https://doi.org/10.1186/s13661-022-01652-8

Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

On some qualitative results in thermodynamics of Cosserat bodies

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Abstract
This paper deals with the linear theory of thermoelastic Cosserat bodies. At the beginning, we formulate the mixed initial-boundary value problem in this context and obtain new theorems of reciprocity in the thermodynamics theory of these media. Then we prove that these new reciprocity relations imply the uniqueness of solution of the mixed problem. Based on the same reciprocal relations, we establish a minimum variational principle, which generalizes those from the theory of classical thermoelasticity.

1 Introduction

The goal of this study is formulating the mixed initial-boundary value problem within the thermodynamics theory of Cosserat bodies and obtaining some qualitative results for the solutions of the formulated problem. One of the reasons why the theory of Cosserat thermoelastic bodies captured the interest of many specialists was that this theory predicts the finite speed of heat signals, as they did most of the nonclassical theories of thermoelasticity. This theory, initiated by the French brothers E. and F. Cosserat [1], introduced a mechanics of continuous solids based on the principle that each point of the body has six degrees of freedom, just like a rigid body. Since the appearance of this theory, but especially in the last period, a lot of works have been published that highlight its advantage over the classical theory of thermoelasticity and also its practical importance. An enumeration of many studies dedicated to the theory, as well as highlighting the contributions of these works, can be discovered in the monographs of Nowacki [2] and Eringen [3]. From the long list of these studies, we list the works [4–20]. Specialists appreciate that a natural fibrous composite such as a human or animal bone has a torsional and bending behavior, which is more faithfully described by the Cosserat elasticity than by the classical elasticity. Results similar to those in this paper were obtained for classical thermoelastic media. In some situations, they were based on the Laplace transform. In other works, these results were possible due to the reformulation of the initial mixed problem so that the initial data are included into the motion and energy equations. Neither of the two procedures is used in our study. The plan of this paper is as follows. In Sect. 2, we synthesize the main equations and conditions that characterize the mixed problem in the theory of the thermodynamics of Cosserat, namely the equations of movement, energy equation, initial data, and boundary conditions. We also specify the regular conditions imposed on the functions we work with, which allow us to obtain the proposed results. In Sect. 3, we formulate and prove the main results of our study. Here we present two results of reciprocity, a uniqueness result, and a variational principle, which extends similar principles from classical thermoelasticity. See [21–26]. Some similar techniques are used in [27–30].

2 Formulation of the problem

We consider an inhomogeneous anisotropic Cosserat body that, beginning at time $t = 0$, occupies the regular domain D of the Euclidian three-dimensional space whose associated vector space is R^3. The domain D is bounded by a piecewise smooth closed surface ∂D. We use both scalar and vector and tensor functions, and these depend on points $x = (x_m)$ of D and temporal variable $t \in [0, \infty)$. A superimposed dot designates partial differentiation with respect to t, whereas the subscript m preceded by a comma designates partial differentiation with respect to the corresponding coordinate x_m. When there is no likelihood of confusion, we can omit writing the space argument and/or time argument of a function.

We will characterize the deformation of a thermoelastic Cosserat body with the help of the following independent variables:

- the displacement vector with components v_m;
- the couple displacement vector with components ϕ_m.

With the help of the independent variables v_m and ϕ_m, we can define the strain tensors with components e_{mn} and ε_{mn} by means of the following kinematic relations:

$$\begin{aligned} e_{mn} &= v_{n,m} + \epsilon_{mnk}\phi_k, \\ \varepsilon_{mn} &= \phi_{n,m}. \end{aligned} \tag{1}$$

Using a procedure similar to that of Green and Lindsay [31] and Eringen [32], we obtain the basic system of field partial differential equations in the linear thermoelasticity of Cosserat bodies (see [32]):

- the motion equations

$$\begin{aligned} \tau_{mn,n} + f_m &= \varrho \ddot{v}_m, \\ \sigma_{mn,n} + \epsilon_{mjk}\tau_{jk} + g_m &= I_{mn}\ddot{\phi}_n; \end{aligned} \tag{2}$$

- the equation of energy

$$q_{m,m} + S = \vartheta_0 \dot{\eta}; \tag{3}$$

- the constitutive relations

$$\begin{aligned} \tau_{mn} &= C_{klmn}e_{kl} + B_{klmn}\varepsilon_{kl} - a_{mn}(\vartheta + \alpha\dot{\vartheta}), \\ \sigma_{mn} &= B_{mnkl}e_{kl} + A_{klmn}\varepsilon_{kl} - b_{mn}(\vartheta + \alpha\dot{\vartheta}), \\ q_m &= \vartheta_0 K_{mn}\vartheta_{,n}, \\ \eta &= a + d\vartheta + h\dot{\vartheta} + a_{mn}e_{mn} + b_{mn}\varepsilon_{mn}. \end{aligned} \tag{4}$$

All these relations hold for $(t, x) \in [0, \infty) \times D$.

Also, in equations (2)–(4) we used the following notations: v_m – the components of displacement, ϕ_m – the components of microrotation, f_m – the body force, g_m – the body couple force, τ_{mn} – the components of the stress tensor, σ_{mn} – the components of the couple stress tensor, q_m – the components of the heat conduction vector, η – the specific entropy, ϑ_0 – the constant reference temperature, I_{mn} – the components of inertia, and ϵ_{mjk} – the alternating symbol.

A superposed dot over a function denotes the differentiation of the function with respect to time t, and a subscript preceded by a comma denotes the differentiation of the function with respect to the corresponding spatial variable.

To obtain relations (4), we used the additional assumption that the reference solid has a center of symmetry at each point; otherwise, it is anisotropic.

The thermoelastic coefficients in (4) are constants, and we have the following symmetry relations satisfied:

$$C_{klmn} = C_{mnkl}, \qquad A_{klmn} = A_{mnkl}, \qquad K_{mn} = K_{nm}. \tag{5}$$

The coefficients of inertia I_{mn} and temperature ϑ_0 are given constants satisfying the conditions

$$I_{mn} > 0, \qquad \vartheta_0 > 0. \tag{6}$$

According to the thermodynamics second law (the entropy production inequality), we obtain the following conditions:

$$d\alpha - h \geq 0, \qquad K_{mn}x_m x_n \geq 0, \quad \forall x_m. \tag{7}$$

Taking into account conditions (7), we can suppose that C_{klmn}, A_{klmn}, and K_{mn} are positive definite tensors, i.e.,

$$\begin{aligned}
&A_{klmn}x_{kl}x_{mn} \geq k_0 x_{kn}x_{kn}, \quad k_0 > 0, \forall x_{mn} = x_{nm}, \\
&C_{klmn}x_{kl}x_{mn} \geq k_1 x_{kn}x_{kn}, \quad k_1 > 0, \forall x_{mn} = x_{nm}, \\
&K_{mn}x_m x_n \geq k_2 x_m x_m, \quad k_2 > 0, \forall x_m.
\end{aligned} \tag{8}$$

Moreover, according to Green and Lindsay [31], we can assume that

$$\alpha > 0, \qquad h > 0, \qquad d\alpha - h > 0. \tag{9}$$

To construct a mixed initial-boundary value problem in our context, we adjoin the following initial data:

$$\begin{aligned}
&v_m(0,x) = v_m^0(x), \qquad \dot{v}_m(0,x) = v_m^1(x), \\
&\phi_m(0,x) = \phi_m^0(x), \qquad \dot{\phi}_m(0,x) = \phi_m^1(x), \\
&\vartheta(0,x) = \vartheta^0(x), \qquad \eta(0,x) = \eta^0(x), \quad x \in \bar{D}.
\end{aligned} \tag{10}$$

Now we intend to adjoin some boundary conditions. To this end, we must define the heat flux and surface tractions t_k and m_k in any regular point of $[0,\infty) \times \partial D$:

$$q = q_k n_k, \qquad t_k = \tau_{kl} n_l, \qquad m_k = \sigma_{kl} n_l, \tag{11}$$

where $n = (n_l)$ is the normal vector to the border ∂D oriented toward its exterior.

Also, the following share of the surface ∂D is needed:

$$\begin{aligned} &\partial D = \Sigma_1 \cup \Sigma_1^c = \Sigma_2 \cup \Sigma_2^c = \Sigma_3 \cup \Sigma_3^c, \\ &\Sigma_1 \cap \Sigma_1^c = \Sigma_2 \cap \Sigma_2^c = \Sigma_3 \cap \Sigma_3^c = \emptyset, \end{aligned}$$

where, clearly, the surfaces Σ_1^c, Σ_2^c, and Σ_3^c are the complements of the surfaces Σ_1, Σ_2, and Σ_3, respectively.

Now we can prescribe the following boundary boundary relations:

$$\begin{aligned} &v_m = \tilde{v}_m \quad \text{on } [0,\infty) \times \Sigma_1, \qquad t_m = \tilde{t}_m \quad \text{on } [0,\infty) \times \Sigma_1^c, \\ &\phi_k = \tilde{\phi}_k \quad \text{on } [0,\infty) \times \Sigma_2, \qquad m_k = \tilde{m}_k \quad \text{on } [0,\infty) \times \Sigma_2^c, \\ &\vartheta = \tilde{\vartheta} \quad \text{on } [0,\infty) \times \Sigma_3, \qquad q = \tilde{q} \quad \text{on } [0,\infty) \times \Sigma_3^c. \end{aligned} \tag{12}$$

Taking into account the constitutive equations (4), from equations (2) and (3) we obtain the following system of partial differential equations:

$$\begin{aligned} \varrho \ddot{v}_m &= C_{klmn} e_{kl,n} + B_{klmn} \varepsilon_{kl,n} + a_{mn}(\vartheta_{,n} + \alpha \dot{\vartheta}_{,n}), \\ I_{mn} \ddot{\phi}_n &= B_{mnkl} e_{kl,n} + A_{klmn} \varepsilon_{kl,n} + b_{mn}(\vartheta_{,n} + \alpha \dot{\vartheta}_{,n}) \\ &\quad + \epsilon_{mjk}\big(A_{jkln} e_{ln} + B_{jkln} \varepsilon_{ln} + a_{jk}(\vartheta + \alpha \dot{\vartheta})\big), \\ h \ddot{\vartheta} &= -d \dot{\vartheta} + a_{mn} \dot{e}_{mn} + b_{mn} \dot{\varepsilon}_{mn} + K_{mn} \vartheta_{,mn} \end{aligned} \tag{13}$$

for $(t,x) \in [0,\infty) \times D$.

We denote by $\mathcal{P}$ the mixed initial-boundary value problem in the context of thermodynamics of Cosserat bodies, consisting of equations (13), initial conditions (11), and boundary relations (12). The qualitative results that we will address next refer to the solutions of the $\mathcal{P}$ problem. The regularity conditions under which we will obtain these results on the solutions are not at all restrictive; they are in fact common in the mechanics of continuous solids.

So the given functions $\mathbf{v} = (v_m)$, $\boldsymbol{\phi} = (\phi_m)$, η^0, and ϑ^0 are continuous on the domain D, the functions $\tilde{\mathbf{t}} = (\bar{t}_m)$, $\tilde{\mathbf{m}} = (\bar{m}_k)$, $\bar{\vartheta}$, and $\bar{q}$ are prescribed and piecewise regular on their domain of definition and continuous in time and on the domain D.

As in [33], a function u is considered to be of class $C^{N,M}$ on cylinder $[0,\infty) \times D$ if u is a continuous function on $[0,\infty) \times D$ and the derivatives

$$\frac{\partial^n}{\partial x_k \partial x_l \cdots \partial x_r}\left(\frac{\partial^m u}{\partial t^m}\right), \quad n \in \{0,1,2,\dots,N\}, m \in \{0,1,2,\dots,M\},$$

are defined and continuous on $[0,\infty) \times D$. Here N and M are positive integers, and the condition $n + m \leq \max\{N, M\}$ is fulfilled. In the particular case $M = N$, we will write C^N instead of $C^{N,N}$. For details, see [33].

3 Results

Our first result is a reciprocity one. To this end, we need the convolution product for two continuous functions. So, if φ and ψ are two scalar functions on $[0,\infty) \times D$ continuous in time, then their convolution product, denoted by $*$, is defined by

$$(\varphi * \psi)(t,x) = \int_0^t \varphi(t-\tau,x)\psi(\tau,x)\,d\tau.$$

Now we introduce the functions $p(t)$ and $r(t)$, useful in the following, defined by

$$p(t) = 1, \qquad r(t) = (p * p)(t) = t, \quad t \in [0,\infty), \tag{14}$$

and we consider the following writing convention:

$$\hat{\varphi}(t,x) = \int_0^t \varphi(\tau,x)\,d\tau = (p * \varphi)(t,x). \tag{15}$$

To obtain a more accessible form of the energy equation (3), we consider the function ω defined on $[0,\infty) \times D$ by the relation

$$\omega = \hat{S} + \vartheta_0(\eta_0 - a), \tag{16}$$

where $\hat{S}$ is defined as in (15).

Also, for a function u of class $C^{0,1}$ on $[0,\infty) \times D$, we define the functions β and γ by

$$\beta u = u + \alpha u, \qquad \gamma u = p * u + \alpha u, \tag{17}$$

where α is from Eqs. (4).

In the following proposition, we formulate the energy equation from (3) in a different manner.

Proposition 1 *If the functions $q_m \in C^{1,0}$ and $\eta \in C^{0,1}$ satisfy the equation of energy (3) and the initial condition $\eta(0,x) = \eta^0(x)$, $x \in D$, then they satisfy the equation*

$$\hat{q}_{m,m} + \omega = \vartheta_0(\eta - a) \quad \forall (t,x) \in [0,\infty) \times D. \tag{18}$$

The reciprocal statement is also true.

Proof Both statements are easy to prove; just have to take into account the writing convention (15). □

A reciprocity relation refers to the connection between two external data systems

$$\mathcal{D}^{(\nu)} = \{f_m^{(\nu)}, g_m^{(\nu)}, S^{(\nu)}, \bar{v}_m^{(\nu)}, \bar{\phi}_m^{(\nu)}, \bar{t}_m^{(\nu)}, \bar{m}_k^{(\nu)}, \bar{\vartheta}^{(\nu)}, \bar{q}^{(\nu)}, v_m^{0,(\nu)}, v_m^{1,(\nu)}, \phi_m^{0,(\nu)}, \phi_m^{1,(\nu)}, \eta^{0,(\nu)}, \vartheta^{0,(\nu)}\},$$

and the solutions corresponding to these data systems

$$\mathcal{S}^{(\nu)} = \{v_m^{(\nu)}, \phi_m^{(\nu)}, \vartheta^{(\nu)}, \tau_{mn}^{(\nu)}, \sigma_{mn}^{(\nu)}, \eta(\nu), q_m^{(\nu)}\}.$$

In both systems, $\nu = 1, 2$.

To simplify the writing of reciprocity relations, we need the following notations:

$$\begin{aligned}
\Gamma_{\nu\mu}(s,r) = {} & \int_{\partial D}\left[t_m^{(\nu)}(s,x)v_m^{(\mu)}(r,x) + m_k^{(\nu)}(s,x)\phi_m^{(\mu)}(r,x) - \frac{1}{\vartheta_0}\beta\bar{q}_m^{(\nu)}(s,x)\vartheta^{(\mu)}(r,x)\right]dA \\
& + \int_D\left[f_m^{(\nu)}(s,x)v_m^{(\mu)}(r,x) + g_m^{(\nu)}(s,x)\phi_m^{(\mu)}(r,x) - \frac{1}{\vartheta_0}\beta\omega^{(\nu)}(s,x)\vartheta^{(\mu)}(r,x)\right]dV \\
& + \int_D\big[\varrho\ddot{v}_m^{(\nu)}(s,x)v_m^{(\mu)}(r,x) + I_{mn}\ddot{\phi}_m^{(\nu)}(s,x)\phi_n^{(\mu)}(r,x) \\
& - h\dot{\vartheta}^{(\nu)}(s,x)\vartheta^{(\mu)}(r,x) - \alpha d\vartheta^{(\nu)}(s,x)\dot{\vartheta}^{(\mu)}(r,x)\big]\,dV \\
& + \frac{1}{\vartheta_0}\int_D \beta\bar{q}_m^{(\nu)}(s,x)\vartheta_{,m}^{(\mu)}(r,x)\,dV, \quad \nu,\mu = 1,2,
\end{aligned} \tag{19}$$

where we used the following convention:

$$\begin{aligned}
& t_m^{(\nu)} = \tau_{mk}^{(\nu)}n_k, \qquad m_k^{(\nu)} = \sigma_{kl}^{(\nu)}n_l, \\
& \omega^{(\nu)} = \bar{S}^{(\nu)} + \vartheta_0\big(\eta^{0,(\nu)} - a\big), \qquad q^{(\nu)} = q_k^{(\nu)}n_k.
\end{aligned} \tag{20}$$

Also, we introduce the following notations:

$$\begin{aligned}
J_{\nu\mu}(s,r) = {} & \tau_{mn}^{(\nu)}(s)v_{m,n}^{(\mu)}(r) + \sigma_{mn}^{(\nu)}(s)\phi_{m,n}^{(\mu)}(r) \\
& - \beta\big[\eta^{(\nu)}(s) - a\big]\vartheta^{(\mu)}(r), \quad \nu,\mu = 1,2,
\end{aligned} \tag{21}$$

and

$$I_{\nu\mu}(s,r) = J_{\nu\mu}(s,r) + h\dot{\vartheta}^{(\nu)}(s)\vartheta^{(\mu)}(r) + \alpha\, d\vartheta^{(\nu)}(s)\dot{\vartheta}^{(\mu)}(r), \quad \nu,\mu = 1,2, \tag{22}$$

where we suppressed the variable x.

Now we formulate and prove the first reciprocity result.

Theorem 1 *If the symmetry relations* (5) *are satisfied, then for any* $s,r \in [0,\infty)$, *we have the equality*

$$\Gamma_{\nu\mu}(s,r) = \Gamma_{\mu\nu}(r,s), \quad \nu,\mu = 1,2. \tag{23}$$

Proof Taking into account the constitutive relations (4) and notations (21), we deduce

$$\begin{aligned}
J_{\nu\mu}(s,r) = {} & C_{klmn}v_{m,n}^{(\nu)}(s)v_{k,l}^{(\mu)}(r) \\
& + B_{klmn}\big[v_{m,n}^{(\nu)}(s)\phi_{k,l}^{(\mu)}(r) + v_{m,n}^{(\mu)}(s)\phi_{k,l}^{(\nu)}(r)\big] + A_{klmn}\phi_{m,n}^{(\nu)}(s)\phi_{k,l}^{(\nu)}(r) \\
& - a_{mn}\big[\beta v_{m,n}^{(\nu)}(s)\vartheta^{(\mu)}(r) + \beta v_{m,n}^{(\mu)}(s)\vartheta^{(\nu)}(r)\big] \\
& - b_{mn}\big[\beta\phi_{m,n}^{(\nu)}(s)\vartheta^{(\mu)}(r) + \beta\phi_{m,n}^{(\mu)}(s)\vartheta^{(\nu)}(r)\big] \\
& - \alpha h\dot{\vartheta}^{(\nu)}(s)\dot{\vartheta}^{(\mu)}(r) - d\vartheta^{(\nu)}(s)\vartheta^{(\mu)}(r) \\
& - h\dot{\vartheta}^{(\nu)}(s)\vartheta^{(\mu)}(r) - \alpha\, d\vartheta^{(\nu)}(s)\dot{\vartheta}^{(\mu)}(r).
\end{aligned} \tag{24}$$

Based on relations (5), (24), and (22), we easily deduce that

$$I_{\nu\mu}(s,r) = I_{\mu\nu}(r,s). \tag{25}$$

Now we consider equations (2) and (18) and notation (21), and so we are led to the relation

$$\begin{aligned} J_{\nu\mu}(s,r) = {} & \left[\tau_{mn}^{(\nu)}(s)v_m^{(\mu)}(r) + \sigma_{mn}^{(\nu)}(s)\phi_m^{(\mu)}(r) - \frac{1}{\vartheta_0}\beta\bar{q}_n^{(\nu)}(s)\vartheta^{(\mu)}(r)\right]_{,n} \\ & + f_m^{(\nu)}(s)v_m^{(\mu)}(r) + g_m^{(\nu)}(s)\phi_m^{(\mu)}(r) - \frac{1}{\vartheta_0}\beta\omega^{(\nu)}(s)\vartheta^{(\mu)}(r) \\ & - \varrho\ddot{v}_m^{(\nu)}(s)v_m^{(\mu)}(r) - I_{mn}\ddot{\phi}_m^{(\nu)}(s)\phi_m^{(\mu)}(r) + \frac{1}{\vartheta_0}\beta\bar{q}_m^{(\nu)}(s)\vartheta_{,m}^{(\mu)}(r). \end{aligned} \tag{26}$$

Now we integrate over D equality (22) and then use relation (25) and the theorem of divergence, so that we are led to the equality

$$\int_D I_{\nu\mu}(s,r)\,dV = \Gamma_{\nu\mu}(r,s),$$

which, together with (25), ensures equality (23). The proof of the theorem is complete. □

To obtain another reciprocity result, we introduce the notations

$$\begin{aligned} & F_m^{(\nu)} = r * \left[f_m^{(\nu)}(s) + g_m^{(\nu)}(s)\right] + \varrho\left[t\dot{v}_m^{1,(\nu)} + v_m^{0,(\nu)}\right], \\ & G_m^{(\nu)} = r * \left[g_m^{(\nu)}(s) + g_m^{(\nu)}(s)\right] + \varrho\left[t\dot{v}_m^{1,(\nu)} + v_m^{0,(\nu)}\right], \\ & R^{(\nu)} = -t\vartheta^{0,(\nu)}, \quad \nu = 1,2. \end{aligned} \tag{27}$$

With the help of Theorem 1 and notations (27), we obtain a new reciprocity result.

Theorem 2 *If the symmetry relations* (5) *are satisfied and* $\mathcal{S}^{(\nu)}$ *is the solution corresponding to the external data system* $\mathcal{S}^{(\nu)}$, $\nu = 1,2$, *then we have the following equality*:

$$\begin{aligned} & \int_{\partial D} r * \left[t_m^{(1)}(s) * v_m^{(2)} + m_k^{(1)} * \phi_k^{(2)} - \frac{1}{\vartheta_0}q^{(1)} * \gamma\vartheta^{(2)}\right] dA \\ & \quad + \int_D \left[F_m^{(1)} * v_m^{(2)} + G_m^{(1)} * \phi_m^{(2)} - \frac{1}{\vartheta_0}p * \vartheta^{(2)} * \gamma\omega^{(1)}\right] dV \\ & \quad - \frac{\alpha}{\vartheta_0}\int_{\partial D} p * q^{(1)} * R^{(2)}\,dA - \frac{\alpha}{\vartheta_0}\int_D \omega^{(1)} * R^{(2)}\,dV \\ & \quad + \int_D \left[(h-\alpha d)R^{(1)} * \vartheta^{(2)} + \alpha p * K_{mn}\vartheta_{,n}^{(1)} * R_{,m}^{(2)}\right] dV \\ & = \int_{\partial D} r * \left[t_m^{(2)}(s) * v_m^{(1)} + m_k^{(2)} * \phi_k^{(1)} - \frac{1}{\vartheta_0}q^{(2)} * \gamma\vartheta^{(1)}\right] dA \\ & \quad + \int_D \left[F_m^{(2)} * v_m^{(1)} + G_m^{(2)} * \phi_m^{(1)} - \frac{1}{\vartheta_0}p * \vartheta^{(1)} * \gamma\omega^{(2)}\right] dV \end{aligned} \tag{28}$$

$$
\begin{aligned}
&-\frac{\alpha}{\vartheta_0}\int_{\partial D} p * q^{(2)} * R^{(1)}\,dA - \frac{\alpha}{\vartheta_0}\int_D \omega^{(2)} * R^{(1)}\,dV \\
&+\int_D \bigl[(h-\alpha d)R^{(2)} * \vartheta^{(1)} + \alpha p * K_{mn}\vartheta^{(2)}_{,n} * R^{(1)}_{,m}\bigr]\,dV.
\end{aligned}
$$

Proof We use the reciprocity relation (23) in which we replace r with τ and s with $t-\tau$, and then integrate the relation obtained over the interval $[0,t]$, so that we obtain the relation

$$
\begin{aligned}
F_{\nu\mu}(t) = {} & \int_{\partial D}\biggl[t^{(\nu)}_m(s) * v^{(\mu)}_m + m^{(\nu)}_k * \phi^{(\mu)}_k - \frac{1}{\vartheta_0} p * q^{(\nu)} * \beta\vartheta^{(\mu)}\biggr]\,dA \\
&+\int_D \biggl[f^{(\nu)}_m * v^{(\mu)}_m + g^{(\nu)}_m * \phi^{(\mu)}_m - \frac{1}{\vartheta_0}\omega^{(\nu)} * \beta\vartheta^{(\mu)}\biggr]\,dV \\
&-\int_D \bigl[\varrho \ddot{v}^{(\nu)}_m * v^{(\mu)}_m + I_{mn}\ddot{\phi}^{(\nu)}_m * \phi^{(\mu)}_n - h\dot{\vartheta}^{(\nu)} * \vartheta^{(\mu)} - \alpha d\vartheta^{(\nu)} * \dot{\vartheta}^{(\mu)}\bigr]\,dV \\
&+\frac{1}{\vartheta_0}\int_D p * q^{(\nu)}_m * \beta\vartheta^{(\mu)}_{,m}\,dV,
\end{aligned}
\tag{29}
$$

where

$$
F_{\nu\mu}(t) = \int_0^t \Gamma_{\nu\mu}(s,t-s)\,ds.
$$

It is no difficult to verify that for arbitrary continuous functions r and p on $[0,\infty)\times D$, we have the following equalities:

$$
\begin{aligned}
& r * \dot{\vartheta}^{(\nu)} * \vartheta^{(\mu)} = p * \bigl(p * \dot{\vartheta}^{(\nu)}\bigr) * \vartheta^{(\mu)} = p * \bigl(\vartheta^{(\nu)} - \vartheta^{(\nu)}_0\bigr) * \vartheta^{(\mu)} \\
&\qquad\qquad = p * \vartheta^{(\nu)} * \vartheta^{(\mu)} + R^{(\nu)} * \vartheta^{(\mu)}, \\
& r * \beta\vartheta^{(\mu)} = p * \bigl(\gamma\vartheta^{(\mu)} - \alpha\vartheta^{(\mu)}_0\bigr) = p * \gamma\vartheta^{(\mu)} + \alpha R^{(\mu)}, \\
& r * \ddot{v}^{(\nu)}_m = v^{(\nu)}_m - t v^{1,(\nu)}_m - v^{0,(\nu)}_m, \\
& r * \ddot{\phi}^{(\nu)}_m = \phi^{(\nu)}_m - t\phi^{1,(\nu)}_m - \phi^{0,(\nu)}_m, \\
& l * \gamma h = l * (p * h + \alpha h) = h * \gamma l.
\end{aligned}
\tag{30}
$$

Based on (30) and (29), we deduce

$$
\begin{aligned}
r * F_{\nu\mu}(t) = {} & \int_{\partial D} r * \biggl[t^{(\nu)}_m(s) * v^{(\mu)}_m + m^{(\nu)}_k * \phi^{(\mu)}_k - \frac{1}{\vartheta_0} q^{(\nu)} * \gamma\vartheta^{(\mu)}\biggr]\,dA \\
&+\int_D \biggl[F^{(\nu)}_m * v^{(\mu)}_m + G^{(\nu)}_m * \phi^{(\mu)}_m - \frac{1}{\vartheta_0} p * \vartheta^{(\mu)} * \gamma\omega^{(\nu)}\biggr]\,dV \\
&-\int_D \bigl[\varrho v^{(\nu)}_m * v^{(\mu)}_m + I_{mn}\phi^{(\nu)}_m * \phi^{(\mu)}_n - (h+\alpha d)p * \vartheta^{(\nu)} * \vartheta^{(\mu)}\bigr]\,dV \\
&+\int_D r * K_{mn}\vartheta^{(\nu)}_{,n} * \bigl(p * \vartheta^{(\mu)}_{,m} + \alpha\vartheta^{(\mu)}_{,m}\bigr)\,dV \\
&-\frac{\alpha}{\vartheta_0}\int_{\partial D} p * q^{(\nu)} * R^{(\mu)}\,dA - \frac{\alpha}{\vartheta_0}\int_D \omega^{(\nu)} * R^{(\mu)}\,dA \\
&+\int_D \bigl[hR^{(\nu)} * \vartheta^{(\mu)} + \alpha\, dR^{(\mu)} * \vartheta^{(\nu)} + \alpha p * K_{mn}\vartheta^{(\nu)}_{,n} * R^{(\mu)}_{,m}\bigr]\,dV.
\end{aligned}
\tag{31}
$$

Finally, from relations (5), (23), (29), and (31) we obtain the desired relation (28). □

To simplify the relations that follow, we will use the notation

$$\begin{aligned} G(s,r) = {}& \int_D \left[f_m(s)v_m(r) + g_m(s)\phi_m(r) - \frac{1}{\vartheta_0}\omega(s)\beta\vartheta(r) \right] dV \\ &+ \int_{\partial D} \left[t_m(s)v_m(r) + m_k(s)\phi_m(r) - \frac{1}{\vartheta_0}\bar{q}(s)\beta\vartheta(r) \right] dA, \quad s,r \in [0,\infty). \end{aligned} \tag{32}$$

The reciprocity relation (23) is also the basis of the following theorem.

Theorem 3 *If the symmetry relations* (5) *are satisfied and* $\mathcal{S}^{(\nu)}$ *is the solution corresponding to the external data system* $\mathcal{S}^{(\nu)}$, $\nu = 1, 2$, *then we have the following equality*:

$$\begin{aligned} &\frac{d}{dt}\left\{ \int_D [\varrho v_m v_m + I_{mn}\phi_m\phi_n + \alpha K_{mn}\bar{\vartheta}_{,m}\bar{\vartheta}_{,n}]\, dV \right\} \\ &\quad + \frac{d}{dt}\left\{ \int_0^t \int_D \left[(\alpha d - h)\vartheta^2 + K_{mn}\bar{\vartheta}_{,n}\bar{\vartheta}_{,m} \right] dV\, ds \right\} \\ &= \int_0^t \left[G(t-\tau, t+\tau) - G(t+\tau, t-\tau) \right] d\tau \\ &\quad + \int_D \left[\varrho\left(\dot{v}_m(2t)v_m(0) + \dot{v}_m(0)v_m(2t) \right] + I_{mn}\left[\dot{\phi}_m(2t)\phi_m(0) + \dot{\phi}_m(0)\phi_m(2t) \right) \right] dV \\ &\quad + \int_D \left[(\alpha d - h)\vartheta(0)\vartheta(2t) + \alpha K_{mn}\bar{\vartheta}_{,n}(2t)\vartheta_{,m}(0) \right] dV. \end{aligned} \tag{33}$$

Proof By equality (23) we obtain

$$\int_0^t \Gamma_{11}(t+\tau, t-\tau)\, d\tau = \int_0^t \Gamma_{11}(t-\tau, t+\tau)\, d\tau, \tag{34}$$

which by relations (23) and (32) results in

$$\begin{aligned} &\int_0^t \Gamma_{11}(t+\tau, t-\tau)\, d\tau \\ &\quad = \int_0^t G(t+\tau, t-\tau)\, d\tau \\ &\qquad - \int_0^t \int_D \left[\varrho\ddot{v}_m(t+\tau)v_m(t-\tau) + I_{mn}\ddot{\phi}_m(t+\tau)\phi_m(t-\tau) \right] dV\, d\tau \\ &\qquad + \int_0^t \int_D \left[h\dot{\vartheta}(t+\tau)\vartheta(t-\tau) + \alpha d\vartheta(t+\tau)\dot{\vartheta}(t-\tau) \right] dV\, d\tau \\ &\qquad + \int_0^t \int_D K_{mn}\bar{\vartheta}_{,n}(t+s)\left[\bar{\vartheta}_{,m}(t-\tau) + \alpha\ddot{\bar{\vartheta}}_{,m}(t-\tau) \right] dV\, d\tau. \end{aligned} \tag{35}$$

Analogously, we deduce

$$\begin{aligned} &\int_0^t \Gamma_{11}(t-\tau, t+\tau)\, d\tau \\ &\quad = \int_0^t G(t-\tau, t+\tau)\, d\tau \end{aligned}$$

$$\begin{aligned}
&-\int_0^t\int_D\big[\varrho\ddot{v}_m(t-\tau)v_m(t+\tau)+I_{mn}\ddot{\phi}_m(t-\tau)\phi_m(t+\tau)\big]\,dV\,d\tau \\
&+\int_0^t\int_D\big[h\dot{\vartheta}(t-\tau)\vartheta(t+\tau)+\alpha\,d\vartheta(t-\tau)\dot{\vartheta}(t+\tau)\big]\,dV\,d\tau \\
&+\int_0^t\int_D K_{mn}\bar{\vartheta}_{,n}(t-s)\big[\dot{\bar{\vartheta}}_{,m}(t+\tau)+\alpha\ddot{\bar{\vartheta}}_{,m}(t+\tau)\big]\,dV\,d\tau.
\end{aligned}\tag{36}$$

After usual integration by parts, we deduce the following formulas:

$$\begin{aligned}
&\int_0^t\ddot{u}(t+\tau)v(t-\tau)\,d\tau=\dot{u}(2t)v(0)-\dot{u}(t)v(t)+\int_0^t\dot{v}((t-\tau)\dot{u}(t+\tau)\,d\tau, \\
&\int_0^t\ddot{v}(t-\tau)u(t+\tau)\,d\tau=\dot{v}(t)u(t)-\dot{v}(0)v(2t)+\int_0^t\dot{v}((t-\tau)\dot{u}(t+\tau)\,d\tau, \\
&\int_0^t u(t+\tau)v(t-\tau)\,d\tau=-v(0)u(2t)+u(t)v(t)+\int_0^t\dot{u}((t+\tau)u(t-\tau)\,d\tau.
\end{aligned}$$

Taking these formulas into account, from (35) we obtain

$$\begin{aligned}
&\int_0^t\Gamma_{11}(t+\tau,t-\tau)\,d\tau \\
&\quad=\int_0^t G(t+\tau,t-\tau)\,d\tau \\
&\qquad-\int_D\big[\varrho\big(\dot{v}_m(2t)v_m(0)-\dot{v}_m(t)v_m(t)\big)+I_{mn}\big(\dot{\phi}_m(2t)\phi_n(0)-\dot{\phi}_m(t)\phi_n(t)\big)\big]\,dV\,d\tau \\
&\qquad-\int_0^t\int_D\big[h\big(\vartheta(2t)\vartheta(0)-\vartheta^2(t)\big)+\alpha d\big(\vartheta^2(2t)-\vartheta(0)\vartheta(2t)\big)\big]\,dV\,d\tau \\
&\qquad-\int_0^t\int_D\big[\varrho\dot{v}_m(t-\tau)\dot{v}_m(t+\tau)+I_{mn}\dot{\phi}_m(t-\tau)\dot{\phi}_n(t+\tau)\big]\,dV\,d\tau \qquad (37)\\
&\qquad+\int_0^t\int_D\big[h\vartheta(t+\tau)\dot{\vartheta}(t-\tau)+\alpha d\dot{\vartheta}(t+\tau)\dot{\vartheta}(t-\tau)\big]\,dV\,d\tau \\
&\qquad+\int_0^t\int_D K_{mn}\big[\dot{\bar{\vartheta}}_{,n}(t+\tau)\bar{\vartheta}_{,m}(t-\tau)+\alpha\vartheta_{,n}(t+\tau)\vartheta_{,m}(t-\tau)\big]\,dV\,d\tau \\
&\qquad+\int_D K_{mn}\big[\bar{\vartheta}_{,n}(2t)(\bar{\vartheta}_{,m}+\alpha\dot{\bar{\vartheta}}_{,m})(0)-\alpha\bar{\vartheta}_{,n}(2t)\vartheta_{,m}(0)\big]\,dV
\end{aligned}$$

Similarly, from (36) we have

$$\begin{aligned}
&\int_0^t\Gamma_{11}(t-\tau,t+\tau)\,d\tau \\
&\quad=\int_0^t G(t-\tau,t+\tau)\,d\tau \\
&\qquad-\int_D\big[\varrho\big(\dot{v}_m(t)v_m(t)-\dot{v}_m(0)v_m(2t)\big)+I_{mn}\big(\dot{\phi}_m(t)\phi_n(t)-\dot{\phi}_m(0)\phi_n(2t)\big)\big]\,dV\,d\tau \\
&\qquad-\int_0^t\int_D\big[\varrho\dot{v}_m(t-\tau)\dot{v}_m(t+\tau)+I_{mn}\dot{\phi}_m(t-\tau)\phi_n(t+\tau)\big]\,dV\,d\tau \\
&\qquad-\int_0^t\int_D\big[h\dot{\vartheta}(t-\tau)\vartheta(t+\tau)+\alpha d\vartheta(t-\tau)\dot{\vartheta}(t+\tau)\big]\,dV\,d\tau \qquad (38)
\end{aligned}$$

$$+\int_0^t\int_D K_{mn}\bar{\vartheta}_{,n}(t-\tau)\dot{\bar{\vartheta}}_{,m}(t+\tau)\,dV\,d\tau-\int_D \alpha K_{mn}\dot{\bar{\vartheta}}_{,m}\bar{\vartheta}_{,m}\,dV$$
$$+\int_0^t\int_D \alpha K_{mn}\vartheta_{,m}(t+\tau)\vartheta_{,n}(t-\tau)\,dV\,ds.$$

Clearly, taking into account relations (5), (34), (37), and (38), we arrive at the desired relation (33). □

Now we can approach the problem of the uniqueness of the solution of problem $\mathcal{P}$.

Theorem 4 *We suppose that*:
- *the symmetry relations* (5) *are satisfied*;
- ϱ *and* $\alpha d - h$ *are strictly positive*;
- *the tensor* K_{mn} *is positive semidefinite*;
- $\alpha \geq 0$.

Then the mixed problem $\mathcal{P}$ *admits at most one solution*.

Proof We do the proof by reduction to the absurd. Suppose that problem $\mathcal{P}$ admits two solutions $(v_m^{(1)}, \phi_m^{(1)}, \vartheta^{(1)})$ and $(v_m^{(2)}, \phi_m^{(2)}, \vartheta^{(2)})$. Due to the linearity of problem $\mathcal{P}$, the difference between the two solutions

$$(v_m, \phi_m, \vartheta) = \left(v_m^{(1)} - v_m^{(2)}, \phi_m^{(1)} - \phi_m^{(2)}, \vartheta^{(1)} - \vartheta^{(2)}\right)$$

is also the solution of problem $\mathcal{P}$, but corresponding to zero initial data and homogeneous boundary conditions. In this situation, equality (33) reduces to

$$\int_D [\varrho v_m v_m + I_{mn}\phi_m\phi_n + \alpha K_{mn}\bar{\vartheta}_{,m}\bar{\vartheta}_{,n}]\,dV$$
$$+\int_0^t\int_D \left[(\alpha d - h)\vartheta^2 + K_{mn}\bar{\vartheta}_{,n}\bar{\vartheta}_{,m}\right]dV\,ds = 0, \quad t \in [0,\infty). \tag{39}$$

Based on the assumptions of the theorem, we have $\varrho > 0$, $\alpha d - h > 0$, and $\alpha \geq 0$, and the tensor K_{mn} is positive semidefinite, so that inequality (39) implies

$$v_m = 0, \qquad \phi_m = 0, \qquad \vartheta = 0,$$

which completes the proof of the theorem. □

As the last our main result, we approach a variational principle for a Cosserat thermoelastic body. In fact, we extend Lebon's principle, formulated in the theory of classical thermoelasticity.

To this aim, we first consider a so-called thermoelastic state of the media with the content

$$\mathcal{A} = (v_m, \phi_m, \vartheta, e_{mn}, \varepsilon_{mn}, \tau_{mn}, \sigma_{mn}, q_m, S),$$

and denote by H the set of states of the media of this form. If we consider the usual addition and multiplication of states with scalars, then we deduce that H is endowed with a structure of linear space.

Now we introduce the functional $\mathcal{F}$ on H by

$$\begin{aligned}\mathcal{F}(t,\mathcal{A}) = \int_D p * \Big[& C_{klmn}e_{kl} * e_{mn} + B_{mnkl}e_{kl} * \varepsilon_{mn} + A_{klmn}\varepsilon_{kl} * \varepsilon_{mn} \\ & - \varrho v_m * v_m + I_{mn}\phi_m * \phi_n - \frac{1}{\vartheta_0} r * K_{mn}\vartheta_{,m} * \vartheta_{,n} - r * q_m * \vartheta_{,m} \\ & + [\varrho v_m - r * \tau_{mn,n} - F_m] * v_m + (I_{mn}\phi_m - p * \sigma_{mn,n} - \epsilon_{mjk}\tau_{jk} - G_m) * \phi_m \\ & + \frac{\vartheta_0}{a} r * (S - a_{mn}e_{mn} - b_{mn}\varepsilon_{mn}) * (S - a_{kl}e_{kl} - b_{kl}\varepsilon_{kl}) \\ & - r * (\tau_{mn} * e_{mn} + \sigma_{mn} * \varepsilon_{mn}) - (p * S + r * q_{m,m} - R) * \vartheta \Big] dV \qquad (40) \\ & + \int_{\Sigma_1} r * t_m * \tilde{v}_m \, dA + \int_{\Sigma_1^c} r * (t_m - \tilde{t}_m) * v_m \, dA \\ & + \int_{\Sigma_2} r * m_k * \tilde{\phi}_k \, dA + \int_{\Sigma_2^c} r * (m_k - \tilde{m}_k) * \phi_k \, dA \\ & + \int_{\Sigma_3} r * q * \tilde{\vartheta} \, dA + \int_{\Sigma_3^c} r * (q - \tilde{q}) * \vartheta \, dA, \quad t \in [0,\infty),\end{aligned}$$

for any $\mathcal{A} = (v_m, \phi_m, \vartheta, e_{mn}, \varepsilon_{mn}, \tau_{mn}, \sigma_{mn}, q_m, S) \in H$.

Theorem 5 *Assume that the symmetry relations* (5) *are satisfied, $a \neq 0$ in the domain D, and the thermoelastic state $\mathcal{A}$ is a solution of the mixed $\mathcal{P}$. Then the variation of the functional $\mathcal{A}$ is zero, i.e.,*

$$\delta\mathcal{F}(t,\mathcal{A}) = 0, \quad t \in [0,\infty). \qquad (41)$$

Proof Let us consider two arbitrary thermoelastic states of the body

$$\begin{aligned}\mathcal{A} &= (v_m, \phi_m, \vartheta, e_{mn}, \varepsilon_{mn}, \tau_{mn}, \sigma_{mn}, q_m, S), \\ \bar{\mathcal{A}} &= (\bar{v}_m, \bar{\phi}_m, \bar{\vartheta}, \bar{e}_{mn}, \bar{\varepsilon}_{mn}, \bar{\tau}_{mn}, \bar{\sigma}_{mn}, \bar{q}_m, \bar{S}).\end{aligned}$$

Since the space H is linear, we can conclude that $\mathcal{A} + \lambda\bar{\mathcal{A}} \in H$ for all λ, which is a real parameter.

It is no difficult to observe that

$$\begin{aligned}\delta_{\bar{\mathcal{A}}}\mathcal{F}(t,\mathcal{A}) = \int_D \Big\{ & r * \big[(C_{klmn}e_{mn} + B_{mnkl}\varepsilon_{mn}) * \bar{e}_{kl} \\ & + (B_{klmn}e_{mn} + A_{klmn}\varepsilon_{mn}) * \bar{\varepsilon}_{kl}\big] \\ & - r * \frac{T_0}{a}(a_{mn} * \bar{e}_{mn} + b_{mn} * \bar{\varepsilon}_{mn})(\varrho S - a_{mn}e_{mn} - b_{mn}\varepsilon_{mn}) \\ & - r * (\tau_{mn} * \bar{e}_{mn} + \sigma_{mn} * \bar{\varepsilon}_{mn}) - r * \left(q_m + \frac{1}{T_0}K_{mn}\vartheta_{,n}\right) * \bar{\vartheta}_{,m} \\ & + p * \left[-T + \frac{T_0}{a} p * (\varrho S - a_{mn}e_{mn} - b_{mn}\varepsilon_{mn})\right] * \varrho\bar{S}\Big\} dV \\ & + \int_D \big\{[\varrho v_m - r * \tau_{mn,n} - F_m] * \bar{v}_m\end{aligned}$$

$$
\begin{aligned}
&+ (I_{mn}\phi_n - p * \epsilon_{mjk}\tau_{jk} - G_m) * \bar{\phi}_m \\
&- (p * \varrho S + r * q_{m,m} - R) * \bar{\vartheta} \Big\} \, dV \\
&- \int_D \Big\{ h * \Big[e_{mn} - \frac{1}{2}(u_{n,m} + u_{m,n}) \Big] * \bar{\tau}_{mn} \\
&+ r * (\varepsilon_{mn} - u_{n,m} + \phi_m) * \bar{\sigma}_{mn} + r * (\vartheta_{,m} - \vartheta_{,m}) * \bar{q}_m \Big\} \, dV \\
&+ \int_{\Sigma_1} r * (\bar{v}_m - v_m) * \bar{t}_m \, dA + \int_{\Sigma_1^c} r * (t_m - \bar{t}_m) * \bar{v}_m \, dA \\
&+ \int_{S_2} r * m_k * \bar{\phi}_k \, dA + \int_{S_2^c} r * (m_k - \bar{m}_k) * \phi_k \, dA \\
&+ \int_{S_3} r * (\bar{\vartheta} - \vartheta) * \bar{q} \, dA + \int_{S_3^c} r * (q - \bar{q}) * \bar{\vartheta} \, dA, \quad \forall t \in [0, \infty).
\end{aligned}
\tag{42}
$$

Taking into account the basic equations (13), relations (27), (33), and (34) and the boundary relations (12), from (42) we deduce that

$$
\delta_{\bar{\mathcal{A}}} \mathcal{F}(t, \mathcal{A}) = 0 \quad \forall \bar{\mathcal{A}} \in H, \tag{43}
$$

whence it follows that

$$
\delta \mathcal{F}(t, \mathcal{A}) = 0, \quad t \geq 0,
$$

which completes the proof of Theorem 5. □

Remark It is not difficult to show that the statement of Theorem 5 is also valid reciprocally (see Gurtin [34]). In other words, if identity (41) is true, then the state $\mathcal{A}$ for which this identity holds is the unique solution of our problem. The idea of the proof, which is also suggested by Gurtin [34], is based on a particular choice of the thermoelastic state $\bar{\mathcal{A}}$. In our case the thermoelastic state proposed by Lebon [21] can be successfully used.

4 Conclusions

We approached the linear theory of thermoelastic Cosserat bodies. First, we formulate the mixed initial-boundary value problem in this context and obtain new theorems of reciprocity in the thermodynamics theory of these media. Then we prove that these new reciprocity relations imply the uniqueness of solution of the mixed problem. Based on the same reciprocal relations, we establish a minimum variational principle, which generalizes those from the theory of classical thermoelasticity. It is important to emphasize that even if in the context of thermoelastic Cosserat media the basic equations of motion and energy are more complicated, most of the important results of the mechanics of continuum media remain valid.

Acknowledgements

The authors thank the anonymous reviewers who have carefully read the manuscript and have proposed pertinent corrections that have led to an improvement of our paper.

Funding

Not applicable.

Availability of data and materials
Not applicable.

Declarations

Competing interests
The authors declare no competing interests.

Author contributions
M.M., I.M.F. and S.V. wrote the main manuscript text and M.M. supervised all calculations. No tables, no figures in our manuscript

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Publisher's Note

Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 26 August 2022 Accepted: 15 September 2022 Published online: 26 September 2022

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