theorem C⁻ defines the concepts of strong singularity, weak singularity, and regularity of the functions v^{-1} and r^{-1} at zero. In the paper [2], the oscillatory properties of a higher-order equation in form (6) are studied in the case of regularity at one endpoint and strong singularity at the other endpoint of the interval. This case can be called "standard", because the order of the corresponding inequality coincides with the number of boundary conditions on f. When the functions v^{-1} and r^{-1} are regular at one endpoint and weakly singular at the other endpoint of the interval, inequality (5) is of the second order, but f satisfies three boundary conditions, so we have the so-called "overdetermined" case. This creates additional difficulties in estimating the least constant in (5), because we need to obtain conditions on weights in (5) that directly determine the oscillation of equation (6). For inequality (5), there are four overdetermined cases, two of which $\{(\mathrm{iv})^+, (\mathrm{iii})^-\}$ and $\{(\mathrm{iii})^+, (\mathrm{iv})^-\}$ are studied in the paper [14], where $(\mathrm{i})^+$ denotes item (i) of Theorem C⁺, $(\mathrm{i})^-$ denotes item (i) of Theorem C⁻, and so on. Here we study the remaining overdetermined cases $\{(\mathrm{iv})^+, (\mathrm{ii})^-\}$ and $\{(\mathrm{ii})^+, (\mathrm{iv})^-\}$.

There are only two works [2] and [14] that investigate the oscillatory properties of equation (6), which, when expanded, has the form

$$r\big(vry''\big)'' + r\big(vr'y'\big)'' + r'\big(vry''\big)' + r'\big(vr'y'\big)' - uy = 0, \quad t \in I.$$

All other works investigate equations in form (8), i.e., when $r = 1$. In a series of works (see, e.g., [7, 8, 26, 27]), the oscillation of equations in form (8) is studied by the known methods of the qualitative theory of differential equations, but there one or both of coefficients in (8) are power functions. This is due to the fact that these methods are effective for studying the oscillatory properties of second-order differential equations but they are difficult to extend to fourth- and higher-order differential equations. Based on suitable inequalities, the method presented in this paper is known as the "variational method", and it differs from the previous methods. This variational method allows to remove restrictions on the coefficients. Using the variational method, the paper [23] studies oscillation of equations in form (8) in the standard case, while the papers [15] and [16] discuss the problem in the overdetermined cases. The spectral properties of the operator of the fourth and higher orders generated by the differential expression in form (9) are considered, e.g., in the works [3, 4], [9, Chaps. 29 and 34], [11, 19, 25].

Note that in the theory of differential equations there are situations similar to the one described above, when techniques that are useful for second-order equations turn out to be useless when solving problems of order higher than two. For example, the recent paper [5] contributes to the development of a possible unitary method for some higher-order elliptic problems, since the known methods for these problems are typically for second-order equations not for higher order.

This paper is organized as follows. Section 2 contains all the auxiliary statements necessary to prove the main results. In Sect. 3, we establish the validity of inequality (5) in the cases $\{(\mathrm{iv})^+, (\mathrm{ii})^-\}$ and $\{(\mathrm{ii})^+, (\mathrm{iv})^-\}$. In Sect. 4, on the basis of the obtained results, we get strong oscillation and nonoscillation conditions for equation (6). In Sect. 5, we consider the spectral properties of the operator L generated by differential expression (7).

2 Preliminaries

Let $-\infty \le a < b \le \infty$. Suppose that $\chi_{(a,b)}(\cdot)$ stands for the characteristic function of the interval (a,b).

Theorem A *Let* $1 < p \le q < \infty$.

(i) *Inequality*

$$\left(\int_a^b \left|u(t)\int_a^t f(s)\,ds\right|^q dt\right)^{\frac{1}{q}} \le C\left(\int_a^b |v(t)f(t)|^p\,dt\right)^{\frac{1}{p}} \tag{10}$$

holds if and only if

$$A^+ = \sup_{z\in(a,b)}\left(\int_z^b u^q(t)\,dt\right)^{\frac{1}{q}}\left(\int_a^z v^{-p'}(s)\,ds\right)^{\frac{1}{p'}} < \infty;$$

in addition,

$$A^+ \le C \le p^{\frac{1}{q}}(p')^{\frac{1}{p'}}A^+,$$

where C *is the least constant in* (10).

(ii) *Inequality*

$$\left(\int_a^b \left|u(t)\int_t^b f(s)\,ds\right|^q dt\right)^{\frac{1}{q}} \le C\left(\int_a^b |v(t)f(t)|^p\,dt\right)^{\frac{1}{p}} \tag{11}$$

holds if and only if

$$A^- = \sup_{z\in(a,b)}\left(\int_a^z u^q(t)\,dt\right)^{\frac{1}{q}}\left(\int_z^b v^{-p'}(s)\,ds\right)^{\frac{1}{p'}} < \infty.$$

In addition,

$$A^- \le C \le p^{\frac{1}{q}}(p')^{\frac{1}{p'}}A^-,$$

where C *is the least constant in* (11). *Let*

$$B_1^-(a,b) = \sup_{z\in(a,b)}\left(\int_z^b u^q(t)\left(\int_z^t r^{-1}(x)\,dx\right)^q dt\right)^{\frac{1}{q}}\left(\int_a^z v^{-p'}(s)\,ds\right)^{\frac{1}{p'}},$$

$$B_2^-(a,b) = \sup_{z\in(a,b)}\left(\int_z^b u^q(t)\,dt\right)^{\frac{1}{q}}\left(\int_a^z\left(\int_s^z r^{-1}(x)\,dx\right)^{p'} v^{-p'}(s)\,ds\right)^{\frac{1}{p'}},$$

$$B^-(a,b) = \max\left\{B_1^-(a,b), B_2^-(a,b)\right\}.$$

The following two statements follow from the results of the work [12].

Theorem B⁻ *Let* $1 < p \le q < \infty$. *The inequality*

$$\left(\int_a^b \left|u(t)\int_a^t\left(\int_s^t r^{-1}(x)\,dx\right)f(s)\,ds\right|^q dt\right)^{\frac{1}{q}} \le C\left(\int_a^b |v(t)f(t)|^p\,dt\right)^{\frac{1}{p}} \tag{12}$$

holds if and only if $B^-(a,b)<\infty$. *In addition,* $B^-(a,b)\le C\le 8p^{\frac{1}{q}}(p')^{\frac{1}{p'}}B^-(a,b)$, *where* C *is the least constant in* (12).

Let

$$B_1^+(a,b)=\sup_{z\in(a,b)}\left(\int_a^z\left(\int_t^z r^{-1}(x)\,dx\right)^q u^q(t)\,dt\right)^{\frac{1}{q}}\left(\int_z^b v^{-p'}(s)\,ds\right)^{\frac{1}{p'}},$$

$$B_2^+(a,b)=\sup_{z\in(a,b)}\left(\int_a^z u^q(t)\,dt\right)^{\frac{1}{q}}\left(\int_z^b\left(\int_z^s r^{-1}(x)\,dx\right)^{p'} v^{-p'}(s)\,ds\right)^{\frac{1}{p'}},$$

$$B^+(a,b)=\max\left\{B_1^+(a,b),B_2^+(a,b)\right\}.$$

Theorem B⁺ *Let* $1<p\le q<\infty$. *Inequality*

$$\left(\int_a^b\left|u(t)\int_t^b\left(\int_t^s r^{-1}(x)\,dx\right)f(s)\,ds\right|^q dt\right)^{\frac{1}{q}}\le C\left(\int_a^b\left|v(t)f(t)\right|^p dt\right)^{\frac{1}{p}} \tag{13}$$

holds if and only if $B^+(a,b)<\infty$. *In addition,* $B^+(a,b)\le C\le 8p^{\frac{1}{q}}(p)^{\frac{1}{p'}}B^+(a,b)$, *where* C *is the least constant in* (13).

3 Inequality (5)

Assume that all the conditions of Theorems C⁻ and C⁺ are given for $p=2$.

As pointed out in the Introduction, depending on the degree of singularity of the functions v^{-1} and r^{-1} at infinity and at zero, for the function $f\in W^2_{p,v}(r,I)$, Theorems C⁻ and C⁺ give all possible cases of the existence of the following limits: $\lim_{t\to\infty}f(t)=f(\infty)$, $\lim_{t\to\infty}D_r^1f(t)=D_r^1f(\infty)$, $\lim_{t\to0^+}f(t)=f(0)$, and $\lim_{t\to0^+}D_r^1f(t)=D_r^1f(0)$. Theorem 6 of [14] lists all pairs of items of Theorems C⁻ and C⁺, under which inequality (5) does not hold, they are the following: $[(\mathrm{i})^+,(\mathrm{i})^-]$, $[(\mathrm{ii})^+,(\mathrm{i})^-]$, $[(\mathrm{iii})^+,(\mathrm{i})^-]$, $[(\mathrm{i})^+,(\mathrm{ii})^-]$, $[(\mathrm{i})^+,(\mathrm{iii})^-]$, and $[(\mathrm{iii})^+,(\mathrm{iii})^-]$. For the pairs $[(\mathrm{i})^+,(\mathrm{iv})^-]$, $[(\mathrm{iv})^+,(\mathrm{i})^-]$, $[(\mathrm{iii})^+,(\mathrm{ii})^-]$, $[(\mathrm{ii})^+,(\mathrm{iii})^-]$, and $[(\mathrm{ii})^+,(\mathrm{ii})^-]$, the function $f\in\mathring{W}^2_{p,v}(r,I)$ has two boundary conditions at the endpoints of the interval I, i.e., the standard case; therefore, second-order inequality (5) is equivalent to the well-known integral inequalities (see [22]). For the pairs $[(\mathrm{iv})^+,(\mathrm{ii})^-]$, $[(\mathrm{ii})^+,(\mathrm{iv})^-]$, $[(\mathrm{iv})^+,(\mathrm{iii})^-]$, and $[(\mathrm{iii})^+,(\mathrm{iv})^-]$, the function $f\in\mathring{W}^2_{p,v}(r,I)$ has three boundary conditions at the endpoints of the interval I, i.e., the overdetermined cases. As mentioned above, in [14], inequality (5) was studied under the conditions of the pairs $[(\mathrm{iv})^+,(\mathrm{iii})^-]$ and $[(\mathrm{iii})^+,(\mathrm{iv})^-]$. Here we study the remaining cases $[(\mathrm{iv})^+,(\mathrm{ii})^-]$ and $[(\mathrm{ii})^+,(\mathrm{iv})^-]$.

Assume that $\rho(t)=\int_0^t r^{-1}(x)\,dx$, $t\in I$. Let $B_1^+(\tau,\infty)\equiv B_1^+(\tau)$, $B_2^+(\tau,\infty)\equiv B_2^+(\tau)$,

$$B_3^+(\tau)=\frac{1}{\rho(\tau)}\left(\int_0^\tau\rho^q(t)u^q(t)\,dt\right)^{\frac{1}{q}}\left(\int_\tau^\infty\left(\int_\tau^s r^{-1}(x)\,dx\right)^{p'} v^{-p'}(s)\,ds\right)^{\frac{1}{p'}},$$

$$F_1^-(\tau)=\sup_{0<z<\tau}\frac{1}{\rho(\tau)}\left(\int_0^z\rho^q(t)u^q(t)\,dt\right)^{\frac{1}{q}}\left(\int_z^\tau\left(\int_s^\tau r^{-1}(x)\,dx\right)^{p'} v^{-p'}(s)\,ds\right)^{\frac{1}{p'}},$$

$$F_2^-(\tau)=\sup_{0<z<\tau}\frac{1}{\rho(\tau)}\left(\int_z^\tau\left(\int_t^\tau r^{-1}(x)\,dx\right)^q u^q(t)\,dt\right)^{\frac{1}{q}}\left(\int_0^z\rho^{p'}(s)v^{-p'}(s)\,ds\right)^{\frac{1}{p'}},$$

$$B^+(\tau)=\max\left\{B_1^+(\tau),B_2^+(\tau)\right\},\qquad \mathcal{B}^+(\tau)=\max\left\{B^+(\tau),B_3^+(\tau)\right\},$$

$$F^-(\tau) = \max\{F_1^-(\tau), F_2^-(\tau)\}, \qquad \mathcal{B}^+F^- = \inf_{\tau\in I}\max\{\mathcal{B}^+(\tau), F^-(\tau)\},$$

where $\tau \in I$.

Let $\bar{\nu}(s) = \rho(s)\nu(s)$, $s \in I$, and $\bar{\nu}^{-1} \in L_{p'}(I)$. Then, for any $\tau \in I$, there exists k_τ such that

$$\int_0^\tau \bar{\nu}^{-p'}(t)\,dt = k_\tau \int_\tau^\infty \bar{\nu}^{-p'}(t)\,dt; \tag{14}$$

in addition, k_τ increases in τ, $\lim_{\tau\to 0^+} k_\tau = 0$, and $\lim_{\tau\to\infty} k_\tau = \infty$. Moreover, there exists $\tau_1 \in I$ such that $k_{\tau_1} = 1$ and $\int_0^{\tau_1} \bar{\nu}^{-p'}(t)\,dt = \int_{\tau_1}^\infty \bar{\nu}^{-p'}(t)\,dt$.

The following theorem uses the ideas of the proof of Theorem 2.2 in [22].

Theorem 1 *Let $1 < p \le q < \infty$ and $\{(\mathrm{iv})^+, (\mathrm{ii})^-\}$ hold, i.e., $r^{-1} \in L_1(I)$, $\nu^{-1} \in L_{p'}(I_\infty)$, $\nu^{-1} \notin L_{p'}(I_0)$, and $\int_0^1 \nu^{-p'}(t)(\int_0^t r^{-1}(x)\,dx)^{p'}\,dt < \infty$. Then, for the least constant C in (5), the estimates*

$$4^{-\frac{1}{p}}\mathcal{B}^+F^- \le C \le 11p^{\frac{1}{q}}(p')^{\frac{1}{p'}}\mathcal{B}F^- \tag{15}$$

and

$$\sup_{\tau\in I}\left(1 + k_\tau^{p-1}\right)^{-\frac{1}{p}} F^-(\tau) \le C \le 11p^{\frac{1}{q}}(p')^{\frac{1}{p'}} F^-(\tau^-) \tag{16}$$

hold, where

$$\tau^- = \inf\{\tau > 0 : \mathcal{B}^+(\tau) \le F^-(\tau)\}. \tag{17}$$

Proof *Sufficiency*. By the conditions of Theorem 1, on the basis of item (iv) of Theorem C^+ and item (ii) of Theorem C^-, we have

$$\mathring{W}_{p,\nu}^2(r,I) = \{f \in W_{p,\nu}^2(r,I) : f(0) = f(\infty) = D_r^1 f(\infty) = 0\}. \tag{18}$$

Let $\tau \in I$. We assume that $f(t) = \int_0^t r^{-1}(x)D_r^1 f(x)\,dx$ for $0 < t < \tau$, $f(t) = -\int_t^\infty r^{-1}(x)D_r^1 f(x)\,dx$ for $t > \tau$, and $D_r^1 f(x) = -\int_x^\infty D_r^2 f(s)\,ds$ for $x \in I$. Then, for $f \in \mathring{W}_{p,\nu}^2(r,I)$, we have

$$\begin{aligned} f(t) &= -\int_0^t r^{-1}(x)\int_x^\infty D_r^2 f(s)\,ds\,dx \\ &= -\int_0^t \rho(s)D_r^2 f(s)\,ds - \rho(t)\int_t^\tau D_r^2 f(s)\,ds - \rho(t)\int_\tau^\infty D_r^2 f(s)\,ds \end{aligned} \tag{19}$$

for $0 < t < \tau$ and

$$f(t) = \int_t^\infty r^{-1}(x)\int_x^\infty D_r^2 f(s)\,ds\,dx = \int_t^\infty \left(\int_t^s r^{-1}(x)\,dx\right) D_r^2 f(s)\,ds \tag{20}$$

for $t > \tau$.

For $f \in \mathring{W}^2_{p,v}(r,I)$ from (18) we get

$$0 = \int_0^\infty r^{-1}(x)D^1_r f(x)\,dx = -\int_0^\infty r^{-1}(x)\int_x^\infty D^2_r f(s)\,ds\,dx$$
$$= -\int_0^\infty D^2_r f(s)\int_0^s r^{-1}(x)\,dx\,ds = -\int_0^\infty \rho(s)D^2_r f(s)\,ds.$$

Assume that $\rho(s)D^2_r f(s) = g(s)$, $s \in I$. Hence, $\int_0^\infty g(s)\,ds = 0$. To the right-hand side of (19) we add $\frac{\rho(t)}{\rho(\tau)}\int_0^\infty g(s)\,ds = 0$. Then, for $f \in \mathring{W}^2_{p,v}(r,I)$, from (19) and (20) we obtain

$$\begin{aligned} f(t) = \chi_{(0,\tau)}(t)\biggl[&\frac{\rho(t)}{\rho(\tau)}\int_\tau^\infty\biggl(\int_\tau^s r^{-1}(x)\,dx\biggr)\frac{g(s)}{\rho(s)}\,ds - \frac{\rho(\tau)-\rho(t)}{\rho(\tau)}\int_0^t g(s)\,ds \\ &- \frac{\rho(t)}{\rho(\tau)}\int_t^\tau\biggl(\int_s^\tau r^{-1}(x)\,dx\biggr)\frac{g(s)}{\rho(s)}\,ds\biggr] \\ &+ \chi_{(0,\infty)}(t)\int_t^\infty\biggl(\int_t^s r^{-1}(x)\,dx\biggr)\frac{g(s)}{\rho(s)}\,ds. \end{aligned} \tag{21}$$

In view of (21), the belonging $f \in \mathring{W}^2_{p,v}(r,I)$ is equivalent to the belonging $g \in L_{p,\bar{v}}(I)$ and $\int_0^\infty g(s)\,ds = 0$. Assume that $\tilde{L}_{p,\bar{v}}(I) = \{g \in L_{p,\bar{v}}(I) : \int_0^\infty g(s)\,ds = 0\}$.

Replacing (21) into the left-hand side of (5), we get it in the form

$$\begin{aligned} \biggl(\int_0^\tau\biggl|&u(t)\frac{\rho(t)}{\rho(\tau)}\int_\tau^\infty\biggl(\int_\tau^s r^{-1}(x)\,dx\biggr)\frac{g(s)}{\rho(s)}\,ds \\ &- u(t)\frac{\rho(\tau)-\rho(t)}{\rho(\tau)}\int_0^t g(s)\,ds - u(t)\frac{\rho(t)}{\rho(\tau)}\int_t^\tau\biggl(\int_s^\tau r^{-1}(x)\,dx\biggr)\frac{g(s)}{\rho(s)}\,ds\biggr|^q\,dt \\ &+ \int_\tau^\infty\biggl|u(t)\int_t^\infty\biggl(\int_t^s r^{-1}(x)\,dx\biggr)\frac{g(s)}{\rho(s)}\,ds\biggr|^q\,dt\biggr)^{\frac{1}{q}} \le C\biggl(\int_0^\infty\bigl|\bar{v}(s)g(s)\bigr|^p\,ds\biggr)^{\frac{1}{p}}. \end{aligned} \tag{22}$$

Now, in the left-hand side of (22), applying Minkowski's inequality for sums, then Hölder's inequality, Theorem A, and Theorem B$^+$, we get

$$\begin{aligned} &\biggl(\int_0^\infty\bigl|u(t)f(t)\bigr|^q\,dt\biggr)^{\frac{1}{q}} \\ &\le p^{\frac{1}{q}}\bigl(p'\bigr)^{\frac{1}{p'}}\bigl(F_1^-(\tau)+F_2^-(\tau)\bigr)\biggl(\int_0^\tau\bigl|v(s)D^2_r f(s)\bigr|^p\,ds\biggr)^{\frac{1}{p}} \\ &\quad + \bigl(8p^{\frac{1}{q}}\bigl(p'\bigr)^{\frac{1}{p'}}B^+(\tau)+B_3^+(\tau)\bigr)\biggl(\int_\tau^\infty\bigl|v(s)D^2_r f(s)\bigr|^p\,ds\biggr)^{\frac{1}{p}} \\ &\le\bigl[\bigl(2p^{\frac{1}{q}}\bigl(p'\bigr)^{\frac{1}{p'}}F^-(\tau)\bigr)^{p'}+\bigl(9p^{\frac{1}{q}}\bigl(p'\bigr)^{\frac{1}{p'}}\mathcal{B}^+(\tau)\bigr)^{p'}\bigr]^{\frac{1}{p'}}\biggl(\int_0^\infty\bigl|v(s)D^2_r f(s)\bigr|^p\,ds\biggr)^{\frac{1}{p}} \\ &\le 11p^{\frac{1}{q}}\bigl(p'\bigr)^{\frac{1}{p'}}\max\bigl\{\mathcal{B}^+(\tau),F^-(\tau)\bigr\}\biggl(\int_0^\infty\bigl|v(s)D^2_r f(s)\bigr|^p\,ds\biggr)^{\frac{1}{p}}. \end{aligned} \tag{23}$$

Since the left-hand side of (23) is independent of $\tau \in I$, (23) implies the right estimate in (15).

The condition $r^{-1} \in L_1(I)$ of Theorem 1 gives that $\lim_{t\to\infty} \rho(t) = \rho(\infty) < \infty$. Therefore, the function $\rho(\tau)B^+(\tau)$ does not increase and $\rho(\tau)F^-(\tau)$ does not decrease. Let $\lim_{\tau\to\infty} \rho(\tau)F^-(\tau) = D$. If $D = \infty$, then for sufficiently large $\tau \in I$, we obviously have that $\rho(\tau)F^-(\tau) > \rho(\tau)B^+(\tau)$. Moreover, $\rho(\tau)F^-(\tau) > \rho(\tau)B_3^+(\tau)$ follows from the relation

$$\begin{aligned}\rho(\tau)B_3^+(\tau) &= \left(\int_0^N \rho^q(t)u^q(t)\,dt\right)^{\frac{1}{q}} \left(\int_\tau^\infty \left(\int_\tau^s r^{-1}(x)\,dx\right)^{p'} v^{-p'}(s)\,ds\right)^{\frac{1}{p'}} \\ &\quad + \left(\int_N^\tau \rho^q(t)u^q(t)\,dt\right)^{\frac{1}{q}} \left(\int_\tau^\infty \left(\int_\tau^s r^{-1}(x)\,dx\right)^{p'} v^{-p'}(s)\,ds\right)^{\frac{1}{p'}} \end{aligned} \tag{24}$$

for large $\tau > N > 0$, where the second term is less than $B_2^+(N)$. If $D < \infty$, then from the estimates $\lim_{\tau\to\infty} \rho(\tau)F_2^-(\tau) \le D$ we find that $(\int_t^\infty r^{-1}(x)\,dx)u(t) \in L_q(I_\infty)$. Hence, $\lim_{\tau\to\infty} \rho(\tau)B_2^+(\tau) = 0$. Moreover, $\lim_{\tau\to\infty} \rho(\tau)B_i^+(\tau) = 0$, $i = 1,3$, follows from (24) and from

$$\rho(\tau)B_1^+(\tau) \le \left(\int_\tau^\infty \left(\int_t^\infty r^{-1}(x)\,dx\right)^q u^q(t)\,dt\right)^{\frac{1}{q}} \left(\int_\tau^\infty v^{-p'}(s)\,ds\right)^{\frac{1}{p'}}.$$

Thus, in some neighborhood of infinity, we have that $\rho(\tau)F^-(\tau) > \rho(\tau)\mathcal{B}^+(\tau)$. Therefore, in relation (17) there exists $\tau^- > 0$ and $F^-(\tau^-) \ge \mathcal{B}^+(\tau^-)$. Consequently,

$$F^-(\tau^-) \ge \mathcal{B}^+F^- = \inf_{\tau\in I} \frac{1}{\rho(\tau)} \max\{\rho(\tau)\mathcal{B}^+(\tau), \rho(\tau)F^-(\tau)\},$$

and the right estimate in (16) holds.

Necessity. By the conditions of Theorem 1, we have that $\bar{v}^{-1} \in L_{p'}(I)$. Therefore, (14) holds. For $\tau \in I$, we consider two sets $\pounds_1 = \{g \in L_{p,\bar{v}}(0,\tau) : g \le 0\}$ and $\pounds_2 = \{g \in L_{p,\bar{v}}(\tau,\infty) : g \ge 0\}$. Further, repeating the steps of the necessary part of Theorem 7 in [14], for each $g_1 \in \pounds_1$ and $g_2 \in \pounds_2$, we construct functions $g_2 \in \pounds_2$ and $g_1 \in \pounds_1$ such that $g(t) = g_1(t)$ for $0 < t \le \tau$ and $g(t) = g_2(t)$ for $t > \tau$ belongs to the set $\tilde{L}_{p,\bar{v}}(I)$. In addition, assuming in both cases $g(t) = g_1(t)$ for $0 < t \le \tau$ and $g(t) = g_2(t)$ for $t > \tau$, we have

$$\begin{aligned}\int_0^\infty \left|\bar{v}(t)g(t)\right|^p dt &= \left(1 + k_\tau^{p-1}\right) \int_0^\tau \left|\bar{v}(t)g_1(t)\right|^p dt \\ &= \left(1 + k_\tau^{1-p}\right) \int_\tau^\infty \left|\bar{v}(t)g_2(t)\right|^p dt < \infty. \end{aligned} \tag{25}$$

Replacing the constructed function $g \in \tilde{L}_{p,v}(I)$ in (22), we obtain

$$\begin{aligned}&\left(\int_0^\tau \left| u(t)\frac{\rho(t)}{\rho(\tau)} \int_\tau^\infty \left(\int_\tau^s r^{-1}(x)\,dx\right) \frac{g_2(s)}{\rho(s)}\,ds \right.\right. \\ &\quad \left. + u(t)\frac{\rho(\tau)-\rho(t)}{\rho(\tau)} \int_0^t \left|g_1(s)\right| ds + u(t)\frac{\rho(t)}{\rho(\tau)} \int_t^\tau \left(\int_s^\tau r^{-1}(x)\,dx\right) \frac{|g_1(s)|}{\rho(s)}\,ds \right|^q dt \\ &\quad \left. + \int_\tau^\infty \left| u(t) \int_t^\infty \left(\int_t^s r^{-1}(x)\,dx\right) \frac{g_2(s)}{\rho(s)}\,ds \right|^q dt \right)^{\frac{1}{q}} \le C\left(\int_0^\infty \left|\bar{v}(s)g(s)\right|^p ds\right)^{\frac{1}{p}}. \end{aligned} \tag{26}$$

In the left-hand side of (26), all terms are nonnegative.

Let the function $g \in \tilde{L}_{p,\bar{v}}(I)$ be generated by the function $g_2 \in \pounds_2$. Then, from (25) and (26), we have

$$\frac{1}{\rho(\tau)}\left(\int_0^\tau \rho^q(t)u^q(t)\,dt\right)^{\frac{1}{q}}\int_\tau^\infty\left(\int_\tau^s r^{-1}(x)\,dx\right)\frac{g_2(s)}{\rho(s)}\,ds$$
$$\leq C\left(1+k_\tau^{1-p}\right)^{\frac{1}{p}}\left(\int_\tau^\infty \left|\bar{v}(t)g_2(t)\right|^p\,dt\right)^{\frac{1}{p}},$$
$$\left(\int_\tau^\infty\left(u(t)\int_t^\infty\left(\int_t^s r^{-1}(x)\,dx\right)\frac{g_2(s)}{\rho(s)}\,ds\right)^q dt\right)^{\frac{1}{q}}$$
$$\leq C\left(1+k_\tau^{1-p}\right)^{\frac{1}{p}}\left(\int_\tau^\infty \left|\bar{v}(t)g_2(t)\right|^p\,dt\right)^{\frac{1}{p}}.$$

Due to the arbitrariness of $g_2 \in \pounds_2$, on the basis of the reverse Hölder inequality and Theorem B^+, we obtain

$$B^+(\tau) \leq C\left(1+k_\tau^{1-p}\right)^{\frac{1}{p}}, \qquad B_3^+(\tau) \leq C\left(1+k_\tau^{1-p}\right)^{\frac{1}{p}},$$

i.e.,

$$\mathcal{B}^+(\tau) \leq C\left(1+k_\tau^{1-p}\right)^{\frac{1}{p}}. \tag{27}$$

Similarly, for the function $g \in \tilde{L}_{p,\bar{v}}(I)$ generated by the function $g_1 \in \pounds_1$, from (25) and (26) we have

$$\frac{1}{\rho(\tau)}\left(\int_0^\tau\left(u(t)\left(\int_t^\tau r^{-1}(x)\,dx\right)\int_0^t \left|g_1(s)\right|ds\right)^q dt\right)^{\frac{1}{q}}$$
$$\leq C\left(1+k_\tau^{p-1}\right)^{\frac{1}{p}}\left(\int_0^\tau \left|\bar{v}(t)g_1(t)\right|^p\,dt\right)^{\frac{1}{p}},$$
$$\frac{1}{\rho(\tau)}\left(\int_0^\tau\left(u(t)\rho(t)\int_t^\tau\left(\int_s^\tau r^{-1}(x)\,dx\right)\frac{|g_1(s)|}{\rho(s)}\,ds\right)^q dt\right)^{\frac{1}{q}}$$
$$\leq C\left(1+k_\tau^{p-1}\right)^{\frac{1}{p}}\left(\int_0^\tau \left|\bar{v}(t)g_1(t)\right|^p\,dt\right)^{\frac{1}{p}}.$$

The latter, due to the arbitrariness of $g_1 \in \pounds_1$, by Theorem A, gives that

$$F^-(\tau) \leq C\left(1+k_\tau^{p-1}\right)^{\frac{1}{p}}. \tag{28}$$

From (27) and (28) we find that

$$\mathcal{B}^+F^- = \inf_{\tau\in I}\max\left\{\mathcal{B}^+(\tau),F^-(\tau)\right\} \leq C\inf_{\tau\in I}\left[\max\left\{\left(1+k_\tau^{p-1}\right)\left(1+k_\tau^{1-p}\right)\right\}\right]^{\frac{1}{p}} \leq 4^{\frac{1}{p}}C,$$

which yields the left estimate in (15). From (28) we get the left estimate in (16). The proof of Theorem 1 is complete. □

Assume that $\bar{\rho}(t) = \int_t^\infty r^{-1}(x)\,dx$, $t \in I$. Let $B_1^-(0,\tau) \equiv B_1^-(\tau)$, $B_2^-(0,\tau) \equiv B_2^-(\tau)$,

$$B_3^-(\tau) = \frac{1}{\bar{\rho}(\tau)} \left(\int_\tau^\infty \bar{\rho}^q(t) u^q(t)\,dt \right)^{\frac{1}{q}} \left(\int_0^\tau \left(\int_s^\tau r^{-1}(x)\,dx \right)^{p'} v^{-p'}(s)\,ds \right)^{\frac{1}{p'}},$$

$$F_1^+(\tau) = \sup_{z>\tau} \frac{1}{\bar{\rho}(\tau)} \left(\int_\tau^z \left(\int_\tau^z r^{-1}(x)\,dx \right)^q u^q(t)\,dt \right)^{\frac{1}{q}} \left(\int_z^\infty \bar{\rho}^{p'}(s) v^{-p'}(s)\,ds \right)^{\frac{1}{p'}},$$

$$F_2^+(\tau) = \sup_{z>\tau} \frac{1}{\bar{\rho}(\tau)} \left(\int_z^\infty \bar{\rho}^q(t) u^q(t)\,dt \right)^{\frac{1}{q}} \left(\int_\tau^z \left(\int_\tau^s r^{-1}(x)\,dx \right)^{p'} v^{-p'}(s)\,ds \right)^{\frac{1}{p'}},$$

$$B^-(\tau) = \max\{B_1^-(\tau), B_2^-(\tau)\}, \qquad \mathcal{B}^-(\tau) = \max\{B^-(\tau), B_3^-(\tau)\},$$

$$F^+(\tau) = \max\{F_1^+(\tau), F_2^+(\tau)\}, \qquad \mathcal{B}^-F^+ = \inf_{\tau\in I} \max\{\mathcal{B}^-(\tau), F^+(\tau)\},$$

where $\tau \in I$.

Theorem 2 *Let $1 < p \le q < \infty$ and $\{(\mathrm{ii})^+, (\mathrm{iv})^-\}$ hold, i.e., $r^{-1} \in L_1(I)$, $v^{-1} \notin L_{p'}(I_\infty)$, $v^{-1} \in L_{p'}(I_0)$, and $\int_1^\infty v^{-p'}(t)(\int_t^\infty r^{-1}(x)\,dx)^{p'}\,dt < \infty$. Then, for the least constant C in (5), the estimates*

$$4^{-\frac{1}{p}} \mathcal{B}^-F^+ \le C \le 11 p^{\frac{1}{q}} (p')^{\frac{1}{p'}} \mathcal{B}^-F^+$$

and

$$\sup_{\tau\in I} \left(1 + k_\tau^{1-p}\right)^{-\frac{1}{p}} F^+(\tau) \le C \le 11 p^{\frac{1}{q}} (p')^{\frac{1}{p'}} F^+(\tau^+)$$

hold, where

$$\tau^+ = \inf\{\tau > 0 : \mathcal{B}^-(\tau) \le F^+(\tau)\}.$$

Proof The conditions of Theorem 2 are symmetric to the conditions of Theorem 1. Therefore, the statement of Theorem 2 follows from the statement of Theorem 1. In inequality (5), under the conditions of Theorem 2, we change the variables $t = \frac{1}{x}$, then we obtain inequality (5) and the conditions of Theorem 1, where $u(x)$ is replaced by $\tilde{u}(x) = u(\frac{1}{x})x^{-\frac{2}{q}}$, $v(x)$ is replaced by $\tilde{v}(x) = v(\frac{1}{x})x^{\frac{2}{p'}}$, and $r(x)$ is replaced by $\tilde{r}(x) = r(\frac{1}{x})x^2$. Thus, the conditions of Theorem 2 turn to the conditions of Theorem 1 for the functions $\tilde{v}$ and $\tilde{r}$. Now, we use Theorem 1 and get the results with respect to the functions $\tilde{u}$, $\tilde{v}$, and $\tilde{r}$. Then, changing the variable to t, we prove Theorem 2. □

4 Oscillation properties of equation (6)

Let us remind that in [14] inequality (5) was studied under the conditions of the pairs $[(\mathrm{iv})^+, (\mathrm{iii})^-]$ and $[(\mathrm{iii})^+, (\mathrm{iv})^-]$, while in this paper it is studied under the conditions of the other pairs $[(\mathrm{iv})^+, (\mathrm{ii})^-]$ and $[(\mathrm{ii})^+, (\mathrm{iv})^-]$. Therefore, the characterizations of inequality (5) obtained here differ from those obtained in [14]. However, the method of applying them to study equation (6) and operator L is the same. In order to give a more complete presentation, in this Sect. 4 and next Sect. 5, we repeat some of the steps from the corresponding Sections of [14].

Two points t_1 and t_2, such that $t_1 \neq t_2$ of the interval I, are called conjugate with respect to equation (6), if there exists a solution y of equation (6) such that $y(t_1) = y_2(t_2) = 0$ and $D_r^1 y(t_1) = D_r^1 y(t_2) = 0$. Equation (6) is called oscillatory at infinity (at zero), if for any $T \in I$, there exist conjugate points with respect to equation (6) to the right (left) of T. Otherwise, equation (6) is called nonoscillatory at infinity (at zero).

On the basis of Theorems 28 and 31 of [9] (see also, e.g., Lemma 2.1 of [26]), we have the following two variational lemmas.

Lemma 1 *Equation* (6) *is nonoscillatory at infinity if and only if there exists* $T > 0$ *and the inequality*

$$\int_T^\infty v(t)\left|D_r^2 f(t)\right|^2 - u(t)\left|f(t)\right|^2 dt \geq 0, \quad f \in C_0^\infty(T,\infty), \tag{29}$$

holds.

Lemma 2 *Equation* (6) *is nonoscillatory at zero if and only if there exists* $T > 0$ *and the inequality*

$$\int_0^T v(t)\left|D_r^2 f(t)\right|^2 - u(t)\left|f(t)\right|^2 dt \geq 0, \quad f \in C_0^\infty(0,T), \tag{30}$$

holds.

Let $T \geq 0$. We consider the inequality

$$\int_T^\infty u(t)\left|f(t)\right|^2 dt \leq C_T \int_T^\infty v(t)\left|D_r^2 f(t)\right|^2 dt, \quad f \in \mathring{W}_{p,v}^2\bigl(r,(T,\infty)\bigr). \tag{31}$$

In the work [14], on the basis of Lemmas 1 and 2, there was proved the following lemma.

Lemma 3 *Let* C_T *be the least constant in* (31).

(i) *Equation* (6) *is nonoscillatory at infinity if and only if there exists a constant* $T > 0$ *such that* $0 < C_T \leq 1$ *holds.*
(ii) *Equation* (6) *is oscillatory at infinity if and only if* $C_T > 1$ *for all* $T \geq 0$.

For the inequality

$$\int_0^T u(t)\left|f(t)\right|^2 dt \leq C_T \int_0^T v(t)\left|D_r^2 f(t)\right|^2 dt, \quad f \in \mathring{W}_{p,v}^2\bigl(r,(0,T)\bigr), \tag{32}$$

we have one more lemma from [14].

Lemma 4 *Let* C_T *be the least constant in* (32).

(i) *Equation* (6) *is nonoscillatory at zero if and only if there exists a constant* $T > 0$ *such that* $0 < C_T \leq 1$ *holds.*
(ii) *Equation* (6) *is oscillatory at zero if and only if* $C_T > 1$ *for all* $T \geq 0$.

The conditions for oscillation and nonoscillation of equation (6) at zero and at infinity directly follow from Lemmas 1, 2 and Theorems 1, 2. Let us present those oscillatory properties, which we need to establish the spectral properties of the operator L, namely the conditions for strong oscillation and nonoscillation of equation (6) with the parameter $\lambda > 0$:

$$D_r^2\big(v(t)D_r^2 y(t)\big) - \lambda u(t)y(t) = 0, \quad t \in I. \tag{33}$$

Equation (33) is called strong oscillatory (nonoscillatory) at zero and at infinity if it is oscillatory (nonoscillatory) for all $\lambda > 0$ at zero and at infinity, respectively.

From inequalities (31) and (32) for equation (33), we respectively have

$$\lambda \int_T^\infty u(t)\big|f(t)\big|^2\,dt \le \lambda C_T \int_T^\infty v(t)\big|D_r^2 f(t)\big|^2\,dt, \quad f \in \mathring{W}^2_{p,v}\big(r,(T,\infty)\big), \tag{34}$$

$$\lambda \int_0^T u(t)\big|f(t)\big|^2\,dt \le \lambda C_T \int_0^T v(t)\big|D_r^2 f(t)\big|^2\,dt, \quad f \in \mathring{W}^2_{p,v}\big(r,(0,T)\big). \tag{35}$$

The following lemma was also proved in [14].

Lemma 5 *Let C_T be the least constant in* (34) ((35)).

(i) *Equation* (33) *is strong nonoscillatory at infinity* (*at zero*) *if and only if* $\lim_{T\to\infty} C_T = 0$ ($\lim_{T\to 0^+} C_T = 0$).

(ii) *Equation* (33) *is strong oscillatory at infinity* (*at zero*) *if and only if* $C_T = \infty$ ($C_T = \infty$) *for any* $T > 0$.

Now, on the basis of Lemma 5, as in [14], we establish criteria of strong oscillation and nonoscillation of equation (33) at zero and at infinity. Let $p = q = 2$ in the expressions $B^-(\tau)$, $F^+(\tau)$, $B^+(\tau)$, and $F^-(\tau)$. We replace u^2 by u and v^{-2} by v^{-1}. In addition, we assume that $\bar{B}^-(T,\tau) = (B^-(\tau))^2$, $\bar{F}^+(\tau) = (F^+(\tau))^2$, $\bar{B}^+(\tau) = (B^+(\tau))^2$, and $\bar{F}^-(\tau) = (F^-(\tau))^2$. Moreover, we take

$$\bar{B}_3^-(\tau) = \frac{1}{\bar{\rho}^2(\tau)} \int_\tau^\infty \bar{\rho}^2(t)u(t)\,dt \int_0^\tau \left(\int_s^\tau r^{-1}(x)\,dx\right)^2 v^{-1}(s)\,ds$$

and

$$\bar{B}_3^+(\tau) = \frac{1}{\rho^2(\tau)} \int_0^\tau \rho^2(t)u(t)\,dt \int_\tau^\infty \left(\int_\tau^s r^{-1}(x)\,dx\right)^2 v^{-1}(s)\,ds$$

instead of $B_3^-(\tau)$ and $B_3^+(\tau)$, respectively.

Theorem 3 *Let $r^{-1} \in L_1(I)$, $v^{-1} \in L_1(I_\infty)$, $v^{-1} \notin L_1(I_0)$, and*

$$\int_0^1 v^{-1}(t)\left(\int_0^t r^{-1}(x)\,dx\right)^2 dt < \infty.$$

(i) *Equation* (33) *is strong nonoscillatory at zero if and only if*

$$\lim_{\tau\to 0^+} \sup_{0<z<\tau} \frac{1}{\rho^2(\tau)} \int_0^z \rho^2(t)u(t)\,dt \int_z^\tau \left(\int_s^\tau r^{-1}(x)\,dx\right)^2 v^{-1}(s)\,ds = 0, \tag{36}$$

$$\lim_{\tau\to 0^+} \sup_{0<z<\tau} \frac{1}{\rho^2(\tau)} \int_z^\tau \left(\int_t^\tau r^{-1}(x)\,dx\right)^2 u(t)\,dt \int_0^z \rho^2(s)v^{-1}(s)\,ds = 0. \tag{37}$$

(ii) *Equation* (33) *is strong oscillatory at zero if and only if*

$$\lim_{\tau\to 0^+}\sup_{0<z<\tau}\frac{1}{\rho^2(\tau)}\int_0^z \rho^2(t)u(t)\,dt\int_z^\tau\left(\int_s^\tau r^{-1}(x)\,dx\right)^2 v^{-1}(s)\,ds=\infty \tag{38}$$

or

$$\lim_{\tau\to 0^+}\sup_{0<z<\tau}\frac{1}{\rho^2(\tau)}\int_z^\tau\left(\int_t^\tau r^{-1}(x)\,dx\right)^2 u(t)\,dt\int_0^z \rho^2(s)v^{-1}(s)\,ds=\infty. \tag{39}$$

Proof (i) Suppose that equation (33) is strong nonoscillatory at zero. Then, by Lemma 5, we have that $\lim_{T\to 0^+} C_T = 0$ for the least constant C_T in inequality (35). From the left estimate in (16) for inequality (35) we have

$$\sup_{0<\tau<T}(1+k_\tau)^{-1}\bar{F}^-(\tau)\le C_T, \tag{40}$$

where $T>0$.

By the definition of k_τ on the interval $(0,T)$, we find that $\lim_{\tau\to 0^+} k_\tau = 0$. Therefore, $0 = \lim_{T\to 0^+} C_T \ge \lim_{\tau\to 0^+}(1+k_\tau)^{-1}\bar{F}^-(\tau) = \lim_{\tau\to 0^+}\bar{F}^-(\tau)$. The latter gives that

$$\lim_{\tau\to 0^+}\bar{F}_i^-(\tau)=0,\quad i=1,2. \tag{41}$$

Consequently, (36) and (37) hold.

Inversely, let (36) and (37) hold, i.e., (41) hold. From the right estimate in (16), for inequality (35), we have

$$C_T\le 22^2\bar{F}^-\left(\tau^-\right), \tag{42}$$

where $\tau^-\in(0,T)$. Since $\lim_{T\to 0^+}\tau^-=0$, then from (41) we get

$$\lim_{T\to 0^+} C_T\le 22^2\lim_{T\to 0^+}\bar{F}^-\left(\tau^-\right)=22^2\lim_{\tau\to 0^+}\bar{F}^-(\tau)=0.$$

Thus, $\lim_{T\to 0^+} C_T = 0$ and, by Lemma 5, equation (33) is strong nonoscillatory at zero.

(ii) Let equation (33) be strong oscillatory at zero. Then, by Lemma 5, for any $T>0$, we have that $C_T=\infty$, where C_T is the least constant in (35). Therefore, from (42) we get $\bar{F}^-(\tau^-)=\infty$ for any $T>0$. Since $\tau^-\in(0,T)$, then $\lim_{\tau\to 0^+}\bar{F}^-(\tau)=\infty$. This means that either (38) or (39) or both hold.

Inversely, let (38) hold, i.e., $\lim_{\tau\to 0^+}\bar{F}_1^-(\tau)=\infty$. Since $\bar{F}^-(\tau)$ does not decrease, then $\bar{F}^-(\tau)=\infty$ for any $\tau\in(0,T)$ and for any $T>0$. Then from (40) we get that $C_T=\infty$ for any $T>0$. Hence, by Lemma 5, equation (33) is strong oscillatory at zero. Similarly, if (39) holds, then from (40) we get that equation (33) is strong oscillatory at zero. The proof of Theorem 3 is complete. □

Now, we assume that the function u together with the function v is positive and sufficiently times continuously differentiable on the interval I. In the theory of oscillatory properties of differential equations, there is the reciprocity principle (see [6]), from which it follows that equation (33) and its reciprocal equation

$$D_r^2\big(u^{-1}(t)D_r^2y(t)\big) - \lambda v^{-1}(t)y(t) = 0, \quad t \in I, \tag{43}$$

are simultaneously oscillatory or nonoscillatory.

On the basis of this reciprocity principle, from Theorem 3 we have the following statement.

Theorem 4 *Let $r^{-1} \in L_1(I)$, $u \notin L_1(I_0)$, $u \in L_1(I_\infty)$, and*

$$\int_0^1 u(t)\left(\int_0^t r^{-1}(x)\,dx\right)^2 dt < \infty.$$

(i) *Equation (33) is strong nonoscillatory at zero if and only if (36) and (37) hold.*
(ii) *Equation (33) is strong oscillatory at zero if and only if (38) or (39) holds.*

Similarly, on the basis of inequality (34), we have the following theorem.

Theorem 5 *Let $r^{-1} \in L_1(I)$, $v^{-1} \notin L_1(I_\infty)$, $v^{-1} \in L_1(I_0)$, and*

$$\int_1^\infty v^{-1}(t)\left(\int_t^\infty r^{-1}(x)\,dx\right)^2 dt < \infty.$$

(i) *Equation (33) is strong nonoscillatory at infinity if and only if*

$$\lim_{\tau\to\infty}\sup_{z>\tau}\frac{1}{\bar\rho^2(\tau)}\int_\tau^z\left(\int_\tau^z r^{-1}(x)\,dx\right)^2 u(t)\,dt\int_z^\infty \bar\rho^2(s)v^{-1}(s)\,ds = 0, \tag{44}$$

$$\lim_{\tau\to\infty}\sup_{z>\tau}\frac{1}{\bar\rho^2(\tau)}\int_z^\infty \bar\rho^2(t)u(t)\,dt\int_\tau^z\left(\int_\tau^s r^{-1}(x)\,dx\right)^2 v^{-1}(s)\,ds = 0. \tag{45}$$

(ii) *Equation (33) is strong oscillatory at infinity if and only if*

$$\lim_{\tau\to\infty}\sup_{z>\tau}\frac{1}{\bar\rho^2(\tau)}\int_\tau^z\left(\int_\tau^z r^{-1}(x)\,dx\right)^2 u(t)\,dt\int_z^\infty \bar\rho^2(s)v^{-1}(s)\,ds = \infty \tag{46}$$

or

$$\lim_{\tau\to\infty}\sup_{z>\tau}\frac{1}{\bar\rho^2(\tau)}\int_z^\infty \bar\rho^2(t)u(t)\,dt\int_\tau^z\left(\int_\tau^s r^{-1}(x)\,dx\right)^2 v^{-1}(s)\,ds = \infty. \tag{47}$$

Using the reciprocity principle, the following statement follows from the application of Theorem 5 to equation (43).

Theorem 6 *Let $r^{-1} \in L_1(I)$, $u \in L_1(I_0)$, $u \notin L_1(I_\infty)$, and*

$$\int_1^\infty u(t)\left(\int_t^\infty r^{-1}(x)\,dx\right)^2 dt < \infty.$$

(i) *Equation (33) is strong nonoscillatory at infinity if and only if (44) and (45) hold.*
(ii) *Equation (33) is strong oscillatory at infinity if and only if (46) or (47) holds.*

5 Spectral characteristics of differential operator *L*

Let the minimal differential operator $L_{\min}$ be generated by the differential expression

$$ly(t) = \frac{1}{u(t)} D_r^2\big(v(t)D_r^2 y\big)$$

in the space $L_{2,u} \equiv L_2(u;I)$ with inner product $(f,g)_{2,u} = \int_0^\infty f(t)g(t)u(t)\,dt$, i.e., $L_{\min}y = ly$ is an operator with the domain $D(L_{\min}) = C_0^\infty(I)$.

It is known that all self-adjoint extensions of the minimal differential operator L have the same spectrum (see [9]).

One of the most important problems in the theory of singular differential operators is to find conditions under which any self-adjoint extension L of the operator $L_{\min}$ has a spectrum, which is discrete and bounded below. These properties (boundedness below and discreteness) guarantee that the singular operator behaves like a regular one (see [10]).

The relationship between the oscillatory properties of equation (33) and spectral properties of the operator L is explained in the following statement.

Lemma 6 ([9]) *The operator L is bounded below and has a discrete spectrum if and only if equation* (33) *is strong nonoscillatory.*

On the basis of Lemma 6, from Theorems 3–6 as corollaries, we obtain the following propositions.

Proposition 1 *Let the conditions of Theorem 3 or 4 hold. Then the operator L is bounded below and has a discrete spectrum if and only if* (36) *and* (37) *hold.*

Proposition 2 *Let the conditions of Theorem 5 or 6 hold. Then the operator L is bounded below and has a discrete spectrum if and only if* (44) *and* (45) *hold.*

The operator $L_{\min}$ is nonnegative. Therefore, it has Friedrich's extension L_F. By Propositions 1 and 2, the operator L_F has a discrete spectrum if and only if (36) and (37) hold under the conditions of Proposition 1, and (44) and (45) hold under the conditions of Proposition 2.

Since for $p = q = 2$ inequality (5) can be rewritten as $(f,f)_2 C^{-2} \le (L_F f,f)_{2,u}$, then from Theorems 1 and 2 we have the following propositions.

Proposition 3 *Let the conditions of Theorem 3 hold. Then the operator L_F is positive-definite if and only if $\bar{\mathcal{B}}^+\bar{F}^- = \inf_{\tau\in I}\max\{\bar{\mathcal{B}}^+(\tau),\bar{F}^-(\tau)\} < \infty$. Moreover, there exist constants $\alpha,\beta : 0 < \alpha < \beta$, and the estimate $\alpha\bar{\mathcal{B}}^+\bar{F}^- \le \lambda_1^{-1} \le \beta\bar{\mathcal{B}}^+\bar{F}^-$ holds for the smallest eigenvalue λ_1 of the operator L_F.*

Proposition 4 *Let the conditions of Theorem 5 hold. Then the operator L_F is positive-definite if and only if $\bar{\mathcal{B}}^-\bar{F}^+ = \inf_{\tau\in I}\max\{\bar{\mathcal{B}}^-(\tau),\bar{F}^+(\tau)\} < \infty$. Moreover, there exist constants $\alpha,\beta : 0 < \alpha < \beta$, and the estimate $\alpha\bar{\mathcal{B}}^-\bar{F}^+ \le \lambda_1^{-1} \le \beta\bar{\mathcal{B}}^-\bar{F}^+$ holds for the smallest eigenvalue λ_1 of the operator L_F.*

Let us note that for the operator L_F, from Theorem 1 under the conditions of Theorem 3, we have the following spectral problem:

$$\begin{cases} D_r^2(v(t)D_r^2 y(t)) = \lambda u(t)y(t), \\ D_r' y(0) = D_r' y(\infty) = y(\infty) = 0, \end{cases}$$

while from Theorem 2 under the conditions of Theorem 5, we have the following spectral problem:

$$\begin{cases} D_r^2(v(t)D_r^2 y(t)) = \lambda u(t)y(t), \\ y(0) = D_r' y(0) = D_r' y(\infty) = 0. \end{cases}$$

Since according to Rellih's lemma (see [20, p. 183]) the operator L_F^{-1} has a discrete spectrum bounded below in $L_{2,u}$ if and only if the space with the norm $(L_F f, f)_{2,u}^{\frac{1}{2}}$ is compactly embedded into the space $L_{2,u}$, then from Propositions 1 and 2 we have one more statement.

Proposition 5 *Let the conditions of Theorem 3 (Theorem 5) hold. Then the embedding $\mathring{W}_{2,v}^2(r,I) \hookrightarrow L_{2,u}$ is compact and the operator L_F^{-1} is uniformly continuous on $L_{2,u}$ if and only if (36) and (37) ((44) and (45)) hold.*

The following statement is from the work [2].

Lemma 7 *Let $H = H(I)$ be a certain Hilbert function space and $C[0,\infty) \cap H$ be dense in it. For any point $x_0 \in I$, we introduce the operator $E_{x_0} f = f(x_0)$ defined on $C[0,\infty) \cap H$, which acts in the space of complex numbers. Let us assume that E_{x_0} is a closure operator. Then the norm of this operator is equal to the value $(\sum_{n=1}^{\infty} |\varphi_n(x_0)|^2)^{\frac{1}{2}}$ (finite or infinite), where $\{\varphi_n(\cdot)\}_{n=1}^{\infty}$ is any complete orthonormal system of continuous functions in H.*

Let

$$D^+(t) = \int_0^t \rho^2(z)v^{-1}(z)\,dz + \rho^2(t)\int_t^\infty v^{-1}(z)\,dz, \quad t \in I.$$

Lemma 8 *Let the conditions of Theorem 3 hold. Then*

$$\left(D^+(t)\right)^{\frac{1}{2}} = \sup_{f \in \mathring{W}_{2,v}^2(r)} \frac{|f(t)|}{\|D_r^2 f\|_{2,v}}, \quad t \in I, \tag{48}$$

where

$$\left\|D_r^2 f\right\|_{2,v} = \left(\int_0^\infty v(t)\left|D_r^2 f(t)\right|^2 dt\right)^{\frac{1}{2}}.$$

Proof Let $\tau \in I$. Assume that

$$D^+(t,\tau) = \Bigg[\chi_{(0,\tau)}(t)\int_\tau^\infty \left(\int_\tau^s r^{-1}(x)\,dx\right)^2 v^{-1}(s)\,ds$$
$$+ \chi_{(0,\tau)}(t)\int_t^\tau \left(\int_s^\tau r^{-1}(x)\,dx\right)^2 v^{-1}(s)\,ds$$

$$+ \chi_{(0,\tau)}(t)\left(\int_t^\tau r^{-1}(x)\,dx\right)^2\int_0^t v^{-1}(s)\,ds$$
$$+ \chi_{(\tau,\infty)}(t)\int_t^\infty\left(\int_t^s r^{-1}(x)\,dx\right)^2 v^{-1}(s)\,ds\Bigg]^{\frac{1}{2}}.$$

From (19) and (20), for the function $f \in \mathring{W}^2_{2,v}(r)$, we have

$$f(t) = \chi_{(0,\tau)}(t)\left[-\int_0^t \rho(s)D_r^2 f(s)\,ds - \rho(t)\int_t^\tau D_r^2 f(s)\,ds - \rho(t)\int_\tau^\infty D_r^2 f(s)\,ds\right]$$
$$+ \chi_{(\tau,\infty)}(t)\int_t^\infty\left(\int_t^s r^{-1}(x)\,dx\right)D_r^2 f(s)\,ds. \tag{49}$$

In (49), taking the modulus in both parts and applying Hölder's inequality in the integrals of each term, we obtain

$$\left|f(t)\right| \le \chi_{(0,\tau)}(t)\left[\int_0^t \rho^2(z)v^{-1}(z)\,dz + \rho^2(t)\int_t^\tau v^{-1}(z)\,dz\right]^{\frac{1}{2}}\left(\int_0^\tau v(s)\left|D_r^2 f(s)\right|^2 ds\right)^{\frac{1}{2}}$$
$$+ \left[\chi_{(0,\tau)}(t)\rho(t)\left(\int_\tau^\infty v^{-1}(z)\,dz\right)^{\frac{1}{2}}\right.$$
$$\left.+ \chi_{(\tau,\infty)}(t)\left(\int_t^\infty\left(\int_t^z r^{-1}(x)\,dx\right)^2 v^{-1}(z)\,dz\right)^{\frac{1}{2}}\right]\left(\int_\tau^\infty v(s)\left|D_r^2 f(s)\right|^2 ds\right)^{\frac{1}{2}}.$$

Therefore,

$$\left|f(t)\right| \le \left[\chi_{(0,\tau)}(t)\int_0^t \rho^2(z)v^{-1}(z)\,dz + \chi_{(0,\tau)}\rho^2(t)\int_t^\infty v^{-1}(z)\,dz\right.$$
$$\left.+ \chi_{(\tau,\infty)}(t)\int_\tau^\infty\left(\int_t^z r^{-1}(x)\,dx\right)^2 v^{-1}(z)\,dz\right]^{\frac{1}{2}}\left\|D_r^2 f\right\|_{2,v},$$

i.e.,

$$\left|f(t)\right| \le \inf_{\tau\in I} D^+(t,\tau)\left\|D_r^2 f\right\|_{2,v}.$$

Since $\left(D^+(t)\right)^{\frac{1}{2}} = \lim_{\tau\to\infty} D^+(t,\tau) \ge \inf_{\tau\in I} D^+(t,\tau)$, then from the last relation we have the estimate from above of the right-hand side of (48):

$$\left(D^+(t)\right)^{\frac{1}{2}} \ge \sup_{f\in\mathring{W}^2_{2,v}(r)} \frac{|f(t)|}{\|D_r^2 f\|_{2,v}}. \tag{50}$$

Now, we need to establish a similar estimate from below. We fix $t \in I$ in (49) and select a function $D_r^2 f$ depending on t as follows:

$$\left(D_r^2 f\right)_t(s) = \begin{cases} -\chi_{(0,t)}(s)\rho(s)v^{-1}(s) & \text{if } 0 < t < \tau, \\ -\chi_{(t,\tau)}(s)\rho(t)v^{-1}(s) & \text{if } 0 < t < \tau, \\ -\chi_{(\tau,\infty)}(s)\rho(t)v^{-1}(s) & \text{if } 0 < t < \tau, \\ \chi_{(t,\infty)}(s)(\int_t^s r^{-1}(x)\,dx)v^{-1}(s) & \text{if } t > \tau. \end{cases}$$

Replacing this function in (49), we get the value of the function $f(D_r^2 f)_t(z)$ at the point $z = t$:

$$f_t(t) = \chi_{(0,\tau)}(t)\int_0^t \rho^2(s)v^{-1}(s)\,ds + \chi_{(0,\tau)}(t)\rho^2(t)\int_t^\infty v^{-1}(s)\,ds \\ + \chi_{(\tau,\infty)}(t)\int_\tau^\infty \left(\int_t^s r^{-1}(x)\,dx\right)^2 v^{-1}(s)\,ds = \left(D^+(t,\tau)\right)^2. \tag{51}$$

Let us calculate the norm $L_{2,u}$ of the function $(D_r^2 f)_t$:

$$\left(\int_0^\infty v(s)\left|\left(D_r^2 f\right)_t(s)\right|^2 ds\right)^{\frac{1}{2}} \\ = \left(\int_0^\tau v(s)\left|\left(D_r^2 f\right)_t(s)\right|^2 ds + \int_\tau^\infty v(s)\left|\left(D_r^2 f\right)_t(s)\right|^2 ds\right)^{\frac{1}{2}} \\ = \left[\chi_{(0,\tau)}(t)\int_0^t \rho^2(s)v^{-1}(s)\,ds + \chi_{(0,\tau)}(t)\rho^2(t)\int_t^\infty v^{-1}(s)\,ds \right. \\ \left. + \chi_{(\tau,\infty)}(t)\int_\tau^\infty \left(\int_t^s r^{-1}(x)\,dx\right)^2 v^{-1}(s)\,ds\right]^{\frac{1}{2}} = D^+(t,\tau). \tag{52}$$

From (51) and (52), we get

$$\sup_{f\in \mathring{W}^2_{2,v}(r)} \frac{|f(t)|}{\|D_r^2 f\|_{2,v}} \geq \frac{|f_t(t)|}{\|(D_r^2 f)_t\|_{2,v}} = D^+(t,\tau)$$

for any $\tau \in I$. Since the left-hand side of this equality does not depend on $\tau \in I$, passing to the limit at its right-hand side, we obtain the lower bound

$$\sup_{f\in \mathring{W}^2_{2,v}(r)} \frac{|f(t)|}{\|D_r^2 f\|_{2,v}} \geq \left(D^+(t)\right)^{\frac{1}{2}},$$

which, together with (50), gives (48). The proof of Lemma 8 is complete. □

Let the operator L_F^{-1} be uniformly continuous on $L_{2,u}$. Let $\{\lambda_k\}_{k=1}^\infty$ be eigenvalues and $\{\varphi_k\}_{k=1}^\infty$ be a corresponding complete orthonormal system of eigenfunctions of the operator L_F^{-1}.

Theorem 7 *Let the conditions of Theorem 3 hold. Let (36) and (37) hold.*

(i) *Then*

$$D^+(t) = \sum_{k=1}^\infty \frac{|\varphi_k(t)|^2}{\lambda_k}, \quad t \in I. \tag{53}$$

(ii) *The operator L_F^{-1} is nuclear if and only if $\int_0^\infty u(t)\,D^+(t)\,dt < \infty$, and for the nuclear norm $\|L_F^{-1}\|_{\sigma_1}$ of the operator L_F^{-1}, the relation*

$$\int_0^\infty u(t)\,D^+(t)\,dt = \left\|L_F^{-1}\right\|_{\sigma_1} = \sum_{k=1}^\infty \frac{1}{\lambda_k} \tag{54}$$

holds.

Proof By the condition of Theorem 7, we have that the operator L_F^{-1} is uniformly continuous on $L_{2,u}$ (see Proposition 5). In Lemma 7, we take $\mathring{W}^2_{2,v}(r,I)$ with the norm $\|D_r^2 f\|_{2,v}$ as the space $H(I)$. Since the system of functions $\{\lambda_k^{-\frac{1}{2}}\varphi_k\}_{k=1}^{\infty}$ is a complete orthonormal system in the space $\mathring{W}^2_{2,v}(r,I)$, then by Lemma 7 we have

$$\|E_t\|^2 = \left(\sup_{f\in\mathring{W}^2_{2,v}(r)} \frac{|f(t)|}{\|D_r^2 f\|_{2,v}}\right)^2 = \sum_{k=1}^{\infty}\frac{|\varphi_k(t)|^2}{\lambda_k},$$

where $E_t f = f(t)$. The latter and (48) give (53). Multiplying both sides of (53) by u and integrating them from zero to infinity, we get (54). The proof of Theorem 7 is complete. □

Let

$$D^-(t) = \int_t^{\infty} \bar\rho^2(z)v^{-1}(z)\,dz + \bar\rho^2(t)\int_0^t v^{-1}(z)\,dz, \quad t\in I.$$

Similarly, we have the following statement.

Theorem 8 *Let the conditions of Theorem 4 hold. Let (44) and (45) hold.*

(i) *Then*

$$D^-(t) = \sum_{k=1}^{\infty}\frac{|\varphi_k(t)|^2}{\lambda_k}, \quad t\in I.$$

(ii) *The operator L_F^{-1} is nuclear if and only if $\int_0^{\infty} u(t)D^-(t)\,dt < \infty$, and for the nuclear norm $\|L_F^{-1}\|_{\sigma_1}$ of the operator L_F^{-1}, the relation*

$$\int_0^{\infty} u(t)D^-(t)\,dt = \left\|L_F^{-1}\right\|_{\sigma_1} = \sum_{k=1}^{\infty}\frac{1}{\lambda_k}$$

holds.

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