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Temporal decay for the highest-order derivatives of solutions of the compressible Hall-magnetohydrodynamic equations

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Abstract

Recently, Gao and Yao established the global existence and temporal decay rates of solutions for a system of compressible Hall-magnetohydrodynamic fluids (Gao and Yao in Discrete Contin. Dyn. Syst. 36: 3077–3106, 2016). However, because of the difficulty of derivative loss in the nonlinear terms, Gao and Yao could not provide the temporal decay for the highest-order derivatives of classical solutions. In this paper, motivated by the decomposition technique of both low and high frequencies of solutions in (Wang and Wen in Sci. China Math. 65: 1199–1228 2022), we further derive the temporal decay for the highest-order derivatives of the strong solutions. Moreover, the decay rate is optimal, since it agrees with the solutions of the linearized system.

Keywords: Compressible Hall-magnetohydrodynamic fluids; Highest-order derivatives; Fourier theory; Optimal time-decay rates

1 Introduction

In this paper, we investigate the temporal decay of the highest-order derivatives of solutions to the Cauchy problem of a compressible Hall-magnetohydrodynamic system, which can be described as follows:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho \mathbf{u}) = 0, \\ \partial_t(\rho \mathbf{u}) + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) - \mu \Delta \mathbf{u} - (\mu + \nu)\nabla \operatorname{div} \mathbf{u} + \nabla P(\rho) = \operatorname{curl} \mathbf{B} \times \mathbf{B}, \\ \partial_t \mathbf{B} - (\mathbf{B} \cdot \nabla)\mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{B} + \mathbf{B} \operatorname{div} \mathbf{u} + \operatorname{curl}[\frac{(\operatorname{curl}\mathbf{B})\times \mathbf{B}}{\rho}] = \Delta \mathbf{B}, \\ \operatorname{div} \mathbf{B} = 0, \end{cases} \tag{1.1}$$

with initial condition

$$(\rho, \mathbf{u}, \mathbf{B})|_{t=0} = (\rho_0, \mathbf{u}_0, \mathbf{B}_0). \tag{1.2}$$

Lets us explain the notations appearing in system (1.1). The functions $\rho = \rho(t,x)$, $\mathbf{u} = \mathbf{u}(t,x)$, and $\mathbf{B} = \mathbf{B}(t,x)$ represent the density, velocity, and magnetic field of a Hall-magnetohydrodynamic fluid, respectively. The pressure function of the fluid $P(\rho)$ depending on the density ρ is a smooth function. The constants μ and ν represent the viscosity coefficients of the fluid and satisfy the following physical conditions

$$\mu > 0, \quad 2\mu + 3\nu \geq 0.$$

Hall-magnetohydrodynamics fluids have attracted more and more attention from plasma physicists. It is thought to be the key to understanding the magnetic reconnection problem. Acheritogaray et al. [1]provided a derivation of the Hall-magnetohydrodynamics system through a set of scale limits in the Euler–Maxwell system of ions and electrons and stated system (1.1) for the Hall-magnetohydrodynamics fluids. The interested readers may further refer to [3, 9, 12, 36, 39, 47] and the referencestherein for the relevant physical progress. When the Hall effect term $\operatorname{curl}(\rho^{-1}(\operatorname{curl}\mathbf{B}) \times \mathbf{B})$ is neglected in (1.1), the Hall-MHD system reduces to the well-known MHD system. At present the MHD system has been extensively investigated from mathematical, physical and numerical aspects; see [6, 7, 13–27, 31, 33, 41, 46, 53] and the references cited therein.

The well-posedness problem for the Hall-MHD system has also been widely investigated; see [4, 42] and the references cited therein. Since we are interested in the temporal decay for the solutions of the compressible system (1.1), we briefly introduce relevant results. The interested readers can refer to [5, 49] and the references therein for the temporal decay of solutions to the incompressible Hall-MHD system.

Fan et al. proved the local existence of strong solutions with positive initial density and global-in-time classical solutions around the rest state $(1, \mathbf{0}, \mathbf{0})$ with small initial perturbation. They also established the optimal time decay rate for classical solutions [8]:

$$\left\|(\rho - 1, \mathbf{u}, \mathbf{B})(t)\right\|_{L^2(\mathbb{R}^3)} \leq C(1+t)^{-\frac{3}{4}}.$$

They required the initial perturbation to be small in $H^3(\mathbb{R}^3)$-norm and bounded in $L^1(\mathbb{R}^3)$-norm. Later, Gao–Yao [11] deduced temporal decay rates for the higher-order spatial derivatives of classical solutions:

$$\begin{aligned} &\left\|\nabla^k(\rho - 1, \mathbf{u})(t)\right\|_{H^{3-k}(\mathbb{R}^3)} \leq C(1+t)^{-\frac{3+2k}{4}}, \\ &\left\|\nabla^k \mathbf{B}\right\|_{H^{3-k}(\mathbb{R}^3)} \leq C(1+t)^{-\frac{3+2m}{4}}, \end{aligned} \tag{1.3}$$

where $0 \leq k \leq 2$ and $0 \leq m \leq 3$. It is easy to see that these decay rates are the same as for the heat equation. However, because of the difficulty of derivative loss in the nonlinear terms, Gao and Yao could not provide the temporal decay for the highest-order derivatives, i.e., $\nabla^3(\rho - 1, \mathbf{u})(t)$. In this paper, we establish the temporal decay for the highest-order derivatives.

For simplicity, we consider the existence of unique strong solutions with small perturbation and establish the temporal decay for the highest-order derivatives of strong solutions by using the decomposition technique of both low and high frequencies of solutions in [45]. Note that it is easy to verify that (1.3) also holds for $k = 3$ by following the proof of our main result.

In addition, recently the temporal decay of solutions to the full compressible Hall-MHD fluids has been also widely investigated; see [10, 30, 40, 43, 44] for examples. Our main result can be easily extended to the full case. Before presenting our main results, we introduce some notations often used throughout this paper.

The letter $C > 0$ represents a generic constant that varies from line to line, and $C_i > 0$ is a fixed constant for $i \in \mathbb{Z}^+$; $\langle \cdot, \cdot \rangle$ stands for the inner product in $L^2(\mathbb{R}^3)$, and $a \lesssim b$ means that $a \le Cb$ for some constant $C > 0$. For simplicity, we also denote $a \approx b$ if $a \lesssim b$ and $a \gtrsim b$. The symbol ∇^l with an integer $l \ge 1$ represents the spatial derivatives of order l. We set $\partial_i = \partial_{x_i}$ $(i = 1, 2, 3)$ and $\partial_x^\alpha = \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \partial_{x_3}^{\alpha_3}$ with multiindices $\alpha = (\alpha_1, \alpha_2, \alpha_3)$.

For simplicity, $\| \cdot \|_{L^p} := \| \cdot \|_{L^p(\mathbb{R}^3)}$ and $\| \cdot \|_{H^s} := \| \cdot \|_{H^s(\mathbb{R}^3)}$, where $1 \le p \le \infty$ and $s \in \mathbb{R}$. By Λ^s we denote the pseudodifferential operator defined by

$$\Lambda^s f = \mathcal{F}^{-1}\big(|\xi|^s \widehat{f}\big) \quad \text{for } s \in \mathbb{R},$$

where $\widehat{f}$ and $\mathcal{F}^{-1}(f)$ stand for the Fourier and inverse Fourier transforms, respectively.

Let $\xi \in \mathbb{R}^3$, and let $\varphi(\xi)$ be a smooth cut-off function satisfying $0 \le \varphi(\xi) \le R_0$ and

$$\varphi(\xi) = \begin{cases} 1, & |\xi| > R_0, \\ 0, & |\xi| < R_0, \end{cases} \tag{1.4}$$

where R_0 satisfies $R_0 > \sqrt{\frac{8}{\mu}}$. Then we can define the frequency distribution for the function $f \in L^2(\mathbb{R}^3)$ as follows:

$$f^L(x) = \varphi(D_x) f(x), f^H(x) = \big(I - \varphi(D_x)\big) f(x),$$

where $D_x := \frac{1}{\sqrt{-1}}(\partial_{x_1}, \partial_{x_2}, \partial_{x_3})$, and $\varphi(D_x)$ is a pseudodifferential operator of $\varphi(\xi)$. Note that $f(x)$ can be expressed as follows:

$$f(x) = f^L(x) + f^H(x). \tag{1.5}$$

Now we introduce the main result in this paper.

Theorem 1.1 *Suppose that* $(\rho_0 - 1, \mathbf{u}_0, \mathbf{B}_0) \in H^2(\mathbb{R}^3)$ *satisfies*

$$\big\|(\rho_0 - 1, \mathbf{u}_0, \mathbf{B}_0)\big\|_{H^2(\mathbb{R}^3)} \le \varepsilon \tag{1.6}$$

for some sufficiently small constant ε. *Then the Cauchy problem* (1.1)–(1.2) *has a unique global-in-time solution* $(\rho, \mathbf{u}, \mathbf{B})$, *which satisfies*

$$\rho - 1 \in C^0\big([0,\infty); H^2\big(\mathbb{R}^3\big)\big) \cap C^1\big([0,\infty); H^1\big(\mathbb{R}^3\big)\big), \tag{1.7}$$

$$\mathbf{u}, \mathbf{B} \in C^0\big([0,\infty); H^2\big(\mathbb{R}^3\big)\big) \cap C^1\big([0,\infty); L^1\big(\mathbb{R}^3\big)\big). \tag{1.8}$$

Furthermore, if the initial data $(\rho_0 - 1, \mathbf{u}_0, \mathbf{B}_0)$ *is bounded in* $L^1(\mathbb{R}^3)$, *then the strong solution* $(\rho, \mathbf{u}, \mathbf{B})$ *enjoys the decay estimates for all* $t \ge 0$,

$$\big\|\nabla^k(\rho - 1, \mathbf{u}, \mathbf{B})(t)\big\|_{L^2(\mathbb{R}^3)} \le C(1+t)^{-\frac{3}{4}-\frac{k}{2}}, \quad k = 0, 1, 2. \tag{1.9}$$

Now we will introduce our main idea for deriving the optimal time-decay rates in (1.9). The main difficulty focuses on how to obtain the energy estimates that include only the highest-order spatial derivative of the solution $\nabla^2(\rho-1,\mathbf{u})$, which is essentially caused by the "degenerate" dissipative structure of the hyperbolic parabolic system. To get the dissipative estimate for $\nabla^2\rho$, the usual energy method in [11] is constructing the interaction energy functional between u and $\nabla\rho$ by using the pressure term in linearized momentum equations; see (3.27). It implies that both the first and second orders of the spatial derivatives of the velocity and the density should be involved in the Lyapunov functional

$$\mathcal{L}(t)=\|\nabla\rho\|_{H^1}^2+\|\nabla\mathbf{u}(t)\|_{H^1}^2+\int_{\mathbb{R}^3}\nabla\mathbf{u}\cdot\nabla\nabla\rho\,\mathrm{d}x\sim\|\nabla(\rho,\mathbf{u})(t)\|_{H^1}^2.$$

Consequently, the L^2-norm of the highest order and the first-order derivative of the solution have the same time-decay rate.

One of the main goals in this paper is developing a way to capture the optimal time-decay rates for the highest-order derivative of the solution to the Cauchy problem (1.1)–(1.2) if the initial perturbation is bounded in $L^1(\mathbb{R}^3)$. Firstly, by using the standard energy method we establish estimate (3.24) of the energy functional $\mathcal{F}_H(t)$ in (3.25). Secondly, motivated by the decomposition technique of both low and high frequencies of solutions in [45], to get rid of the obstacle from the term $\int_{\mathbb{R}^3}\nabla u\cdot\nabla\rho\,\mathrm{d}x$, we will remove the low-medium-frequency part of the term from $\mathcal{F}_H(t)$ (see (4.12)), which requires a new estimate for the low-medium-frequency term (see Lemma 4.1 for more detail). This method can also be seen, for example, in [50] for the two-phase flow and in [51] for the MHD system.

The rest of this paper is organized as follows. In Sect. 2, for the convenience of calculation, we transform the original system (1.1) into a perturbation form (2.5). In Sect. 3, we establish a priori estimates of solutions and provide the global-in-time existence and uniqueness of the solutions for the Hall-MHD system. Finally, in Sect. 4, as in [45], we obtain the optimal time decay rate for the nonhomogeneous system (2.5) by the decomposition technique of both low and high frequencies of solutions.

2 Reformulation

For the convenience of the subsequent calculations, we rewrite system (1.1) as follows, Since $\operatorname{div}\mathbf{B}=0$, we have

$$\operatorname{curl}\mathbf{B}\times\mathbf{B}=\mathbf{B}\cdot\nabla\mathbf{B}-\mathbf{B}\cdot\nabla^{\mathrm{T}}\mathbf{B}=\operatorname{div}(\mathbf{B}\otimes\mathbf{B})-\frac{1}{2}\nabla|\mathbf{B}|^2,\tag{2.1}$$

$$\operatorname{curl}\operatorname{curl}\mathbf{B}=\nabla\operatorname{div}\mathbf{B}-\Delta\mathbf{B}=-\Delta\mathbf{B},\tag{2.2}$$

where $\nabla^{\mathrm{T}}\mathbf{B}$ denotes the transposed matrix of $\nabla\mathbf{B}$, and

$$\begin{cases}\partial_t\rho+\operatorname{div}(\rho\mathbf{u})=0,\\ \partial_t\mathbf{u}+\mathbf{u}\cdot\nabla\mathbf{u}-\mu\frac{\Delta\mathbf{u}}{\rho}-(\mu+\nu)\frac{\nabla\operatorname{div}\mathbf{u}}{\rho}+\frac{P'(\rho)}{\rho}\nabla\rho=\frac{\mathbf{B}\cdot\nabla\mathbf{B}}{\rho}-\frac{\mathbf{B}\cdot\nabla^{\mathrm{T}}\mathbf{B}}{\rho},\\ \partial_t\mathbf{B}-(\mathbf{B}\cdot\nabla)\mathbf{u}+(\mathbf{u}\cdot\nabla)\mathbf{B}+\mathbf{B}(\operatorname{div}\mathbf{u})-\Delta\mathbf{B}+\operatorname{curl}[\frac{(\operatorname{curl}\mathbf{B})\times\mathbf{B}}{\rho}]=0,\\ \operatorname{div}\mathbf{B}=0.\end{cases}\tag{2.3}$$

Letting

$$\omega=\rho-1,\qquad\mathbf{u}=\mathbf{u},\qquad\mathbf{B}=\mathbf{B},\tag{2.4}$$

we can further rewrite system (1.1)–(1.2) in the perturbation from:

$$\begin{cases} \omega_t + \operatorname{div} \mathbf{u} = \mathcal{H}_1, \\ \mathbf{u}_t - \mu\Delta\mathbf{u} - (\mu+\nu)\nabla \operatorname{div} \mathbf{u} + \nabla\omega = \mathcal{H}_2, \\ \mathbf{B}_t - \Delta\mathbf{B} = \mathcal{H}_3, \\ \operatorname{div} \mathbf{B} = 0, \\ (\omega, \mathbf{u}, \mathbf{B})|_{t=0} = (\omega_0, \mathbf{u}_0, \mathbf{B}_0) = (\rho_0 - 1, \mathbf{u}_0, \mathbf{B}_0), \end{cases} \tag{2.5}$$

where the nonlinear terms $\mathcal{H}_1$–$\mathcal{H}_3$ are defined by

$$\begin{cases} \mathcal{H}_1 := -\operatorname{div}(\omega\mathbf{u}), \\ \mathcal{H}_2 := -\mathbf{u}\cdot\nabla\mathbf{u} - h_1(\omega)\nabla\omega + g_1(\omega)(\mathbf{B}\cdot\nabla\mathbf{B} - \mathbf{B}\cdot\nabla^{\mathrm{T}}\mathbf{B}) \\ \qquad - g_2(\omega)[\mu\Delta\mathbf{u} + (\mu+\nu)\nabla\operatorname{div}\mathbf{u}], \\ \mathcal{H}_3 := (\mathbf{B}\cdot\nabla)\mathbf{u} - (\mathbf{u}\cdot\nabla)\mathbf{B} - (\operatorname{div}\mathbf{u})\mathbf{B} - \operatorname{curl}[g_1(\omega)(\mathbf{B}\cdot\nabla\mathbf{B} - \mathbf{B}\cdot\nabla^{\mathrm{T}}\mathbf{B})], \end{cases} \tag{2.6}$$

with the nonlinear functions

$$g_1(\omega) := \frac{1}{\omega+1}, \tag{2.7}$$

$$g_2(\omega) := \frac{\omega}{\omega+1}, \tag{2.8}$$

$$h_1(\omega) := \frac{P'(\omega+1)}{\omega+1} - 1. \tag{2.9}$$

3 Global-in-time unique solvability for the nonlinear system

In this section, we focus on the global(-in-time) existence and uniqueness of solutions for the Hall-MHD equations. The local strong solutions can be extended to the global solutions by the standard continuity method and global a priori estimates.

3.1 Global existence of solutions

First, we define the work space for system (2.5) by

$$\begin{aligned} \Omega(0,T) := \bigl\{(\omega,\mathbf{u},\mathbf{B}) | \omega \in C^0\bigl((0,T);H^2\bigl(\mathbb{R}^3\bigr)\bigr) \cap C^1\bigl((0,T);H^1\bigl(\mathbb{R}^3\bigr)\bigr), \\ \mathbf{u},\mathbf{B} \in C^0\bigl((0,T);H^2\bigl(\mathbb{R}^3\bigr)\bigr) \cap C^1\bigl((0,T);L^2\bigl(\mathbb{R}^3\bigr)\bigr), \\ \nabla\omega \in L^2\bigl((0,T);H^1\bigl(\mathbb{R}^3\bigr)\bigr), \nabla\mathbf{u}, \nabla\mathbf{B} \in L^2\bigl((0,T);H^2\bigl(\mathbb{R}^3\bigr)\bigr)\bigr\} \end{aligned} \tag{3.1}$$

for $0 \le T \le +\infty$. Then, motivated by [6, 35], we can obtain the following local existence result of a unique solutions to system (2.5).

Proposition 3.1 (Local existence) *Let* $(\omega_0, \mathbf{u}_0, \mathbf{B}_0) \in H^2(\mathbb{R}^3)$ *and suppose that*

$$\inf\{\omega_0 + 1\} > 0.$$

Then there exists a positive constant T_0*, only depending on* $\|(\omega_0, \mathbf{u}_0, \mathbf{B}_0)\|_{H^2(\mathbb{R}^3)}$*. such that the Cauchy problem* (2.5) *has a unique solution* $(\omega, \mathbf{u}, \mathbf{B}) \in \Omega(0, T_0)$*, which satisfies*

$$\inf_{x\in\mathbb{R}^3, 0\le t\le T_0} \{\omega + 1\} > 0$$

and

$$\big\|(\omega,\mathbf{u},\mathbf{B})(t)\big\|_{H^2},\left(\int_0^t \big\|\nabla(\mathbf{u},\mathbf{B})(\tau)\big\|_{H^2}^2\,\mathrm{d}\tau\right)^{\frac{1}{2}} \le \sqrt{C_1}\big\|(\omega_0,\mathbf{u}_0,\mathbf{B}_0)\big\|_{H^2},$$

where C_1 is a positive constant.

Proof The statement can be easily obtained by an iterative method and a fixed point theorem; we refer to [6, 35] for examples. □

Proposition 3.2 (A priori estimates) *Suppose that system* (2.5) *has a solution* $(\omega,\mathbf{u},\mathbf{B}) \in \Omega(0,T)$ *with constant* $T > 0$. *Then there exists a sufficiently small constant* $\varepsilon_0 > 0$ *such that if*

$$\sup_{0\le t\le T}\big\|(\omega,\mathbf{u},\mathbf{B})(t)\big\|_{H^2} \le \varepsilon_0, \tag{3.2}$$

then for all $t \in [0,T]$, *we have that*

$$\big\|(\omega,\mathbf{u},\mathbf{B})(t)\big\|_{H^2} + \int_0^t \left(\big\|\nabla\omega(\tau)\big\|_{H^1}^2 + \big\|\nabla(\mathbf{u},\mathbf{B})(\tau)\big\|_{H^2}^2\right)\mathrm{d}\tau \le C_2\big\|(\omega_0,\mathbf{u}_0,\mathbf{B}_0)(t)\big\|_{H^2}^2, \tag{3.3}$$

where C_2 is a positive constant independent of T.

Proof The proof of the proposition will be given in Sect. 3.2. □

According to Propositions 3.1–3.2, we can derive the following theorem, which implies the global existence of unique solutions.

Theorem 3.1 (Global existence) *Assume that* $(\omega_0,\mathbf{u}_0,\mathbf{B}_0) \in H^2(\mathbb{R}^3)$. *Then there exists a positive constant* ε *such that if, additionally,*

$$\mathfrak{C}_0 < \min\{\varepsilon/\sqrt{C_1}, \varepsilon/\sqrt{C_1C_2}\} < \infty, \tag{3.4}$$

then the initial-value problem (2.5) *admits a unique solution* $(\omega,\mathbf{u},\mathbf{B})$, *which satisfies the following estimate for all* $t > 0$:

$$\big\|(\omega,\mathbf{u},\mathbf{B})(t)\big\|_{H^2} + \int_0^t \left(\big\|\nabla\omega(\tau)\big\|_{H^1}^2 + \big\|\nabla(\mathbf{u},\mathbf{B})(\tau)\big\|_{H^2}^2\right)\mathrm{d}\tau \le C_2\mathfrak{C}_0^2, \tag{3.5}$$

where $\mathfrak{C}_0 := \|(\omega_0,\mathbf{u}_0,\mathbf{B}_0)\|_{H^2}$.

Proof Since Theorem 3.1 can be deduced from Propositions 3.1–3.2 by a classical method, we omit the trivial. We refer the interested readers to [35, 38]. □

Remark 3.1 By the Sobolev imbedding inequality we have

$$\frac{1}{2} \le \rho + 1 \le \frac{3}{2}.$$

Therefore, under the assumptions in Proposition 3.2, we obtain, for $k \geq 1$,

$$\big|(g_1, g_2, h_1)(\omega)\big| \leq C|\omega| \tag{3.6}$$

and

$$\big|\big(g_1^{(k)}, g_2^{(k)}, h_1^{(k)}\big)(\omega)\big| \leq C, \tag{3.7}$$

where C is a positive constant.

3.2 Proof of Proposition 3.2

In this section, we complete the proof of Proposition 3.2. The key step is the energy method used to derive the estimates of the both lower- and highest-order derivatives of the solution $(\omega, \mathbf{u}, \mathbf{B})$ for the transformed Cauchy problem (2.5).

Lemma 3.1 *We have*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}_L(t) + \frac{\gamma_1}{4}\|\nabla\omega\|_{L^2}^2 + \frac{1}{2}\|\nabla\mathbf{u}\|_{H^1}^2 + \frac{1}{2}\|\operatorname{div}\mathbf{u}\|_{H^1}^2 + \frac{1}{2}\|\nabla\mathbf{B}\|_{H^1}^2 \leq 0, \tag{3.8}$$

where

$$\mathcal{F}_L(t) := \frac{1}{2}\big(\|\omega\|_{H^1}^2 + \|\mathbf{u}\|_{H^1}^2 + \|\mathbf{B}\|_{H^1}^2\big) + \gamma_1\int_{\mathbb{R}^3}\nabla\omega\cdot\mathbf{u}\,\mathrm{d}x, \tag{3.9}$$

and $\gamma_1 < \frac{1}{4}$ is a positive constant.

Proof Multiplying $\nabla^k(2.5)_1$, $\nabla^k(2.5)_2$, and $\nabla^k(2.5)_3$ by $\nabla^k\omega$, $\nabla^k\mathbf{u}$, and $\nabla^k\mathbf{B}$, respectively, and integrating over $\mathbb{R}^3$ by parts, we have

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(\big\|\nabla^k\omega\big\|_{L^2}^2 + \big\|\nabla^k\mathbf{u}\big\|_{L^2}^2 + \big\|\nabla^k\mathbf{B}\big\|_{L^2}^2\big)\\
&\quad + (\mu+\nu)\big\|\nabla^k\operatorname{div}\mathbf{u}\big\|_{L^2}^2 + \mu\big\|\nabla^k\nabla\mathbf{u}\big\|_{L^2}^2 + \big\|\nabla^k\nabla\mathbf{B}\big\|_{L^2}^2\\
&\quad = \big\langle\nabla^k\omega, \nabla^k\mathcal{H}_1\big\rangle + \big\langle\nabla^k\mathbf{u}, \nabla^k\mathcal{H}_2\big\rangle + \big\langle\nabla^k\mathbf{B}, \nabla^k\mathcal{H}_3\big\rangle.
\end{aligned}\tag{3.10}$$

By $\langle\nabla(2.5)_1, \mathbf{u}\rangle + \langle(2.5)_2, \nabla\omega\rangle$ we get

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\nabla\omega\cdot\mathbf{u}\,\mathrm{d}x + \|\nabla\omega\|_{L^2}^2\\
&\quad = \|\operatorname{div}\mathbf{u}\|_{L^2}^2 + \mu\int_{\mathbb{R}^3}\nabla\omega\cdot\Delta\mathbf{u}\,\mathrm{d}x + (\mu+\nu)\int_{\mathbb{R}^3}\nabla\omega\cdot\nabla\operatorname{div}\mathbf{u}\,\mathrm{d}x\\
&\qquad + \int_{\mathbb{R}^3}\nabla\mathcal{H}_1\cdot\mathbf{u}\,\mathrm{d}x + \int_{\mathbb{R}^3}\mathcal{H}_2\cdot\nabla\omega\,\mathrm{d}x.
\end{aligned}\tag{3.11}$$

Then using Young's inequality, we get the following inequalities for some fixed constant γ_1:

$$\gamma_1\mu\int_{\mathbb{R}^3}\nabla\omega\cdot\Delta\mathbf{u}\,\mathrm{d}x \leq \frac{\gamma_1}{4}\|\nabla\omega\|_{L^2}^2 + \gamma_1\mu^2\|\Delta\mathbf{u}\|_{L^2}^2 \tag{3.12}$$

and

$$\gamma_1(\mu+\nu)\int_{\mathbb{R}^3}\nabla\omega\cdot\nabla\operatorname{div}\mathbf{u}\,\mathrm{d}x\le\frac{\gamma_1}{4}\|\nabla\omega\|_{L^2}^2+\gamma_1(\mu+\nu)^2\|\nabla\operatorname{div}\mathbf{u}\|_{L^2}^2. \tag{3.13}$$

Summing up the two identities $\gamma_1\times$ (3.11) and $\sum_{0\le k\le 1}$ (3.10) and then using (3.12)–(3.13), we arrive at

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\{\|\omega\|_{H^1}^2+\|\mathbf{u}\|_{H^1}^2+\|\mathbf{B}\|_{H^1}^2+2\gamma_1\int_{\mathbb{R}^3}\nabla\omega\cdot\mathbf{u}\,\mathrm{d}x\right\}\\
&\quad+\frac{\gamma_1}{2}\|\nabla\omega\|_{L^2}^2+\mu\|\nabla\mathbf{u}\|_{H^1}^2+(\mu+\nu)\|\operatorname{div}\mathbf{u}\|_{H^1}^2+\|\nabla\mathbf{B}\|_{H^1}^2\\
&\le\gamma_1\mu^2\|\Delta\mathbf{u}\|_{L^2}^2+\gamma_1(\mu+\nu)^2\|\nabla\operatorname{div}\mathbf{u}\|_{L^2}^2+\gamma_1\|\operatorname{div}\mathbf{u}\|_{L^2}^2+\int_{\mathbb{R}^3}\omega\cdot\mathcal{H}_1\,\mathrm{d}x\\
&\quad+\int_{\mathbb{R}^3}\nabla\omega\cdot\nabla\mathcal{H}_1\,\mathrm{d}x+\int_{\mathbb{R}^3}\mathbf{u}\cdot\mathcal{H}_2\,\mathrm{d}x+\int_{\mathbb{R}^3}\nabla\mathbf{u}\cdot\nabla\mathcal{H}_2\,\mathrm{d}x+\int_{\mathbb{R}^3}\mathbf{B}\cdot\mathcal{H}_3\,\mathrm{d}x\\
&\quad+\int_{\mathbb{R}^3}\nabla\mathbf{B}\cdot\nabla\mathcal{H}_3\,\mathrm{d}x+\gamma_1\int_{\mathbb{R}^3}\mathbf{u}\cdot\nabla\mathcal{H}_1\,\mathrm{d}x+\gamma_1\int_{\mathbb{R}^3}\nabla\omega\cdot\mathcal{H}_2\,\mathrm{d}x\\
&:=\gamma_1\mu^2\|\Delta\mathbf{u}\|_{L^2}^2+\gamma_1(\mu+\nu)^2\|\nabla\operatorname{div}\mathbf{u}\|_{L^2}^2+\gamma_1\|\operatorname{div}\mathbf{u}\|_{L^2}^2+\sum_{i=1}^{8}\mathcal{K}_i.
\end{aligned}\tag{3.14}$$

The nonlinear terms $\mathcal{K}_i$ $(1\le i\le 8)$ on the right-hand side of (3.14) can be bounded as follows. Using Hölder's inequality, Young's inequality, Lemmas A.1–A.2, integration by parts, and (3.2), we get

$$\begin{aligned}
\mathcal{K}_1&=-\int_{\mathbb{R}^3}\omega\cdot\operatorname{div}(\omega\mathbf{u})\,\mathrm{d}x\\
&\le C\|\omega\|_{L^6}\left\|\nabla(\omega\mathbf{u})\right\|_{L^{\frac{6}{5}}}\\
&\le C\|\omega\|_{L^6}\left(\|\nabla\omega\|_{L^2}\|\mathbf{u}\|_{L^3}+\|\omega\|_{L^3}\|\nabla\mathbf{u}\|_{L^2}\right)\\
&\le C\varepsilon_0\left\|\nabla(\omega,\mathbf{u})\right\|_{L^2}^2
\end{aligned}\tag{3.15}$$

and

$$\begin{aligned}
\mathcal{K}_2&=-\int_{\mathbb{R}^3}\nabla\omega\cdot\nabla\operatorname{div}(\omega\mathbf{u})\,\mathrm{d}x\\
&\le C\left\|\nabla^2\omega\right\|_{L^2}\left\|\nabla(\omega\mathbf{u})\right\|_{L^2}\\
&\le C\left\|\nabla^2\omega\right\|_{L^2}\left(\|\nabla\omega\|_{L^2}\|\mathbf{u}\|_{L^\infty}+\|\nabla\mathbf{u}\|_{L^2}\|\omega\|_{L^\infty}\right)\\
&\le C\varepsilon_0\left(\left\|\nabla^2\omega\right\|_{L^2}^2+\left\|\nabla(\mathbf{u},\omega)\right\|_{L^2}^2\right).
\end{aligned}\tag{3.16}$$

Then, thanks to Hölder's inequality, Young's inequality, (3.6)–(3.7), assumption (3.2), and the definition of $\mathcal{H}_2$, we obtain that

$$\begin{aligned}
\mathcal{K}_3&=\int_{\mathbb{R}^3}\mathbf{u}\cdot\left(-\mathbf{u}\cdot\nabla\mathbf{u}-h_1(\omega)\nabla\omega\right)\mathrm{d}x\\
&\quad+\int_{\mathbb{R}^3}g_1(\omega)\mathbf{u}\cdot\left(\mathbf{B}\cdot\nabla\mathbf{B}-\mathbf{B}\cdot\nabla^{\mathrm{T}}\mathbf{B}\right)\mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
&- \int_{\mathbb{R}^3} g_2(\omega)\mathbf{u} \cdot \big(\mu \Delta \mathbf{u} + (\mu + \nu)\nabla \operatorname{div} \mathbf{u}\big)\,\mathrm{d}x \\
&\leq C\|\mathbf{u}\|_{L^6}\big(\|\mathbf{u}\|_{L^3}\|\nabla \mathbf{u}\|_{L^2} + \big\|h_1(\omega)\big\|_{L^3}\|\nabla\omega\|_{L^2}\big) \\
&\quad + C\|\mathbf{u}\|_{L^6}\big(\big\|g_1(\omega)\big\|_{L^3}\|\mathbf{B}\|_{L^\infty}\|\nabla \mathbf{B}\|_{L^2}\big) \\
&\quad + C\|\mathbf{u}\|_{L^6}\big(\big\|g_2(\omega)\big\|_{L^3}\big\|\nabla^2\mathbf{u}\big\|_{L^2}\big) \\
&\leq C\varepsilon_0\big(\big\|\nabla(\omega, \mathbf{u}, \mathbf{B})\big\|_{L^2}^2 + \big\|\nabla^2\mathbf{u}\big\|_{L^2}^2\big)
\end{aligned}
\tag{3.17}
$$

and

$$
\begin{aligned}
\mathcal{K}_4 &= \int_{\mathbb{R}^3} \nabla \mathbf{u} \cdot \nabla\big(-\mathbf{u}\cdot\nabla\mathbf{u} - h_1(\omega)\nabla\omega\big)\,\mathrm{d}x \\
&\quad + \int_{\mathbb{R}^3} \nabla \mathbf{u} \cdot \nabla\big(g_1(\omega)\big(\mathbf{B}\cdot\nabla\mathbf{B} - \mathbf{B}\cdot\nabla^{\mathrm{T}}\mathbf{B}\big)\big)\,\mathrm{d}x \\
&\quad - \int_{\mathbb{R}^3} \nabla \mathbf{u} \cdot \nabla\big(g_2(\omega)\big(\mu\Delta\mathbf{u} + (\mu+\nu)\nabla \operatorname{div}\mathbf{u}\big)\big)\,\mathrm{d}x \\
&\leq C\big\|\nabla^2\mathbf{u}\big\|_{L^2}\big(\|\nabla\mathbf{u}\|_{L^2}\|\mathbf{u}\|_{L^\infty} + \big\|h_1(\omega)\big\|_{L^\infty}\|\nabla\omega\|_{L^2}\big) \\
&\quad + C\big\|\nabla^2\mathbf{u}\big\|_{L^2}\big(\big\|g_1(\omega)\big\|_{L^\infty}\|\mathbf{B}\|_{L^\infty}\|\nabla\mathbf{B}\|_{L^2}\big) \\
&\quad + C\big\|\nabla^2\mathbf{u}\big\|_{L^2}\big(\big\|g_2(\omega)\big\|_{L^\infty}\big\|\nabla^2\mathbf{u}\big\|_{L^2}\big) \\
&\leq C\varepsilon_0\big(\big\|\nabla(\omega,\mathbf{u},\mathbf{B})\big\|_{L^2}^2 + \big\|\nabla^2\mathbf{u}\big\|_{L^2}^2\big).
\end{aligned}
\tag{3.18}
$$

For the term $\mathcal{K}_5$, integrating by parts and using Hölder's inequality and Sobolev inequalities, we first obtain the estimate

$$
\begin{aligned}
&\int_{\mathbb{R}^3} -\operatorname{curl}\big[g_1(\omega)(\mathbf{B}\cdot\nabla\mathbf{B})\big]\mathbf{B}\,\mathrm{d}x \\
&\quad = -\int_{\mathbb{R}^3} g_1(\omega)(\mathbf{B}\cdot\nabla\mathbf{B})\operatorname{curl}\mathbf{B}\,\mathrm{d}x \\
&\quad \leq \|\nabla\mathbf{B}\|_{L^2}\|\mathbf{B}\|_{L^\infty}\big\|g_1(\omega)\big\|_{L^\infty}\|\operatorname{curl}\mathbf{B}\|_{L^2} \\
&\quad \leq C\varepsilon_0\|\nabla\mathbf{B}\|_{L^2}^2.
\end{aligned}
$$

Hence from the above we get

$$
\begin{aligned}
\mathcal{K}_5 &= \int_{\mathbb{R}^3} \mathbf{B}\cdot\big((\mathbf{B}\cdot\nabla)\mathbf{u} - (\mathbf{u}\cdot\nabla)\mathbf{B} - \mathbf{B}\operatorname{div}\mathbf{u}\big)\,\mathrm{d}x \\
&\quad - \int_{\mathbb{R}^3} \operatorname{curl}\big[g_1(\omega)(\mathbf{B}\cdot\nabla\mathbf{B}) - \big(\mathbf{B}\cdot\nabla^T\mathbf{B}\big)\big]\mathbf{B}\,\mathrm{d}x \\
&\leq C\|\mathbf{B}\|_{L^6}\big(\|\mathbf{B}\|_{L^3}\|\nabla\mathbf{u}\|_{L^2} + \|\mathbf{u}\|_{L^3}\|\nabla\mathbf{B}\|_{L^2}\big) + C\varepsilon_0\|\nabla\mathbf{B}\|_{L^2}^2 \\
&\leq C\varepsilon_0\big\|\nabla(\mathbf{u},\mathbf{B})\big\|_{L^2}^2.
\end{aligned}
\tag{3.19}
$$

Similarly to (3.19), we deduce

$$
\mathcal{K}_6 = \int_{\mathbb{R}^3} \nabla\mathbf{B}\cdot\nabla\big((\mathbf{B}\cdot\nabla)\mathbf{u} - (\mathbf{u}\cdot\nabla)\mathbf{B} - \mathbf{B}\operatorname{div}\mathbf{u}\big)\,\mathrm{d}x
$$

$$
\begin{aligned}
&-\int_{\mathbb{R}^3} \nabla \operatorname{curl}\big[\big(g_1(\omega)\big(\mathbf{B}\cdot\nabla\mathbf{B}-\mathbf{B}\cdot\nabla^T\mathbf{B}\big)\big]\nabla\mathbf{B}\,\mathrm{d}x \\
&\le C\big\|\nabla^2\mathbf{B}\big\|_{L^2}\big(\|\mathbf{B}\|_{L^\infty}\|\nabla\mathbf{u}\|_{L^2}+\|\mathbf{u}\|_{L^\infty}\|\nabla\mathbf{B}\|_{L^2}\big)+C\varepsilon_0\big\|\nabla^2\mathbf{B}\big\|_{L^2}^2 \\
&\le C\varepsilon_0\big(\big\|\nabla(\mathbf{u},\mathbf{B})\big\|_{L^2}^2+\|\nabla^2\mathbf{B}|_{L^2}^2\big).
\end{aligned}
\tag{3.20}
$$

For the last two terms, using integration by parts, Lemmas A.1–A.2, and Hölder's inequality, we find that

$$
\begin{aligned}
\mathcal{K}_7 &= -\gamma_1\int_{\mathbb{R}^3}\operatorname{div}\mathbf{u}\cdot\mathcal{H}_1\,\mathrm{d}x \\
&\le C\gamma_1\|\operatorname{div}\mathbf{u}\|_{L^2}\|\mathcal{H}_1\|_{L^2} \\
&\le C\gamma_1\|\nabla\mathbf{u}\|_{L^2}\big(\|\nabla\mathbf{u}\|_{L^2}\|\omega\|_{L^\infty}+\|\mathbf{u}\|_{L^\infty}\|\nabla\omega\|_{L^2}\big) \\
&\le C\gamma_1\varepsilon_0\big\|\nabla(\omega,\mathbf{u})\big\|_{L^2}^2
\end{aligned}
\tag{3.21}
$$

and

$$
\begin{aligned}
\mathcal{K}_8 &= C\gamma_1\|\nabla\omega\|_{L^2}\|\mathcal{H}_2\|_{L^2} \\
&\le C\gamma_1\|\nabla\omega\|_{L^2}\big(\|\mathbf{u}\|_{L^\infty}\|\nabla\mathbf{u}\|_{L^2}+\big\|h_1(\omega)\big\|_{L^\infty}\|\nabla\omega\|_{L^2}\big) \\
&\quad + C\gamma_1\|\nabla\omega\|_{L^2}\big(\big\|g_1(\omega)\big\|_{L^\infty}\|\mathbf{B}\|_{L^\infty}\|\nabla\mathbf{B}\|_{L^2}\big) \\
&\quad + C\gamma_1\|\nabla\omega\|_{L^2}\big(\big\|g_2(\omega)\big\|_{L^\infty}\big\|\nabla^2\mathbf{u}\big\|_{L^2}\big) \\
&\le C\gamma_1\varepsilon_0\big(\big\|\nabla(\omega,\mathbf{u},\mathbf{B})\big\|_{L^2}^2+\big\|\nabla^2\mathbf{u}\big\|_{L^2}^2\big).
\end{aligned}
\tag{3.22}
$$

Putting (3.15)–(3.16) and (3.17)–(3.22) into (3.14) yields

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\{\|\omega\|_{H^1}^2+\|\mathbf{u}\|_{H^1}^2+\|\mathbf{B}\|_{H^1}^2+2\gamma_1\int_{\mathbb{R}^3}\nabla\omega\cdot\mathbf{u}\,\mathrm{d}x\right\} \\
&\quad+\frac{\gamma_1}{2}\|\nabla\omega\|_{L^2}^2+\mu\|\nabla\mathbf{u}\|_{H^1}^2+(\mu+\nu)\|\operatorname{div}\mathbf{u}\|_{H^1}^2+\|\nabla\mathbf{B}\|_{H^1}^2 \\
&\le \gamma_1\mu^2\|\Delta\mathbf{u}\|_{L^2}^2+\gamma_1(\mu+\nu)^2\|\nabla\operatorname{div}\mathbf{u}\|_{L^2}^2+\gamma_1\|\operatorname{div}\mathbf{u}\|_{L^2}^2 \\
&\quad+C(1+\gamma_1)\varepsilon_0\big(\big\|\nabla(\omega,\mathbf{u},\mathbf{B})\big\|_{L^2}^2+\big\|\nabla^2(\omega,\mathbf{u},\mathbf{B})\big\|_{L^2}^2\big).
\end{aligned}
\tag{3.23}
$$

Taking a fixed constant $0<\gamma_1<\frac{1}{4}$, we get the desired estimate from (3.23). The proof of the lemma is complete. □

Now we exploit the energy method to establish an estimate for the highest-order derivatives of the solution $(\omega,\mathbf{u},\mathbf{B})$.

Lemma 3.2 *We have*

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}_H(t)+\frac{\gamma_2}{4}\|\nabla\nabla\omega\|_{L^2}^2+\frac{\mu+\nu}{2}\big\|\nabla^2\operatorname{div}\mathbf{u}\big\|_{L^2}^2+\frac{\mu}{2}\big\|\nabla^3\mathbf{u}\big\|_{L^2}^2+\frac{1}{2}\big\|\nabla^3\mathbf{B}\big\|_{L^2}^2 \\
&\le\frac{1}{4}\|\nabla\operatorname{div}\mathbf{u}\|_{L^2}^2+C\varepsilon_0\big\|\nabla^2(\mathbf{u},\mathbf{B})\big\|_{L^2}^2,
\end{aligned}
\tag{3.24}
$$

where

$$\mathcal{F}_H(t) := \frac{1}{2}\left(\left\|\nabla^2\omega\right\|_{L^2}^2 + \left\|\nabla^2\mathbf{u}\right\|_{L^2}^2 + \left\|\nabla^2\mathbf{B}\right\|_{L^2}^2\right) + \gamma_2\int_3 \nabla\nabla\omega\cdot\nabla\mathbf{u}\,\mathrm{d}x, \tag{3.25}$$

and $\gamma_2 \leq \{\frac{1}{4}, \frac{1}{8\mu}, \frac{1}{8(\mu+\nu)}\}$ *is a given positive constant.*

Proof Multiplying $\nabla^2(2.5)_1 - \nabla^2(2.5)_3$ by $\nabla^2\omega$, $\nabla^2\mathbf{u}$, $\nabla^2\mathbf{B}$, respectively, and integrating the three resulting identities over $\mathbb{R}^3$, we get

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\{\left\|\nabla^2\omega\right\|_{L^2}^2 + \left\|\nabla^2\mathbf{u}\right\|_{L^2}^2 + \left\|\nabla^2\mathbf{B}\right\|_{L^2}^2\right\}\\
&\qquad + \mu\left\|\nabla^2\nabla\mathbf{u}\right\|_{L^2}^2 + (\mu+\nu)\left\|\nabla^2\operatorname{div}\mathbf{u}\right\|_{L^2}^2 + \left\|\nabla^2\nabla\mathbf{B}\right\|_{L^2}^2\\
&\quad = \left\langle\nabla^2\omega, \nabla^2\mathcal{H}_1\right\rangle + \left\langle\nabla^2\mathbf{u}, \nabla^2\mathcal{H}_2\right\rangle + \left\langle\nabla^2\mathbf{B}, \nabla^2\mathcal{H}_3\right\rangle.
\end{aligned}\tag{3.26}$$

Multiplying $\nabla(2.5)_2$ by $\nabla\nabla\omega$ and then exploiting $\nabla^2(2.5)_1$ and Young's inequality, we deduce that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\nabla\nabla\omega\cdot\nabla\mathbf{u}\,\mathrm{d}x + \int_{\mathbb{R}^3}|\nabla\nabla\omega|^2\,\mathrm{d}x\\
&\quad = \mu\int_{\mathbb{R}^3}\nabla\nabla\omega\cdot\nabla\Delta\mathbf{u}\,\mathrm{d}x + (\mu+\nu)\int_{\mathbb{R}^3}\nabla\nabla\omega\cdot\nabla\nabla\operatorname{div}\mathbf{u}\,\mathrm{d}x\\
&\qquad + \int_{\mathbb{R}^3}|\nabla\operatorname{div}\mathbf{u}|^2\,\mathrm{d}x + \langle\nabla u, \nabla\nabla\mathcal{H}_1\rangle + \langle\nabla\nabla\omega, \nabla\mathcal{H}_2\rangle\\
&\quad \leq \frac{1}{2}\|\nabla\nabla\omega\|_{L^2}^2 + \mu^2\|\nabla\Delta\mathbf{u}\|_{L^2}^2 + (\mu+\nu)^2\|\nabla\nabla\operatorname{div}\mathbf{u}\|_{L^2}^2\\
&\qquad + \|\nabla\operatorname{div}\mathbf{u}\|_{L^2}^2 + \langle\nabla\mathbf{u}, \nabla\nabla\mathcal{H}_1\rangle + \langle\nabla\nabla\omega, \nabla\mathcal{H}_2\rangle.
\end{aligned}\tag{3.27}$$

Summing up (3.26) and $\gamma_2\times$ (3.27) with fixed constant γ_2, we have

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\{\left\|\nabla^2\omega\right\|_{L^2}^2 + \left\|\nabla^2\mathbf{u}\right\|_{L^2}^2 + \left\|\nabla^2\mathbf{B}\right\|_{L^2}^2 + 2\gamma_2\int_{\mathbb{R}^3}\nabla\nabla\omega\cdot\nabla\mathbf{u}\,\mathrm{d}x\right\}\\
&\qquad + \frac{\gamma_2}{2}\|\nabla\nabla\omega\|_{L^2}^2 + (\mu+\nu)\left\|\nabla^2\operatorname{div}\mathbf{u}\right\|_{L^2}^2 + \mu\left\|\nabla^2\nabla\mathbf{u}\right\|_{L^2}^2 + \left\|\nabla^2\nabla\mathbf{B}\right\|_{L^2}^2\\
&\quad \leq \gamma_2\mu^2\|\nabla\Delta\mathbf{u}\|_{L^2}^2 + \gamma_2(\mu+\nu)^2\|\nabla\nabla\operatorname{div}\mathbf{u}\|_{L^2}^2 + \gamma_2\|\nabla\operatorname{div}\mathbf{u}\|_{L^2}^2\\
&\qquad + \left\langle\nabla^2\omega, \nabla^2\mathcal{H}_1\right\rangle + \left\langle\nabla^2\mathbf{u}, \nabla^2\mathcal{H}_2\right\rangle + \left\langle\nabla^2\mathbf{B}, \nabla^2\mathcal{H}_3\right\rangle\\
&\qquad + \langle\nabla\mathbf{u}, \nabla\nabla\mathcal{H}_1\rangle + \langle\nabla\nabla\omega, \nabla\mathcal{H}_2\rangle.
\end{aligned}\tag{3.28}$$

Now we estimate the nonlinear terms on the right-hand side of (3.28). Thanks to integration by parts, Lemmas A.1–A.2, Lemma A.5, Hölder's inequality, and Young's inequality, we get

$$\begin{aligned}
\left\langle\nabla^2\omega, \nabla^2\mathcal{H}_1\right\rangle &\leq C\left\|\nabla^2\omega\right\|_{L^2}\left(\|\nabla^2\omega\|_{L^2}\|\operatorname{div}\mathbf{u}\|_{L^\infty} + \left\|\nabla^2\operatorname{div}\mathbf{u}\right\|_{L^2}\|\omega\|_{L^\infty}\right)\\
&\qquad + C\left\|\nabla^2\omega\right\|_{L^2}^2\|\operatorname{div}\mathbf{u}\|_{L^\infty} + C\left\|\nabla^2\omega\right\|_{L^2}\left\|\nabla^2(\mathbf{u}\cdot\nabla\omega) - \nabla^2\nabla\omega\cdot\mathbf{u}\right\|_{L^2}\\
&\leq C\left\|\nabla^2\omega\right\|_{L^2}^2\|\operatorname{div}\mathbf{u}\|_{L^\infty} + C\left\|\nabla^2\omega\right\|_{L^2}\|\omega\|_{H^2}\left\|\nabla^2\operatorname{div}\mathbf{u}\right\|_{L^2}
\end{aligned}$$

$$
\begin{aligned}
&+C\|\nabla^2\omega\|_{L^2}\big(\|\nabla^2\mathbf{u}\|_{L^6}\|\nabla^2\omega\|_{L^3}+\|\nabla\mathbf{u}\|_{L^\infty}\|\nabla^2\omega\|_{L^2}\big)\\
&\le C\varepsilon_0\big(|\nabla^3\mathbf{u}\|_{L^2}^2+\|\nabla^2(\omega,\mathbf{u})\|_{L^2}^2\big).
\end{aligned}
\tag{3.29}
$$

Using Young's inequality, Hölder's inequality, Lemma A.5, integration by parts, and (3.6)–(3.7), we get that

$$
\begin{aligned}
\langle\nabla^2\mathbf{u},\nabla^2\mathcal{H}_2\rangle &\le C\big(\big|\langle\nabla^3\mathbf{u},\nabla(\mathbf{u}\cdot\nabla\mathbf{u})\rangle\big|+\big|\langle\nabla^3\mathbf{u},\nabla\big(h_1(\omega)\nabla\omega\big)\rangle\big|\big)\\
&\quad+C\big(\big|\langle\nabla^3\mathbf{u},\nabla\big(g_1(\omega)\mathbf{B}\cdot\nabla\mathbf{B}\big)\rangle\big|+\big|\langle\nabla^3\mathbf{u},\nabla\big(g_1(\omega)\mathbf{B}\cdot\nabla^T\mathbf{B}\big)\rangle\big|\big)\\
&\quad+C\big(\big|\langle\nabla^3\mathbf{u},\nabla\big(g_2(\omega)\Delta\mathbf{u}\big)\rangle\big|+\big|\langle\nabla^3\mathbf{u},\nabla\big(g_2(\omega)\nabla\operatorname{div}\mathbf{u}\big)\rangle\big|\big)\\
&\le C\|\nabla^3\mathbf{u}\|_{L^2}\big(\|\nabla\mathbf{u}\|_{L^6}\|\nabla\mathbf{u}\|_{L^3}+\|\mathbf{u}\|_{L^\infty}\|\nabla^2\mathbf{u}\|_{L^2}\big)\\
&\quad+C\|\nabla^3\mathbf{u}\|_{L^2}\big(\|h_1(\omega)\|_{L^\infty}\|\nabla^2\omega\|_{L^2}+\|\nabla h_1(\omega)\|_{L^6}\|\nabla\omega\|_{L^3}\big)\\
&\quad+C\|\nabla^3\mathbf{u}\|_{L^2}\big(\|g_1(\omega)\|_{L^\infty}\|\nabla(\mathbf{B}\cdot\nabla\mathbf{B})\|_{L^2}+\|\nabla g_1(\omega)\|_{L^6}\|\mathbf{B}\cdot\nabla\mathbf{B}\|_{L^3}\big)\\
&\quad+C\|\nabla^3\mathbf{u}\|_{L^2}\big(\|g_2(\omega)\|_{L^\infty}\|\nabla^3\mathbf{u}\|_{L^2}+\|\nabla g_2(\omega)\|_{L^\infty}\|\nabla^2\mathbf{u}\|_{L^2}\big)\\
&\le C\varepsilon_0\big(\|\nabla^2(\omega,\mathbf{u},\mathbf{B})\|_{L^2}^2+\|\nabla^3\mathbf{u}\|_{L^2}^2\big).
\end{aligned}
\tag{3.30}
$$

By similar estimates we easily get

$$
\begin{aligned}
\langle\nabla^2\mathbf{B},\nabla^2\mathcal{H}_3\rangle &\le C\big|\langle\nabla^3\mathbf{B},\nabla(\mathbf{B}\cdot\nabla)\mathbf{u}\rangle\big|+C\big|\langle\nabla^3\mathbf{B},\nabla(\mathbf{u}\cdot\nabla)\mathbf{B}\rangle\big|\\
&\quad+C\big|\langle\nabla^3\mathbf{B},\nabla(\mathbf{B}\operatorname{div}\mathbf{u})\rangle\big|+C\big|\langle\nabla^2\mathbf{B},\nabla^2\operatorname{curl}\big[g_1(\omega)\big(\mathbf{B}\cdot\nabla\mathbf{B}-\mathbf{B}\cdot\nabla^T\mathbf{B}\big)\big]\rangle\big|\\
&\le C\|\nabla^3\mathbf{B}\|_{L^2}\big(\|\nabla\mathbf{B}\|_{L^6}\|\nabla\mathbf{u}\|_{L^3}+\|\mathbf{B}\|_{L^\infty}\|\nabla^2\mathbf{u}\|_{L^2}\big)\\
&\quad+C\|\nabla^3\mathbf{B}\|_{L^2}\big(\|\nabla\mathbf{u}\|_{L^6}\|\nabla\mathbf{B}\|_{L^3}+\|\mathbf{u}\|_{L^\infty}\|\nabla^2\mathbf{B}\|_{L^2}\big)\\
&\quad+C\varepsilon_0\big(\|\nabla^3\mathbf{B}\|_{L^2}^2+\|\nabla^2\omega\|_{L^2}^2\big)\\
&\le C\varepsilon_0\big(\|\nabla^3\mathbf{B}\|_{L^2}^2+\|\nabla^2(\mathbf{u},\mathbf{B},\omega)\|_{L^2}^2\big),
\end{aligned}
\tag{3.31}
$$

where

$$
\begin{aligned}
&\int_{\mathbb{R}^3}-\nabla^2\operatorname{curl}\big[g_1(\omega)(\mathbf{B}\cdot\nabla)\mathbf{B}\big]\nabla^2\mathbf{B}\,\mathrm{d}x\\
&\quad=-\int_{\mathbb{R}^3}\nabla^2\big[g_1(\omega)(\mathbf{B}\cdot\nabla)\mathbf{B}\big]\nabla^2\operatorname{curl}\mathbf{B}\,\mathrm{d}x\\
&\quad\le\big(\|\nabla^2g_1(\omega)\|_{L^2}\|B\|_{L^\infty}\|\nabla^2B\|_{L^\infty}+\|\nabla g_1(\omega)\|_{L^6}\|\nabla B\|_{L^6}\|\nabla B\|_{L^6}\big)\|\nabla^2\operatorname{curl}B\|_{L^2}\\
&\qquad+\big(\|\nabla g_1(\omega)\|_{L^6}\|B\|_{L^6}\|\nabla^2B\|_{L^6}+\|g_1(\omega)\|_{L^\infty}\|\nabla B\|_{L^3}\|\nabla^2B\|_{L^6}\big)\|\nabla^2\operatorname{curl}B\|_{L^2}\\
&\qquad+\|g_1(\omega)\|_{L^\infty}\|B\|_{L^\infty}\|\nabla^3B\|_{L^2}\|\nabla^2\operatorname{curl}B\|_{L^2}\\
&\quad\le C\varepsilon_0\big(\|\nabla^3\mathbf{B}\|_{L^2}^2+\|\nabla^2\omega\|_{L^2}^2\big)
\end{aligned}
$$

and

$$
\begin{aligned}
\langle\nabla\mathbf{u},\nabla\nabla\mathcal{H}_1\rangle &\le-\gamma_2\int_3\nabla\operatorname{div}\mathbf{u}\cdot\nabla\mathcal{H}_1\,\mathrm{d}x\\
&\le C\gamma_2\|\nabla\operatorname{div}\mathbf{u}\|_{L^2}\|\nabla\mathcal{H}_1\|_{L^2}
\end{aligned}
$$

$$\begin{aligned}
&\leq C\gamma_2\left\|\nabla^2\mathbf{u}\right\|_{L^2}\left(\left\|\nabla^2\mathbf{u}\right\|_{L^2}\|\omega\|_{L^\infty}+\left\|\nabla^2\omega\right\|_{L^2}\|\mathbf{u}\|_{L^\infty}\right)\\
&\leq C\gamma_2\varepsilon_0\left\|\nabla^2(\omega,\mathbf{u})\right\|_{L^2}^2.
\end{aligned} \tag{3.32}$$

For the last term on the right-hand side of (3.28), using Hölder's inequality, integration by parts, Young's inequality, Lemma A.5, and (3.6)–(3.7), we obtain

$$\begin{aligned}
\langle\nabla\nabla\omega,\nabla\mathcal{H}_2\rangle &\leq C\gamma_2\|\nabla\nabla\omega\|_{L^2}\|\nabla\mathcal{H}_2\|_{L^2}\\
&\leq C\gamma_2\left\|\nabla^2\omega\right\|_{L^2}\left(\|\mathbf{u}\|_{L^\infty}\left\|\nabla^2\mathbf{u}\right\|_{L^2}+\|\nabla\mathbf{u}\|_{L^6}\|\nabla\mathbf{u}\|_{L^3}\right)\\
&\quad+C\gamma_2\left\|\nabla^2\omega\right\|_{L^2}\left(\left\|h_1(\omega)\right\|_{L^\infty}\left\|\nabla^2\omega\right\|_{L^3}+\left\|\nabla h_1(\omega)\right\|_{L^\infty}\|\nabla\omega\|_{L^2}\right)\\
&\quad+C\gamma_2\left\|\nabla^2\omega\right\|_{L^2}\left\|g_1(\omega)\right\|_{L^\infty}\left(\|\mathbf{B}\|_{L^\infty}\left\|\nabla^2\mathbf{B}\right\|_{L^2}+\|\nabla\mathbf{B}\|_{L^6}\|\nabla\mathbf{B}\|_{L^3}\right)\\
&\quad+C\gamma_2\left\|\nabla^2\omega\right\|_{L^2}\left(\left\|g_2(\omega)\right\|_{L^\infty}\left\|\nabla^3\mathbf{u}\right\|_{L^2}+\left\|\nabla g_2(\omega)\right\|_{L^3}\left\|\nabla^2\mathbf{u}\right\|_{L^6}\right)\\
&\leq C\gamma_2\varepsilon_0\left(\left\|\nabla^2(\omega,\mathbf{u},\mathbf{B})\right\|_{L^2}^2+\left\|\nabla^3\mathbf{u}\right\|_{L^2}^2\right).
\end{aligned} \tag{3.33}$$

Putting (3.29)–(3.33) into (3.28), we derive that

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\{\left\|\nabla^2\omega\right\|_{L^2}^2+\left\|\nabla^2\mathbf{u}\right\|_{L^2}^2+\left\|\nabla^2\mathbf{B}\right\|_{L^2}^2+2\gamma_2\int_{\mathbb{R}^3}\nabla\nabla\omega\cdot\nabla\mathbf{u}\,\mathrm{d}x\right\}\\
&\quad+\frac{\gamma_2}{4}\|\nabla\nabla\omega\|_{L^2}^2+\frac{(\mu+\nu)}{2}\left\|\nabla^2\operatorname{div}\mathbf{u}\right\|_{L^2}^2+\frac{\mu}{2}\left\|\nabla^2\nabla\mathbf{u}\right\|_{L^2}^2+\frac{1}{2}\left\|\nabla^2\nabla\mathbf{B}\right\|_{L^2}^2\\
&\leq\frac{1}{4}\|\nabla\operatorname{div}\mathbf{u}\|_{L^2}^2+C\varepsilon_0\left\|\nabla^2(\mathbf{u},\mathbf{B})\right\|_{L^2}^2,
\end{aligned} \tag{3.34}$$

where $0<\gamma_2\leq\{\frac{1}{4},\frac{1}{8\mu},\frac{1}{8(\mu+\nu)}\}$ is a fixed positive constant. Consequently, we complete the proof of (3.24). □

With Lemmas 3.1–3.2 in hand, it is easy to further obtain Proposition 3.2. As a matter of fact, keeping in mind the definitions of $\mathcal{F}_L$ and $\mathcal{F}_H$ and Young's inequality, we have

$$\frac{1}{C_4}\left\|(\omega,\mathbf{u},\mathbf{B})\right\|_{H^2}^2\leq\mathcal{F}_L(t)+\mathcal{F}_H(t)\leq C_4\left\|(\omega,\mathbf{u},\mathbf{B})\right\|_{H^2}^2, \tag{3.35}$$

which yields

$$\mathcal{F}_L(t)+\mathcal{F}_H(t)\approx\left\|(\omega,\mathbf{u},\mathbf{B})\right\|_{H^2}^2, \tag{3.36}$$

where $C_4>0$ is a constant. Thus integrating the two inequalities in the two above lemmas over $[0,t]$, we obtain (3.3) for the smallness of ε_0. This completes the proof of Proposition 3.2.

4 Decay-in-time rates of the solution

In this section, we derive the decay-in-time rates for the solution in the previous section. We divide the proof into two subsections.

4.1 Cancelation of the low-frequency part

Inspired by the observation of canceling the low-frequency part of the solution in [45], we have the following conclusion.

Lemma 4.1 *We have*

$$\begin{aligned}\big\|\nabla^2(\omega,\mathbf{u},\mathbf{B})\big\|_{L^2}^2 &\le Ce^{-C_3t}\big\|\nabla^2(\omega_0,\mathbf{u}_0,\mathbf{B}_0)\big\|_{L^2}^2\\ &\quad + C\int_0^t e^{-C_3(t-\tau)}\big\|\nabla^2\big(\omega^L,\mathbf{u}^L,\mathbf{B}^L\big)(\tau)\big\|_{L^2}^2\,\mathrm{d}\tau,\end{aligned}\tag{4.1}$$

where $C>0$ *is a constant.*

Proof Multiplying $\nabla(2.5)_2$ by $\nabla\nabla\omega^L$ in L^2 and then integration by parts and $(2.5)_1$, we get

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\nabla\nabla\omega^L\cdot\nabla\mathbf{u}\,dx\\ &\quad=\mu\int_{\mathbb{R}^3}\nabla\nabla\omega^L\cdot\nabla\Delta\mathbf{u}\,\mathrm{d}x+(\mu+\nu)\int_{\mathbb{R}^3}\nabla\nabla\omega^L\cdot\nabla\nabla\operatorname{div}\mathbf{u}\,\mathrm{d}x\\ &\qquad+\int_{\mathbb{R}^3}\big(\nabla\operatorname{div}\mathbf{u}\cdot\nabla\operatorname{div}\mathbf{u}^L-\nabla\nabla\omega^L\nabla\nabla\omega\big)\,\mathrm{d}x\\ &\qquad-\int_{\mathbb{R}^3}\big(\nabla\mathcal{H}_1^L\cdot\nabla\operatorname{div}\mathbf{u}-\nabla\nabla\omega^L\cdot\nabla\mathcal{H}_2\big)\,\mathrm{d}x.\end{aligned}\tag{4.2}$$

Similarly to (3.12)–(3.13), using Young's inequality, we find that

$$\begin{aligned}&-\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\nabla\nabla\omega^L\cdot\nabla\mathbf{u}\,\mathrm{d}x\\ &\quad\le\frac{\mu}{2}\|\nabla\Delta\mathbf{u}\|_{L^2}^2+\frac{(\mu+\nu)}{2}\|\nabla\nabla\operatorname{div}\mathbf{u}\|_{L^2}^2+\|\nabla\operatorname{div}\mathbf{u}\|_{L^2}^2\\ &\qquad+\frac{1}{2}\big\|\nabla\operatorname{div}\mathbf{u}^L\big\|_{L^2}^2+\left(2+\frac{1+2\mu+\nu}{2}\right)\big\|\nabla\nabla\omega^L\big\|_{L^2}^2\\ &\qquad+\frac{1}{8}\|\nabla\nabla\omega\|_{L^2}^2+\frac{1}{2}\big\|\nabla\mathcal{H}_1^L\big\|_{L^2}^2+\frac{1}{2}\|\nabla\mathcal{H}_2\|_{L^2}^2.\end{aligned}\tag{4.3}$$

By the Plancherel theorem, Lemma A.2, and (3.33) we obtain

$$\big\|\nabla\mathcal{H}_1^L\big\|_{L^2}^2+\|\nabla\mathcal{H}_2\|_{L^2}^2\le C\varepsilon_0\big(\big\|\nabla^2(\omega,\mathbf{u},\mathbf{B})\big\|_{L^2}^2+\big\|\nabla^3\mathbf{u}\big\|_{L^2}^2\big).\tag{4.4}$$

Adding up $\gamma_2\times(4.3)$ and (3.24) in Lemma 3.2 and then using (4.4) and Lemma A.3, we get

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathcal{F}_H(t)-\gamma_2\int_{\mathbb{R}^3}\nabla\nabla\omega^L\cdot\nabla\mathbf{u}\,\mathrm{d}x\right)+\frac{\gamma_2}{8}\big\|\nabla^2\omega\big\|_{L^2}^2\\ &\qquad+\frac{\mu}{4}\big\|\nabla^2\mathbf{u}^H\big\|_{L^2}^2+\frac{\mu}{4}\big\|\nabla^2\nabla\mathbf{u}\big\|_{L^2}^2+\frac{\mu+\nu}{2}\big\|\nabla^2\operatorname{div}\mathbf{u}\big\|_{L^2}^2\\ &\qquad+\frac{1}{4}R_0^2\big\|\nabla^2\mathbf{B}^H\big\|_{L^2}^2+\frac{1}{4}\big\|\nabla^3\mathbf{B}\big\|_{L^2}^2\\ &\quad\le\left(\frac{1}{4}+\gamma_2\right)\|\nabla\operatorname{div}\mathbf{u}\|_{L^2}^2+\frac{\gamma_2\mu}{2}\|\nabla\Delta\mathbf{u}\|_{L^2}^2+\frac{\gamma_2(\mu+\nu)}{2}\|\nabla\nabla\operatorname{div}\mathbf{u}\|_{L^2}^2\\ &\qquad+C\gamma_2\big(\big\|\nabla\nabla\omega^L\big\|_{L^2}^2+\big\|\nabla\operatorname{div}\mathbf{u}^L\big\|_{L^2}^2\big)+C\varepsilon_0(1+\gamma_2)\big\|\nabla^2(\omega,\mathbf{u},\mathbf{B})\big\|_{L^2}^2.\end{aligned}\tag{4.5}$$

In addition, using decomposition (1.5), we further put $\frac{\mu}{4}R_0^2\|\nabla^2\mathbf{u}^L\|_{L^2}^2+\frac{1}{4}R_0^2\|\nabla^2\mathbf{B}^L\|_{L^2}^2$ on the both sides of (4.5) to get

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathcal{F}_H(t)-\gamma_2\int_{\mathbb{R}^3}\nabla\nabla\omega^L\cdot\nabla\mathbf{u}\,\mathrm{d}x\right)+\frac{\gamma_2}{8}\left\|\nabla^2\omega\right\|_{L^2}^2+\frac{\mu}{8}R_0^2\left\|\nabla^2\mathbf{u}\right\|_{L^2}^2\\
&\quad+\frac{\mu}{4}\left\|\nabla^2\nabla\mathbf{u}\right\|_{L^2}^2+\frac{\mu+\nu}{2}\left\|\nabla^2\operatorname{div}\mathbf{u}\right\|_{L^2}^2+\frac{1}{4}R_0^2\left\|\nabla^2\mathbf{B}\right\|_{L^2}^2+\frac{1}{4}\left\|\nabla^3\mathbf{B}\right\|_{L^2}^2\\
&\quad\le\left(\frac{1}{4}+\gamma_2\right)\|\nabla\operatorname{div}\mathbf{u}\|_{L^2}^2+\frac{\gamma_2\mu}{2}\|\nabla\Delta\mathbf{u}\|_{L^2}^2+\frac{\gamma_2(\mu+\nu)}{2}\|\nabla\nabla\operatorname{div}\mathbf{u}\|_{L^2}^2\\
&\quad+C\gamma_2\left\|\nabla\nabla\omega^L\right\|_{L^2}^2+\left(\frac{1}{4}R_0^2+C\gamma_2\right)\left\|\nabla^2\mathbf{u}^L\right\|_{L^2}^2\\
&\quad+\frac{1}{4}R_0^2\left\|\nabla^2\mathbf{B}^L\right\|_{L^2}^2+C\varepsilon_0(1+\gamma_2)\left\|\nabla^2(\omega,\mathbf{u},\mathbf{B})\right\|_{L^2}^2.
\end{aligned}\tag{4.6}$$

Moreover, noting that $\gamma_2<\frac{1}{4}$ and $R_0^2>\frac{8}{\mu}$ and using the smallness of ε_0, we obviously get

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathcal{F}_H(t)-\gamma_2\int_{\mathbb{R}^3}\nabla\nabla\omega^L\cdot\nabla\mathbf{u}\,\mathrm{d}x\right)+\frac{\gamma_2}{16}\left\|\nabla^2\omega\right\|_{L^2}^2+\frac{\mu}{16}R_0^2\left\|\nabla^2\mathbf{u}\right\|_{L^2}^2\\
&\quad+\frac{\mu}{8}\left\|\nabla^3\mathbf{u}\right\|_{L^2}^2+\frac{\mu+\nu}{8}\left\|\nabla^2\operatorname{div}\mathbf{u}\right\|_{L^2}^2+\frac{1}{16}R_0^2\left\|\nabla^2\mathbf{B}\right\|_{L^2}^2+\frac{1}{4}\left\|\nabla^3\mathbf{B}\right\|_{L^2}^2\\
&\quad\le C\left\|\nabla^2\left(\omega^L,\mathbf{u}^L,\mathbf{B}^L\right)\right\|_{L^2}^2.
\end{aligned}\tag{4.7}$$

Recalling the frequency decomposition (1.5), we have

$$\begin{aligned}
&\mathcal{F}_H(t)-\gamma_2\int_{\mathbb{R}^3}\nabla\nabla\omega^L\cdot\nabla\mathbf{u}\,\mathrm{d}x\\
&\quad=\frac{1}{2}\left(\left\|\nabla^2\omega\right\|_{L^2}^2+\left\|\nabla^2\mathbf{u}\right\|_{L^2}^2+\left\|\nabla^2\mathbf{B}\right\|_{L^2}^2\right)+\gamma_2\int_{\mathbb{R}^3}\nabla\nabla\omega^H\cdot\nabla\mathbf{u}\,\mathrm{d}x.
\end{aligned}\tag{4.8}$$

For the term of $\gamma_2\int_{\mathbb{R}^3}\nabla\nabla\omega^H\cdot\nabla\mathbf{u}\,\mathrm{d}x$, exploiting Lemma A.3, Young's inequality, and integration by parts, we deduce that

$$\begin{aligned}
-\gamma_2\int_{\mathbb{R}^3}\nabla\nabla\omega^H\cdot\nabla\mathbf{u}\,\mathrm{d}x&=\gamma_2\int_{\mathbb{R}^3}\nabla\omega^H\cdot\nabla^2\mathbf{u}\,\mathrm{d}x\\
&\le\frac{\gamma_2}{2}\left\|\nabla\omega^H\right\|_{L^2}^2+\frac{\gamma_2}{2}\|\nabla\operatorname{div}\mathbf{u}\|_{L^2}^2\\
&\le\frac{\gamma_2}{2}\|\nabla\omega\|_{L^2}^2+\frac{\gamma_2}{2}\left\|\nabla^2\mathbf{u}\right\|_{L^2}^2,
\end{aligned}\tag{4.9}$$

where we have used the fact that $0<\gamma_2<\frac{1}{8}$.

Now combining (4.8) with (4.9) yields

$$\mathcal{F}_H(t)-\gamma_2\int_{\mathbb{R}^3}\nabla\nabla\omega^L\cdot\nabla\mathbf{u}\,\mathrm{d}x\approx\left\|\nabla^2(\omega,\mathbf{u},\mathbf{B})\right\|_{L^2}^2.\tag{4.10}$$

Thanks (4.10), we derive from (4.7) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathcal{F}_H(t)-\gamma_2\int_{\mathbb{R}^3}\nabla\nabla\omega^L\cdot\nabla\mathbf{u}\,\mathrm{d}x\right)+C_3\left(\mathcal{F}_H(t)-\gamma_2\int_{\mathbb{R}^3}\nabla\nabla\omega^L\cdot\nabla\mathbf{u}\,\mathrm{d}x\right)$$
$$\le C\left\|\nabla^2\left(\omega^L,\mathbf{u}^L,\mathbf{B}^L\right)\right\|_{L^2}^2. \tag{4.11}$$

Consequently, using Gronwall's inequality, we conclude that

$$\mathcal{F}_H(t)-\gamma_2\int_{\mathbb{R}^3}\nabla\nabla\omega^L\cdot\nabla\mathbf{u}\,\mathrm{d}x$$
$$\le Ce^{-C_3t}\left(\mathcal{F}_H(0)-\gamma_2\int_3\nabla\nabla\omega_0^L\cdot\nabla\mathbf{u}_0\,\mathrm{d}x\right)$$
$$+C\int_0^t e^{-C_3(t-\tau)}\left\|\nabla^2\left(\omega^L,\mathbf{u}^L,\mathbf{B}^L\right)(\tau)\right\|_{L^2}^2\,\mathrm{d}\tau. \tag{4.12}$$

This completes the proof of the lemma. □

4.2 Decay estimate of the low-frequency part

We will give an estimate of the low-frequency part of the solution by analyzing the structure of the semigroup of the Cauchy problem (2.5). To this end, by the Hausdorff decomposition in [2] we first adopt the following notations:

$$m=\Lambda^{-1}\operatorname{div}\mathbf{u},\qquad M=\Lambda^{-1}\operatorname{curl}\mathbf{u},$$

where $(\operatorname{curl}\mathbf{u})_{ij}=\partial_j\mathbf{u}^i-\partial_i\mathbf{u}^j$. Then decoupling the Cauchy problem (2.5), we obtain the following two systems:

$$\begin{cases}\omega_t+\Lambda m=\mathcal{H}_1,\\ m_t-(2\mu+\nu)\Delta m-\Lambda\omega=\mathcal{I}_2,\\ \mathbf{B}_t-\Delta\mathbf{B}=\mathcal{H}_3,\\ (\omega,m,\mathbf{B})|_{t=0}=(\omega_0,m_0,\mathbf{B}_0)(x),\end{cases} \tag{4.13}$$

and

$$\begin{cases}M_t-\mu\Delta M=\Lambda^{-1}\operatorname{curl}\mathcal{H}_2,\\ M(0,x)=M_0(x),\end{cases} \tag{4.14}$$

where $\mathcal{I}_2:=\Lambda^{-1}\operatorname{div}\mathcal{H}_2$, $m_0:=\Lambda^{-1}\operatorname{div}\mathbf{u}_0$, and $M_0:=\Lambda^{-1}\operatorname{curl}\mathbf{u}_0$. Then we can directly obtain the following lemma by a simple calculation; see [45, 54] for examples.

Lemma 4.2 *Let $\mathbf{B}(t,x)$ and $M(t,x)$ be the solutions to linearized equations of $(4.13)_3$ and (4.14), respectively. Then, for all $|\xi|^2\ge 0$, we have that*

$$\left|\widehat{\mathbf{B}}(t,\xi)\right|^2\le Ce^{-|\xi|^2t}\left|\widehat{\mathbf{B}}(0,\xi)\right|^2 \tag{4.15}$$

and

$$\left|\widehat{M}(t,\xi)\right|^2 \le Ce^{-\mu|\xi|^2 t}\left|\widehat{M}(0,\xi)\right|^2, \tag{4.16}$$

where $C > 0$ *is a constant, and* $\widehat{\mathbf{B}}$ *and* $\widehat{M}$ *are the Fourier transforms of* $\mathbf{B}$ *and* M*, respectively.*

From the linearized system $(4.13)_1$–$(4.13)_2$, by the Fourier transform we can easily obtain the following system:

$$\begin{cases} \widehat{\omega}_t = -|\xi|\widehat{m}, \\ \widehat{m}_t = |\xi|\widehat{\omega} - (2\mu + \nu)|\xi|^2\widehat{m}, \end{cases} \tag{4.17}$$

which can be rewritten as

$$\widehat{\mathbf{U}}_t = \widehat{\mathcal{A}}\big(|\xi|\big)\widehat{\mathbf{U}}, \tag{4.18}$$

where $\widehat{\mathbf{U}} := (\widehat{\omega}, \widehat{m})$ and

$$\widehat{\mathcal{A}}\big(|\xi|\big) := \begin{bmatrix} 0 & -|\xi| \\ |\xi| & -(2\mu + \nu)|\xi|^2 \end{bmatrix}. \tag{4.19}$$

According to the theory of ODEs, there exists a solution of system (4.17), which can be expressed by

$$\widehat{\mathbf{U}} = e^{t\widehat{\mathcal{A}}(|\xi|)}\widehat{\mathbf{U}}(0). \tag{4.20}$$

Taking the inverse Fourier transform on the both sides of (4.20), we obtain the solution

$$\mathbf{U} = A(t)\mathbf{U}(0), \tag{4.21}$$

where $\mathbf{A}(t)\mathbf{U} := \mathcal{F}^{-1}(e^{t\widehat{\mathcal{A}}(|\xi|)}\widehat{\mathbf{U}}(\xi))$.

We easily compute the characteristic polynomial of the matrix $\widehat{\mathcal{A}}(|\xi|)$:

$$\det\big(\widehat{\mathcal{A}}\big(|\xi|\big) - \lambda\mathbf{I}\big) = \begin{vmatrix} -\lambda & -|\xi| \\ |\xi| & -(2\mu + \nu)|\xi|^2 - \lambda \end{vmatrix} = \lambda^2 + (2\mu + \nu)|\xi|^2\lambda + |\xi|^2; \tag{4.22}$$

The eigenvalues $\lambda_i(\xi)$ $(i = 1, 2)$ of $\widehat{\mathcal{A}}(|\xi|)$ can be calculated by (4.22) as follows:

$$\begin{cases} \lambda_1(|\xi|) = -(\mu + \frac{1}{2}\nu)|\xi|^2 + i|\xi|\sqrt{\frac{(2\mu+\nu)^2}{4}|\xi|^2 - 1}, \\ \lambda_2(|\xi|) = -(\mu + \frac{1}{2}\nu)|\xi|^2 - i|\xi|\sqrt{\frac{(2\mu+\nu)^2}{4}|\xi|^2 - 1}. \end{cases} \tag{4.23}$$

Based on the semigroup decomposition theory proposed in [32], we get

$$e^{t\widehat{\mathcal{A}}(|\xi|)} = e^{\lambda_1 t}P_1(\xi) + e^{\lambda_2 t}P_2(\xi), \tag{4.24}$$

where

$$P_i(\xi) = \prod_{j\neq i} \frac{A(|\xi|) - \lambda_j \mathbf{I}}{\lambda_i - \lambda_j} \quad (i,j = 1,2) \tag{4.25}$$

is a set of projection operators.

Then we get asymptotic expansions of $\lambda_i(\xi)$ $(i = 1,2)$, $P_i(\xi)$ $(i = 1,2)$, and $e^{t\widehat{\mathcal{A}}(|\xi|)}$ in the case of different frequency situations. More precisely, we have the following:

Lemma 4.3 *For any* $|\xi| \le 1$, $\lambda_i(\xi)$ $(i = 1,2)$ *has the Taylor series expansion*

$$\begin{cases} \lambda_1 = -b|\xi|^2 + i(|\xi| + O(|\xi|^3)), \\ \lambda_2 = -b|\xi|^2 - i(|\xi| + O(|\xi|^3)), \end{cases} \tag{4.26}$$

where b is a constant.

Proof We refer to [29] for the proof. □

According to the lemma, we can obtain a time-decay estimate of the low-frequency part of the solution of the linear system (4.17).

Lemma 4.4 *For* $1 \le p \le 2$, *we have*

$$\left\| \nabla^k \left(\omega^L, m^L\right)(t) \right\|_{L^2} \le C(1+t)^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{2})-\frac{k}{2}} \left\| (\omega_0, m_0) \right\|_{L^p} \tag{4.27}$$

for any integer $k \ge 0$.

Proof Thanks to expressions (4.23)–(4.24) and the Fourier transform, we can obtain the following specific expression of the Green matrix $e^{t\mathcal{A}}$:

$$\widehat{D}\left(|\xi|\right) = e^{t\widehat{\mathcal{A}}(\xi)} = \begin{pmatrix} f_1(\lambda_1,\lambda_2) & -|\xi| f_2(\lambda_1,\lambda_2) \\ |\xi| f_2(\lambda_1,\lambda_2) & f_1(\lambda_1,\lambda_2) - 2b|\xi|^2 f_2(\lambda_1,\lambda_2) \end{pmatrix}, \tag{4.28}$$

where

$$\begin{cases} f_1(\lambda_1,\lambda_2) = \frac{\lambda_1 e^{\lambda_2 t} - \lambda_2 e^{\lambda_1 t}}{\lambda_1 - \lambda_2}, \\ f_2(\lambda_1,\lambda_2) = \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2}. \end{cases} \tag{4.29}$$

For any $|\xi| \le R_0$, by simple calculation we have

$$\begin{aligned} \left|f_1(\lambda_1,\lambda_2)\right| &= e^{\lambda_2 t} + \frac{\lambda_2}{\lambda_1 - \lambda_2}\left(e^{\lambda_2 t} - e^{\lambda_1 t}\right) \\ &= e^{-b|\xi|^2 t} \cos\left(\left(|\xi| + O\left(|\xi|^3\right)\right)t\right) \\ &\quad - e^{-b|\xi|^2 t}\left[\left(\frac{-b|\xi|^2}{|\xi| + O(|\xi|^3)}\right) \sin\left(\left(|\xi| + O\left(|\xi|^3\right)\right)t\right)\right] \\ &\lesssim e^{-b|\xi|^2 t}. \end{aligned} \tag{4.30}$$

Similarly, we obtain

$$\begin{aligned}\left|f_2(\lambda_1,\lambda_2)\right| &= \frac{e^{\lambda_1 t}-e^{\lambda_2 t}}{\lambda_1-\lambda_2}\\ &= \frac{e^{-b|\xi|^2 t}}{|\xi|+O(|\xi|^3)}\sin\bigl(\bigl(|\xi|+O\bigl(|\xi|^3\bigr)\bigr)t\bigr)\lesssim |\xi|^{-1}e^{-b|\xi|^2 t}.\end{aligned} \tag{4.31}$$

From these two estimates we can derive that

$$\left|f_1(\lambda_1,\lambda_2)-2b|\xi|^2 f_2(\lambda_1,\lambda_2)\right|\lesssim e^{-b|\xi|^2 t}. \tag{4.32}$$

For any $|\xi|\le 1$, we combine (4.20) and (4.30)–(4.32) to get

$$\begin{aligned}|\widehat{\omega}| &\lesssim |\widehat{D_{11}}|\cdot|\widehat{\omega_0}|+|\widehat{D_{12}}|\cdot|\widehat{m_0}|\\ &\lesssim \left|f_1(\lambda_1,\lambda_2)\right|\left|\widehat{\omega_0}\right|+\left|\left|\xi\right|f_2(\lambda_1,\lambda_2)\right|\left|\widehat{m_0}\right|\\ &\lesssim e^{-b|\xi|^2 t}\bigl(|\widehat{\omega_0}|+|\widehat{m_0}|\bigr)\end{aligned} \tag{4.33}$$

and

$$\begin{aligned}|\widehat{m}| &\lesssim |\widehat{D_{21}}|\cdot|\widehat{\omega_0}|+|\widehat{D_{22}}|\cdot|\widehat{m_0}|\\ &\lesssim \left||\xi|g_2(\lambda_1,\lambda_2)\right|\left|\widehat{\omega_0}\right|+\left||\xi|g_2(\lambda_1,\lambda_2)\right|\left|\widehat{m_0}\right|\\ &\lesssim e^{-b|\xi|^2 t}\bigl(|\widehat{\omega_0}|+|\widehat{m_0}|\bigr).\end{aligned} \tag{4.34}$$

Thanks to the Plancherel theorem, (4.20), (4.33), and (4.34), we obtain

$$\begin{aligned}\left\|\nabla^k\bigl(\omega^L,m^L\bigr)(t)\right\|_{L^2} &= \left\|(i\xi)^k\bigl(\widehat{\omega^L},\widehat{m^L}\bigr)\right\|_{L^2_\xi}\\ &= \left(\int_{\mathbb{R}^3}\left|(i\xi)^k\bigl(\widehat{\omega^L},\widehat{m^L}\bigr)(t,\xi)\right|^2\,\mathrm{d}\xi\right)^{\frac12}\\ &\le C\left(\int_{|\xi|\le R_0}|\xi|^{2k}\left|(\widehat{\omega},\widehat{m})(t,\xi)\right|^2\,\mathrm{d}\xi\right)^{\frac12}\\ &\le C\left(\int_{|\xi|\le R_0}|\xi|^{2k}e^{-b|\xi|^2 t}\left|\bigl(\widehat{\omega^L},\widehat{m^L}\bigr)(0,\xi)\right|^2\,\mathrm{d}\xi\right)^{\frac12}.\end{aligned} \tag{4.35}$$

Applying Hausdorff–Young's and Hölder's inequalities to (4.35), we have

$$\begin{aligned}\left\|\nabla^k\bigl(\omega^L,m^L\bigr)(t)\right\|_{L^2} &\le C(1+t)^{-\frac32(\frac12-\frac1q)-\frac k2}\left\|(\widehat{\omega},\widehat{m})(0,\xi)\right\|_{L^q_\xi}\\ &\le C(1+t)^{-\frac32(\frac1p-\frac12)-\frac k2}\left\|(\omega_0,m_0)\right\|_{L^p},\end{aligned} \tag{4.36}$$

which ends the proof of Lemma 4.4. □

Based on Lemmas 4.2 and 4.4, we get the following estimates.

Proposition 4.1 *Let $1 \le p \le 2$. For any integer $k \ge 0$, we have*

$$\left\|\nabla^k\left(\omega^L,\mathbf{u}^L,\mathbf{B}^L\right)(t)\right\|_{L^2} \le C(1+t)^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{2})-\frac{k}{2}}\left\|(\omega_0,m_0,\mathbf{B}_0)\right\|_{L^p}.$$

Proof Thanks to the two estimates in Lemma 4.2, we can follow the arguments of (4.35) and (4.36) to get

$$\left\|\nabla^k\mathbf{B}^L(t)\right\|_{L^2} \le C(1+t)^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{2})-\frac{k}{2}}\|\mathbf{B}_0\|_{L^p} \tag{4.37}$$

and

$$\left\|\nabla^k M^L(t)\right\|_{L^2} \le C(1+t)^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{2})-\frac{k}{2}}\|M_0\|_{L^p}. \tag{4.38}$$

Recalling that

$$\mathbf{u} = \Delta^{-1}(\nabla\operatorname{div}\mathbf{u} - \operatorname{curl}\operatorname{curl}\mathbf{u}) = -\Lambda^{-1}\nabla m + \Lambda^{-1}\operatorname{curl}M,$$

we have

$$\left\|\nabla^k\mathbf{u}^L(t)\right\|_{L^2} = \left\|\nabla^k\left(m^L,M^L\right)(t)\right\|_{L^2},$$

which, together with (4.27) and (4.38), implies that

$$\left\|\nabla^k\mathbf{u}^L(t)\right\|_{L^2} \le C(1+t)^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{2})-\frac{k}{2}}\|\mathbf{u}_0\|_{L^p}. \tag{4.39}$$

The combination of (4.27), (4.37), and (4.39) ends the proof of Proposition 4.1. □

4.3 Decay rates for the nonlinear system

Now we are in the position to derive the optimal time-decay rate of the solution of nonlinear system (2.5). Let us redefine

$$\mathbf{D}(t) := \left(\omega(t),\mathbf{u}(t),\mathbf{B}(t)\right)^T$$

and

$$\mathbf{K} = \begin{pmatrix} 0 & \operatorname{div} & 0 \\ \nabla & -\mu\Delta-(\mu+\nu)\nabla\operatorname{div} & 0 \\ 0 & 0 & -\Delta \end{pmatrix}.$$

In other words, system (2.5) can be expressed as follows:

$$\mathbf{D}_t + \mathbf{K}\mathbf{D} = \mathcal{H}(\mathbf{D}) \tag{4.40}$$

with the initial data

$$\mathbf{D}|_{t=0} = \mathbf{D}(0), \tag{4.41}$$

where $\mathcal{H}(V)$ is defined by

$$\mathcal{H}(\mathbf{D}) := (\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3)^T.$$

Thanks to Duhamel's principle and the initial data $\mathbf{K}(0)\mathbf{D}(0)$ of the solution to the linearized system of (2.5), we can express the solution of the ordinary differential equation as

$$\mathbf{D}(t) = \mathbf{K}(0)\mathbf{D}(0) + \int_0^t \mathbf{K}(t-\tau)\mathcal{H}(\mathbf{D})(\tau)\,\mathrm{d}\tau. \tag{4.42}$$

In addition, thanks to Proposition 4.1, we can obtain the following estimate of the low-frequency part of the solution to the nonlinear problem.

Lemma 4.5 *Suppose that* $1 \le p \le 2$. *Then for any integer* $k \ge 0$,

$$\begin{aligned}\left\|\nabla^k \mathbf{D}^L(t)\right\|_{L^2} &\le C_6(1+t)^{-(\frac{3}{4}+\frac{k}{2})}\left\|\mathbf{D}(0)\right\|_{L^1} \\ &\quad + C_6\int_0^{\frac{t}{2}}(1+t-\tau)^{-(\frac{3}{4}+\frac{k}{2})}\left\|\mathcal{H}(\mathbf{D})(\tau)\right\|_{L^1}\mathrm{d}\tau \\ &\le C_6\int_{\frac{t}{2}}^t(1+t-\tau)^{-\frac{k}{2}}\left\|\mathcal{H}(\mathbf{D})(\tau)\right\|_{L^2}\mathrm{d}\tau,\end{aligned} \tag{4.43}$$

where $C_6 > 0$ *is a constant.*

With Lemmas 4.1 and 4.5 in hand, we can further establish the optimal time-decay rate for the solution.

Lemma 4.6 (Optimal time-decay rates) *Under the assumptions in Theorem 1.1, we have*

$$\left\|\nabla^k(\omega, \mathbf{u}, \mathbf{B})(t)\right\|_{L^2} \le C(1+t)^{-(\frac{3}{4}+\frac{k}{2})}, \quad k = 0, 1, 2, \tag{4.44}$$

for any $t \in [0, \infty)$.

Proof We introduce the nondecreasing Lyapunov function

$$\mathcal{R}(\tau) := \sup_{0\le\tau\le t}\sum_{k=0}^{2}(1+\tau)^{\frac{3}{4}+\frac{k}{2}}\left\|\nabla^k(\omega, \mathbf{u}, \mathbf{B})(\tau)\right\|_{L^2}, \tag{4.45}$$

where, for $0 \le k \le 2$,

$$\left\|\nabla^k(\omega, \mathbf{u}, \mathbf{B})(\tau)\right\|_{L^2} \le C_7(1+\tau)^{-(\frac{3}{4}+\frac{k}{2})}\mathcal{R}(\tau), \quad 0 \le \tau \le t. \tag{4.46}$$

Here the constant $C_7 > 0$ is independent of ε_0.

From Hölder's inequality and (4.46) we have

$$\begin{aligned}\left\|\mathcal{H}(\mathbf{D})(\tau)\right\|_{L^1} &\lesssim \left\|(\omega, \mathbf{u}, \mathbf{B})\right\|_{L^2}\left\|\nabla(\omega, \mathbf{u}, \mathbf{B})\right\|_{L^2} \\ &\quad + \|\omega\|_{L^2}\left\|\nabla^2\mathbf{u}\right\|_{L^2} + \|\nabla\mathbf{u}\|_{L^2}^2 + \|\nabla\mathbf{B}\|_{L^2}^2\end{aligned}$$

$$
\begin{aligned}
&+ \|\omega\|_{L^2}\left\|\nabla^2\mathbf{B}\right\|_{L^2} \\
&\lesssim \varepsilon_0\mathcal{R}(t)(1+\tau)^{-\frac{5}{4}}.
\end{aligned}
\tag{4.47}
$$

Similarly to (4.47), we obtain

$$
\begin{aligned}
\left\|\mathcal{H}(\mathbf{D})(\tau)\right\|_{L^2} &\lesssim \left\|(\omega,\mathbf{u},\mathbf{B})\right\|_{L^3}\left\|\nabla(\omega,\mathbf{u},\mathbf{B})\right\|_{L^6} \\
&\quad + \|\omega\|_{L^\infty}\left\|\nabla^2\mathbf{u}\right\|_{L^2} + \left\|\nabla(\mathbf{u},\mathbf{B})\right\|_{L^3} + \left\|\nabla(\mathbf{u},\mathbf{B})\right\|_{L^6} \\
&\lesssim \left\|(\omega,\mathbf{u},\mathbf{B})\right\|_{H^1}\left\|\nabla^2(\omega,\mathbf{u},\mathbf{B})\right\|_{L^2} + \|\omega\|_{H^2}\left\|\nabla^2\mathbf{u}\right\|_{L^2} \\
&\quad + \left\|\nabla(\mathbf{u},\mathbf{B})\right\|_{H^1} + \left\|\nabla^2(\mathbf{u},\mathbf{B})\right\|_{L^2} \\
&\lesssim \varepsilon_0^{1-\vartheta}\mathcal{R}^{1+\vartheta}(t)(1+\tau)^{-(\frac{7}{4}+\frac{3}{4}\vartheta)},
\end{aligned}
\tag{4.48}
$$

where $\vartheta \in (0,\frac{1}{2})$ is a given constant. Thanks to (4.43) and Lemma A.6, we get

$$
\begin{aligned}
\left\|\nabla^k\mathbf{D}^L(t)\right\|_{L^2} &\leq C(1+t)^{-(\frac{3}{4}+\frac{k}{2})}\left\|\mathbf{D}(0)\right\|_{L^1} \\
&\quad + C\int_0^{\frac{t}{2}}(1+t-\tau)^{-(\frac{3}{4}+\frac{k}{2})}\varepsilon_0\mathcal{R}(\tau)(1+\tau)^{-\frac{5}{4}}\,\mathrm{d}\tau \\
&\quad + C\int_{\frac{t}{2}}^{t}(1+t-\tau)^{-\frac{k}{2}}\varepsilon_0^{1-\vartheta}\mathcal{R}^{1+\vartheta}(\tau)(1+\tau)^{-(\frac{7}{4}+\frac{3}{4}\vartheta)}\,\mathrm{d}\tau \\
&\leq C(1+t)^{-(\frac{3}{4}+\frac{k}{2})}\left(\left\|\mathbf{D}(0)\right\|_{L^1} + \varepsilon_0\mathcal{R}(\tau) + \varepsilon_0^{1-\vartheta}\mathcal{R}^{1+\vartheta}(\tau)\right),
\end{aligned}
\tag{4.49}
$$

where $0 \leq k \leq 2$. Substituting the above two inequalities into (4.1) yields

$$
\begin{aligned}
\left\|\nabla^2\mathbf{D}(t)\right\|_{L^2}^2 &\leq Ce^{-C_3t}\left\|\nabla^2\mathbf{D}(0)\right\|_{L^2}^2 \\
&\quad + C\left(\left\|\mathbf{D}(0)\right\|_{L^1}^2 + \varepsilon_0^2\mathcal{R}^2(t)\right)\int_0^t e^{-C_3(t-\tau)}(1+t)^{-\frac{7}{2}}\,\mathrm{d}\tau \\
&\quad + C\varepsilon_0^{2-2\vartheta_1}\mathcal{R}^{2+2\vartheta}(t)\int_0^t e^{-C_3(t-\tau)}(1+t)^{-\frac{7}{2}}\,\mathrm{d}\tau.
\end{aligned}
\tag{4.50}
$$

By (4.50) and Lemma A.6 we obtain

$$
\left\|\nabla^2\mathbf{D}(t)\right\|_{L^2}^2 \leq C(1+t)^{-\frac{7}{2}}\left(\left\|\mathbf{D}(0)\right\|_{H^2\cap L^1}^2 + \varepsilon_0^2\mathcal{R}^2(t) + \varepsilon_0^{2-2\vartheta}\mathcal{R}^{2+2\vartheta}(t)\right).
\tag{4.51}
$$

Using Lemma A.3 and (1.5), we have

$$
\begin{aligned}
\left\|\nabla^k\mathbf{D}(t)\right\|_{L^2}^2 &\leq C\left\|\nabla^k\mathbf{D}^L(t)\right\|_{L^2}^2 + C\left\|\nabla^k\mathbf{D}^H(t)\right\|_{L^2}^2 \\
&\leq C\left\|\nabla^k\mathbf{D}^L\right\|_{L^2}^2 + C\left\|\nabla^2\mathbf{D}\right\|_{L^2}^2.
\end{aligned}
\tag{4.52}
$$

From the above calculation we deduce that for $0 \leq k \leq 2$,

$$
\left\|\nabla^k\mathbf{D}(t)\right\|_{L^2}^2 \leq C(1+t)^{-(\frac{3}{2}+k)}\left(\left\|\mathbf{D}(0)\right\|_{H^2\cap L^1}^2 + \varepsilon_0^2\mathcal{R}^2(t) + \varepsilon_0^{2-2\vartheta}\mathcal{R}^{2+2\vartheta}(t)\right).
\tag{4.53}
$$

Then, for a sufficiently small ε_0 and a constant C_8, which is independent of ε_0, we can derive that

$$\mathcal{R}^2(t) \le \frac{C_8}{2}\Big(\big\|(\omega,\mathbf{u},\mathbf{B})(0)\big\|^2_{H^2\cap L^1} + {\varepsilon_0}^2\mathcal{R}^2(t) + {\varepsilon_0}^{2-2\vartheta}\mathcal{R}^{2+2\vartheta}(t)\Big). \tag{4.54}$$

By Young's inequality we obtain

$$C_8\varepsilon^{2-2\vartheta}\mathcal{R}^{2+2\vartheta}(t) \le \frac{1-\vartheta}{2}C_8^{\frac{2}{1-\vartheta}} + \frac{1+\vartheta}{2}{\varepsilon_0}^{\frac{4(1-\vartheta)}{1+\vartheta}}\mathcal{R}^4(t). \tag{4.55}$$

Thus we have

$$\mathcal{R}^2(t) \le \mathcal{J}_0 + C_{\varepsilon_0}\mathcal{R}^4(t), \tag{4.56}$$

where $C_{\varepsilon_0} := \frac{1+\vartheta}{2}{\varepsilon_0}^{\frac{4(1-\vartheta)}{1+\vartheta}}$ and $\mathcal{J}_0 := C_8\|(\omega,\mathbf{u},\mathbf{B})(0)\|^2_{H^2\cap L^1} + \frac{1-\vartheta}{2}C_8^{\frac{2}{1-\vartheta}}$.

Suppose $\mathcal{R}^2(t) > 2\mathcal{J}_0$ for any $t \ge t_1$ with a positive constant t_1. Since $\mathcal{R}(t) \in C^0[0,+\infty)$ and $\mathcal{R}^2(0)$ is small, we have that

$$\mathcal{R}^2(t_0) = 2\mathcal{J}_0 \tag{4.57}$$

with some $t_0 \in (0,t_1)$. By (4.56) we have $\mathcal{R}^2(t_0) \le \mathcal{J}_0 + C_{\varepsilon_0}\mathcal{R}^4(t_0)$, which implies

$$\mathcal{R}^2(t_0) \le \frac{\mathcal{J}_0}{1-C_{\varepsilon_0}\mathcal{R}^2(t_0)}. \tag{4.58}$$

Assume that the small constant ε_0 satisfies $C_{\varepsilon_0} < \frac{1}{4\mathcal{J}_0}$, which leads to $C_{\varepsilon_0}\mathcal{R}^2(t_0) < \frac{1}{2}$. This fact, together with (4.58), implies $\mathcal{R}^2(t_0) < 2\mathcal{J}_0$, which it contradicts with (4.57). Therefore we get $\mathcal{R}^2(t) \le 2\mathcal{J}_0$ for all $t \ge t_1$. Keeping in mind that $\mathcal{R}(t)$ is nondecreasing, we have $\mathcal{R}(t) \le C$ for all $t \in [0,+\infty)$. This completes this proof. □

Thanks to Lemma 4.6, we complete the proof of Theorem 1.1.

Appendix: Analytic tools

This appendix is devoted to providing some important mathematical results, which have been used in the previous sections.

Lemma A.1 ([34]) *Let $f \in H^2(\mathbb{R}^3)$. Then we have*

(i) $\|f\|_{L^p} \lesssim \|f\|_{H^1}$ *for* $2 \le p \le 6$;
(ii) $\|f\|_{L^\infty} \lesssim \|\nabla f\|^{1/2}_{L^2}\|\nabla f\|^{1/2}_{H^1} \lesssim \|\nabla f\|_{H^1}$;
(iii) $\|f\|_{L^6} \lesssim \|\nabla f\|_{L^2}$.

Lemma A.2 ([28]) *We have*

$$\big\|\nabla^k(fg)\big\|_{L^q} \lesssim \|f\|_{L^{q_1}}\big\|\nabla^k g\big\|_{L^{q_2}} + \big\|\nabla^k f\big\|_{L^{q_3}}\|g\|_{L^{q_4}} \tag{A.1}$$

for $k \ge 1$, where $1 \le q_i \le +\infty$, and

$$\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q_3} + \frac{1}{q_4}. \tag{A.2}$$

Lemma A.3 ([45]) *For any integers r, s, and t, we have*

$$\|\nabla^s f^L\|_{L^2} \le r_0^{s-r}\|\nabla^r f^L\|_{L^2}, \qquad \|\nabla^s f^H\|_{L^2} \le \frac{1}{R_0^{t-s}}\|\nabla^t f^H\|_{L^2}, \tag{A.3}$$

$$\|\nabla^s f^L\|_{L^2} \le \|\nabla^t f\|_{L^2} \quad \text{and} \quad \|\nabla^s f^H\|_{L^2} \le \|\nabla^t f\|_{L^2}, \tag{A.4}$$

where $f \in H^n(\mathbb{R}^3)$ *and* $r \le s \le t \le n$. *Moreover,*

$$r_0^s\|f^n\|_{L^2} \le \|\nabla^s f^n\|_{L^2} \le R_0^s\|f^n\|_{L^2} \tag{A.5}$$

for some constants $r_0 > 0$ *and* $R_0 > 0$.

Next, we introduce the Gagliardo–Nirenberg inequality.

Lemma A.4 ([48]) *Let* $\psi(\omega)$ *be a smooth function of* ω *with bounded derivatives of any order. If* $\|\omega\|_{L^\infty(\mathbb{R}^3)} \le 1$, *then for any integer* $j \ge 1$, *we have*

$$\|\nabla^j \psi(\omega)\|_{L^q(\mathbb{R}^3)} \lesssim \|\nabla^j \omega\|,$$

where $1 \le q \le \infty$.

Lemma A.5 ([37]) *Suppose* $0 \le i, j \le k$. *Then we have*

$$\|\nabla^i h\|_{L^p} \lesssim \|\nabla^j h\|_{L^{p_1}}^{1-\sigma}\|\nabla^k h\|_{L^{p_2}}^{\sigma},$$

where $0 \le \sigma \le 1$, *and*

$$\frac{i}{3} - \frac{1}{p} = \left(\frac{j}{3} - \frac{1}{p_1}\right)(1-\sigma) + \left(\frac{k}{3} - \frac{1}{p_2}\right)\sigma.$$

In particular, if $p = \infty$, *then* $0 < \sigma < 1$ *is required.*

For decay estimates of solutions, we further introduce the following basic inequalities.

Lemma A.6 ([52]) *Suppose* $b_1, b_2, b_3 \in \mathbb{R}^3$ *and* $b_1 > 0$, $0 \le b_1 \le b_2$, $b_3 > 0$. *Then for* $t \in \mathbb{R}_+$,

$$\int_0^t (1+t-t)^{-b_1}(1+t)^{-b_2}\,dt \le C(b_1,b_2)(1+t)^{-b_1},$$

and

$$\int_0^t (1+\tau)^{-b_1}e^{-b_3(t-\tau)}\,d\tau \le C(b_1,b_3)(1+t)^{-b_1}, \tag{A.6}$$

where $C(b_1,b_2) > 0$ *and* $C(b_1,b_3) > 0$ *are constants depending only on* b_1, b_2, b_3.

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The authors declare no competing interests.

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This work was carried out in collaboration between the three authors. Weiwei Wang designed the study and guided the research. Yuting Guo and Rui Sun performed the analysis and wrote the first draft of the manuscript. Yuting Guo, Rui Sun and Weiwei Wang managed the analysis of the study. The three authors read and approved the final manuscript.

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